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H4.SMR 539/10

Coarsening Techniques

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$$(a u_x)_x + (c u_y)_y = F$$

Special cases:

Anisotropic $\quad a \gg c, \; a \ll c$

Variable coefficients:

Smooth

Highly oscillatory

Strongly discontinuous

(orders-of-magnitude jumps).

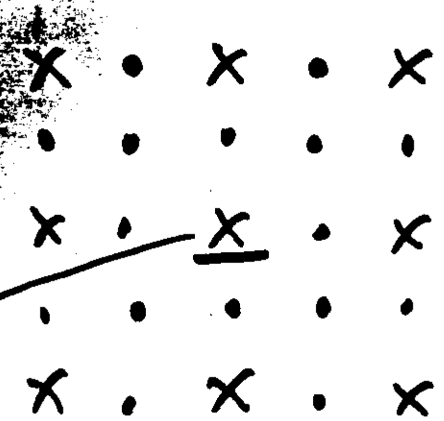
Convection added $\quad + q u_x + p u_y$

Nonlinearities $\quad a, c \ldots, \; F = F(x, y, u, u_x, \ldots)$

Mixed terms $\quad + (b u_x)_y$

Indefinite $\quad + r u; \; a, c, r > 0$

Weighting Residuals: $R^h = F^h - L_h u^h$



$$\begin{matrix} \omega_2 & \omega_1 & \omega_2 \\ \omega_1 & \underline{\omega_0} & \omega_1 \\ \omega_2 & \omega_1 & \omega_2 \end{matrix}$$

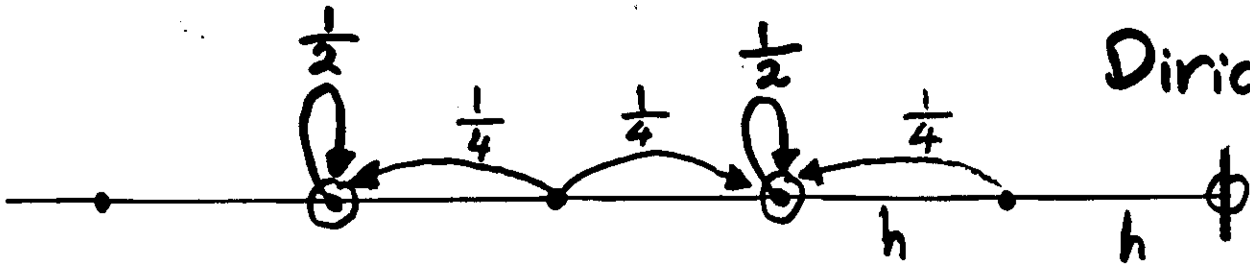
$$\begin{aligned} I_h^{2h} R_{ij}^h = \; & \omega_0 R_{ij} \\ & + \omega_1 \left(R_{i+1,j} + R_{i,j+1} + R_{i-1,j} + R_{i,j-1} \right) \\ & + \omega_2 \left(R_{i+1,j+1} + R_{i+1,j-1} + R_{i-1,j+1} + R_{i-1,j-1} \right) \end{aligned}$$

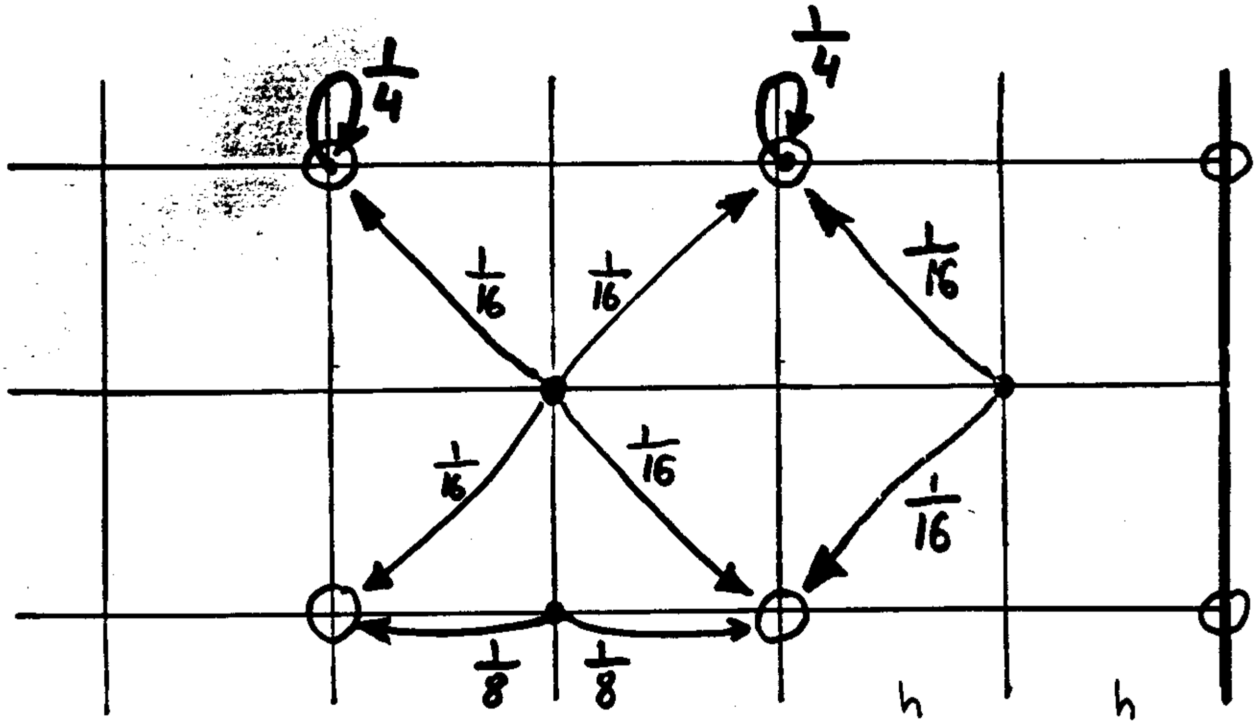
Injection: $\omega_0 = 1$, $\omega_1 = \omega_2 = 0$

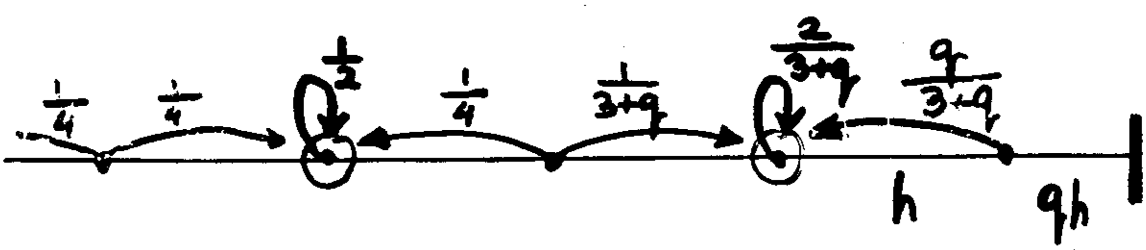
Full weighting: $\omega_0 = \frac{1}{4}$, $\omega_1 = \frac{1}{8}$, $\omega_2 = \frac{1}{16}$

B.C.

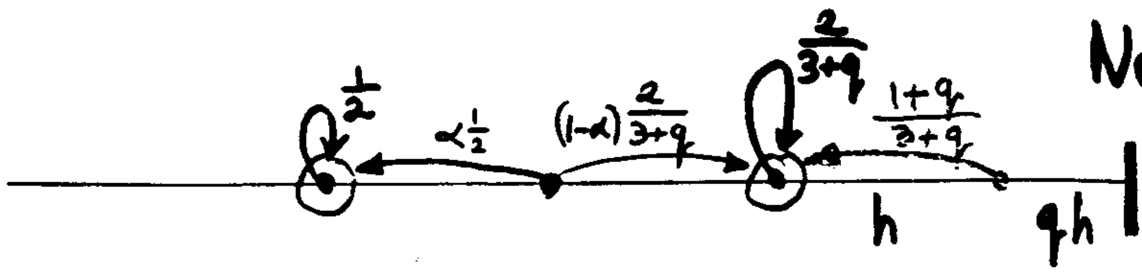
Dirichlet







Neumann



FULL RESIDUAL WEIGHTING

$$L^h V^h = R^h \equiv F^h - L^h U^h$$

$$L^{2h} V^{2h} = I_h^{2h} R^h$$

L^h comparable to L^{2h}
(e.g., both in divided form)

$\Rightarrow$

$$\int I_h^{2h} R^h \, dx\,dy = \int R^h \, dx\,dy$$

for any function R^h

Variational Coarsening

$AU = b$ on grid h, A symm. pos. def.

$\Updownarrow$

$\frac{1}{2} U^T A U - U^T b = \min!$

$I_{2h}^h V^{2h}$ coarse-grid correction to u: $U \approx u + I_{2h}^h V^{2h}$

$\Downarrow$

minimize

$\frac{1}{2}(u + I_{2h}^h V^{2h})^T A (u + I_{2h}^h V^{2h}) - (u + I_{2h}^h V^{2h})^T b$

$\frac{1}{2} V^{2hT} \underline{I_{2h}^{hT} A I_{2h}^h} V^{2h} - V^{2hT} \underline{I_{2h}^{hT}(b - Au)}$

$\Updownarrow$

$A^{2h} V^{2h} = R^{2h}$ where

$A^{2h} = I_{2h}^{hT} A I_{2h}^h$, $R^{2h} = I_{2h}^{hT}(b - Au)$

, Symmetric, Conservative Full Weighting

$$(aU_x)_x + (bU_y)_y = F$$

$$\bar{a}_{i-1/2} \qquad \bar{a}_{i+1/2}$$

$$U_{i-1} \qquad U_i \qquad U_{i+1}$$

$$\frac{\bar{a}_{i+1/2}\left(U_{i+1}-U_i\right)-\bar{a}_{i-\frac{1}{2}}\left(U_i-U_{i-1}\right)}{h^2}$$

Symmetric matrix.

Conservation form better approximat. (flux is smoother)

Strongly discontinuous coeff.

$$L^h u \equiv (a u_x)_x + (b u_y)_y = f(x, y, u)$$

$\varepsilon < a(x,y,u),\ b(x,y,u) < \frac{1}{\varepsilon}$ discontinuously

Alcouffe
Brandt
Dendy
Painter

Interpolation: Based on relaxed error V. $LV \approx 0$.
Linear interpolation when u_x continuous
but here: continuity of the flux ~~au_x~~ aV_x

$$I_{2h}^h V^{2h}{}_i = \frac{\bar{a}_{i-1/2} V^{2h}_{i-1} + \bar{a}_{i+1/2} V^{2h}_{i+1}}{\bar{a}_{i-1/2} + \bar{a}_{i+1/2}}$$

$\bar{a}$ properly averaged

Coarse-grid problem: $\left(I_h^{2h} L^h I_{2h}^h\right) V^{2h} = I_h^{2h} R^h, \quad I_h^{2h} = I_{2h}^{h\ T}$

$$\Leftrightarrow \min \mathcal{F}\left(u^h + I_{2h}^h V^{2h}\right), \qquad \mathcal{F}(w^h) = \left(L^h w^h, w^h\right) - 2\left(f^h, w^h\right)$$

$\Downarrow$

Black-Box MG: for any symmetric 9-point L^h. Dendy

Algebraic MG (AMG): for general positive-type symmetric
Brandt McCormick Ruge

Systems: e.g.
$$L^h P_n - \alpha(P_n - P_w) = f$$
$$L^h P_w - \alpha(P_w - P_n) = g$$