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THEORY OF ACTIVE MODE LOCKING FOR A HOMOGENEOUS LINE

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$-\frac{\langle N_1\rangle}{\tau_r}$ (B.20a)

$-\frac{\langle N_2\rangle}{\tau_r}$ (B.20b)

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Theory of Active Mode Locking for a Homogeneous Line

We have seen in Section 5.4.5 that, in mode-locked operation, a single ultrashort light pulse is produced that travels back and forth in the laser cavity. According to the discussion already presented in Section 5.4.5.1, a theory of cw mode locking can be developed, in the time domain, by requiring that the pulse reproduce itself after each cavity round trip. We will limit our discussion here to the case of a homogeneous line, since, in this case, the problem can be handled in a simple and elegant way.[1] We consider the laser configuration of Fig. C.1 and assume that the electric field of the light pulse before entering the amplifier, $E_1(t)$, can be described by a generalized Gaussian form, i.e., [see also (5.118)],

$$E_1(t) = E_0 \exp[-\alpha t^2 + i(\omega_0 t + \beta t^2)] \tag{C.1}$$

where ω_0 is the carrier frequency and α and β, respectively, describe the time behavior of the field amplitude and frequency. To be more precise, the pulse intensity has a time width (FWHM) given by

$$\tau_p = [(2\ln 2)/\alpha]^{1/2} \tag{C.2}$$

while its frequency (which increases linearly with time) is given by $\omega_0 + \beta t$.

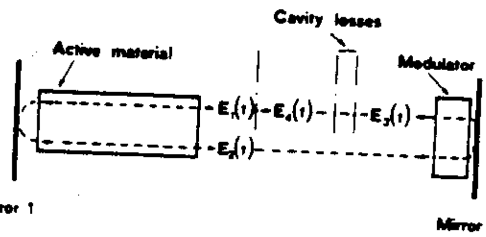

FIG. C.1. Experimental arrangement considered for the theoretical analysis of active mode locking.

We will also assume that the pulse width τ_p is much smaller than $2l/c$, where l is the length of the active material so that, when the pulse is traveling in the active material, there is no overlap with its own reflection from mirror 1. Note that (C.1) can be expressed more conveniently as

$$E_1(t) = E_0 \exp(-\Gamma t^2) \exp(i\omega_0 t) \tag{C.3}$$

where

$$\Gamma = \alpha - i\beta \tag{C.4}$$

is the complex Gaussian pulse parameter. In the analysis that follows we require the pulse to maintain the generalized Gaussian shape of (C.3) while traveling through the active material and through the modulator. We will therefore make various simplifying assumptions to ensure that this is so.

After these preliminary remarks, let us begin by considering AM mode locking. We let $g(\omega)$ be the amplitude (i.e., electric field) gain per pass in the active material under saturation conditions. With the assumption that the upper level decay time is much longer than the cavity round-trip time, it can be shown[2] that

$$g(\omega) = [\exp - i(\omega l/c)] \exp\{(g_0/2)/[1 + 2i(\omega - \omega_0)/\Delta\omega_0]\} \tag{C.5}$$

where l is the length of the active material, g_0 is the single pass saturated power gain at the center frequency of the transition, ω_0, and $\Delta\omega_0$ is the width (FWHM) of the laser line. Note that, according to (C.5), the power gain $G(\omega)$ is given by

$$G(\omega) = |g(\omega)|^2 = \exp(g) \tag{C.6}$$

where the gain g is given by

$$g = g_0/\{1 + [2(\omega - \omega_0)/\Delta\omega_0]^2\} \tag{C.7}$$

i.e., it shows the expected Lorentzian shape of a homogeneous line. Now, since the time behavior of the electric-field $E_1(t)$ of the laser pulse is Gaussian, its Fourier transform $E_1(\omega)$ will also be Gaussian and given by

$$E_1(\omega) = \frac{E_0}{2}\left(\frac{1}{\pi\Gamma}\right)^{1/2} \exp\left[-\frac{(\omega - \omega_0)^2}{4\Gamma}\right] \tag{C.8}$$

After traveling through the active material, the Fourier transform of the pulse will then be $E_1(\omega)g(\omega)$. For this quantity to be a Gaussian function we require $g(\omega)$ to be a Gaussian function. We therefore expand the expression appearing as the argument of the second exponential function in (C.5) as a power series of $(\omega - \omega_0)$. This gives

$$\begin{aligned} g(\omega) &= \exp(-i\{(\omega l/c) + [g_0(\omega - \omega_0)/\Delta\omega_0]\}) \\ &\times \exp(g_0/2)\{1 - [2(\omega - \omega_0)/\Delta\omega_0]^2\} \end{aligned} \tag{C.9}$$

The imaginary terms in the first exponential function correspond to a phase term $\phi = \phi(\omega)$, which gives the time delay experienced by the pulse after traveling in the active material (due to the finite group velocity of the pulse; see Section 8.5) as

$$\tau_d = -\frac{d\phi}{d\omega} = \frac{l}{c} + \frac{g_0}{\Delta\omega_0} \tag{C.10}$$

Note that this delay is not simply l/c because the gain line gives an additional finite contribution to the refractive index of the medium. This contribution must be taken into account when considering the requirement that the round-trip pulse propagation time be set equal to the period of the loss modulation. For simplicity we will not consider the effect of this delay any further. We therefore ignore the phase term in (C.9) and write

$$g(\omega) = \exp(g_0/2)\{1 - [2(\omega - \omega_0)/\Delta\omega_0]^2\} \tag{C.11}$$

We also ignore the fact that there is a finite value of the reflectivity of mirror 1, since this will be taken into account in the overall cavity losses. After passing once more through the active material the light pulse will experience another gain factor $g(\omega)$ as given by (C.11). The electric field $E_2(\omega)$ after one round trip in the laser material is then

$$\begin{aligned} E_2(\omega) &= E_1(\omega)[g(\omega)]^2 \\ &= \left[\frac{E_0}{2}\left(\frac{1}{\pi\Gamma}\right)^{1/2} \exp(g_0 l)\right] \exp\left[-\frac{(\omega - \omega_0)^2}{4\Gamma'}\right] \end{aligned} \tag{C.12}$$

where, according to (C.8) and (C.11), Γ' is such that

$$\frac{1}{\Gamma'} = \frac{1}{\Gamma} + \frac{16 g_0}{\Delta\omega_0^2} \tag{C.13}$$

The corresponding electric field in the time domain, $E_2(t)$, can be obtained by taking the Fourier transform of (C.12). We get

$$E_2(t) = E_0 \left(\frac{\Gamma'}{\Gamma}\right)^{1/2} \exp(g_0 l) \exp(-\Gamma' t^2 + i\omega_0 t) \tag{C.14}$$

which is again a Gaussian function.

It must be noted that the approximate expressions (C.9) and (C.11) hold provided the spectral width of the light pulse is much less than the width $\Delta\omega_0$ of the gain curve. The following analysis is, therefore, valid only if

$$(\tau_p \Delta\omega_0) \gg 1 \tag{C.15}$$

Within the same approximation the change in pulsewidth as a result of the pulse passing through the laser material is very small. We then have $\Gamma' \cong \Gamma$ and, accordingly, (C.13) can be approximated to

$$\Gamma' = \Gamma - 16 \frac{g_0}{\Delta\omega_0^2} \Gamma^2 \tag{C.16}$$

Also within the same approximation, (C.14) can be approximated to

$$E_2(t) = E_0 \exp(g_0) \exp(-\Gamma' t^2 + i\omega_0 t) \tag{C.17}$$

Note that, from (C.16) one finds that $\mathrm{Re}(\Gamma') < \mathrm{Re}(\Gamma)$, where Re stands for real part. We then see that, according to (C.4) and (C.2), the pulse broadens after passing through the amplifier.

Let us now consider the passage of the pulse through the modulator. We assume the modulator placed as near as possible to mirror 2 and also that the modulator length is much smaller than the length, $c\tau_p$, of the pulse. Neglecting the finite reflectivity of mirror 2 we then consider the effect produced by a double pass of the pulse through the modulator. If we let the double pass modulator loss be $\gamma_m(t)$ we can write

$$\gamma_m = \delta(1 - \cos \omega_m t) = 2\delta \sin^2(\omega_m t/2) \tag{C.18}$$

where 2δ is the maximum loss introduced by the modulator and ω_m, the modulator frequency, is assumed to be such that the modulator period is equal to the round trip time of the light pulse in the laser cavity. For small losses the modulator transmission T_m can then be written as

$$T_m = 1 - \gamma_m \simeq \exp(-\gamma_m) = \exp[-2\delta \sin^2(\omega_m t/2)] \tag{C.19}$$

It was shown in Section 5.4.5.1 that the pulse passes through the modulator when the loss is zero (i.e., at time $t = 0$). Since the pulsewidth is also assumed to be much smaller than the cavity round trip time (i.e., $\tau_p\omega_m \ll 1$), then (C.19) can be approximated as

$$T_m \simeq \exp(-\delta\omega_m^2 t^2/2) \tag{C.20}$$

which is now a Gaussian function. After passing through the modulator the pulse $E_3(t)$ is given by

$$E_3(t) = E_2(t) T_m(t) \tag{C.21}$$

From (C.21), (C.20), and (C.17) we then get

$$E_3(t) = E_0 \exp(g_0) \exp[-\Gamma'' t^2 + i\omega_0 t] \tag{C.22}$$

where

$$\Gamma'' = \Gamma' + \frac{\delta\omega_m^2}{2} \tag{C.23}$$

Note that, since $\mathrm{Re}(\Gamma'') > \mathrm{Re}(\Gamma)$, the pulse is narrowed when passing through the modulator.

To account for the fixed cavity losses arising from finite mirror reflectivities and internal losses, the pulse $E_4(t)$ after one round trip is written as

$$E_4(t) = [\exp(-\gamma)]E_3(t) \tag{C.24}$$

where γ, the logarithmic power loss per pass, is given by (5.8). We now set the self-consistency condition $E_4(t) = E_1(t)$. From (C.24), (C.22), and (C.3) we immediately get

$$g_0 = \gamma \tag{C.25a}$$

and

$$\Gamma'' = \Gamma \tag{C.25b}$$

From (C.25b), with the help of (C.23) and (C.16), we get

$$\frac{16 g_0}{\Delta\omega_0^2}\Gamma^2 = \frac{\delta\omega_m^2}{2} \tag{C.26}$$

This shows that the pulse broadening in the amplifier must be balanced by the pulse narrowing in the modulator. Equation (C.26) also shows that Γ is, in this case, a real quantity so that, according to (C.4), we have

$$\beta = -I_m(\Gamma) = 0 \tag{C.27a}$$

$$\alpha = \mathrm{Re}(\Gamma) = \left(\frac{\delta}{2g_0}\right)^{1/2}\frac{\omega_m \Delta\omega_0}{4} \tag{C.27b}$$

Equations (C.25a) and (C.27) together provide a complete solution of our problem. Note that (C.25a) shows that, within the approximation $\Gamma' \cong \Gamma$, the threshold for mode-locked operation corresponds to the cw saturated gain g_0 being equal to the cavity losses. Note that, according to (C.27a), the pulse has no frequency sweep. Equation (C.27b) with the help of (C.2) gives the pulse duration. Putting $\nu_m = \omega_m/2\pi$ and $\Delta\nu_0 = \Delta\omega_0/2\pi$ we then get

$$\tau_p = \left(\frac{2\sqrt{2}\ln 2}{\pi^2}\right)^{1/2}\left(\frac{g_0}{\delta}\right)^{1/4}\left(\frac{1}{\nu_m \Delta\nu_0}\right)^{1/2} \tag{C.28}$$

We note that the first factor on the right-hand side of (C.28) is approximately equal to 0.45. The second factor, as a result of the 1/4th power, is approximately equal to unity. From (C.28) we then obtain the result

$$\tau_p \cong 0.45/(\nu_m \Delta\nu_0)^{1/2} \tag{C.29}$$

i.e., (5.122).

The case of FM mode locking can be treated in a similar way. We again assume the electric field of the pulse and the amplitude gain to be given by (C.3) and (C.11), respectively. The modulator now produces a time-varying phase shift $\Delta\phi$. For sinusoidal modulation we put

$$\Delta\phi = \delta \cos(\omega_m t) \tag{C.30}$$

In this case a self-consistent solution is obtained only when the pulse passes through the modulator while the phase shift $\Delta\phi$ is either maximum or minimum (i.e., it is stationary). We will therefore assume the pulse to pass through the modulator when $t = 0$. The modulator transmission can then be written as

$$T_m = \exp(i\Delta\phi) \cong C \exp[-i\delta(\omega_m t)^2/2] \tag{C.31}$$

where $C = \exp(i\delta)$. Since T_m now has the form of a Gaussian function, the pulse, after the modulator, will again be given by (C.22) where now

$$\Gamma'' = \Gamma' + \frac{i\delta\omega_m^2}{2} \tag{C.32}$$

Using (C.24) again together with the condition $E_4(t) = E_1(t)$ we find in this case that

$$\alpha \cong \beta = \left(\frac{\delta}{2g_0}\right)^{1/2} \left(\frac{\omega_m \Delta\omega_0}{4}\right) \tag{C.33a}$$

$$g_0 = \gamma \tag{C.33b}$$

A comparison of (C.33a) with (C.27b) shows that, for the same value of $(\delta/2g_0)$, i.e., for the same values of δ and, according to (C.33b), the same cavity loss γ, the quantity α and hence the pulse width τ_p is the same for both AM and FM mode locking. For the latter case, however, since β is nonzero, the pulse frequency shows a linear sweep.

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Velocity of light
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Stefan-Boltzmann