INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.545/11

WORKSHOP ON MATHEMATICAL PHYSICS AND GEOMETRY

(4 - 15 March 1991)

Four-manifolds and gauge theory

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FOUR-MANIFOLDS AND GAUGE THEORY

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In these lectures I will describe, in general terms, the basic results and techniques of Donaldson theory. By Donaldson theory I mean the application of ideas from gauge theory to the classification of four-manifolds. My aim is to give a general picture of how gauge theory can be used to prove theorems about 4-manifolds and so there are essentially no detailed proofs; however in several places I have tried to convey some of the main lines in the proofs. The principal reference is the book by Donaldson and Kronheimer [4] which contains full proofs and further references. Many of the basic results are also proved in the books by Freed and Uhlenbeck [5] and Lawson [7].

§1 THE MAIN THEOREMS

In this first section I will describe the main results on the classification of 4-manifolds which arise from Donaldson theory and set them in the general context of the theory of 4-manifolds. Throughout we use the term closed manifold to mean one which is compact and has no boundary. Let X be a simply-connected, closed, oriented 4-manifold — the term simply-connected 4-manifold will, unless specified otherwise, mean a 4-manifold which satisfies these hypotheses. There is a significant difference between the results for topological manifolds and for smooth manifolds so it is important to carefully specify whether we are working in the topological category (topological manifolds up to homeomorphism) or in the smooth category (smooth manifolds up to diffeomorphism). Associated to X is a basic invariant, its **intersection form**

$$Q = Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}.$$

This is a bilinear form which is **symmetric** and **unimodular**: symmetric means that $Q(x,y) = Q(y,x)$ for all $x, y \in H_2(X;\mathbb{Z})$ and unimodular means that if we choose a basis $e_1, \dots, e_r$ for the free abelian group $H_2(X;\mathbb{Z})$ and express Q as the symmetric matrix

$$A = (a_{ij}), \qquad a_{ij} = Q(e_i, e_j)$$

then $\det A = \pm 1$. The fact that $H_2(X;\mathbb{Z})$ is a free abelian group follows from the hypotheses on X and it follows from Poincaré duality that Q is unimodular.

Two natural questions immediately present themselves:

The Realisation Question. Given a symmetric unimodular form Q is it the intersection form of some simply connected 4-manifold?

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The Classification Question. Classify 4-manifolds with given intersection form.

The first theorem in the subject is the following result of [8].

Theorem of Milnor and Whitehead (1958). *Let X and Y be simply connected 4-manifolds. Then if $Q_X \cong Q_Y$ it follows that X and Y are homotopy equivalent.*

This is the crudest possible classification of 4-manifolds and we would like more refined results which classify topological 4-manifolds up to homeomorphism, and smooth 4-manifolds up to diffeomorphism. Milnor's paper [8] is the place where many of the themes in the topology of 4-manifolds first appear, in particular the importance of the intersection form and the fact that there are significant differences between the classification of 4-manifolds and the analogous classification problem in higher dimensions.

Let us briefly digress to introduce some of the terminology of bilinear forms and the basic invariants associated to symmetric unimodular forms Q defined over $\mathbb{Z}$. More details can be found in [9] and [13]

(1) The **rank** of Q is the rank of the group on which Q is defined. In terms of a matrix representation of Q it is the size of the matrix.
(2) The form Q can be diagonalised over $\mathbb{R}$ and we define $b^+ = b^+(Q)$ to be the number of positive entries which occur when Q is diagonalised over $\mathbb{R}$ and $b^- = b^-(Q)$ to be the number of negative entries.
(3) The **signature** of Q is defined by

$$\sigma(Q) = b^+ - b^-.$$

(4) The **type** of Q is defined in the following roundabout way. We say that Q has type II, or is **even**, if $Q(x,x)$ is always even. Then Q has type I, or is **odd**, if it does not have type II.
(5) If $Q(x,x) \geq 0$ for all x, and $Q(x,x) = 0$ if and only if $x = 0$, we say that Q is **positive definite**; Q is **negative definite** if $Q(x,x) \leq 0$ for all x, and $Q(x,x) = 0$ if and only if $x = 0$. We say that Q is **definite** if it is either positive definite or negative definite.

There is a basic algebraic fact about even definite forms, see [9] or [13].

Lemma. *Suppose Q is an even definite symmetric unimodular form over $\mathbb{Z}$; then $\sigma(Q)$ is divisible by 8.*

There is an even definite symmetric unimodular form over $\mathbb{Z}$ with signature 8, this is E_8. The matrix of E_8 is given in [9] and [13] and also in Narasimhan's lecture [10]. It is a pleasant exercise to diagonalise this matrix over $\mathbb{R}$ and check that it has signature 8. It can also be shown, see for example [9] and [13], that E_8 cannot be diagonalised over the integers.

The second classical theorem about the intersection forms of 4-manifolds is the following result proved in [12].

Rohlin's Theorem (1952). *Let X be a smooth simply connected 4-manifold; then $\sigma(Q_X)$ is divisible by 16.*

This theorem shows that there are genuine restrictions on the intersection forms of smooth 4-manifolds. For example E_8 cannot be the intersection form of a smooth simply connected 4-manifold. The significance of Rohlin's theorem for the classification of 4-manifolds is discussed in Milnor's paper [8].

Before describing some of answers to the general Realisation and Classification Problems let us discuss other "classical invariants" of smooth simply connected 4-manifolds. Since we are now assuming that the manifold is smooth the other source of invariants of X is the tangent bundle T_X and, in particular, its characteristic classes. The tangent bundle has two basic characteristic classes, the **Stiefel-Whitney class**

$$w_2 \in H^2(X; \mathbb{Z}/2)$$

and the **Pontryagin class**

$$p_1 \in H^4(X; \mathbb{Z}).$$

These can be computed from Q_X as follows.

The function

$$H^2(X; \mathbb{Z}) \to \mathbb{Z}/2, \qquad x \mapsto Q(x,x) \mod 2$$

is linear and so, by the unimodularity of Q, it is given by

$$x \mapsto Q(c,x) \mod 2$$

for some $c \in H^2(X;\mathbb{Z})$. The homomorphism $\mathbb{Z} \to \mathbb{Z}/2$ of coefficients, given by reduction modulo 2, induces a homomorphism $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}/2)$ of cohomology and under this homomorphism $c \mapsto w_2$.

By applying the Hirzebruch signature theorem we deduce that

$$\langle p_1, [X] \rangle = 3(b^+ - b^-)$$

where $\langle , \rangle$ is the pairing between cohomology and homology and $[X] \in H^4(X;\mathbb{Z})$ is the fundamental class of the oriented 4-manifold X. Since $H^4(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and the isomorphism is given by $x \mapsto \langle x, [X] \rangle$ it follows that p_1 is determined by the intersection form.

Thus we see that the classical invariants of X are all determined by the intersection form and if we follow the analogy with the classification of manifolds in dimensions 5 or more it should now follow that the intersection form of X essentially determines X. Indeed if we classify 4-manifolds up to homeomorphism this is indeed true. The main theorem in the purely topological study of simply connected 4-manifolds is the following result proved in [6].

Freedman's Theorem (1982).

(1) *Suppose X and Y are smooth 4-manifolds such that $Q_X \cong Q_Y$; then X and Y are homeomorphic.*

(2) *Let Q be a symmetric unimodular form over $\mathbf{Z}$; then there is a topological 4-manifold X with $Q_X \cong Q$.*
(3) *Suppose X and Y are topological 4-manifolds with $Q_X \cong Q_Y \cong Q$. If Q has type II then X and Y are homeomorphic. If Q has type I then there are precisely two topological manifolds, up to homeomorphism, with intersection form Q.*

In part (3) of Freedman's theorem the two manifolds with the same intersection form are distinguished by their **Kirby-Siebenmann** invariant; this is an invariant $k(X) \in \mathbf{Z}/2$ and it vanishes if and only if $X \times S^1$ has a smooth structure. In particular if X is smooth $k(X) = 0$ and we see the relation between part (1) and part (3). This result completely settles the Realisation and Classification Questions for simply connected topological 4-manifolds up to homeomorphism.

Let us now turn to smooth manifolds and, therefore, to Donaldson's theorems—here there are several surprises waiting for us. I will divide Donaldson's work into three parts.

Definite forms. Donaldson proves the following theorem which gives very dramatic restrictions on the possible definite forms which arise as the intersection forms of simply-connected smooth 4-manifolds.

Theorem. *Suppose that X is a smooth simply connected 4-manifold such that Q_X is definite; then Q_X is diagonal.*

The original reference is [1] and the theorem is discussed, very carefully, in [4]. It should be contrasted with Freedman's theorem which tells us that, given a symmetric unimodular form defined over $\mathbf{Z}$, there always exists a simply-connected topological 4-manifold with this intersection form. Donaldson's theorem tells us that if the form is definite and not diagonal then the manifold given by Freedman's theorem cannot be smooth. The theory of definite symmetric unimodular forms is a difficult part of classical number theory, see for example [9] and [13] but Donaldson's theorem tells us that none of these forms, apart from the simple diagonal forms, can occur as the intersection forms of smooth 4-manifolds.

One of the consequences of the combination of this theorem and Freedman's theorem is that there must exist a fake $\mathbb{R}^4$—this is a smooth manifold which is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to $\mathbb{R}^4$. A very clear description of why this must follow is given in the book by Freed and Uhlenbeck [5]. The existence of a fake $\mathbb{R}^4$ is proved by an implicit argument—the only way to account for the fact that Freedman's methods must break down in the smooth category is that there is a fake $\mathbb{R}^4$—and there is no known way of constructing a fake $\mathbb{R}^4$ directly.

Donaldson's theorem proves, for example, that

$$nE_8 = E_8 \oplus \cdots \oplus E_8$$

(where there are n-summands) cannot be the intersection form of a smooth simply connected 4 manifold. Note that if n is odd then this also follows from Rohlin's theorem.

Now let K be the **Kummer surface**

$$K = \{[z_0, z_1, z_2, z_3] : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3$$

where $[z_0, z_1, z_2, z_3]$ are the homogeneous coordinates of a point in 3-dimensional complex projective space $\mathbb{CP}^3$. Then K is a smooth 4-manifold; Milnor shows in [8] that K is simply-connected and

$$Q_K = -E_8 \oplus -E_8 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On the other hand we know from Donaldson's theorem that $-E_8 \oplus -E_8$ cannot be the intersection form of a smooth simply-connected 4-manifold. It is natural to look for the dividing line between the non-existence results and the intersection form of K. For this we need to study indefinite forms.

Indefinite forms. There is a classification of indefinite symmetric unimodular forms over $\mathbf{Z}$; this is given by the **Hasse-Minkowski** theorem, see [9] and [13]. Such forms are classified by their rank $r = b^+ + b^-$, signature $\sigma = b^+ - b^-$, and type. The forms fall into two distinct families:

(1) type I,

$$n(1) \oplus m(-1)$$

where, $r = n + m$, $\sigma = n - m$, $b^+ = n$,

(2) type II,

$$-nE_8 \oplus m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where, $r = 8n + 2m$, $\sigma = -8n$, $b^+ = m$.

In the first family we assume that $n, m \geq 1$ to ensure the forms are indefinite and in the second we assume that $m \geq 1$.

Each of the forms in the first family is the interesection form of a smooth 4-manifold. The intersection form of $\mathbb{CP}^2$ with it usual orientation is just (1) and the intersection form of $\overline{\mathbb{CP}}^2$, by which we mean $\mathbb{CP}^2$ with the opposite orientation, is (-1). Now by taking the connected sum of n copies of $\mathbb{CP}^2$ with m copies of $\overline{\mathbb{CP}}^2$ we get intersection form $n(1) \oplus m(-1)$.

In the second family the case $n = 0$ is easy to handle. Let S be the product $S^2 \times S^2$ so that

$$Q_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

By taking the connected sum of m copies of S we can realise the case $n = 0$ and m arbitrary. However the example of the Kummer surface K requires us to look at the question of whether the forms in the second family with $n \neq 0$ can be realised as the intersection forms of smooth 4-manifolds.

Theorem. *Suppose X is a simply-connected smooth 4-manifold with even indefinite intersection form; then*

$$b^+ = 1 \implies Q_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$b^+ = 2 \implies Q_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The original reference is [2] and the theorem is discussed in careful detail in [4]. Thus it follows, by combining this theorem with Rohlin's theorem, that in the second family of even indefinite unimodular forms the minimal, in the obvious sense, form with non-zero n which can occur as the intersection form of a smooth simply-connected 4-manifold is

$$Q_K = -2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus K is indecomposable and it is tempting to believe it is one of the basic building blocks of smooth 4-manifolds in the sense of the following conjecture.

Conjecture. *The only even indefinite unimodular forms defined over $\mathbb{Z}$ which can be the intersection forms of smooth simply-connected 4-manifolds are*

$$pQ_K \oplus qQ_S.$$

If this conjecture is true then we get a complete answer to the realisation question for smooth simply-connected 4-manifolds. There are four indecomposable pieces

$$S, \quad \mathbb{CP}^2, \quad \overline{\mathbb{CP}}^2, \quad K$$

and every smooth 4-manifold is homeomorphic to a connected sum of these indecomposable pieces. The only intersection forms which can occur are given by direct sums of

$$Q_S, \quad Q_{\mathbb{CP}^2}, \quad Q_{\overline{\mathbb{CP}}^2}, \quad Q_K.$$

Now, let us turn our attention to the Classification Question for smooth simply-connected 4-manifolds.

Polynomial Invariants. The classification question for smooth 4-manifolds is to classify smooth 4-manifolds up to diffeomorphism. We will see that it is considerably more complicated than the classification up to homeomorphism given by Freedman's theorem. Indeed one of the conclusions of Donaldson theory is that in many cases there are an infinite number of smooth manifolds with a fixed intersection form. In view of Freedman's theorem we can express this by saying that in many cases there are an infinite number of smooth manifolds within each homeomorphism class. To distinguish between these smooth manifolds we need more invariants and these are provided by Donaldson's **polynomial invariants.**

These polynomial invariants are defined under the following hypothesis: X is a smooth simply connected 4-manifold with indefinite intersection form and b^+ is odd. The invariants are polynomial functions

$$\Phi_k = \Phi_k(X) : H_2(X;\mathbb{Z}) \times \cdots \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$$

where there are k factors $H_2(X;\mathbb{Z})$. To say that they are invariants means that if $f : X \to Y$ is a an orientation preserving diffeomorphism and

$$f_* : H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$$

is the induced isomorphism on homology, then

$$\Phi_k(Y)(f_*(x_1), \ldots, f_*(x_k)) = \Phi_k(X)(x_1, \ldots, x_k).$$

There are two main theorems about the polynomials Φ_k.

Theorem 1. *Suppose that X is a connected sum $X = X_1 \# X_2$ where $b^+(X_1)$, $b^+(X_2)$ are both odd, and Q_{X_1}, Q_{X_2} are both indefinite; then $\Phi_k(X) = 0$.*

Theorem 2. *If Z is an algebraic surface with b^+ odd and indefinite intersection form, then, for large enough k, $\Phi_k(Z)$ is non-zero.*

The original reference is [3] and the construction of the polynomial invariants and the proof of these theorems is very carefully discussed in [4].

Here is an example. Let S_d be a smooth algebraic surface in $\mathbb{CP}^3$ of degree d. So S_d is the zero set of a homogeneous polynomial in 4 variables of degree d. Then by repeating the method Milnor used to compute the intersection form of the Kummer surface K we deduce that

$$b^+ = \alpha_d = \frac{1}{3}(d-1)(d-2)(d-3)$$
$$b^- = \beta_d = \frac{2}{3}(d-1)(2d^2-4d+3)$$

If d is odd the intersection form of S_d has type I and if d is large enough it follows that the form is indefinite and type I. Thus, by the classification of such forms, it must be isomorphic to

$$\alpha_d(1) \oplus \beta_d(-1).$$

Therefore, by Freedman's theorem, it follows that S_d is homeomorphic to a connected sum of α_d copies of $\mathbb{CP}^2$ and β_d copies of $\overline{\mathbb{CP}}^2$. By theorem 1, provided d is large enough, all the polynomial invariants of this connected sum vanish. Since S_d is an algebraic surface, theorem 2, shows the polynomial invariants of S_d do not all vanish. Therefore it follows that S_d cannot be diffeomorphic to a connected sum of α_d copies of $\mathbb{CP}^2$ and β_d copies of $\overline{\mathbb{CP}}^2$. This shows that, provided d is large enough, there are at least two smooth manifolds with interecsection form

$$\alpha_d(1) \oplus \beta_d(-1).$$

A more careful application of the method shows that, up to diffeomorphism, there are an infinite number of smooth manifolds homeomorphic to a conecected sum of one copy of $\mathbb{CP}^2$ and nine copies of $\overline{\mathbb{CP}}^2$. This result is discussed in [4].

§2 Gauge theory

Now I will outline some of the ideas which go into the proofs of Donaldson's theorems. The theme running through Donaldson's work is to treat the spaces of solutions of the Yang-Mills equations as invariants of the underlying manifold X. From now on we will assume that X is a smooth simply-connected 4-manifold equipped with a Riemannian metric.

Let P be a principal $SU(2)$ bundle over X. Then such principal bundles are classified by their Chern class $c_2(P) \in H^4(X)$. From now on H^p will denote integral homology. Since X is closed and oriented, $H^4(X) \cong \mathbb{Z}$ so we can identify the Chern class $c_2(P)$ with an integer. Thus we write

$$k = k(P) = -\langle c_2(P), [X] \rangle$$

where $[X] \in H_4(X)$ is the fundamental class. We refer to k as the **Chern number** of P. This minus sign may look rather strange but, following Donaldson, it fits in best with later orientation conventions. Now we write P_k for a bundle determined by the integer k. Let A be a connection on P_k. Thus locally, on an open set U in X on which the bundle is trivialised, such a connection is given by

$$A_U = A_1(x)dx_1 + A_2(x)dx_2 + A_3(x)dx_3 + A_4(x)dx_4$$

where the A_i are functions on U which take their values in $\mathfrak{su}(2)$, the Lie algebra of $SU(2)$. The Lie algebra $\mathfrak{su}(2)$ is the set of skew-adjoint 2×2 complex matrices with trace zero and so the A_i are matrix valued functions. On $U \cap V$, A_U and A_V are related by

$$A_U = g^{-1} A_V g + g^{-1} dg$$

where $g : U \cap V \to SU(2)$ is the transition function of the bundle P_k. To make sense of this equation remember that both g and the A_i are matrix valued functions.

Now let $\mathcal{A}_k$ be the space of connections on P_k and let $\mathcal{G}_k$ be the group of automorphisms of P_k which are the identity on the base space X. This group $\mathcal{G}_k$ is often referred to as the **group of gauge transformations** of P_k. This group $\mathcal{G}_k$ acts on $\mathcal{A}_k$ by pull-back of connections. Locally this is given by

$$g^*(A) = g^{-1} A g + g^{-1} dg$$

where A is the connection and, since we are working locally, the gauge transformation g becomes a function with values in $SU(2)$.

The action of $\mathcal{G}_k$ on $\mathcal{A}_k$ and various facts about connections, in particular the notion of an **irreducible connection**, has been described in more detail in the lectures given by Ramadas and we refer to the notes from those lectures [11] for more details.

Now let $\mathcal{A}_k^*$ be the space of irreducible connections on P_k. We can form the two quotient spaces

$$\mathcal{B}_k = \mathcal{A}_k / \mathcal{G}_k, \qquad \mathcal{B}_k^* = \mathcal{A}_k^* / \mathcal{G}_k.$$

We use the notation $[A]$ for the element of $\mathcal{B}_k$ defined by a connection A: this is the **gauge equivalence class** of A. For technical reasons it is often easier to work with the subgroup $\mathcal{G}_k^0$ of $\mathcal{G}_k$ consisting of those automorphisms of P_k which induce the identity at a fixed point x_0 in X. The advantage of passing to this subgroup is that $\mathcal{G}_k^0$ acts freely on $\mathcal{A}_k$ and this action has local slices. Thus the projection

$$\mathcal{A}_k \to \mathcal{A}_k / \mathcal{G}_k^0 = \mathcal{B}_k^0$$

is a principal bundle with structure group $\mathcal{G}_k^0$. In fact this space $\mathcal{B}_k^0$ is homotopy equivalent to a familiar function space.

Lemma. *There is a homotopy equivalence*

$$\mathcal{B}_k^0 \simeq \mathrm{Map}_k(X, \mathbb{HP}^\infty)$$

In the statement of the lemma Map means base point preserving maps, $\mathbb{HP}^\infty$ is infinite dimensional quaternionic projective space, and Map_k means the component of the mapping space consisting of those maps f such that the induced homomorphism

$$f_* : H_4(X) \cong \mathbb{Z} \to H_4(\mathbb{HP}^\infty) \simeq \mathbb{Z}$$

is multiplication by k.

There is a natural principal $SU(2)$-bundle

$$\mathcal{P}_k \to \mathcal{B}_k^0$$

defined as follows. The group $\mathcal{G}_k^0$ acts freely on $\mathcal{A}_k$; it also acts on P_k since, by definition it is a group of automorphisms of P_k. Thus we may form the quotient

$$\mathcal{P}_k = \mathcal{A}_k \times_{\mathcal{G}_k^0} P_k.$$

Since P_k is a principal $SU(2)$ bundle over X it follows that $\mathcal{P}_k$ is a principal $SU(2)$ bundle over

$$\mathcal{A}_k \times_{\mathcal{G}_k^0} X = \mathcal{B}_k^0 \times X$$

where the last equality follows from the fact that $\mathcal{G}_k^0$ acts trivially on X.

In terms of function spaces we can describe this bundle $\mathcal{P}_k$ as follows. There is a natural evaluation map

$$\mathrm{Map}_k(X, \mathbb{HP}^\infty) \times X \to \mathbb{HP}^\infty$$

and $\mathcal{P}_k$ is the bundle over $\mathcal{B}_k^0 \times X \simeq \mathrm{Map}_k(X, \mathbb{HP}^\infty) \times X$ induced from the canonical principal $SU(2)$ bundle over $\mathbb{HP}^\infty$ by this map.

Now let

$$c = c_2(\mathcal{P}_k) \in H^4(\mathcal{B}_k^0 \times X)$$

be the second Chern class of $\mathcal{P}_k$. We can use the Künneth theorem (together with our standing hypotheses on X) to decompose $H^4(\mathcal{B}_k^0 \times X)$ as

$$H^0(\mathcal{B}_k^0) \otimes H^4(X) \oplus H^2(\mathcal{B}_k^0) \otimes H^2(X) \oplus H^4(\mathcal{B}_k^0) \otimes H^0(X).$$

With respect to this decomposition we write

$$c^{2,2} \in H^2(\mathcal{B}_k^0) \otimes H^2(X)$$

for the appropriate component of c. Now we use $c^{2,2}$ to define a homomorphism

$$\mu_0 : H_2(X) \to H^2(\mathcal{B}_k^0)$$

in the natural way. There is a pairing

$$H^2(X) \otimes H_2(X) \to \mathbb{Z}$$

and this gives a pairing

$$H^2(\mathcal{B}_k^0) \otimes H^2(X) \otimes H_2(X) \to H^2(\mathcal{B}_k^0)$$

which we denote by $\langle , \rangle$. Then μ_0 is defined by

$$\mu_0(u) = \langle c^{2,2}, u \rangle.$$

In fact this homomorphism μ_0 descends to a homomorphism

$$\mu : H_2(X) \to H^2(\mathcal{B}_k^*)$$

The relation between $\mathcal{B}_k$ and $\mathcal{B}_k^*$ is as follows. By definition $\mathcal{B}_k^{*,0}$ is a subspace of $\mathcal{B}_k^0$ and it can be checked that $\mathcal{B}_k^{*,0}$ is the total space of a principal $SO(3)$ bundle over $\mathcal{B}_k^*$,

$$SO(3) \to \mathcal{B}_k^{*,0} \xrightarrow{\pi} \mathcal{B}_k^*.$$

By arguing directly with this bundle it is possible to prove that there is a commutative diagram

$$\begin{array}{ccc} H_2(X) & \xrightarrow{\mu_0} & H^2(\mathcal{B}_k^0) \\ {\scriptstyle \mu}\downarrow & & \downarrow \\ H^2(\mathcal{B}_k^*) & \xrightarrow[\pi^*]{} & H^2(\mathcal{B}_k^{0,*}) \end{array}$$

I will not try to describe the proof of this; it is given in [4]. This map

$$\mu : H_2(X) \to H^2(\mathcal{B}_k^*)$$

is one of the important ingredients in the theory.

Now it is time to introduce the self-duality equations and the Yang-Mills moduli space. Given a connection A on P_k we can form the curvature $F_A \in \Omega^2(X; \mathfrak{su}(2))$ – the space of 2-forms on X with values in the bundle $\mathfrak{su}(2)$ defined by

$$\mathfrak{su}(2) = P_k \times_{SU(2)} \mathfrak{su}(2)$$

where $SU(2)$ acts on $\mathfrak{su}(2)$ by the adjoint representation. Locally the curvature is given by the formula

$$F_A = dA + A \wedge A.$$

In this local formula remember that A is a matrix of 1-forms so dA is the matrix of 2-forms obtained by applying the exterior derivative d to each of the entries of A, and $A \wedge A$ is defined by the combination of matrix multiplication and the exterior product of forms.

Now suppose that X has a metric. Then the metric and the orientation define the **Hodge star** operator

$$* : \Omega^2(X; \mathfrak{su}(2)) \to \Omega^2(X; \mathfrak{su}(2)).$$

On $\mathbb{R}^4$ with its usual metric and orientation $*$ is given by

$$*(dx_i \wedge dx_j) = \pm dx_k \wedge dx_l$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and the sign is $+$ if $(1, 2, 3, 4) \mapsto (i, j, k, l)$ is an even permutation and $-$ if it is odd. This operator is extended to matrix valued forms by applying it to each entry of the matrix.

I can now describe the Yang-Mills equations.

The self-duality equations.

$$*F_A = F_A$$

The anti-self duality equations.

$$*F_A = -F_A$$

To understand these equations better it is a very good exercise work them out explicitly in terms of the above local description of connections, curvature, and the Hodge star operator. They are first order non-linear equations for the connection A. It is straightforward to check that if A satisfies one of these equations then so does $g^*(A)$ where $g \in \mathcal{G}_k$. Self-duality or anti-self duality is a matter of orientation conventions. Here I will follow Donaldson and concentrate on the ASD (anti-self dual) equations and refer to a connection whose curvature satisfies the ASD equations as an **ASD connection**. Now we define the **moduli space** of ASD connections

$$\mathcal{M}_k = \frac{\text{ASD connections}}{\mathcal{G}_k}.$$

We use the obvious notation $\mathcal{M}_k^*$ for the moduli space of irreducible ASD connections.

The structure of the moduli spaces. First we discuss the local structure of the moduli space. The main result is that, for a generic metric, the space $\mathcal{M}_k^*$ is a smooth manifold of dimension

$$8k - 3(1 + b^+).$$

In particular note that if k is negative, there are no ASD connections. The proof of this result is given in [4] and also [5] and [7].

In general there are singularities in $\mathcal{M}_k$ corresponding to reducible ASD connections. However it is possible to analyse the local structure of $\mathcal{M}_k$ in a neighbourhood of these singularities, see [4], [5], and [7]. There are two special cases where there are no reducible ASD connections.

Lemma. *Suppose that either*

(1) *the intersection form Q_X is indefinite, or*
(2) *Q_X is even and $k = 1$.*

Then there are no reducible ASD connections on X and $\mathcal{M}_k$ is a smooth manifold.

For the proof see [4] or [3]. This gives us a complete description of the local structure of $\mathcal{M}_k$ so we now look at its global structure.

The moduli space $\mathcal{M}_k$ is not compact so we should analyse what happens as we "go off to infinity" in $\mathcal{M}_k$. To deal with this precisely we introduce the following definition.

Definition. An **ideal ASD connection** with Chern number k consists of a pair

$$([A]; \{x_1, \ldots, x_l\})$$

where $[A] \in \mathcal{M}_{k-l}$ and $\{x_1, \ldots, x_l\}$ is an unordered l tuple of points in X. The curvature of the ideal connection $([A]; \{x_1, \ldots, x_l\})$ is the measure

$$|F_A|^2 + 8\pi^2 \sum_{i=1}^{l} \delta_{x_i}$$

where $|F_A|^2$ is the pointwise norm of the curvature F_A.

Here $|F_A|^2 + 8\pi^2 \sum \delta_{x_i}$ is the measure which, for any continuous function f on X, gives the integral

$$\int_X f|F_A|^2 d\mu + 8\pi^2 \sum_{i=1}^{l} f(x_i)$$

where $d\mu$ is the measure on X defined by the metric. Note that we allow the possibility that $l = 0$, in which case we have a genuine ASD connection. We also allow the possibility $l = k$, in which case we have a flat connection on the product bundle on X and a set of k points in X; since X is simply connected it must follow that the flat connection is the trivial connection and we simply identify the ideal ASD connection with the set of points $\{x_1, \ldots, x_k\}$.

Definition. A sequence of ASD connections $[A_\alpha]$ **converges weakly** to the ideal ASD connection $([A]; \{x_1, \ldots, x_l\})$ if

(1) The sequence $|F_{A_\alpha}|^2$ converges to $|F_A|^2 + 8\pi^2 \sum \delta_{x_r}$ as measures.
(2) There are bundle isomorphisms

$$\rho_\alpha : P_l|_{X_0} \to P_k|_{X_0},$$

where $X_0 = X \backslash \{x_1, \ldots, x_l\}$, such that the sequence of connections $\rho_\alpha^* A_\alpha$ converges to A in the C^∞ topology on compact sets.

Here part (1) means that for each continuous function f on X

$$\int_X f|F_{A_\alpha}|^2 d\mu \to \int_X f|F_A|^2 d\mu + 8\pi^2 \sum_{i=1}^{l} f(x_i).$$

Now we have the following version of Uhlenbeck's weak compactness theorem.

Theorem. *Let $[A_\alpha]$ be a sequence of connections. Then there is a subsequence which converges weakly to an ideal ASD connection.*

The proof of this theorem is given in each of the main references. There is a very simple analogy which may help to understand ideal ASD connections and the weak compactness theorem. Let Rat_k be the space of meromorphic functions on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$; equivalently the space of holomorphic map $S^2 \to S^2$. Then such a function is completely determined, up to a constant, by its zeroes $\{z_1, \ldots, z_k\}$ and its poles $\{p_1, \ldots, p_k\}$. We can examine the behaviour of a sequence of such functions f_α where the poles $\{p_1, \ldots, p_k\}$ remain constant, one zero, say $z_1(\alpha)$, converges to one of the poles, say p_1, and the other zeroes $\{z_2, \ldots, z_k\}$ remain constant. Then this sequence does not converge to an element of Rat_k; rather it converges weakly, in exactly the sense described above, to the "ideal rational function" $(f; p_1)$ where the zeroes and poles of f are

$$\{z_2, \ldots, z_k\}, \qquad \{p_2, \ldots, p_k\}.$$

Here the role of the curvature is played by the energy density $|df|^2$.

The weak compactness theorem is used in many places in the theory. One immediate application is that it gives a compactification of the moduli spaces $\mathcal{M}_k$ as follows. Define $S^l(X)$, the **l-th symmetric product** of X, to be the space

$$S^l(X) = X^l / \Sigma_l$$

where X^l is the l-fold Cartesian product of X and the symmetric group Σ_l acts on X^l by permuting factors. Now define the **space of ideal ASD connections** to be

$$\mathcal{IM}_k = \bigcup_{l=0}^{k} \mathcal{M}_{k-l} \times S^l(X)$$

topologised so that sequences converge if and only if they converge weakly in the sense of the above definition. The weak compactnes theorem tells us that the space $\mathcal{IM}_k$ is compact. Now define the **compactified moduli space** $\bar{\mathcal{M}}_k$ to be the closure of $\mathcal{M}_k$ in $\mathcal{IM}_k$.

§3 Even intersection forms

Now suppose that X is a smooth simply connected 4-manifold with even intersection form. If X has definite intersection form we assume that $k = 1$. It follows that, for a generic metric on X, there are no reducible ASD connections, so $\mathcal{M}_k \subset \mathcal{B}_k^*$, and the moduli spaces $\mathcal{M}_k$ are smooth manifolds of dimension $8k - 3(1 + b^+)$. We can now form

$$H_2(X) \xrightarrow{\mu} H^2(\mathcal{B}_k^*) \to H^2(\mathcal{M}_k)$$

the second homomorphism is induced by the inclusion of $\mathcal{M}_k$ in $\mathcal{B}_k^*$. We still use the notation

$$\mu : H_2(X) \to H^2(\mathcal{M}_k)$$

for this homomorphism.

Recall that, geometrically, p-dimensional closed submanifolds of a manifold M define p-dimensional homology classes in M. On the other hand, codimension q submanifolds (which must have no boundary but need not be compact) define q dimensional cohomology classes in M. Each 2 dimensional homology class u in the 4-manifold X can be represented by a 2-dimensional surface $\Sigma_u \subset X$ and we now describe how to represent the cohomology class $\mu(u) \in H^2(\mathcal{M}_k)$ by a codimension 2 submanifold $V_u \subset \mathcal{M}_k$ and how this submanifold V_u is related to Σ_u.

Main Technical Lemma. *Let $\Sigma \subset X$ be a compact orientable surface with no boundary and let $u \in H_2(X)$ be the homology class represented by Σ. Let N_Σ be a sufficiently small tubular neighbourhood of Σ. Then we can find a smooth codimension 2 submanifold $V_\Sigma^{(k)} \subset \mathcal{M}_k$ with the following properties:*

(1) *The submanifold $V_\Sigma^{(k)} \subset \mathcal{M}_k$ represents the cohomology class $\mu(u) \in H^2(\mathcal{M}_k)$.*
(2) *Given surfaces $\Sigma_1, \dots, \Sigma_r \subset X$ in general position, the submanifolds $V_{\Sigma_i}^{(k)} \subset \mathcal{M}_k$ are in general position.*
(3) *Let $[A_\alpha]$ be a sequence of connections in $V_\Sigma^{(k)}$ which converges to an ideal connection $([A]; \{x_1, \dots, x_l\})$. Then either one of the points x_i must lie in the tubular neighbourhood N_Σ or the connection $[A]$ lies in $V_\Sigma^{(k-l)} \subset \mathcal{M}_{k-l}$.*

Assuming this result, which is proved in [2] and [4], we go on to outline the proof of the following theorem.

Theorem. *Let X be a smooth, simply connected 4-manifold with even intersection form Q_X.*

(1) *If Q_X is definite then $H_2(X) = 0$.*
(2) *Suppose that Q_X is indefinite, then*

$$b^+ = 1 \implies Q_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$b^+ = 2 \implies Q_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Proof of part (1). Our assumptions are that Q_X is definite and even. By changing the orientation of X if necessary we can assume that the intersection form of X is negative definite and so $b^+ = 0$. Then, for a generic metric on X, there are no irreducible ASD connections and so the moduli space $\mathcal{M}_1$ is a smooth 5-dimensional manifold.

Pick two surfaces $\Sigma_1, \Sigma_2 \subset X$ in general position which represent homology classes $u_1, u_2 \in H_2(X)$. Pick suitably small tubular neighbourhoods N_i of the surfaces Σ_i. Now the main lemma shows that we can find codimension 2 submanifolds $V_1, V_2 \subset \mathcal{M}_1$ which represent the classes $\mu(u_1), \mu(u_2) \in H^2(\mathcal{M}_1)$ and are in general position. Let

$$L = V_1 \cap V_2$$

so, since V_1 and V_2 are both 3-dimensional submanifolds of a 5-dimensional manifold it follows that L has dimension 1.

Now we count the number of ends of L. Recall the definition of an **end** of a topological space Y. Intuitively the number of ends of Y is the number of components of $Y \setminus C$ where C is a sufficiently large compact set. The precise definition is as follows. If C, D are compact sets with $D \subset C$ we get an inclusion

$$Y \setminus C \subset Y \setminus D$$

and this inclusion induces a map

$$\pi_0(Y \setminus C) \to \pi_0(Y \setminus D)$$

where π_0 means the set of (path) components. The number of ends of Y is the inverse limit

$$\varprojlim \pi_0(Y \setminus C)$$

and an end of Y is a component of the topological space

$$\varprojlim Y \setminus C.$$

If we take a sequence $[A_\alpha]$ of connections in L which converges to an ideal connection then, since $k = 1$, the only possibility is that it converges to the ideal connection given by the trivial connection on the product bundle and a single point in X. In view of part (3) of the main technical lemma this point must lie in $N_1 \cap N_2$. Now a direct geometrical argument proves the following lemma.

Lemma. *There is precisely one end of L for each component of $N_1 \cap N_2$.*

To prove this lemma, more generally to analyse the ends of the moduli spaces $\mathcal{M}_k$, it is necessary to use the "glueing construction" of Taubes. We will not go into this construction in detail, see Taubes's paper [14] and the basic references [4], [5], and [7] for details. The proof of the above lemma is given in [2] and [4].

Let us now complete the proof of part (1) of the theorem. The surfaces $\Sigma_1, \Sigma_2 \subset X$ are in general position so they meet in a finite number of points. Since Σ_i represents $u_i \in H^2(X)$ it follows that

$$Q_X(u_1, u_2) = |\Sigma_1 \cap \Sigma_2| \mod 2$$

where $|\Sigma_1 \cap \Sigma_2|$ is the number of points in the finite set $\Sigma_1 \cap \Sigma_2$. The neighbourhoods N_1 and N_2 can be chosen small enough so that the number of components of $N_1 \cap N_2$ is equal to the number of points of intersection of Σ_1 and Σ_2. The number of components of $N_1 \cap N_2$ is equal to the number of ends of L and since L is 1-dimensional it must have an even number of ends. Putting these facts together leads to the following conclusion: for all $u_1, u_2 \in H^2(X)$

$$Q_X(u_1, u_2) = 0 \mod 2.$$

Notice that our assumption is that Q_X is even, that is $Q_X(u, u) = 0 \mod 2$ for all $u \in H^2(X)$, and the conclusion is that $Q_X(u_1, u_2) = 0 \mod 2$ for all $u_1, u_2 \in H^2(X)$.

Now suppose that $H_2(X) \neq 0$ and pick a non-zero $u \in H_2(X)$. Then since Q_X is unimodular there must exist another element $v \in H_2(X)$ such that $Q_X(u, v) = 1$. But we have just established that $Q_X(u, v)$ is even and this contradiction shows that $H_2(X) = 0$.

Notice how the above argument contains three main steps:

(1) Use the given information about Q_X to determine the dimension of the moduli space.
(2) Now look at the intersection L of codimension 2 submanifolds of the form V_Σ and count the number of ends of L geometrically.
(3) Finally count the number of ends of L algebraically.

We now outline how to prove part (2) of the theorem by repeating the above steps.

Proof of part (2). First we consider the case where Q_X is even and indefinite, and $b^+ = 1$. In this case we use $\mathcal{M}_2$, which is a smooth manifold of dimension 10, and consider the intersections of codimension 2 submanifolds V_Σ. The contradiction comes from looking at four surfaces $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \subset X$ in general position and the corresponding codimension 2 submanifolds $V_1, V_2, V_3, V_4 \subset \mathcal{M}_2$ in general position. So we analyse the ends of

$$L = V_1 \cap V_2 \cap V_3 \cap V_4.$$

Let $[A_\alpha]$ be a sequence of connections in L which converges to an ideal connection. Since $k = 2$ there are two possibilities to consider:

(1) The limit ideal ASD connection is of the form $([A]; \{x\})$ with $[A] \in \mathcal{M}_1$ and $x \in X$.
(2) The limit is the product connection on the trivial bundle and a set two points $x, y \in X$.

We now use part (3) of the main technical lemma to show that the first case cannot happen. Since the surfaces Σ_i are in general position no three of them intersect and we can assume that the tubular neighbourhoods N_i are chosen sufficiently small so that no three of the N_i intersect. Thus the point x can lie in at most two of the N_i. For convenience let us suppose that x does not lie in N_3 nor in N_4. Now part (3) of the main technical lemma shows that, using the obvious notation, the connection $[A]$ must lie

$$V_3^{(1)} \cap V_4^{(1)} \subset \mathcal{M}_1.$$

But now we count dimensions; the dimension of $\mathcal{M}_1$ is 2 and so $V_3^{(1)}$ and $V_4^{(1)}$ are codimension 2 submanifolds of a 2-dimensional manifold which are in general position. Therefore

$$V_3^{(1)} \cap V_4^{(1)} = \varnothing$$

and so the first possibility cannot happen.

Thus the sequence $[A_\alpha]$ must converge to two points $x, y \in X$. Where can the points x, y lie? Since the surfaces are in general position no three of them intersect. We can suppose the neighbourhoods N_i are chosen small enough so that no three of them intersect and, for $i \neq j$, the number of components of $N_i \cap N_j$ is the same as the number of points of intersection of Σ_i and Σ_j. In this case the main technical lemma shows that each of the N_i must contain one of the points and we have just shown that the intersection of any three of the N_i must be empty. We can assume, by interchanging x and y if necessary, that $x \in N_1$ and then one of the following possibilities must hold:

(1) $x \in N_1 \cap N_2$, $y \in N_3 \cap N_4$
(2) $x \in N_1 \cap N_3$, $y \in N_2 \cap N_4$
(3) $x \in N_1 \cap N_4$, $y \in N_2 \cap N_3$.

Notice how the argument shows that if we had used five surfaces then

$$V_1 \cap V_2 \cap V_3 \cap V_4 \cap V_5 = \varnothing$$

where the V_i are the corresponding codimension 2 submanifolds of $\mathcal{M}_2$. Therefore this intersection is a compact 0 dimensional submanifold of $\mathcal{M}_2$ and so consists of a finite number of points. This fact leads to the definition of the Donaldson polynomials but in our present context it shows that we cannot use five surfaces in the present proof. Next we analyse the ends of L geometrically to prove the following result.

Lemma.

(1) *There is precisely one end of L for each (unordered) pair $\{C, D\}$ where C is a component of $N_i \cap N_j$, D is a component of $N_k \cap N_l$, and $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*
(2) *There is a compact set $K \subset L$ and a homeomorphism*

$$L \setminus K \to (0,1) \times \coprod \Lambda_{C,D}$$

where the disjoint union is taken over all (unordered) pairs $\{C, D\}$ as in part (1) and each $\Lambda_{C,D}$ is a compact 1-manifold.
(3) *There is a cohomology class $w_1 \in H^1(\mathcal{M}_2; \mathbb{Z}/2)$ such that*

$$\langle w_1, [\Lambda_{C,D}] \rangle = 1$$

where $[\Lambda_{C,D}]$ is the homology class defined by the compact 1-manifold $\Lambda_{C,D}$.

The proof of this lemma comes from the direct analysis of the ends of the moduli space and is given in detail in [2] and [4]. By part (3) we can truncate the space L by by removing open cylinders around the ends $\Lambda_{C,D}$ to produce a compact 2-manifold N with boundary such that

$$\partial N = \coprod \Lambda_{C,D}.$$

Thus our geometric analysis shows two things:

(1) The number of ends of L, counted modulo 2, is

$$Q_X(u_1,u_2)Q_X(u_3,u_4) + Q_X(u_1,u_3)Q_X(u_2,u_4) + Q_X(u_1,u_4)Q_X(u_2,u_3)$$

where $u_i \in H_2(X)$ is the homology class represented by the surface $\Sigma_i \subset X$.
(2) There is a cohomology class $w_1 \in H^1(\mathcal{M}_2; \mathbb{Z}/2)$ such that

$$\langle w_1, \partial N \rangle = \sum \langle w_1, [\Lambda_{C,D}] \rangle$$

and thus $\langle w_1, \partial N \rangle$ is the same, modulo 2, as the number of ends of L.

But, necessarily,

$$\langle w_1, \partial N \rangle = 0$$

and so we conclude that

$$Q_X(u_1,u_2)Q_X(u_3,u_4) + Q_X(u_1,u_3)Q_X(u_2,u_4) + Q_X(u_1,u_4)Q_X(u_2,u_3) = 0 \mod 2.$$

Now suppose that Q_X has rank > 2. Our hypothesis is that Q_X is indefinite and even and it follows that we can find elements $u_1, u_2, u_3, u_4 \in H_2(X)$ such that

$$Q_X(u_1,u_2) = Q_X(u_3,u_4) = 1 \mod 2$$
$$Q_X(u_1,u_3) = Q_X(u_2,u_4) = Q_X(u_1,u_4) = Q_X(u_2,u_3) = 0 \mod 2$$

One (rather crude) way to see this is to use the classification of even indefinite forms. Another, more direct, way is to work mod 2 and prove directly that any non-singular symmetric bilinear form Q over $\mathbb{Z}/2$ must have even rank, say $2r$, over $\mathbb{Z}/2$ and we can choose a basis $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$ such that

$$Q(\alpha_i, \beta_j) = \delta^i_j$$
$$Q(\alpha_i, \alpha_j) = 0$$
$$Q(\beta_i, \beta_j) = 0$$

where δ^i_j is the Kronecker δ.

Thus if the rank of Q_X is different from 2 we have a contradiction and, since Q_X is even and indefinite it follows that

$$Q_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This proves the result in the case $b^+ = 1$.

In the case $b^+ = 2$ a similar argument with the $k = 3$ moduli space and six codimension 2 submanifolds V_Σ gives a contradiction. If we now try the argument in the case $b^+ = 3$ with the $k = 4$ moduli space and eight codimension 2 submanifolds V_Σ the argument breaks down. In this case, if we take a sequence of connections $[A_\alpha]$ in L, the intersection of the eight codimension 2 submanifolds, which converges to an ideal ASD connection we can no longer conclude that the only possibility is that the limiting ideal ASD connection consists of four points in X and the trivial flat connection on the product bundle. Of course the argument must break down because of the existence of the Kummer surface.

§4 Some final remarks

To finish these notes I will make a couple of very brief remarks concerning the proof of Donaldson's theorem concerning definite intersection forms and the construction of the Donaldson polynomials.

Definite intersection forms. Suppose now that Q_X is definite—we can choose the orientation of X so that it is negative definite. Then we examine the moduli space $\mathcal{M}_1$. This moduli space has singularities, one singularity for each pair $\{u, -u\}$ where $u \in H_2(X)$ and

$$Q_X(u, u) = Q_X(-u, -u) = -1.$$

Let r be the number of such pairs. Note that if Q_X is even, which is the case we examined in some detail in the previous section, there are no singularities. In a neighbourhood of a non-singular point $\mathcal{M}_1$ is a smooth manifold of dimension 5. In a neighbourhood of a singular point the space $\mathcal{M}_1$ is homeomorphic to a cone on $\mathbb{CP}^2$ where the singularity corresponds to the cone point. Now we analyse the ends of $\mathcal{M}_1$ to prove that there is a compact set K, which includes the singularities, such that

$$\mathcal{M}_1 \setminus K \cong (0, 1) \times X.$$

Finally we must prove that the 5-dimensional manifold $\mathcal{M}_1^*$ is orientable. These facts are proved in detail in each of the three main references [4], [5], and [7].

Now we truncate the space $\mathcal{M}_1$ by removing an open neighbourhood of each of the singular points and cutting off the end to obtain a compact 5-dimensional oriented manifold N whose boundary consists of r-copies of $\mathbb{CP}^2$ together with a single copy of X. This gives a cobordism from X to r copies of $\mathbb{CP}^2$.

Now we use the fact that, by the Hirzebruch signature theorem, the signature is a cobordism invariant. Let us suppose that in the r copies of $\mathbb{CP}^2$ which occur in the boundary of N there are p-copies with the standard orientation, that is intersection form (1), and q with the opposite orientation, with intersection form (-1), where $p + q = r$. It follows that

$$\sigma(Q_X) = p - q.$$

However, from the definition of r, and the fact that Q_X is negative definite, it follows that

$$-\sigma(Q_X) \geq r = p + q.$$

Thus $q - p \geq q + p$ and so $p = 0$, $q = r$, and

$$\sigma(Q_X) = -r.$$

But it now follows, by a direct algebraic argument, that if we pick one element u_i, $1 \leq r$, from each of the r-pairs $\{u, -u\}$ with $Q_X(u, u) = -1$ then $u_1, \ldots, u_r$ is an integral basis for $H_2(X)$ in which the form is diagonal.

This argument is also described in Narasimhan's lecture [10].

The definition of Donaldson polynomials. Once more I will restrict to the case where Q_X is indefinite so there are no singularities in the moduli spaces $\mathcal{M}_k$. Suppose also that b^+ is odd, then the dimension of $\mathcal{M}_k$ is even, say

$$\dim \mathcal{M}_k = 2l.$$

Now pick l homology classes $u_1, \ldots, u_l \in H_2(X)$. Then we can form the cohomology class

$$\mu(u_1) \cdots \mu(u_l) \in H^{2l}(\mathcal{M}_k)$$

where the product is the cup product in cohomology. This is now a top-dimensional cohomology class in $\mathcal{M}_k$ and we would like to get an integer by evaluating this cohomology class on "the fundamental cycle" of $\mathcal{M}_k$. Of course we cannot do this without some further work since $\mathcal{M}_k$ is not compact.

One approach is to argue with the codimension 2 submanifolds $V_i \subset \mathcal{M}_k$ representing the classes $\mu(u_i)$ and their intersection

$$L = V_1 \cap \cdots \cap V_l.$$

The V_i are in general position so they intersect in a 0-dimensional manifold. Now we repeat the analysis of the ends of L, which was so important in the previous section, to prove that in this case, provided k is large enough, L has no ends. Therefore L is compact and it consists of a finite number of points. Next we must deal with orientations to attach a sign to each of the points in L. Finally we define $\Phi_l(u_1, \ldots, u_l)$ to be the number of points in L counted with signs. This argument is carried out in detail in [4] and [3].

Another approach to the definition of Φ_l is to compactify the moduli space $\mathcal{M}_k$ as in §2 to get $\bar{\mathcal{M}}_k$ and then to prove that the cohomology classes $\mu(u_i)$ extend over $\bar{\mathcal{M}}_k$. Here we must use the analysis of the ends of the moduli space which, in the previous approach led to the proof that L is finite. Next we must check that even though $\bar{\mathcal{M}}_k$ is not a manifold it is a "pseudo-manifold"; in particular the singularities of $\bar{\mathcal{M}}_k$ have codimension at least 2 and therefore there is a fundamental class

$$[\bar{\mathcal{M}}_k] \in H^{2l}(\bar{\mathcal{M}}_k).$$

Now we can define

$$\Phi_l(u_1, \ldots, u_l) = \langle \mu(u_1) \cdots \mu(u_l), [\bar{\mathcal{M}}_k] \rangle.$$

This second approach is also discussed in [4] and [3].

The most difficult part of the definition of the polynomials Φ_l is the proof that they do not depend on the metric on X. The idea is most easily expressed using the codimension 2 submanifolds and the finite set of points L. Then the strategy of the proof is that a path of metrics joining a metric g to a metric g' will provide a cobordism between the corresponding finite sets L and L'. Once more the details are given in [4] and [3].

References

1. S. K. Donaldson, *An application of gauge theory to four dimensional topology*, Journal of Differential Geometry **18** (1983), 279–315.
2. S. K. Donaldson, *Connections, cohomology and the intersection forms of four manifolds*, Journal of Differential Geometry **24** (1986), 275–341.
3. S. K. Donaldson, *Polynomial invariants for smooth 4-manifolds*, Topology **29** (1990), 257–315.
4. S. K. Donaldson and P. B. Kronheimer, *The geometry of four manifolds*, Oxford University Press, Oxford, UK, 1990.
5. D. S. Freed and K. K. Uhlenbeck, *Instantons and four-manifolds*, MSRI publications, Vol 1, Springer-Verlag, New York, 1984.
6. M. H. Freedman, *The topology of four-dimensional manifolds*, Journal of Differential Geometry **17** (1982), 357–454.
7. H. B. Lawson, *The theory of gauge fields in four dimensions*, CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1985.
8. J. Milnor, *On simply connected 4-manifolds*, Symposium Internacionale Topolgia Algebraica, Mexico, 1958, pp. 122–128.
9. J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Springer-Verlag, Berlin, 1973.
10. M. S. Narasimhan, *Notes from a lecture at this workshop*.
11. T. R. Ramadas, *Notes from lectures at this workshop*.
12. V. A. Rohlin, *New results in the theory of four dimensional manifolds*, Dok. Akad. Nauk. USSR **84** (1952), 221–224.
13. J.-P. Serre, *A course in arithmetic*, Springer-Verlag, Berlin, 1973.
14. C. H. Taubes, *Self-dual connections over non-self-dual four manifolds*, Journal of Differential Geometry **17** (1982), 139–170.

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