INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.545/2

WORKSHOP ON MATHEMATICAL PHYSICS AND GEOMETRY

(4 - 15 March 1991)

# Generalities on $A/G$, An Introduction to K. Uhlenbeck's Theorems

T.R. Ramadas
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay - 400 005
India

These are preliminary lecture notes, intended only for distribution to participants

## Generalities on $\mathcal{A}/\mathcal{G}$, An Introduction to K.Uhlenbeck's Theorems.

T. R. RAMADAS

School of Mathematics, TIFR, Homi Bhabha Road, Bombay 400 005, India.

### §1: Prerequisites.

In this section we recall definitions and set up notation, and close by previewing the remaining material.

A quick introduction to the relevant material is [N]. *For the present we only consider smooth "objects". In particular all connections and gauge transformations will be smooth until further notice.*

### §1.a Connections, gauge transformations.

Let $M$ be a compact connected oriented manifold of dimension $n$, $G$ a compact Lie group, $P$ a principal $G$-bundle on $M$. For example $M$ could be $S^3$ in which case $P \sim M \times G$, or $M = S^4$ and $G = SU(2)$ in which case $P$ is classified up to equivalence by its second chern number. Denote by $\pi$ the projection $P \to M$. Given $g \in G$ denote by $R_g$ the corresponding map $P \to P$, and for $x \in P$, let $x.g \equiv R_g(x)$.

Recall the following equivalent definitions of a *connection*:

(1) We have on $P$ an exact sequence of $G$-vector bundles: $0 \to T_\pi P \to TP \to \pi^* TM \to 0$ where $T_\pi P$ is the "tangent bundle along the fibres". A connection is $G$-equivariant splitting of this sequence.

(2) A connection is a equivariant *Lie* $G$-valued 1-form $A$ on $P$ satisfying $A(\sigma(X)) = X$ for $X \in Lie\ G$. Here "equivariant" means $R_g^* A = ad_{g^{-1}} A$ and $\sigma(X)$ denotes the vector field on $P$ given by the differential of the $G$-action.

Exercise 1.1: Check the equivalence of the two definitions above. Prove that connections exist.

Recall the following equivalent definitions of a *gauge transformation*:

(1) A gauge transformation is a map $s : P \to G$ satisfying

$$s(x.g) = g^{-1} s(x) g. \tag{1-1}$$

(2) A gauge transformation is an automorphism $\phi : P \to P$ of principal bundles that is trivial on the base. That $\phi$ is an automorphism means that $\phi(x.g) = \phi(x)g$; "trivial on the base" means that $\phi$ leaves each fibre of $\pi$ invariant.

The correspondence $s \leftrightarrow \phi_s$ is given by $\phi_s(x) = x.s(x)$. Note that gauge transformations form a group.

Exercise 1.2: Check the equivalence of the two definitions above. Note that $\phi_{s_1 s_2} = \phi_{s_1} \circ \phi_{s_2}$. Define $Ad\ P$ to be the bundle associated to the adjoint action of $G$ on itself; check that a gauge transformation can be regarded as a section of $Ad\ P$.

(Recall that given a left action $(q, g) \mapsto g.q$ of $G$ on a space $Q$, the *associated bundle with fibre* $Q$ is defined to be the space $P \times Q$ modulo the equivalence relation $(x, q) \sim (x.g, g^{-1}q)$. The adjoint action of $G$ on itself is given by $(h, g) \mapsto gh \equiv Ad_g h = ghg^{-1}$.)

Action of gauge transformations on connections.

We have the following formula:

$$(\phi^* A)(x) = ad_{s^{-1}(x)} A(x) + s^{-1}(x) ds(x) \tag{1-2}$$

where $s^{-1}(x)ds(x)$ denotes the (left-invariant) Maurer-Cartan form on $G$ pulled back to $P$ *via* the map $s$. (If $G$ is a matrix group this expression for the Maurer-Cartan form can be interpreted literally; note also that in this case $ad_{s^{-1}(x)} A(x) = s^{-1}(x) A(x) s(x)$.)

PROOF: Write $\phi$ as the composite $P \xrightarrow{I \times s} P \times G \xrightarrow{R} P$ where $I$ is the identity map of $P$ and $R$ is the (right) action of $G$.

Lectures at the Workshop on Mathematical Physics and Geometry, Trieste, March 1991

### §1.b Exterior algebra of *Lie* $G$-valued forms.

Let $N$ be a manifold (In our case it will be either $M$ or $P$). Let $\Lambda_N^p \otimes Lie\ G$ be the bundle of $p$-forms on $N$ with values in $Lie\ G$. We have then

(1) a multiplication

$$(\Lambda_N^p \otimes Lie\ G) \times (\Lambda_N^q \otimes Lie\ G) \\ \to \Lambda_N^{p+q} \otimes Lie\ G,$$

and

(2) a differential $\Lambda_N^p \otimes Lie\ G \to \Lambda_N^{p+1} \otimes Lie\ G$.

If $A, B, C$ are forms of degree $p$, $q$, $r$ respectively and we denote by $[.,.]$ the multiplication we have

(1) $[A, B] = (-1)^{pq}[B, A]$,
(2) $(-1)^{pr}[A, [B, C]] + (\text{cyclic permutations}) = 0$, and
(3) $d[A, B] = [dA, B] + (-1)^p[A, dB]$.

Given a connection $A$ on $P$ we let $F_A$ denote its ***curvature two-form*** $dA + \frac{1}{2}[A, A]$.

Exercise 1.3: We recall the definition of the product $[.,.]$. If $A = \sum_I A_I\omega^I$ and $B = \sum_J B_J\omega^J$, where $\{\omega^I\}_I$ and $\{\omega^J\}_J$ are bases for $p$-forms and $q$-forms respectively and the $A_I$ and $B_J$ are elements of $Lie\ G$, then

$$[A, B] = \sum_{I,J}[A_I, B_J]\omega^I \wedge \omega^J.$$

(a) Check (1), (2) and (3) above.
(b) Let $\omega$ be the Maurer-Cartan form on $G$. Check that $d\omega + \frac{1}{2}[\omega, \omega] = 0$.
(c) For any connection $A$ on $P$ check the *Bianchi identity*: $dF_A + [A, F_A] = 0$.
(d) Given a gauge transformation $s$, check that $F_{A^s} = Ad_{s^{-1}}F_A$.
(e) (Local expression for $F_A$:) If in a co-ordinate chart $\{x_i\}$ we write $A = \sum_i A_i dx^i$, then $F_A = \frac{1}{2}\sum_{i,j} F_{A,ji}dx^j dx^i$. where

$$F_{A,ji} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} + [A_j, A_i].$$

### §1.c Forms on $M$ with values in associated bundles; covariant differentiation.

If $T : G \to End\ V$ is a representation of $G$ we let $V_T$ denote the associated vector bundle. We have a natural isomorphism between sections of $\Lambda_M^p \otimes V_T$ and $G$-equivariant horizontal $p$-forms on $P$. (Equivariance of such a form $\alpha$ means that $R_g^*\alpha = T(g^{-1})\alpha$ and horizontal means that the interior product with any vertical vector is zero. Such forms are also called *basic*.)

We let $ad\ P$ denote the vector bundle associated to the adjoint representation of $G$. Note that if $E$ is any associated vector bundle, we have a natural map $ad\ P \to End\ E$.

Fix a connection $A$. Given a basic $V$-valued p-form $\alpha$ its *covariant differrential with respect to $A$* is defined to be $d_A\alpha = H \circ d\alpha$ where $H$ is the projection operator on the dual of the "horizontal bundle".

Exercise 1.4:
(a) Check that $d_A\alpha = d\alpha + A \wedge_T \alpha$, where $\wedge_T$ is defined using the representation $T$.
(b) Check that $d_A^2\chi = [F, \chi]$. (*Ricci's identity*).

### §1.d Transition functions.

By local triviality of $P$ and compactness of $M$ we can find a finite family $(U_\mu, \{y_{i,\mu}\}, \{\tau_\mu\}, D_\mu)$ where the $U_\mu$ are open subsets of $M$ and the $\tau_\mu$ are sections of $P$ over the $U_\mu$, and we assume for later purposes that

(1) for each fixed $\mu$ the $\{y_{i,\mu}\}$ are a set of co-ordinates on the $U_\mu$.
(2) for each $\mu$, $D_\mu$ is a relatively compact open subset of $U_\mu$ with smooth boundary such that the $D_\mu$ form an open cover of $M$.

It is easily seen that there exist, for every $(\mu,\nu)$, $G$-valued functions $g_{\mu\nu}$ on $U_\mu \cap U_\nu$ defined by $\tau_\nu(y) = \tau_\mu(y)g_{\mu\nu}(y)$, $y \in U_\mu \cap U_\nu$.

Exercise 1.5:

(a) The transition functions satisfy $g_{\mu\nu}(y)g_{\nu\rho}(y)g_{\rho\mu}(y) = 1$, $y \in U_\mu \cap U_\nu$.

(b) A section of a vector bundle $V_T$ associated to $P$ *via* a representation $T : G \to End\ V$ is given by functions $\sigma_\mu$ on each $U_\mu$ satisfying $\sigma_\mu = T(g_{\mu\nu})\sigma_\nu$.

(c) A gauge transformation is given by functions $\{s_\mu\}$ satisfying $s_\nu = g_{\mu\nu}s_\mu g_{\mu\nu}^{-1}$.

(d) A connection is given by *Lie* $G$-valued 1-forms $\{A_\mu\}$ satisfying $A_\nu = ad_{g_{\mu\nu}^{-1}}A_\mu + g_{\mu\nu}^{-1}dg_{\mu\nu}$.

### §1.e Irreducible connections.

The space $\mathcal{A}$ of connections on $P$ is an affine space modelled on $\Gamma(\Lambda^1_M \otimes Lie\ G)$. In particular it is contractible (though we haven't yet endowed it with a toplogy — this we will do soon). Note, however, that the group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}$: $A \underset{\mathcal{R}_s}{\mapsto} A^s$ where

$$A^s = (\phi_s^* A)(x) = Ad_{s^{-1}(x)}A(x) + s^{-1}(x)ds(x) \tag{1-3}$$

It is easily checked that this is a right action.

Note that the center of $G$, which we denote by $Z(G)$, is a subgroup of $\mathcal{G}$: if $s : P \to Z(G)$ is a constant map, it obviously satisfies the condition (1-1). Note also that for such $s$ we have $A^s = A$. Thus $Z(G)$ acts trivially on $\mathcal{A}$. Which are the connections with nontrivial automorphisms ? The answer is given by

LEMMA 1.6. *Given $A \in \mathcal{A}$, the isotropy group at $A$ is isomorphic to the centraliser of the holonomy group of $A$ at $x \in P$.*

Recall that the holonomy group of $A$ at $x$, which we will denote $H(A,x)$, is defined as the subgroup of $G$: $\{g \in G \mid \exists$ a path $x : [0,1] \to P$ with $x(0) = x$, $x(1) = x.g$ and $x^*A = 0.\}$. Holonomy groups at different points are conjugate in $G$.

PROOF: Exercise.

Definition 1.7: A connections $A$ such that the centraliser of $H(A,x)$ is $Z(G)$ is said to be *irreducible*.

Remark 1.8: This definition will not work for nonsmooth connections, with which we will eventually have to deal. At that point we will use another characterisation.

Exercise 1.9: Describe the set of reducible connections on: (1) a principal $SU(2)$ bundle on $S^4$, and (2) a principal $U(2)$ bundle on a two-dimensional manifold $M$.

### §1.f Preview of remaining material.

The group $\mathcal{G}/Z(G)$ acts freely on the set $\hat{\mathcal{A}}^p_{k-1}$ of irreducible connections. We will prove in §2 that under suitable hypotheses this yields an infinite-dimensional principal bundle over an infinite-dimensional manifold $\mathcal{M}$.

One of the lessons of recent years is that a lot of information about the topological and differential structure of $M$ is encoded in sub-manifolds of $\mathcal{M}$. These submanifolds are finite-dimensional; they are defined as solution-spaces of (gauge-invariant) nonlinear partial differential equations involving connections. An essential tool to study these moduli spaces are some theorems due to K. Uhlenbeck [U]. In §3 we give an introduction to this paper.

## §2: Preliminaries.

We will need the apparatus of Sobolev spaces. The reason is that we would like to deal with Banach manifolds where the theory is quite analogous to the case of finite-dimensional manifolds — in particular the inverse function theorem is valid [L].

### §2.a Preliminaries on Sobolev Spaces.

For a quick treatment of this topic see [F]. All the results we need (and more) are summarised in [P, Chapter 9]

Let $p \geq 1$ be a real number, $k \geq 0$ be an integer. Let $D$ be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\bar{D}$, $\bar{D}$ the closure of $D$. We define the Sobolev space $L^p_k(D)$ as the normed linear space

$$L^p_k(D) = \{f \in L^p(D) \mid D^s f \in L^p(D)\ \forall\ multi-index\ s\ such\ that\ |s| \leq k\}.$$

By a multi-index $s$ we mean an n-tuple $\{s_1, \ldots, s_n\}$ of nonnegative integers; the differential in the above definition is in the sense of distributions.

$L^p_k$ is a Banach space. We define two closed subspaces:

(1) $L^p_k(\bar{D})$ = closure of $C^\infty(\bar{D})$ in $L^p_k(D)$.
(2) $L^p_{k,0}(D)$ = closure of $\mathcal{D}(D)$ (the space of $C^\infty$ functions with compact support) in $L^p_k(D)$.

Since $D$ has smooth boundary in fact $L^p_k(D) = L^p_k(\bar{D})$. These definitions can be extended to nonintegral nonnegative $k$ (and in fact more generally, with care over boundary conditions.)

We have then the Sobolev embedding

THEOREM 2.1.
(A) $L^p_k(\bar{D}) \subset L^q_l(\bar{D})$ *if* $k - n/p \geq l - n/q$ *and* $k \geq l$, *and*
(B) $L^p_k(\bar{D}) \subset C^l(\bar{D})$ *if* $k - n/p \geq l$.

We also have the Rellich

LEMMA 2.2. *The embedding in* (A) *above is compact if the strict inequality* $k - n/p > l - n/q$ *holds. Similarly the embedding in* (B) *is compact if* $k - n/p > l$.

Recall that a bounded linear map $T : V \to W$ of Banach spaces is said to be *compact* if it takes bounded sets to precompact sets, or, equivalently if for every bounded sequence $\{v_n\} \subset V$, the sequence $\{Tv_n\}$ has a convergent subsequence.

We illustrate the part (B) of the Theorem in an elementary situation. Let $D$ be a bounded open interval $(a, b)$ in $\mathcal{R}$ — then $L^2_1 \subset C^0$. We will prove $L^2_{1,0} \subset C^0$; from this the more general result can be deduced. It is clearly enough to bound the sup norm of any function $f \in \mathcal{D}(D)$ in terms of its $L^2_1$ norm. We have $f(x) = \int_a^x f'(y)dy$ which yields by Cauchy-Schwartz $sup\ |f| \leq (b-a)^{\frac{1}{2}}(\int_a^b |f'(y)|^2 dy)^{\frac{1}{2}}$.

We also have:

LEMMA 2.3.
(a) *For* $k > n/p$, $L^p_k$ *is a Banach algebra. If* $ql > n$, $k - n/p \geq l - n/q$ *and* $k \geq l$ *the inclusion* $L^p_k \subset L^q_l$ *makes* $L^q_l$ *a module over* $L^p_k$.
(b) *If, for* $i = 1, 2$, $k_i p_i < n$, $k_i \geq k$ *and*

$$\sum_i (k_i - n/p_i) \geq (k - n/p)$$

*then multiplication* $L^{p_1}_{k_1} \otimes L^{p_2}_{k_2} \to L^p_k$ *is well-defined and continuous.*

The proof of these statements uses only the embedding theorem and Hölder inequalities. For example in statement (b) if we take $k_i = k = 0$, and we take we take $1/p_1 + 1/p_2 = 1/p$ the statement is the Hölder inequality. The case $1/p_1 + 1/p_2 < 1/p$ follows because the domain $D$ has finite measure.

We use the notation $L^p_k(\bar{D})$ to emphasize that "in good cases the functions extend up to the boundary", i.e., a restriction to the boundary can be defined. A typical result:

LEMMA 2.4. *The restriction map $C^\infty(\bar{D}) \to C^\infty(\partial\bar{D})$ extends to a bounded linear map $L^p_k(\bar{D}) \to L^p_{k-1/p}(\partial\bar{D}$ provided $k - 1/p > 0$.*

A basic fact about Sobolev spaces is that elliptic operators between them, with appropriate boundary conditions, are Fredholm.

It is straightforward to define Sobolev spaces of functions with values in finite-dimensional real inner product spaces. We skip the details.

### §2.b Topologies on $\mathcal{A}$ and $\mathcal{G}$.

The results in this section are from [NR] and [UF]. (See also [MV],[S].)

We begin by considering an arbitrary real vector bundle $E$ on $M$. Then for any real number $p \geq 1$ and integer $k \geq 0$ we can define a vector space $L^p_k(M, E)$. An element of this space is a represented by a measurable section $\sigma$ of $E$ satisfying:

- ⋆ Given an open set $U \subset M$, a set $\{y_i\}$ of co-ordinates on $U$, a trivialisation $e_j$ of $E$ over $U$, and a relatively compact open $D \subset U$ with smooth boundary, the components of $\sigma$ with respect to the trivialisation are in $L^p_k(\bar{D})$.

*In the last expression $D$ is thought of as a domain in* $\mathbb{R}^n$ via *the co-ordinates $\{y_i\}$. This identification will be implicit in many places below.*

One can similarly define, for any fibre bundle $F$ over $M$, a manifold $L^p_k(M, F)$ of sections of Sobolev class $L^p_k$, *provided $kp > n$*. The necessity for this assumption is obvious — we need to measure the difference between maps in terms of co-ordinate charts on the total space $F$, only when the maps involved are continuous can we be sure that the image of a small enough open set in $M$ is contained in a co-ordinate chart in $F$. The sufficiency follows from the invariance of the relevant Sobolev spaces under diffeomorphisms.

It is useful to have more concrete definitions. Recall (§1.d) the definition of a section in terms of transition functions Choose a finite family $(U_\mu, \{y_{i,\mu}\}, \{\tau_\mu\}, D_\mu)$ such that the $D_\mu$ form an open cover of $M$. Then

$$L^p_k(M, E) = \{\phi_\mu \mid \phi_\mu = g_{\mu\nu}\phi_\nu\} \underset{closed\ subspace}{\subset} \bigoplus_\mu L^p_k(D_\mu, V)$$

where the $g_{\mu\nu}$ are the transition functions.

Exercise 2.5: (a) The vector space $L^p_k(M, E)$ endowed with the norm

$$|\phi| = \sum_\mu |\phi_{j,\mu}|^p_k$$

is a Banach space.

(b) Any other family $(U_\nu, \{y_{i,\nu}\}, \{e_{j,\nu}\}, D_\nu)$, such that the $D_\nu$ form an open cover of $M$, yields an equivalent norm.

We define $\mathcal{A}^p_k$ as the affine subspace

$$\mathcal{A}^p_k = \{A_\nu \mid A_\nu = ad_{g^{-1}_{\mu\nu}} A_\mu + g^{-1}_{\mu\nu} dg_{\mu\nu}\} \underset{closed\ affine\ subspace}{\subset} \bigoplus_\mu L^p_k(D_\mu, \Lambda^p_D \otimes Lie\ G)$$

For integers $k, p$ satisfying $kp > n$ we now let $\mathcal{G}^p_k \equiv L^p_k(M, Ad\ P)$. This is a particular example of a Banach manifold of functions taking values in a manifold (which we have not defined), but one can give slightly *ad hoc* definition which work for the groups of interest, namely $G = SO(n)$ or $G = SU(m)$. Let $T : G \to V$ be the defining representation (thus $V$ is respectively $\mathbb{R}^n$ or $\mathbb{C}^m$) and let $V_T$ be the associated vector bundle. This bundle carries an inner product (respectively real or hermitian) and we can identify $Ad\ P \hookrightarrow End\ V_T$ where the image consists respectively of special orthogonal and special unitary endomorphisms. We now define $L^p_k(M, Ad\ P)$ as the subset of $L^p_k(M, End\ V_T)$ consisting of sections taking values in $Ad\ P$.

Exercise 2.6:

(a) Use the implicit function theorem, valid for differential maps of Banach manifolds, to check that this definition makes $L^p_k(M, Ad\ P)$ a closed submanifold of $L^p_k(M, End\ V_T)$. (A convenient reference is [L, Corollary 2s., page 17] — note that in contrast to the finite-dimensional case one has to assume that the ***kernel of the differential map splits*** ; this is automatic in the case of Hilbert manifolds but has to be checked otherwise.)

(b) We have, in terms of transition functions:

$$L^p_k(M, Ad\ P) = \{s_\mu \mid s_\nu = g_{\mu\nu} s_\mu g^{-1}_{\mu\nu}\} \underset{closed\ submanifold}{\subset} \bigoplus_\mu L^p_k(D_\mu, G)$$

where again $L^p_k(D_\mu, G)$ can be defined as (the Banach submanifold of) functions in $L^p_k(D_\mu, End\ V)$ taking values in $G$.

Note that by the Sobolev theorem elements of $\mathcal{G}^p_k$ give $C^l$-automorphisms of $P$ where $l < k - n/p$.

We have

PROPOSITION 2.7. *$\mathcal{G}^p_k$ is a Banach Lie Group.*

PROOF: By Lemma 2.3(a) the multiplication in $L^p_k(D_\mu, End\ V)$ is a smooth map (multiplication in a Banach algebra is smooth.). Thus the restriction to $L^p_k(D_\mu, G)$ is smooth, as is the inverse, being the restriction of the linear map $s \mapsto s^T$ or $s \mapsto s^\dagger$. Thus $L^p_k(D_\mu, G)$ is a Banach Lie group. The result now follows from 2.6(b). ∎

Exercise 2.8 The Lie algebra of $\mathcal{G}^p_k$ if $L^p_k(M, ad\ P)$.

We can now prove in a routine way, by a technique similar to the last proof,

PROPOSITION 2.9. *Assume $kp > n$. Then $\mathcal{G}^p_k$ acts smoothly on $\mathcal{A}^p_{k-1}$.*

and

PROPOSITION 2.10. *The curvature operator $\mathcal{A}^p_{k-1} \to L^p_{k-2}(M, \Lambda^2_M \otimes ad\ P)$ is smooth.*

Let $\hat{\mathcal{A}}^p_{k-1}$ denote the set of irreducible connections on $P$ of Sobolev class $L^p_k$ — this is an open dense submanifold of $\mathcal{A}^p_k$ (exercise). The group $\hat{\mathcal{G}}^p_k \equiv \mathcal{G}^p_k/Z(G)$ acts freely on this set. The rest of this section will be devoted to the following

THEOREM 2.11. *Assume $kp > n$. The quotient space $\hat{\mathcal{A}}^p_{k-1}/\hat{\mathcal{G}}^p_k$ is a Hausdorff Banach manifold. $\hat{\mathcal{A}}^p_{k-1}$ is a principal $\hat{\mathcal{G}}^p_k$ bundle over this quotient.*

We show:

(1) The action of $\mathcal{G}^p_k$ on $\mathcal{A}^p_{k-1}$ is proper. This will prove [B1] that the quotient is a Hausdorff space.

(2) For $A \in \hat{\mathcal{A}}^p_{k-1}$ the map $\hat{\mathcal{G}}^p_k$ to $\hat{\mathcal{A}}^p_{k-1}$ given by $s \mapsto A^s$ is an injective immersion (i.e., has a closed immersion with topological supplement). By [B2] this will prove the Theorem.

LEMMA 2.12. *Assume $kp > n$. Let $(A_i, s_i)$ be a sequence in $\mathcal{A}^P_{k-1} \times \mathcal{G}^P_k$ such that $A_i \to A$ and $B_i \equiv A_i^{s_i} \to B$. Then there exists a subsequence $s_j$ which tends to a limit $s$ (so that $A^s = B$.)*

PROOF: We will outline a proof under the assumption that $k(p-1) >> n$ so that the relevant connections are actually continuous. For the general case see [UF, Proposition A.5].

Note that in a chart $U_\mu$ we can write

$$ds_i = s_i B_i - A_i s_i \tag{2-1}$$

where $s_i, A_i, B_i$ are regarded as $End\ V$-valued forms. By compactness of $G$ one can find a subsequence $s_j$ converging at some point $p \in U_\mu$. Integrating (2-2) "radially" away from $p$ along straight lines and using standard results about the dependence of solutions of an ODE on coefficient (functions) we get the uniform convergence of the $s_j$ on $D_\mu$. A routine patching now gives this everywhere.

One can now "bootstrap" using (2-2) to get the desired result. ∎

This result proves properness (because the isotropy groups are compact.)

We turn next to (2). The differential of the map $s \mapsto A^s$ at $s = Identity$ is the operator $d_A : L^p_k(ad\ P) \to L^p_{k-1}(\Lambda^1_M \otimes ad\ P)$. By the definition of irreducibility of $A$ this is an injection. One proves that the a topological supplement is given by $ker\ d^*_A$; in fact that

$$L^p_{k-1}(\Lambda^1_M \otimes ad\ P) = Im(d_A) \oplus ker\ d^*_A.$$

This is standard when $A$ is smooth, but needs work when it is not.

## §3: A "Good" Gauge.

This section is essentially an *exposé* of [U]. I have skipped details which can easily be read off from that paper.

### §3.a The Compactness Theorem.

We start with some functional analysis [RS]. Let $V$ be a Banach space, denote by $V^*$ its dual. The *weak topology* on $V$ is the weakest topology such that each linear functional $\ell \in V^*$ is continuous. Recall the following special case of the Banach-Alaoglu

THEOREM 3.1. *Suppose $V$ is reflexive, that is, $(V^*)^* = V$. Then the unit ball is compact in the weak topology.*

Exercise 3.2: The space $L^p_k(M, E)$ (§2.a) is reflexive for $1 < p < \infty$.

To motivate the next theorem consider the case $G = \mathbb{R}$. This is not a compact group, but no matter. In fact we can drop the restriction $kp > n$ (in fact we take $p = 2$, $k = 2$); and we consider the trivial bundle with $R$ as structure group. Then $\mathcal{A}^2_1 = L^2_1(M, \Lambda^1)$, (i.e., 1-forms of Sobolev class $L^2_1$) and $\mathcal{G}^2_2 = L^2_2(M)$ (i.e., 0-forms with Sobolev class $L^2_2$). The action of a 0-form $s$ on a 1-form $A$ is $A \mapsto A + ds$. The function $A \mapsto \int_M |dA|^2\tau$ (where $\tau$ is a volume element on $M$) is gauge-invariant and descends to the quotient modulo $\mathcal{G}^2_2$.

Exercise 3.3(a): Assume $H^1(M, \mathbb{R}) = 0$. Use Hodge Theory to prove: Given a sequence $A_i \in \mathcal{A}^2_1$ of 1-forms such that $\int_M |dA_i|^2\tau \leq B$, there is a subsequence $\{j\} \subset \{i\}$ and gauge transformations $s_j$ in $\mathcal{G}^2_2$ such that $A_j^{s_j}$ is weakly convergent in $\mathcal{A}^2_1$. The weak limit $A$ satisfies $\int_M |dA|^2\tau \leq B$.

Exercise 3.3(b): What happens when $M$ is not simply connected?

Fix a Riemannian volume element $\tau$ on $M$. Consider, on $\mathcal{A}^p_1$, the function $A \mapsto \int_M |F_A|^p\tau$. This is clearly gauge-invariant, and therefore descends to a function on $\mathcal{A}^p_{k-1}/\mathcal{G}^p_k$. The next theorem [U, 1.5] asserts, roughly speaking, that this function is a "norm" on $\mathcal{A}^p_{k-1}/\mathcal{G}^p_k$.

THEOREM 3.4. *Let $p > n/2$, $B$ a nonnegative real number. Suppose $A_i \in \mathcal{A}^p_1$ is a sequence of connections with $\int_M |F_{A_i}|^p\tau \leq B$. Then there is a subsequence $\{j\} \subset \{i\}$ and gauge transformations $s_j$ in $\mathcal{G}^p_2$ such that $A_j^{s_j}$ is weakly convergent in $\mathcal{A}^p_1$. The weak limit $A$ satisfies $\int_M |F_A|^p\tau \leq B$.*

We shall prove a local Theorem from which the above result can be deduced.

### §3.b Existence of a "good gauge".

In this subsection $M = B^n$, the unit ball in $\mathbb{R}^n$. We set $\int = \int_{B^n} dy$ unless another domain of integration is explicitly given. We let $\mathcal{U} = \mathcal{A}^p_1$, $\mathcal{G} = \mathcal{G}^p_2$ and (for $\kappa \geq 0$) $\mathcal{U}_\kappa = \{A \in \mathcal{U} \mid \int |F_A|^{n/2} \leq \kappa\}$. Note that $\mathcal{U}_\kappa$ is invariant under $\mathcal{G}$.

THEOREM 3.5. *Let $n > p > n/2$. Then $\exists\ \kappa = \kappa(n)$ and $c = c(n)$ such that every connection $\tilde{A} \in \mathcal{U}_\kappa$ is gauge equivalent to a connection $A$ where $A$ satisfies*

(a) $d^*A = \sum_i \partial_i A_i = 0$,

(b) $\sum_i y_i A_i = 0$ *on* $S^{n-1}$, *and*

(c) $\|A\|_{q,1} \leq \|F_A\|_q$ *for* $n/2 \leq q \leq p$.

Remark 3.6:

(i) $\sum_i y_i A_i$ is a *Lie* $G$-valued function in $L^p_1(B^n)$. The claim in (b) is that it is in fact in $L^p_{1,0}(B^n)$.

(ii) A version of the Theorem holds in the case $p = n/2$ and gives a regularity result for solutions of Yang-Mills equations.

PROOF: The proof is *via* the continuity method. One shows that $\mathcal{U}_\kappa$ is connected, then that the subset satisfying (a)-(c) (which we henceforth denote $\mathcal{W}_{\kappa,c}$) is both open and closed. The constants will be determined in the course of the proof.

Step 1: Connectedness of $\mathcal{U}_\kappa$ (for $p \geq n/2$)

Define, for $0 \le t \le 1$, $D_t : B^n \to B^n$ by $D_t(x) = tx$. For $A \in \mathcal{U}$ let $A_t = D_t^* A$. Then $F_t \equiv F_{A_t} = D_t^* F_A$ and

$$\int_{B^n} |F_t(y)|^p dy = \int_{B^n} |F(ty)|^p t^p dy$$
$$= t^{p-n} \int_{tB^n} |F(y)|^p dy.$$

The second equality follows from the change of variables formula. The first is left as an exercise.

For fixed $A$, $t \mapsto A_t$ gives a curve in $\mathcal{U}$ connecting $A$ to the zero form, and the above computation shows that the curve stays in $\mathcal{U}_\kappa$ if $A \in \mathcal{U}_\kappa$.

Step 2: The set $\mathcal{W}_{\kappa,c}$ is closed in $\mathcal{U}_\kappa$ for $n > p > n/2$.

Let $\tilde{A}_i \in \mathcal{W}_{\kappa,c}$, $\tilde{A}_i \to \tilde{A}$, and let $s_i$ be gauge transformations such that $A_i \equiv \tilde{A}_i^{s_i}$ satisfies (a)-(c). First, since $\tilde{A}_i \to \tilde{A}$, $\int |F_{\tilde{A}_i}|^p$ is uniformly bounded (by Proposition 2.9). Hence — using gauge-invariance of the integral and (c) — the $A_i$'s form a bounded set in $L_1^p$. By Theorem (3.1) there is a weakly convergent subsequence $A_j \cdots \to A$. We now show (i) $\tilde{A}$ is gauge-equivalent to $A$, and (ii) A satifies (a)-(c).

We first show (ii). That conditions (a) and (b) are preserved under weak limits is clear. As for (c) note that $A_j \cdots \to A$ in $L_1^q$ for $q \le p$ since $L_1^p \to L_1^q$ is norm continuous (and hence preserves weak convergence). Thus $||A||_{q,1} \le \liminf ||A_j||_{q,1} \le c(n)||F_{A_j}||_q = c(n)||F_{\tilde{A}_j}||_q = c(n)||F_{\tilde{A}}||_q = c(n)||F_A||_q$

It remains to show (i). We have $\tilde{A}_j^{s_j} = A_j$, or thinking of the $s_j$ as matrix-valued functions, $ds_j = s_j A_j - \tilde{A}_j s_j$. Since $n > p$ we have $r > p$ such that $1/n - 1/p + 1/r = 0$. Since $G$ is compact the $s_i$ are in $L^\infty$ so $||ds_j||_r \le c_1(||A_j||_r + ||\tilde{A}_j||_r) \le c_2(||A_j||_{p,1} + ||\tilde{A}_j||_{p,1})$ where in the second step we have Sobolev embedding. Thus the $s_j$ are uniformly bounded in $L_1^r$. Pick a weakly convergent sequence $s_k \cdots \to s$ in $L_1^p$.

Step 3: An *à priori* estimate.

We prove: *There exists* $d(n) > 0$ *such that if* $A \in \mathcal{U}$, $||A||_n < k(n)$, *and* $A$ *satisfies* (a)-(b) *then it satisfies* (c).

Let $\nabla A = \sum_{ij} \partial_i A_j dy^y j$. Note first that if $d^* A = 0$ and $\sum_i y_i A_i = 0 on S^{n-1}$ then

(3-1)
$$\int |\nabla A|^2 + \int \sum_i |A_i|^2 = 1/2 \int |dA|^2.$$

The proof is by integration by parts. This holds for any $A \in C^\infty(\bar{B}^n)$ and by Lemma (2.4) for $A \in \mathcal{U}$.

By the results of [ADN***] we have for $A \in \mathcal{U}$ satisfying $\sum_i y_i A_i = 0 on S^{n-1}$, $1 < p < \infty$,

(3-2)
$$||A||_{p,1} \le c\{||dA||_p + ||d^* A||_p + ||A||_p\}.$$

We claim this implies: If $A \in \mathcal{U}$ satisfies (a) and (b) then

(3-3)
$$||A||_{p,1} \le k'(n)||dA||_p \quad (1 < p < \infty)$$

(i.e., there is 'no cohomology'). Suppose the contrary. Then there is a sequence $A_i$ satisfying (a) and (b), with $||A_i||_{p,1} = 1$ and $||dA_i||_p \to 0$. We can suppose (by going to a subsequence if necessary) that $A_n \cdots \to A$, for some $A$ which will then satisfy (a), (b) and $dA = 0$. On the other hand $A_n \to A$ in $L^p$ and $\liminf ||A_i||_p > 0$ by (3-2). Thus we have located $A \ne 0 \in \mathcal{U}$ with $dA = 0$, $d^* A = 0$, and $\sum_i y_i A_i = 0$ on $S^{n-1}$. But (3-1) shows that such an $A$ must be 0 and (3-3) is proved.

An application of the Hölder and Sobolev inequalities yields the required result.

Step 4: Existence of a local gauge for "small" fields.

We prove: *Suppose $A \in \mathcal{U}$ satisfies (a) and (b) with $||A||_n \leq k(n)$. Then there exists $\epsilon > 0$ such that if $||\tilde{A}' - A||_{p,1} \leq \epsilon$ and $\tilde{A}'$ satisfies (b) then $\tilde{A}'$ is gauge equivalent to $A'$ satisfying $d^*A' = 0$. The solution depends smoothly on $\tilde{A}'$.*

The idea is to use the implicit function theorem. Introduce spaces $\mathcal{U}_\nu = \{B \in \mathcal{U} \mid \sum_i y_i B_i = 0 \text{ on } S^{n-1}\}$, $\mathcal{G}_\nu = \{s \in \mathcal{G} \mid \sum_i y_i (ds)_i = 0 \text{ on } S^{n-1}\}$. Look at the map $(\tilde{A}', s) \mapsto d^* \tilde{A}^s$ of $\mathcal{U}_\nu \times \mathcal{G}_\nu$ to $L^{p,\perp}(B^n, ad\ G) = \{\phi \in L^p(B^n, ad\ G) \mid \int \phi = 0\}$. The linearisation of this at $A$, restricted to the tangent space to $\mathcal{G}_n u$ is an isomorphism for small enough $||A||_n$. Now apply the implicit function theorem.

<u>Step 5:</u> **The set $\mathcal{W}_{\kappa,c}$ is open in $\mathcal{U}_\kappa$.**

We prove: ***Suppose $\tilde{A} \in \mathcal{U}_\kappa$ is gauge-equivalent to $A$ satisfying (a)-(c). Then if $\kappa$ is sufficiently small there exists an open neighbourhood of $\tilde{A}$ satisfying (a)-(c).***

We can clearly take $A = \tilde{A}$. Then the previous step gives us what we want if we can prove the following <u>Claim</u>: Suppose $A \in \mathcal{U}_\kappa$ satisfying (a) and (b) with $||A||_n \leq k(n)$. Then for $\kappa$ small enough there exists $\epsilon' > 0$ such that if $||\tilde{A}' - A||_{p,1} \leq \epsilon'$ then $\tilde{A}'$ is gauge equivalent to $\tilde{A}$ satisfying $\sum_i y_i \tilde{A}_i|_{S^{n-1}} = 0$.

We refer the reader to the paper for the short proof.

The theorem is proved. ∎

## References


[B1] N. Bourbaki, "Topologie Générale, Chapt. 3-4," Hermann, Paris, 1960.

[B2] N. Bourbaki, "Variétés Différentielles et Analytiques (Fascicule des Resultats)," Hermann, Paris, 1967.

[F] A. Friedman, "Partial Differential Equations," Holt, Rinehart and Winston, Inc., New York, 1969.

[L] S. Lang, "Differential Manifolds," Addison-Wesley, Reading, Massachusetts, 1972.

[MV] P. K. Mitter and C. M. Viallet, *On the Bundle of Connections and the Gauge Orbit Manifold in Yang-Mills Theory*, Commun. Math. Phys. 79, 457-472.

[N] M. S. Narasimhan, *Differential Geometry*, Lectures delivered at the Summer Workshop on Fibre Bundles and Differential Geometry, Trieste (July 1982)..

[NR] M. S. Narasimhan and T. R. Ramadas, *Geometry of SU(2) Gauge Fields*, Commun. Math. Phys. 67 (1979), 121-136.

[P] R. S. Palais, "Foundations of Global Nonlinear Analysis," W. A. Benjamin, Inc., New York, 1968.

[RS] M. Reed and B. Simon, "Methods of Modern Mathematical Physics I: Functional Analysis," Academic Press, New York and London, 1972.

[S] I. M. Singer, *Some Remarks on the Gribov Ambiguity*, Commun. Math. Phys. 60 (1978), 7-12.

[UF] K. K. Uhlenbeck and D. S. Freed, "Instantons and Four-Manifolds," MSRI Publications, Springer, New York, 1984.