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Algebraic Geometry, Fay's Identity and Correlation Functions

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ABSTRACT

The aim of these lectures is to show how algebraic geometry can be used to formulate and analyse a problem in physics. We shall study a particular quantum field theory defined over a compact Riemann surface. The model we shall discuss first arose in string theory, but has been subsequently studied by physicists as an interesting model in its own right. We shall investigate whether the so-called *correlation functions* of the system (defined as meromorphic sections of line bundles over the product of copies of the Riemann surface) are determined by physical data, viz. certain zeros and poles which they are required to have. Algebraic geometry not only helps in answering this question, but also in determining explicit expressions for the correlation functions. This analysis gives a new proof of an important identity for theta functions due to Fay. We shall also show how algebraic geometry enables us to deal with more complicated physical situations, such as when zero modes, branch point singularities, or different statistics are involved. We shall try to show how the essential physics is brought out by the algebraic geometry formulation, which provides a very natural language as well as effective computational tools.

CONTENTS

1. Formulation of the problem

We shall consider a model quantum field theory, defined over a Riemann surface, and show how algebraic geometry can be used effectively in its study. The model is one which arises naturally in *string theory*, but no knowledge of physics will be assumed nor required. We shall only introduce some terminology from physics as motivation.

The model consists of a pair of *quantum fields* b, c on a compact connected Riemann surface M of genus $g \geq 0$. A quantum field can in general be thought of as a kind of generalised *random variable*. Quantum fields are extremely singular objects and so we shall not deal directly with them here, but only with certain functionals of them, known as *correlation functions*. These are, in fact, the quantities of interest and we shall show how they can be determined in this model from some physical input, using only algebraic geometry, without dealing directly with the fields.

The general correlation function of the system is written *symbolically* as

$$C(m,n) \equiv \langle b(Q_1)\dots b(Q_m)c(P_1)\dots c(P_n)\rangle\,, \tag{1.1}$$

where the Q's and P's are arbitrary points on M. Intuitively, a correlation function of this form should represent the expectation of finding m particles of the b field at $Q_1,\dots,Q_m$ and n of the c field at $P_1,\dots,P_n$ in their ground state. This should not be taken too literally here as (1.1) is merely an amplitude appearing in string theory calculations.

We shall denote by K the holomorphic cotangent bundle of M and by $Pic^d(M)$ the set of holomorphic line bundles over M of degree (or Chern class) $d \in \mathbb{Z}$. Experience with this model system on the complex plane suggests that, over the Riemann surface M, the $C(m,n)$ should have the following properties:

(P1) $C(m,n)$ is a meromorphic section of $\alpha \in Pic^{g-1}(M)$ in each P-variable and of $K\otimes\alpha^{-1} \in Pic^{g-1}(M)$ in each Q-variable, where $H^0(M,\alpha) = 0$.

(P2) $C(m,n)$ has a simple zero when the arguments of two b fields, or of two c fields, coincide (other variables being in general position).

(P3) $C(m,n)$ has a simple pole when the arguments of a b field and a c field coincide (other variables being in general position).

(P4) $C(m,n)$ has no poles other than those required by (P3).

While the mathematical meaning of these conditions on $C(m,n)$ is reasonably clear, some explanation is surely required. The first condition (P1) is a generalisation of the situation in the physics problem, where b and c are normally taken to be 'sections' (operator-valued) of a ***holomorphic spin bundle*** or ***theta characteristic***, i.e. one of the 4^g elements of $Pic^{g-1}(M)$ whose tensor square is K. In physics parlance a field associated to a theta characteristic is said to have ***conformal spin*** 1/2. We generalise this to the '***twisted spin 1/2***' case by associating any $\alpha \in Pic^{g-1}(M)$ to the field c, while associating $K \otimes \alpha^{-1}$ to the ***conjugate field*** b. This is consistent with the ***action principle*** (i.e. a variational principle whose Euler-Lagrange equations give the equations of motion) of the classical version of our quantum system, viz.

$$S \sim \int_M b\bar{\partial}c, \tag{1.2}$$

and so we naturally require $b\bar{\partial}c$ to be a volume form on M.

The condition that α should have no holomorphic section can be thought of as requiring that the generalised random variables b and c have zero mean value, i.e. that the one point correlation functions $C(1,0), C(0,1)$ vanish:

$$H^0(M,\alpha) = 0 = H^0(M, K \otimes \alpha^{-1}) \tag{1.3a}$$

$$< b >= 0 =< c > . \tag{1.3b}$$

As we can see, the condition (P1) contains a lot of rather detailed restrictions coming from the physical system. We shall re-examine it in section 5 and strive for a deeper understanding.

Condition (P2) should be regarded as a way of realising the condition that the fields b, c are *fermionic*, i.e. (P2) realises the ***exclusion principle***. This is not the only way it can be done and we shall come back to the question of statistics in section 7, but it turns out that (P2) is a very interesting way to realise this condition.

Condition (P3) comes from the notion that b and c are ***conjugate fields***, i.e. that the coefficients of their Laurent expansions satisfy $[b_n, c_m]_\pm = \delta_{nm}$, where + (resp. −) indicates

the anticommutator (resp. commutator), for the *fermionic* (resp. *bosonic*) case. Condition (P4) says that all poles must have this physical origin.

It is instructive to look at the solution of the model on the complex plane, where it represents a system of free particles. In that case it is known that

$$C(m,n) = \delta_{mn} det\ |\langle b(Q_i)c(P_j)\rangle|_{i,j=1}^n \tag{1.4a}$$

where the two point function is given by

$$< b(Q)c(P) >= \frac{1}{Q-P} . \tag{1.4b}$$

Note from (1.4) that $C(m,n) = 0$ for $m \neq n$. The physical reason for this is ***charge conservation***: the fields b and c are supposed to carry equal and opposite charge. For $m = n$ the ***2n-point function*** is a determinant of two point functions. This is known as ***Wick's theorem for a system of free fermions***. We can regard it as the definition of a free fermion system. It is the analogue for fermions of the condition for random variables to be uncorrelated.

We thus see that the conditions (P1) - (P4) that we have imposed on the $C(m,n)$ have a definite physical origin and meaning. At the same time they have a definite mathematical meaning. It is not clear at this point whether more input is needed from physics in order to determine the $C(m,n)$. To investigate that we must first put conditions (P1) - (P4) in a more convenient and concise form.

Let $\{M_i | 1 \leq i \leq m+n\}$ be $m+n$ copies of M and denote by $M^{m+n} = \prod_1^{m+n} M_i$ the product manifold of the $m+n$ copies. Let

$$\begin{aligned} p_i : M^{m+n} &\to M \\ (z_1, \ldots, z_i, \ldots, z_{m+n}) &\mapsto z_i \end{aligned} \tag{1.5}$$

denote the i-th canonical projection. We denote by Θ the subset of $Pic^{g-1}(M)$ consisting of holomorphic line bundles on M of degree $g-1$ having at least one nonzero holomorphic section (it is the empty set if $g = 0$). Θ is often called the ***canonical theta divisor***. Then choosing $\alpha \in Pic^{g-1}(M) - \Theta$ we define the holomorphic line bundle

$$\mathcal{F}_\alpha(m,n) = p_1^*(K \otimes \alpha^{-1}) \otimes \ldots \otimes p_m^*(K \otimes \alpha^{-1}) \otimes p_{m+1}^*(\alpha) \otimes \ldots \otimes p_{m+n}^*(\alpha) \tag{1.6}$$

over M^{m+n}.

Let Δ_{ij} denote the diagonal of $M_i \times M_j$ and

$$\begin{aligned} pr_{ij} &: M^{m+n} \to M_i \times M_j \\ (z_1, \ldots, z_i, \ldots, z_j, \ldots, z_{m+n}) &\to (z_i, z_j) \end{aligned} \tag{1.7}$$

the canonical projection to $M_i \times M_j$. Then

$$D_{ij} \equiv pr_{ij}^{-1}(\Delta_{ij}) \tag{1.8}$$

is an element of $Div(M^{m+n})$, the ***divisor group*** of M^{m+n}. Now condition (P2) defines a certain element $D_z(m,n) \in Div(M^{m+n})$, which we call the ***divisor of physical zeros.*** Clearly,

$$D_z(m,n) = \sum\nolimits' D_{ij} + \sum\nolimits'' D_{ij}, \tag{1.9a}$$

where $\sum'$ (resp. $\sum''$) runs over $1 \le i < j \le m$ (resp. $m+1 \le i < j \le m+n$). Condition (P3) similarly gives us $D_p(m,n)$, the ***divisor of physical poles:***

$$D_p(m,n) = \sum D_{ij}, \tag{1.9b}$$

where $\sum$ runs over $1 \le i \le m, m+1 \le j \le m+n$. Then

$$D(m,n) \equiv D_z(m,n) - D_p(m,n) \tag{1.9c}$$

is the total divisor of physical zeros and poles. In the following we shall denote by $\{D_z(m,n)\}$ (resp. $\{D_p(m,n)\}$) the set of D_{ij} appearing on the right-hand side of (1.9a) (resp. (1.9b)).

As is well known, a divisor defines a holomorphic line bundle and a meromorphic section whose divisor of zeros and poles is the given divisor. We denote the line bundle on M^{m+n} defined by $D(m,n)$ as $\mathcal{O}(D(m,n))$. Define

$$\mathcal{M}_\alpha(m,n) \equiv \mathcal{F}_\alpha(m,n) \otimes \mathcal{O}(-D(m,n)). \tag{1.10}$$

Then condition (P4) tells us that $C(m,n)$ is determined by an element of $H^0(M^{m+n}, \mathcal{M}_\alpha(m,n))$. To determine to what extent $C(m,n)$ is fixed by our four conditions, we must clearly determine $\dim H^0(M^{m+n}, \mathcal{M}_\alpha(m,n))$. This is, however, a well defined mathematical problem to which we now turn our attention.

2. Two basic lemmas

Our computations of $H^0(M^{m+n}, \mathcal{M}_\alpha(m,n))$ are inductive and ultimately depend on the fact that $H^0(M,\alpha) = 0$ and $dim\ H^0(M,\mathcal{O}) = 1$. The reason that induction works is explained by Lemmas 2.1 and 2.2 below, which bring out the remarkable structure of $\mathcal{M}_\alpha(m,n)$.

Lemma 2.1. $H^0(M^{m+n}, \mathcal{M}_\alpha(m,n)) = H^0(D_p(m,n), \mathcal{M}_\alpha(m,n)|D_p(m,n))$.

Proof. We have the canonical short exact sequence

$$0 \to \mathcal{O}(-D_p(m,n)) \to \mathcal{O} \to \mathcal{O}|D_p(m,n) \to 0 \tag{2.1}$$

between the ideal sheaf of the closed subscheme $D_p(m,n)$, the structure sheaf of M^{m+n} and the structure sheaf of $D_p(m,n)$. Tensoring (2.1) by $\mathcal{M}_\alpha(m,n)$ (exactness is preserved) and passing to cohomology, we see that we need to prove that

$$H^i(M^{m+n}, \mathcal{F}_\alpha(m,n) \otimes \mathcal{O}(-D_z(m,n))) = 0 \text{ for } i = 0,1. \tag{2.2}$$

The case $i=0$ in (2.2) is trivial, since $H^i(M^{m+n}, \mathcal{F}_\alpha(m,n)) = 0$ for $i = 0,1$ by (1.3a) and the Künneth formula. To prove (2.2) for $i = 1$, we replace $D_p(m,n)$ in (2.1) by $D_z(m,n)$ and tensor the new short exact sequence by $\mathcal{F}_\alpha(m,n)$. Passing to cohomology gives us

$$H^1(M^{m+n}, \mathcal{F}_\alpha(m,n) \otimes \mathcal{O}(-D_z(m,n))) = H^0(D_z(m,n), \mathcal{F}_\alpha(m,n)|D_z(m,n)).$$

By direct computation and the Künneth formula we find that:

$$H^0(D_{ij}, \mathcal{F}_\alpha(m,n)|D_{ij}) = 0 \text{ for each } D_{ij} \in \{D_z(m,n)\}.$$

This completes the proof of (2.2) and hence of Lemma 2.1.

Let us define pr_{ij}^D to be the canonical projection

$$pr_{ij}^D : D_{ij} \equiv \Delta_{ij} \times \prod_{\substack{p=1 \\ p \neq i,j}}^{m+n} M_p \to \Delta_{ij} \tag{2.3a}$$

and π_{ij}^D to be the canonical projection

$$\pi_{ij}^D \;:\; D_{ij} \equiv \Delta_{ij} \times \prod_{\substack{p=1\\ p\neq i,j}}^{m+n} M_p \to \prod_{\substack{p=1\\ p\neq i,j}}^{m+n} M_p \tag{2.3b}$$

Lemma 2.2. Let $m+n>2$. Then for any $D_{ij} \in \{D_p(m,n)\}$ we have

$$H^0(D_{ij},\ \mathcal{M}_\alpha(m,n)|D_{ij}) = H^0(M^{m+n-2},\ \mathcal{M}_\alpha(m-1,n-1))$$

where

$$M^{m+n-2} = \prod_{\substack{p=1\\ p\neq i,j}}^{m+n} M_p.$$

Proof. We have

$$\mathcal{F}_\alpha(m,n)|D_{ij} = \pi_{ij}^{D*}(\mathcal{F}_\alpha(m-1,n-1)) \otimes pr_{ij}^{D*}(K_{ij})$$

where K_{ij} denotes the canonical bundle on Δ_{ij}.

$$D(m,n) = D(m-1,n-1) - D_{ij} + \sum_{\substack{p=1\\ p\neq i}}^{m} (D_{pi} - D_{pj}) + \sum_{\substack{p=m+1\\ p\neq j}}^{m+n} (D_{pj} - D_{ip})$$

$$\mathcal{O}(-D(m,n))|D_{ij} = \pi_{ij}^{D*}(\mathcal{O}(-D(m-1,n-1)) \otimes \mathcal{O}(D_{ij})|D_{ij} \otimes \mathcal{O}\left(\sum_{\substack{p=1\\ p\neq i}}^{m}(D_{pi} - D_{pj}) + \sum_{\substack{p=m+1\\ p\neq j}}^{m+n}(D_{pj} - D_{ip})\right)|D_{ij}$$

$$= \pi_{ij}^{D*}(\mathcal{O}(-D(m-1,n-1))) \otimes pr_{ij}^{D*}(K_{ij}^{-1})$$

Thus

$$\mathcal{M}_\alpha(m,n)|D_{ij} = \pi_{ij}^{D*}(\mathcal{M}_\alpha(m-1,n-1)).$$

Hence by the Künneth formula

$$H^0(D_{ij}, \mathcal{M}_\alpha(m,n)|D_{ij}) = H^0\left(M^{m+n-2}, \mathcal{M}_\alpha(m-1,n-1)\right).$$

3. Computation of $H^0(M^{m+n},\ \mathcal{M}_\alpha(m,n))$

We now consider the computation of $H^0(M^{m+n},\ \mathcal{M}_\alpha(m,n))$. It turns out that the case when $m \neq n$ is much simpler than when $m=n$ and so we take these two cases separately. The first case is in fact an easy consequence of Lemmas 2.1 and 2.2.

Theorem 3.1. Let $m \neq n$. Then

$$H^0\left(M^{m+n},\ \mathcal{M}_\alpha(m,n)\right) = 0.$$

Proof. From (1.3a) and the postulates (P1) - (P4), we see immediately that for $m \neq 0$,

$$H^0(M^m, \mathcal{M}_\alpha(m,0)) = 0 = H^0(M^m, \mathcal{M}_\alpha(0,m)). \tag{3.1}$$

Thus since $m \neq n$ we need to do the computation of $H^0(M^{m+n}, \mathcal{M}_\alpha(m,n))$ only for $m \neq 0$, $n \neq 0$ and $m+n>2$.

By Lemma 2.1 we have to compute $H^0(D_p(m,n), \mathcal{M}_\alpha(m,n)|D_p(m,n))$.

By Lemma 2.2, however, we have

$$H^0(D_{ij}, \mathcal{M}_\alpha(m,n)|D_{ij}) = H^0\left(M^{m+n-2}, \mathcal{M}_\alpha(m-1,n-1)\right) \tag{3.2}$$

for each $D_{ij} \in \{D_p(m,n)\}$. The calculation is thus reduced inductively to the case when m or n vanishes so that (3.1) applies. Thus the left-hand side of (3.2) vanishes for each $D_{ij} \in \{D_p(m,n)\}$. Hence the result.

Remark 3.2. From Theorem 3.1 we deduce that the correlation function $C(m,n) = 0$ for $m \neq n$. This means that we have obtained from our postulates a result obtained in the physics literature from current conservation, as explained in Sect. 1.

We now consider the case when $m = n$. Our approach will again be inductive. We shall start the induction at $n = 1$ and so we first have to consider $\mathcal{M}_\alpha(1,1)$, which determines the *two point function* $C(1,1)$.

Proposition 3.3. Let Δ denote the diagonal of $M \times M$. Then

$$H^0(M \times M, \mathcal{M}_\alpha(1,1)) = H^0(\Delta, \mathcal{O}|\Delta) = \mathbb{C}. \tag{3.3}$$

Proof. We have the canonical short exact sequence

$$0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}|\Delta \to 0, \tag{3.4}$$

where $\mathcal{O}$ is the structure sheaf of $M \times M$. Tensoring (3.4) by $\mathcal{M}_\alpha(1,1)$ and passing to cohomology we obtain (3.3).

The next proposition completes the computation. Since it will be proved inductively it will be convenient on occasion to use the same symbol $D_{ij} \equiv pr_{ij}^{-1}(\Delta_{ij})$ even when the domain of pr_{ij} is M^{2n-2} or M^{2n-4}. This will be clear from the context. We shall also denote the diagonal of $M_i \times M_j \times M_k$ by Δ_{ijk} and of M^{2n} by Δ_{2n}.

Proposition 3.4. Let D_{ij}, D_{rs} be any two distinct elements of $\{D_p(m,n)\}$. Then we have the following commutative diagram of canonical isomorphisms:

$$\begin{array}{ccccc} H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) & \xrightarrow{a} & H^0(D_{ij}, \mathcal{M}_\alpha(n,n)|D_{ij}) & \xrightarrow{b} & H^0(\Delta_{2n}, \mathcal{O}|\Delta_{2n}) \\ & & \searrow^{c} & \nearrow_{d} & \\ & & & H^0(D_{ij} \cap D_{rs}, \mathcal{M}_\alpha(n,n)|D_{ij} \cap D_{rs}) & \end{array}$$

Proof. The proof is by induction and so let us assume that it holds up to $n-1$. (Since we have Prop. 3.3 for $n = 1$, we take $n \geq 2$ in the following). Then,

$$H^0(D_{ij}, \mathcal{M}_\alpha(n,n)|D_{ij}) = H^0(M^{2n-2}, \mathcal{M}_\alpha(n-1,n-1)) \qquad \text{(Lemma 2.2)}$$

$$\begin{aligned} &= H^0(\Delta_{2n-2}, \mathcal{O}|\Delta_{2n-2}) \qquad (ba \text{ for } n-1) \\ &= H^0(\Delta_{2n}, \mathcal{O}|\Delta_{2n}) \end{aligned}$$

Thus arrow b is an isomorphism.

To consider arrows c and d and the commutativity of the triangle of isomorphisms, it is necessary to consider two distinct cases for D_{rs}. Let us first choose $r = k$, $s = \ell$ where $i \neq j \neq k \neq \ell$. Then

$$D_{ij} \cap D_{k\ell} = \Delta_{ij} \times \Delta_{k\ell} \times M^{2n-4}$$

where

$$M^{2n-4} = \prod_{\substack{p=1 \\ p \neq i,j,k,\ell}}^{2n} M_p.$$

Then

$$\begin{aligned} H^0(D_{ij} \cap D_{k\ell}, \mathcal{M}_\alpha(n,n)|D_{ij} \cap D_{k\ell}) &= H^0(M^{2n-4}, \mathcal{M}_\alpha(n-2,n-2)) \\ &= H^0(\Delta_{2n-4}, \mathcal{O}|\Delta_{2n-4}) \\ &= H^0(\Delta_{2n}, \mathcal{O}|\Delta_{2n}), \end{aligned} \tag{3.5}$$

which proves d is an isomorphism in this case.

$$\begin{aligned} H^0(D_{ij}, \mathcal{M}_\alpha(n,n)|D_{ij}) &= H^0(M^{2n-2}, \mathcal{M}_\alpha(n-1,n-1)) \\ &= H^0(D_{k\ell}, \mathcal{M}_\alpha(n-1,n-1)|D_{k\ell}) \qquad (a \text{ for } n-1) \\ &= H^0(M^{2n-4}, \mathcal{M}_\alpha(n-2,n-2)) \\ &= H^0(D_{ij} \cap D_{k\ell}, \mathcal{M}_\alpha(n,n)|D_{ij} \cap D_{k\ell}) \quad \text{(by (3.5))} \end{aligned}$$

which proves c is an isomorphism and completes the proof of the commutativity of the triangle of isomorphisms in this case.

The only other distinct case we need to consider is when $r = j,\ s = k$. Then

$$D_{ij} \cap D_{jk} = \Delta_{ijk} \times M^{2n-3},$$

where

$$M^{2n-3} = \prod_{\substack{p=1 \\ p \neq i,j,k}}^{2n} M_p$$

$$\begin{aligned} H^0(D_{ij} \cap D_{jk}, \mathcal{M}_\alpha(n,n)|D_{ij} \cap D_{jk}) \\ = H^0(\Delta_{ijk} \times M^{2n-3}, \mathcal{M}_\alpha(n-1,n-1)|\Delta_{ijk} \times M^{2n-3}) \quad (3.6) \end{aligned}$$

where $\mathcal{M}_\alpha(n-1,n-1)|\Delta_{ijk} \times M^{2n-3}$ denotes the pullback of $\mathcal{M}_\alpha(n-1,n-1)$ defined over $M^{2n-2} = \prod_{\substack{t=1 \\ t \neq i,j}}^{2n} M_t$ by the canonical isomorphism

$$\Delta_{ijk} \times M^{2n-3} \to M^{2n-2} = M_k \times M^{2n-3}.$$

Thus the right-hand side of (3.6) is canonically isomorphic to

$$\begin{aligned} H^0(M^{2n-2}, \mathcal{M}_\alpha(n-1,n-1)) &= H^0(\Delta_{2n-2}, \mathcal{O}|\Delta_{2n-2}) \\ &= H^0(\Delta_{2n}, \mathcal{O}|\Delta_{2n}), \end{aligned}$$

which proves that d is an isomorphism in this case.

$$\begin{aligned} H^0(D_{ij}, \mathcal{M}_\alpha(n,n)|D_{ij}) &= H^0(M^{2n-2}, \mathcal{M}_\alpha(n-1,n-1)) \\ &= H^0(\Delta_{ijk} \times M^{2n-3}, \mathcal{M}_\alpha(n-1,n-1)|\Delta_{ijk} \times M^{2n-3}) \\ &= H^0(D_{ij} \cap D_{jk}, \mathcal{M}_\alpha(n,n)|D_{ij} \cap D_{jk}), \end{aligned}$$

which proves that c is an isomorphism and that the triangle is commutative in this case as well.

The arrow a alone remains to be discussed. For that it is clearly sufficient to prove isomorphism for the composite ba.

By Lemma 2.1 we have

$$H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) = H^0(D_p(n,n), \mathcal{M}_\alpha(n,n)|D_p(n,n)).$$

Since Δ_{2n} is contained in each $D_{ij} \in \{D_p(n,n)\}$ we have a natural map

$$H^0(D_p(n,n), \mathcal{M}_\alpha(n,n)|D_p(n,n)) \to H^0(\Delta_{2n}, \mathcal{M}_\alpha(n,n)|\Delta_{2n}).$$

This map is injective, since a holomorphic section s of $\mathcal{M}_\alpha(n,n)|D_p(n,n)$ vanishes if it vanishes on Δ_{2n}. To see this, assume such a s is given. Then, by the horizontal arrow b, $s|D_{ij}$ vanishes for each of the divisors whose sum is $D_p(n,n)$. Hence $s = 0$. This implies that (since $\mathcal{M}_\alpha(n,n)|\Delta_{2n} = \mathcal{O}|\Delta_{2n}$)

$$dim\ H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) \leq dim\ H^0(\Delta_{2n}, \mathcal{O}|\Delta_{2n}) = 1 \qquad (3.7)$$

To show that equality holds, take a nonzero section t of $\mathcal{O}|\Delta_{2n}$. Let t_{ij} be the section of $\mathcal{M}_\alpha(n,n)|D_{ij}$ which goes into t under the isomorphism b. Then by the commutativity of the triangle which we have established, t_{ij} and t_{rs} coincide on $D_{ij} \cap D_{rs}$ for any two distinct elements D_{ij}, D_{rs} of $\{D_p(n,n)\}$. Hence the $\{t_{ij}\}$ can be patched consistently to give a nonzero section. Thus

$$H^0(D_p(n,n), \mathcal{M}_\alpha(n,n)|D_p(n,n)) \cong H^0(\Delta_{2n}, \mathcal{O}|\Delta_{2n}). \qquad (3.8)$$

From the horizontal arrows of the commutative diagram of Prop. 3.4 we get:

Theorem 3.5. *$dim\ H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) = 1$.*

Remark 3.6. Theorem 3.5 proves that $\mathcal{M}_\alpha(n,n)$ has a unique nonzero holomorphic section. Thus the correlation function $C(n,n)$ is also unique (up to a multiplicative constant).

4. Wick's Theorem and identities of Cauchy, Frobenius and Fay

We have shown that the $2n$-point function $C(n,n)$ is uniquely determined by our conditions (P1) - (P4) for each $n \geq 1$. Now the two point function $< b(Q)c(P) >$ is a meromorphic section of $p_1^*(K \otimes \alpha^{-1}) \otimes p_2^*(\alpha)$ whose divisor of physical zeros and poles is simply the polar divisor $-\Delta_{12}$. Consider the determinant

$$det\,(\langle b(Q_i)c(P_j)\rangle)\Big|_{i,j=1}^{n} \qquad (4.1)$$

It is clear that this is a meromorphic section of $\mathcal{F}_\alpha(n,n)$ and, on examining conditions (P1) - (P4), it is clear that it satisfies each of them. By our uniqueness theorem, this means that with appropriate normalisation the $2n$-point function $C(n,n)$ is given by the determinant of its two point functions. Recalling the discussion of (1.4), we see that Wick's theorem holds for our system on the compact Riemann surface M. Thus in physical terms we have proved that our system is one of free fermions on M. This conclusion is independent of the genus g of M. We shall now show how this observation leads to interesting identities for each of the three cases (a) $g = 0$, (b) g=1, $(c) g \geq 2$.

(a) Case $g = 0$.

In this case we have:

Proposition 4.1. $\mathcal{M}_\alpha(n,n)$ is the trivial line bundle on M^{2n}.

Proof. An easy consequence of the fact that for $g = 0$, $Pic(M^{2n}) = \mathbb{Z}^{2n}$.

As a consequence

$$\mathcal{F}_\alpha(n,n) = \mathcal{O}(D(n,n)) \qquad (4.2)$$

as line bundles on M^{2n}. Thus the $2n$-point function $C(n,n)$ is given by the canonical meromorphic section of $\mathcal{O}(D(n,n))$ with divisor $D(n,n)$. For $g = 0$, M can be identified with the *complex projective line* $\mathbb{P}^1$, which can itself be identified with two copies U_0, U_∞ of the complex affine line $\mathbf{A}^1$ glued together. We shall write down $C(n,n)$ explicitly on one of the affine lines. Now $\mathcal{O}(D(n,n))$ is simply a combination of $\mathcal{O}(D_{ij})$ and the latter is simply a pullback of $\mathcal{O}(\Delta_{ij})$ on $M_i \times M_j$. The holomorphic section of the latter with

divisor Δ_{ij} is simply $(z_i - z_j)$ in affine coordinates on U_0. We thus get

$$C(n,n) = \frac{\prod_{1 \leq i<j \leq n}(Q_i - Q_j)(P_j - P_i)}{\prod_{1 \leq i,j \leq n}(Q_i - P_j)} . \qquad (4.3)$$

on $(U_o)^{2n}$. On the other hand, we have shown that $C(n,n)$ has a determinantal form as well and so

$$C(n,n) = const. \times det\left(\frac{1}{(Q_i - P_j)}\right)\Big|_{i,j=1}^{n} . \qquad (4.4)$$

It is easy to check that the constant in (4.4) is unity. We thus get:

Theorem 4.2 (Cauchy's identity):

$$\frac{\prod_{1 \leq i<j \leq n}(Q_i - Q_j)(P_j - P_i)}{\prod_{1 \leq i,j \leq n}(Q_i - P_j)} = det\left(\frac{1}{Q_i - P_j}\right)\Big|_{i,j=1}^{n} \qquad (4.5)$$

(b) The case $g = 1$.

In this case the Riemann surface M and its Jacobian variety $Pic^0(M)$ are isomorphic under the Abel map. We shall consequently identify them when convenient. Moreover, K is the trivial line bundle on M and Θ consists of just one element in $Pic^0(M)$, viz. the neutral element corresponding to the trivial line bundle. Corresponding to Prop. 4.1 we have:

Proposition 4.3. Let ξ be any element of $Pic^0(M)$ and let ϕ_ξ^n denote the map

$$\phi_\xi^n : M^{2n} \to Pic^0(M)$$
$$(Q_1, \ldots, Q_n, P_1, \ldots, P_n) \to \mathcal{O}\Big(\sum_1^n Q_i - \sum_1^n P_i\Big) \otimes \xi$$

Then

$$\mathcal{M}_\xi(n,n) = \phi_\xi^{n*}(\mathcal{O}(\Theta)).$$

We can understand the principle of the proof by confining overselves to the case when $n = 1$.

Corollary 4.4. Let ξ be an arbitrary element of $Pic^0(M)$ and let ϕ^1_ξ denote the map

$$\begin{aligned}\phi^1_\xi : M \times M &\to Pic^0(M)\\ (Q,P) &\to \mathcal{O}(Q-P)\otimes\xi.\end{aligned}$$

Then

$$\mathcal{M}_\xi(1,1) = p_1^*(\xi^{-1})\otimes p_2^*(\xi)\otimes\mathcal{O}(\Delta) = \phi^{1*}_\xi(\mathcal{O}(\Theta)).$$

Proof. We can use the ***seesaw principle*** in the following form: if L is a holomorphic line bundle on $X\times Y$, where X and Y are compact connected complex manifolds, such that for each $x\in X$ the restriction $L|\{x\}\times Y$ is trivial and for each $y\in Y$ the restriction $L|X\times\{y\}$ is trivial, then L is trivial.

Restricting $\mathcal{M}_\xi(1,1)$ to $\{Q\}\times M$ we get $\xi\otimes\mathcal{O}(Q)$. The restriction of $\phi^{1*}_\xi(\mathcal{O}(\Theta))$ is the pullback of $\mathcal{O}(\Theta)$ by the map $P\to\xi\otimes\mathcal{O}(Q-P)$. Recalling that Θ is the trivial line bundle on M it is easy to check that this is also $\xi\otimes\mathcal{O}(Q)$. Similarly for the other side of the seesaw.

Corollary 4.5. Let ϕ^1_0 denote the map

$$\begin{aligned}\phi^1_0 : M \times M &\to Pic^0(M)\\ (Q,P) &\to \mathcal{O}(Q-P)\end{aligned}$$

Then,

$$\mathcal{O}(\Delta) = \phi^{1*}_0(\mathcal{O}(\Theta)).$$

Proof. Put ξ as the trivial line bundle in Corollary 4.4.

We remark that $\mathcal{O}(\Delta)$ has a one dimensional space of holomorphic sections for $g\geq 1$. As is well known the unique holomorphic section of $\mathcal{O}(\Theta)$ can be written in terms of the theta function. We refer to the treatise of Griffiths and Harris for details. As a result we see from Corollary 4.5 that the normalised holomorphic section of $\mathcal{O}(\Delta)$ is given by

$$E(Q,P) = \frac{\theta_1(Q-P)}{\theta_1'(0)}\,. \tag{4.6}$$

Putting $\xi = \alpha$ in Corollary 4.4, where α is any element of $Pic^0(M)-\Theta$, we see that the

two point function $< b(Q)c(P) >$ is given by the so-called *Szegő kernel*:

$$< b(Q)c(P) >= S_\alpha(Q,P) = \frac{\theta_1(Q-P-\alpha)}{\theta_1(-\alpha)E(Q,P)}\,. \tag{4.7}$$

where $E(Q,P)$ is given by (4.6).

Then from Prop. 4.3, putting $\xi=\alpha$, we see that

$$C(n,n) = \frac{\theta_1\left(\sum_1^n Q_i - \sum_1^n P_i - \alpha\right)}{\theta_1(-\alpha)}\,\frac{\prod_{1\leq i<j\leq n}E(Q_i,Q_j)E(P_j,P_i)}{\prod_{1\leq i,j\leq n}E(Q_i,P_j)}\,. \tag{4.8}$$

But we have also proved that $C(n,n)$ is the determinant of its two point functions, i.e.

$$C(n,n) = const.\times det\left(\frac{\theta_1(Q_i-P_j-\alpha)}{\theta_1(-\alpha)}\,\frac{1}{E(Q_i,P_j)}\right)\Bigg|^n_{i,j=1} \tag{4.9}$$

Comparing (4.8) and (4.9), it is easy to see that the constant is unity and so we have proved:

Theorem 4.6 (Frobenius' identity). For $\alpha\neq 0$,

$$\begin{aligned}&\frac{\theta_1\left(\sum_1^n Q_i - \sum_1^n P_i + \alpha\right)}{\theta_1(\alpha)}\,\frac{\prod_{1\leq i<j\leq n}\theta_1(Q_i-Q_j)\theta_1(P_j-P_i)}{\prod_{1\leq i,j\leq n}\theta_1(Q_i-P_j)}\\ &\quad = det\left(\frac{\theta_1(Q_i-P_j+\alpha)}{\theta_1(Q_i-P_j)\theta_1(\alpha)}\right)\Bigg|^n_{i,j=1}.\end{aligned} \tag{4.10}$$

In fact Frobenius wrote his identity in terms of the Weierstrass σ-function:

$$\begin{aligned}&\sigma(\alpha)^{n-1}\sigma\left(\alpha+\sum_1^n(Q_i+P_i)\right)\,\frac{\prod_{1<i<j\leq n}\sigma(Q_i-Q_j)\sigma(P_i-P_j)}{\prod_{1<i,j<n}\sigma(Q_i+P_j)}\\ &\quad = det\left(\frac{\sigma(Q_i+P_j+\alpha)}{\sigma(Q_i+P_j)}\right)\Bigg|^n_{i,j=1}\end{aligned} \tag{4.11}$$

where we have corrected a small error in the published paper of Frobenius. It is easy to see that (4.11) is equivalent to (4.10), either by using the well known relation between σ and θ_1, or by reflecting on the fact that σ is just as good a theta function for writing the section of $\mathcal{O}(\Theta)$ as θ_1!

(c) The case $g \geq 2$.

We shall first show how to obtain an explicit expression for the unique holomorphic section of $\mathcal{M}_\alpha(n,n)$, whose existence was proved in Theorem 3.5. We shall first show that the holomorphic line bundle $\mathcal{M}_\alpha(n,n)$ is isomorphic to a pullback to M^{2n} of $\mathcal{O}(\Theta)$, where Θ is the canonical theta divisor in $Pic^{g-1}(M)$. Our analysis is based on the following lemma, which is a modern interpretation by geometers of a result of Riemann!

Lemma 4.7. Let η be a line bundle on M of degree $g-2$ and let

$$I_\eta \,:\, M \to Pic^{g-1}(M)$$

be the map defined by

$$P \to \eta \otimes \mathcal{O}(P).$$

Then

$$I_\eta^*(\mathcal{O}(\Theta)) = K \otimes \eta^{-1}.$$

We shall first discuss $\mathcal{M}_\xi(1,1)$, where ξ is now an *arbitrary* element of $Pic^{g-1}(M)$.

Proposition 4.8. Let ξ be any element of $Pic^{g-1}(M)$. Consider the map

$$\begin{aligned} \phi_\xi^1 \,&:\, M_1 \times M_2 \to Pic^{g-1}(M) \\ &(Q,P) \to \mathcal{O}(Q-P)\otimes\xi \end{aligned}$$

Then

$$\mathcal{M}_\xi(1,1) \equiv p_1^*(K\otimes\xi^{-1})\otimes p_2^*(\xi)\otimes\mathcal{O}(\Delta_{12}) = \phi_\xi^{1*}\,(\mathcal{O}(\Theta)). \tag{4.12}$$

Proof. An easy application of Lemma 4.7 and the *seesaw principle.*

Theorem 4.9. For $\xi \in Pic^{g-1}(M)$ consider the map

$$\begin{aligned} \phi_\xi^n \quad &: \quad M^{2n} \to Pic^{g-1}(M) \\ (Q_1,\cdots,Q_n,P_1,\cdots,P_n) &\to \mathcal{O}(\sum_1^n Q_i - \sum_1^n P_i)\otimes\xi \end{aligned}$$

Then

$$\mathcal{M}_\xi(n,n) = \phi_\xi^{n*}(\mathcal{O}(\Theta)). \tag{4.13}$$

Proof. An application of the seesaw theorem applied inductively from $n=1$ (Prop. 4.8).

We now choose a symplectic basis of a-cycles and b-cycles for M and a dual basis of holomorphic 1-forms $\mathbf{w} \equiv (w_1,\cdots,w_g)$ so that the period matrix of M is now defined. Then the Riemann constant $\kappa \in Pic^{g-1}(M)$, which defines an isomorphism between $Pic^{g-1}(M)$ and the Jacobian $J(M)$, is defined as is also the Riemann theta function $\theta(z)$.

We shall adopt the notation $\theta[\xi](z)$, where $\xi \in Pic^{g-1}(M)$, for the theta function with characteristics $\kappa\otimes\xi^{-1}$. We shall also follow standard practice by writing in the argument of a theta function the expression $\Sigma_1^n Q_i - \Sigma_1^n P_i$ for the sum of n line integrals

$$\int_{P_1+\cdots+P_n}^{Q_1+\cdots+Q_n} \mathbf{w}$$

Then from Theorem 4.9 and the isomorphism between $Pic^0(M)$ and $Alb(M)$ we obtain:

Corollary 4.10. The holomorphic section of $\mathcal{M}_\alpha(n,n)$, unique up to a multiplicative constant, is given by

$$\frac{\theta[\alpha](\sum_1^n Q_i - \sum_1^n P_i)}{\theta[\alpha](0)}. \tag{4.14}$$

The normalization constant $\theta[\alpha](0) \neq 0$ iff $H^0(M,\alpha) = 0$, i.e. iff (1.3a) holds.

Our aim now is to write down an expression for the correlation function $C(n,n)$, since from Theorem 3.1 we know that $C(m,n) = 0$ for $m \neq n$. From eqn. (1.10) and Corollary 4.10 it is clear that what remains to be done is to obtain an expression for the unique meromorphic section of $\mathcal{O}(D(n,n))$ with divisor $D(n,n)$. We should like to have this section written in terms of familiar objects such as theta functions and prime forms. For this we do not deal with $\mathcal{O}(D(n,n))$ directly, but use Prop. 4.8 and the multiplicative structure of $Pic(M^{2n})$ to obtain an isomorphic line bundle. We explain the procedure in detail for the case $n = 1$ (when $D(1,1) = -\Delta_{12}$), since this will be our basic building block.

Tensoring the short exact sequence (2.1) by $\mathcal{O}(\Delta)$and proceeding to cohomology, we see immediately that $\mathcal{O}(\Delta)$ has a one dimensional space of holomorphic sections. From (4.12) we have

$$\mathcal{O}(\Delta_{12}) = \phi_\xi^{1*}(\mathcal{O}(\Theta)) \otimes \mathcal{F}_\xi(1,1)^{-1} \tag{4.15}$$

For the study of the two point function $C(1,1)$, we must choose $\xi = \alpha$. However, the left-hand side of (4.15) does not depend on ξ and so we can make a second choice for ξ. The choice we shall make is to put $\xi = \beta$, where β is an odd theta characteristic with $dim\ H^0(M,\beta) = 1$. Then $\beta = \mathcal{O}(z_1 + \cdots + z_{g-1})$ for some $z_1, \cdots, z_{g-1}$ on M. Let h_β be the unique holomorphic section of β (normalization will be fixed later). Then by the Künneth formula, $h_\beta(Q)h_\beta(P)$ is a holomorphic section of $\mathcal{F}_\beta(1,1) \cong p_1^*(K \otimes \beta^{-1}) \otimes p_2^*(\beta)$. Clearly the divisor of h_β is $z_1 + \cdots + z_{g-1}$. We now need the following lemma:

Lemma 4.11. With the above notation, $\theta[\beta](Q-P) = 0$ iff (a) $Q = P$, or (b) $P =$ some z_i, or (c) $Q =$ some z_i.

From Lemma 4.11 and the preceding discussion we have:

Corollary 4.12 The *prime form*

$$E(Q,P) = \frac{\theta[\beta](Q-P)}{h_\beta(Q)h_\beta(P)} \tag{4.16}$$

is a holomorphic section of a line bundle isomorphic to $\mathcal{O}(\Delta)$ and it vanishes linearly on, and only on, the diagonal Δ of $M \times M$. $E(Q,P)$ can be normalised to vanish like $(Q-P)$ near Δ. In addition $E(Q,P) = -E(P,Q)$.

From the isomorphism

$$\phi_\alpha^{1*}(\mathcal{O}(\Theta)) \otimes \mathcal{F}_\alpha(1,1)^{-1} \cong \phi_\beta^{1*}(\mathcal{O}(\Theta)) \otimes \mathcal{F}_\beta(1,1) \tag{4.17}$$

and Coroll. 4.10 and Coroll. 4.12 we now deduce:

Proposition 4.13. Let $\alpha \in Pic^{g-1}(M)$ be in the complement of the theta divisor so that $\theta[\alpha](0) \neq 0$. Then the normalized two point function $C(1,1)$ is given by the *Szegö kernel*

$$S_\alpha(Q,P) = \frac{\theta[\alpha](Q-P)}{\theta[\alpha](0)} \frac{1}{E(Q,P)} \tag{4.18}$$

From the abelian group structure of $Pic(M^{2n})$, it is clear that $\mathcal{O}(D(n,n))$ is simply a product of pullbacks $pr_{ij}^*(\mathcal{O}(\Delta_{ij}))$ and so we can easily write an isomorphism between $\mathcal{O}(D(n,n))$ and products of pullbacks of the right-hand side of (4.15) with $\xi = \beta$. It is then easy to write down the desired section with divisor $D(n,n)$ in terms of prime forms. Recalling Coroll. 4.10 we can then write down the 2n-point function $C(n,n)$.

Theorem 4.14. The unique normalised 2n-point correlation function $C(n,n)$ is given by

$$\frac{\theta[\alpha](\sum_1^n Q_i - \sum_1^n P_i)}{\theta[\alpha](0)} \frac{\prod_{i<j} E(Q_i,Q_j)E(P_j,P_i)}{\prod_{i,j} E(Q_i,P_j)}. \tag{4.19}$$

The discussion in Sect. 1 leads us to formulate:

Theorem 4.15. *(Fay's identity).* Let $\alpha \in Pic^{g-1}(M)$ be in the complement of the theta divisor, so that $\theta[\alpha](0) \neq 0$. Then

$$\frac{\theta[\alpha](\sum_1^n Q_i - \sum_1^n P_i)}{\theta[\alpha](0)} \frac{\prod_{i<j} E(Q_i,Q_j)\ E(P_j,P_i)}{\prod_{i,j} E(Q_i,P_j)} = det\ [S_\alpha(Q_i,P_j)] \tag{4.20}$$

Proof. Clearly $det\ [S_\alpha(Q_i,P_j)]$ can be regarded as a section of $\mathcal{F}_\alpha(n,n)$ which satisfies postulates (P1) - (P4). Hence

$$det\ [S_\alpha(Q_i,P_j)] \frac{\prod_{i,j} E(Q_i,P_j)}{\prod_{i<j} E(Q_i,Q_j)\ E(P_j,P_i)}$$

is a holomorphic section of $\mathcal{M}_\alpha(n,n)$. However, this latter section is unique up to a multiplicative constant and is given by Coroll. 4.10. The constant is easily seen to be unity by a residue argument.

The above identity was discovered by Fay. The case when $n = 2$ is particularly interesting since the identities for $n > 2$ can in fact be deduced from it as pointed out by Fay. It is amusing to note that this too is in agreement with physical intuition, since it is believed in physics that if the four point function agrees with the free particle case, then so do all the higher point functions. Multiplying (4.17) out in the case $n = 2$, we get the *trisecant identity*:

$$\begin{aligned}&\theta[\alpha](Q_1 - P_1)\theta[\alpha](Q_2 - P_2)E(Q_1, P_2)E(P_1, Q_2)\\ &+ \theta[\alpha](Q_1 - P_2)\theta[\alpha](Q_2 - P_1)E(Q_1, P_1)E(Q_2, P_2)\\ &= \theta[\alpha](Q_1 + Q_2 - P_1 - P_2)\theta[\alpha](0)E(Q_1, Q_2)E(P_1, P_2).\end{aligned} \tag{4.21}$$

5. The generalized b-c system

So far we have discussed only *a 'twisted'* version of the conformal spin $1/2$ fermionic b-c system. There are several closely related systems considered in the physics literature. One is the spin $(1 - J), J$ system, in which the field c is associated to the line bundle $K^{\otimes J}$ and b to $K^{\otimes(1-J)}$, where J is an integer or half integer. In the case when J is a half integer we must first choose a square root line bundle of K, or *theta characteristic*. In the $J = 1/2$ case we have considered so far, we could assume that α had no holomorphic sections. This is no longer possible if $J \geq 1$. Hence we must understand the role of such holomorphic sections in our analysis, since we made crucial use of their absence in our earlier analysis.

Another generalisation is to consider the case when there are special points on the Riemann surface M around which the b and c fields have specified rational monodromy, thus drastically changing the analytic behaviour of the correlation functions. We shall discuss this in the next section. Another related system is the *bosonic b-c system*, usually called the *β-γ system*. This will be taken up in section 7.

We shall now discuss the two point function $< b(Q)c(P) >$ of a *generalised b-c system*. This will be relevant for understanding the spin $(1 - J), J$ system, the monodromy complications to be discussed in the next section, as well as for the β-γ system, since the two point function is independent of statistics.

Definition 5.1. Let $\alpha \in Pic^p(M), \beta \in Pic^q(M)$ where $p + q = 2g - 2$, with no other restrictions on α and β. Let $p_i : M \times M \to M (i = 1, 2)$ be the canonical projections onto the first and second factors of $M \times M$. We shall say that the generalised b-c system has a *two point function* (not necessarily unique), denoted $< b(Q)c(P) >$, if there is at least one nonvanishing meromorphic section of $p_1^*(\beta) \otimes p_2^*(\alpha)$ whose only singularity is a simple pole along the diagonal Δ of $M \times M$.

Remark 5.2. If $p = q = g - 1$ and $\beta \otimes \alpha = K$ with $H^0(M, \alpha) = 0$, we are in the situation of the original system we discussed. If $p = 2(1 - J)(g - 1), q = 2J(g - 1), \beta \otimes \alpha = K$, then we have a 'twisted' generalisation of the spin $(1 - J), J$ system.

The following rather surprising result can be proved:

Theorem 5.3. Necessary and sufficient conditions for the generalised b-c system to have a two point function in the sense of Definition 5.1 are that $p = q = g - 1, \beta \otimes \alpha = K$, and α (hence also β) has no holomorphic sections. Under these conditions the two point function is also unique.

Theorem 5.3 provides a basis for understanding condition (P1) of section 1. It may be recalled that we had to give rather complicated, model-dependent reasons for the conditions making up (P1). The requirement of having a two point function in the sense of Definition 5.1 is quite as satisfactory a basis for the physicist, if not more satisfactory than our earlier justification. Now Theorem 5.3 shows the real origin of these constraints.

While we cannot go into the proof, let us see the mathematical problem of which Theorem 5.3 is the solution. We have the canonical short exact sequence between the ideal sheaf of Δ in $M \times M$, the structure sheaf $\mathcal{O}$ of $M \times M$ and the quotient sheaf:

$$0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}|\Delta \to 0 \tag{5.1}$$

Let $\mathcal{F}_{\alpha\beta} \equiv p_1^*(\beta) \otimes p_2^*(\alpha)$ and $\mathcal{M}_{\alpha\beta} \equiv \mathcal{F}_{\alpha\beta} \otimes \mathcal{O}(\Delta)$. Tensoring (5.1) by $\mathcal{M}_{\alpha\beta}$ we get the exact sequence

$$0 \to \mathcal{F}_{\alpha\beta} \to \mathcal{M}_{\alpha\beta} \to \mathcal{M}_{\alpha\beta}|\Delta \to 0 \tag{5.2}$$

Passing to cohomology we get

$$0 \to H^0(M \times M, \mathcal{F}_{\alpha\beta}) \xrightarrow{i} H^0(M \times M, \mathcal{M}_{\alpha\beta}). \tag{5.3}$$

The condition for the generalised b-c system to have a two point function in the sense of Definition 5.1 is that the injective map i in (5.3) should not be an isomorphism. What we have to do is to prove that if the conditions of Theorem 5.3 are not satisfied, then i is an isomorphism. If they are satisfied, we have already seen that there is a unique two point function.

For the higher spin b-c system we associate the field c to a line bundle $\zeta \in Pic^{2J(g-1)}(M)$ and the field b to $K \otimes \zeta^{-1}$. Then for $J > 1$, ζ has $N = (2J-1)(g-1)$ holomorphic sections, while $K \otimes \zeta^{-1}$ has none. By Theorem 5.3 it cannot have a two point function in the sense of Definition 5.1 (which itself came from condition (P3)): the two point function must have extra singularities apart from the pole along the diagonal.

What physicists do in this situation is to introduce a set of $N = (2J-1)(g-1)$ points $w_1, \ldots, w_N$ on M and assume that $< b(Q)c(P) >$ has a zero when $P \in \{w_1, \ldots, w_N\}$ and a pole when $Q \in \{w_1, \ldots, w_N\}$. In our formulation this means that the two point function is given by a holomorphic section of

$$p_1^*(K \otimes \zeta^{-1}) \otimes p_2^*(\zeta) \otimes \mathcal{O}(\Delta) \otimes p_1^*(\mathcal{O}(W)) \otimes p_2^*(\mathcal{O}(-W)), \qquad (5.4)$$

where $W = \sum_1^n w_i$. Let $\alpha = \zeta \otimes \mathcal{O}(-W)$. Note that $\alpha \in Pic^{g-1}(M)$. Then (5.4) can be written as

$$p_1^*(K \otimes \alpha^{-1}) \otimes p_2^*(\alpha) \otimes \mathcal{O}(\Delta). \qquad (5.5)$$

Since the $w_1, \ldots, w_N$ can be taken to be in general position, we can assume that $H^0(M, \alpha) = 0$. Thus we are back in the twisted spin 1/2 case. In this way the two point correlation function can be written down.

In the case of higher point functions, we must first specify the statistics. In the fermionic case, it is easy to see that Wick's theorem continues to hold. This leads to an identity for $g = 0, g = 1$ and $g \geq 2$, which reduces to the identities of Cauchy, Frobenius and Fay, respectively, after cancelling some extra factors on the two sides.

6. The b-c system in the presence of a 'twist structure'

A generalization of the b-c system that has been studied in the literature is when we are given a distinguished set of points on M around which the b field and c field have a given (rational) monodromy. Such problems arise when there are so-called '*twist fields*' in the problem, a special case being that of '*spin fields*', or when dealing with *orbifolds*.

We shall show in this section how our methods can be generalized to deal rigorously with the most general problem of this kind. We give a complete solution to this problem for the b-c system. We thereby not only give a precise meaning to formal expressions appearing in the literature, but also give a rigorous proof that certain correlation functions which are usually not considered, or ruled out on heuristic grounds, indeed do not occur.

We define a *twist structure* on M to be an assignment of N_+ positive rational numbers $\mu_i (1 \leq i \leq N_+)$ to N_+ distinct distinguished points $x_1, \ldots, x_{N_+}$ of M and N_- negative rational numbers $(-\nu_j)$ $(1 \leq j \leq N_-)$ to N_- points $y_1, \ldots, y_{N_-}$ of M (where the y's are distinct from each other as well as from the x's) such that

$$\sum_{i=1}^{N_+} \mu_i - \sum_{j=1}^{N_-} \nu_j = \ell, \qquad (6.1)$$

where ℓ is a (positive or negative) integer called the *total twist*. The b field and c field are required to have the following behaviour in the neighbourhood of these points:

$$\begin{aligned} b(z) &\sim (z - x_i)^{-\mu_i} \quad (1 \leq i \leq N_+) \\ &\sim (z - y_j)^{\nu_j} \quad (1 \leq j < N_-) \end{aligned} \qquad (6.2a)$$

$$\begin{aligned} c(z) &\sim (z - x_i)^{\mu_i} \quad (1 \leq i < N_+) \\ &\sim (z - y_j)^{-\nu_j} \quad (1 < j < N_-) \end{aligned} \qquad (6.2b)$$

Equations (6.2a,b) must be taken to refer to the behaviour of the correlation functions $C(m,n)$ in the neighbourhood of such points. The problem we shall now consider is that of obtaining the two point function $< b(Q)c(P) >$ in the presence of a given twist structure, our system being defined by (P3), (P4) and (6.2a,b). It will be evident to the reader that the problem is not posed precisely. All sources of ambiguity will be spelled out as we proceed.

If we want maximum generality, we should start once again with the generalised *b-c* system of section 5. This would make our exposition tedious and is unnecessary since our aim is to make contact with the existing literature. One possible use of greater generality would be if a treatment of the spin $(1-J), J$ system were required in which the effect of holomorphic sections were to be eliminated by a twist structure, rather than by poles as in section 5. The modifications that would have to be made to the discussion below are obvious (put $\ell = (2J-1)(g-1)$ in Theorem 6.2 below).

We shall, therefore, take c (resp. b) to be a section of α (resp. $K \otimes \alpha^{-1}$), where $deg\ (\alpha) = g-1$. We put no further restrictions on α at present. We expect the two point function $< b(Q)c(P) >$ to be, in some sense, a *multi-valued section* of $p_1^*(K \otimes \alpha^{-1}) \otimes p_2^*(\alpha)$ and so we must determine a *covering space* $\tilde{M}$ from the given data on which we can interpret it as a *meromorphic* section of a line bundle.

In the following we shall write $E(z,x)$ for the unique holomorphic section of the line bundle $\mathcal{O}(x)$ over M, with divisor x, for any $x \in M$. It is then easy to see that our problem of making precise the behaviour of the two point function near the x_i, y_j, as given by (6.2a,b), is really one of making sense of the formal expression

$$\xi(z) \equiv \prod_{i=1}^{N_+}(E(z,x_i))^{\mu_i} / \prod_{j=1}^{N_-}(E(z,y_j))^{\nu_j}. \tag{6.3}$$

For if (6.3) were defined, $< b(Q)c(P) > E(Q,P)\ \xi(Q)/\xi(P)$ would be holomorphic in each variable and we could hope to use algebraic geometry methods to count the number of such sections.

Now, we can certainly write $\mu_i = p_i/d\ (1 \le i \le N_+), \nu_j = q_j/d\ (1 \le j \le N_-)$, where the p_i, q_j, d are positive integers. Then $\xi(z)$ is the 'd-th root' of

$$\prod_{i=1}^{N_+}(E(z,x_i))^{p_i} / \prod_{j=1}^{N_-}(E(z,y_j))^{q_j}, \tag{6.4}$$

which is the canonical meromorphic section of the line bundle associated to the divisor

$$\sum_{i=1}^{N_+} p_i x_i - \sum_{j=1}^{N_-} q_j y_j\ . \tag{6.5}$$

Our problem is to construct a covering space $\tilde{M}$ of M on which the d-th root of the

section (6.4) can be interpreted as a meromorphic section of a line bundle pulled up from M. Our solution to this problem is based on the following lemma:

Lemma 6.1. *Let ζ be a holomorphic line bundle over a compact, connected Riemann surface M and D an effective (i.e. positive) divisor on M such that $\zeta^d = \mathcal{O}(D)$ for some positive integer d. Let σ denote the canonical holomorphic section of $\mathcal{O}(D)$ with divisor D. Then there is a d-fold cyclic covering $\pi : \tilde{M} \to M$, ramified precisely over the support of D, such that $\pi^*(\zeta)$ admits a holomorphic section τ satisfying*

$$\tau^d = \pi^*(\sigma) \text{ in } H^0(\tilde{M}, \pi^*(\mathcal{O}(D))). \tag{6.6}$$

Let $D = \sum_1^k m_i P_i$, where the P_i are distinct points of M and the m_i are positive integers. Then $\tilde{M}$ is irreducible if and only if the greatest common divisor of $(d, m_i; 1 \le i \le k)$ is unity and nonsingular if and only if $m_i = 1$ for $1 \le i \le k$.

Lemma 6.1 is a known result in algebraic geometry, but it is not directly applicable to our situation since (6.5) is not a positive divisor. We can, however, add a suitable positive divisor to (6.5) and subtract it later. The freedom in choosing the divisor to be added to (6.5) introduces a certain arbitrariness in the construction of $\tilde{M}$, which is intrinsic to the problem since physics only gives us the nonpositive divisor (6.5). Since $\tilde{M}$ has merely an auxiliary role, this is of no importance. So choose positive integers n_j such that $n_j - \nu_j$ is positive $(1 \le j \le N_-)$ and consider the effective divisor

$$D = \sum_{i=1}^{N_+} p_i x_i + \sum_{j=1}^{N_-}(n_j d - q_j) y_j. \tag{6.7}$$

In view of Lemma 6.1, if we want $\tilde{M}$ to be irreducible we must require the g.c.d. of d and the integer coefficients in (6.7) to be unity. While irreducibility is not strictly essential, it is reasonable to say that, if this is not satisfied, then the original problem has been badly posed. The fact that by Lemma 6.1 the covering space $\tilde{M}$ will be a singular curve unless the coefficients appearing in (6.7) are all unity is of no importance: $\tilde{M}$ can always be replaced by its canonical normalization if desired, but this step is not necessary to interpret our formula for the two point function. In the case of *spin fields* each $p_i = 1$, each $q_j = 1$, $d = 2$ and we can choose each $n_j = 1$. Then the integer coefficients in (6.7) are all unity and $\tilde{M}$ is nonsingular as well as irreducible.

The line bundle $\mathcal{O}(D)$ has a canonical holomorphic section $\sigma(z)$ with divisor D, which can be written as a product of prime forms:

$$\sigma(z) = \prod_{i=1}^{N_+}(E(z,x_i))^{p_i}\prod_{j=1}^{N_-}(E(z,y_j))^{(n_j d - q_j)}. \tag{6.8}$$

Now note that

$$deg\ (\mathcal{O}(D)) = d(\ell + \sum_{j=1}^{N_-} n_j), \tag{6.9}$$

i.e. it is a multiple of d. Hence, by the divisibility of the group $Pic^0(M)$ for $g \geq 1$ and the fact that $Pic(M) = \mathbb{Z}$ for $g = 0$, we can find a line bundle ζ such that

$$\zeta^d = \mathcal{O}(D). \tag{6.10}$$

In fact, there are d^{2g} such line bundles and so we have to make a choice if $g \geq 1$. The choice of ζ plays no role in the present abstract discussion, but should be kept in mind when interpreting the final formula for the two point function. It reflects the fact that our original problem was not well posed and so there is an ambiguity in the answer.

We now have the data, viz. $(\zeta, d, \mathcal{O}(D), \sigma)$, to apply Lemma 6.1. We can thus give a rigorous meaning to

$$\prod_{i=1}^{N_+}(E(z,x_i))^{\mu_i}\prod_{j=1}^{N_-}(E(z,y_j))^{(n_j-\nu_j)} \tag{6.11}$$

as a multi-valued section of ζ, which is properly defined as a holomorphic section of the pullback of ζ to the covering space $\tilde{M}$. Now, defining

$$\gamma = \zeta \otimes \mathcal{O}(-\sum_{j=1}^{N_-} n_j y_j), \tag{6.12}$$

we see that we have given a precise meaning to $\xi(z)$ of (6.3) as a multi-valued section of

γ, which is properly defined as a meromorphic section of $\pi^*(\gamma)$ over $\tilde{M}$. Note that

$$deg\ (\gamma) = \ell \equiv \text{'total twist'} \tag{6.13}$$

Then from (P3), (P4) and (6.2), we see that

$$< b(Q)c(P) > = E(Q,P)\xi(Q)/\xi(P) \tag{6.14}$$

is the pullback to $\tilde{M} \times \tilde{M}$ of a holomorphic section of the following line bundle over $M \times M$:

$$\begin{aligned} &p_1^*(K \otimes \alpha^{-1}) \otimes p_2^*(\alpha) \otimes \mathcal{O}(\Delta) \otimes p_1^*(\gamma) \otimes p_2^*(\gamma^{-1}) \\ &= p_1^*(K \otimes \tilde{\alpha}^{-1}) \otimes p_2^*(\tilde{\alpha}) \otimes \mathcal{O}(\Delta), \text{ where } \tilde{\alpha} = \alpha \otimes \gamma^{-1}. \end{aligned} \tag{6.15}$$

Since

$$deg\ (\tilde{\alpha}) + deg\ (K \otimes \tilde{\alpha}^{-1}) = g - 1 - \ell + g - 1 + \ell = 2g - 2, \tag{6.16}$$

we see that we are dealing with the generalized *b-c* system of section 5. Thus the generalized system we discussed in section 5 is precisely necessary to deal with the ordinary *b-c* system in the presence of a twist structure. We can then apply Theorem 5.3 to conclude that to have a nonzero two point function we must have $\tilde{\alpha} \in Pic^{g-1}(M) - \Theta$. As a consequence *the total twist ℓ of the twist structure must be zero.* We have thus proved:

Theorem 5.2. *Given a twist structure on M defined by the data (D, γ, ℓ) the twisted spin $\frac{1}{2}$ b-c system over M (i.e. c is a section of $\alpha \in Pic^{g-1}(M)$, b of $K \otimes \alpha^{-1}$) has a nonzero two point function if and only if the total twist ℓ is zero and $\tilde{\alpha} = \alpha \otimes \gamma^{-1}$ lies in the complement of the theta divisor Θ. When these conditions are satisfied the two point function is unique and given by*

$$\frac{\xi(P)}{\xi(Q)}\ S_{\tilde{\alpha}}(Q,P). \tag{6.17}$$

The methods of this section can be easily combined with those of section 7 to obtain higher point functions in the presence of a twist structure. We can also consider the fermionic *b-c* system, as discussed by us in Sects. [1-4], in the presence of a twist structure of zero total twist. It is easy to see that the theorem proved there that the $2n$-point function is a determinant of two point functions remains valid in the presence of such a twist structure. This leads to an identity which reduces, after cancelling common factors, to Fay's identity for $g \geq 2$ and to Cauchy's and Frobenius' identities for $g = 0$ and 1, respectively.

7. The b-c system with general statistics

Of the 4 conditions (P1) - (P4) that we wrote down in section 1, only one, viz. (P2), specifically restricted the *statistics* of the fields b and c to be *fermionic*. This condition, however, played a very vital role in our calculations and so it is not immediately clear that it can be removed. In fact (P2) cannot be simply removed, but must be replaced by another condition which does not impose any constraint on particle statistics.

At the level of the two point function $< b(Q)c(P) >$ there is no difference between the fermionic b-c system and the (bosonic) β-γ system. The question of statistics comes into question only for higher point functions. In keeping with our aim of studying the constraints imposed by analyticity constraints originating in the physics of the problem, we shall determine the vector space of all possible correlation functions with a given number of b fields and c fields, compatible with the physical analyticity constraints. We shall then study how the imposition of statistics selects a suitable one dimensional subspace.

For the sake of brevity we discuss only the space $V(n,n)$ of $2n$-point functions, i.e. when the number of b fields and c fields are equal, since the same analysis shows that the other cases do not appear. We recall that our system is described by (P1), (P3) and (P4). In fact physics imposes a stronger condition than (P3). We did not need to use the stronger form of (P3) earlier due to the special properties of (P2).

Consider an arbitrary nonzero element $v \in V(n,n)$ and let D_{ij} belong to its polar divisor, i.e. $D_{ij} \in \{D_p(n,n)\}$. Note that

$$D_{ij} = \Delta_{ij} \times M^{2n-2}, M^{2n-2} \equiv \prod_{\substack{k=1 \\ k\neq i,j}}^{2n} M_k \tag{7.1}$$

Then we require that

$$vE(z_i,z_j)|M^{2n-2} \in V(n-1,n-1). \tag{7.2}$$

This condition came for free earlier, because of (P2). Now we are required to impose it.

We must now rewrite conditions (P1), (P3), (P4) and (7.2) in a more convenient form. For this we have to introduce some notation. Let $\mathcal{S}_n$ denote the group of permutations of $\{1,\ldots,n\}$. For each $\sigma \in \mathcal{S}_n$ we define the divisor

$$D_\sigma(n,n) \equiv \sum_{i=1}^{n} D_{\sigma(i),n+i}. \tag{7.3}$$

Thus $D_\sigma(n,n) \subset D_p(n,n)$ for each $\sigma \in \mathcal{S}_n$. The line bundle $\mathcal{O}(D_\sigma(n,n))$ on M^{2n} has a canonical holomorphic section $E_\sigma(n,n)$, where

$$E_\sigma(n,n) \equiv \prod_{i=1}^{n} E(z_{\sigma(i)}, z_{n+i}). \tag{7.4}$$

We shall denote the intersection of the D_{ij} appearing on the right-hand side of (7.3) by $\cap D_\sigma(n,n)$. Our postulates are now as follows:

Every element $v \in V(n,n)$ must satisfy:

$\mathcal{P}1$) v is a meromorphic section of $\mathcal{F}_\alpha(n,n)$ such that $vE_p(n,n)$ is a holomorphic section of $\mathcal{F}_\alpha(n,n) \otimes \mathcal{O}(D_p(n,n))$, where $E_p(n,n) \equiv \prod_{\substack{1\leq i\leq n \\ n+1\leq j\leq 2n}} E(z_i,z_j)$

$\mathcal{P}2$) For each $\sigma \in \mathcal{S}_n$ the restriction of $vE_\sigma(n,n)$ to $\cap D_\sigma(n,n)$ is an element of the vector space

$$U_\sigma(n,n) \equiv H^0(\cap D_\sigma(n,n), \mathcal{F}_\alpha(n,n) \otimes \mathcal{O}(D_\sigma(n,n))| \cap D_\sigma(n,n)) \tag{7.5}$$

Morever, the only element for which this restriction is the zero vector of $U_\sigma(n,n)$ for every $\sigma \in \mathcal{S}_n$ is the zero vector of $V(n,n)$.

Postulate $\mathcal{P}1$ summaries the content of conditions P1, P3 and P4 of section 1, while $\mathcal{P}2$ replaces condition P2. Postulate $\mathcal{P}2$ is simply condition (7.2) applied repeatedly until all possible poles are removed. Note that since we have not imposed any statistics so far, we cannot deduce from (7.2) that we can determine lower point functions from a given $2n$-point function by examining its behaviour at its poles.

Rather surprisingly postulates $\mathcal{P}1$ and $\mathcal{P}2$ suffice to determine $V(n,n)$:

Theorem 7.1. The vector space $V(n,n)$ is the linear span of its $n!$ linearly independent basis elements

$$\left\{ \prod_{i=1}^{n} S_{\alpha}\left(Q_{\sigma(i)}, P_i\right) \middle| \sigma \in \mathcal{S}_n \right\}.$$

Remark 7.2. Theorem 7.1 can be generalised to the spin $(1-J), J$ system by replacing the Szegö kernel $S_{\alpha}(Q,P)$ by the corresponding two point function. If there is also a twist structure with zero total twist, the two point function should be multiplied by the factor $\xi(P)/\xi(Q)$ as in (6.17).

A b-c system will be completely specified if we can associate to it a unique set of correlation functions $\{C(n,n); n \in \mathbb{N}\}$. Thus for each $n \in \mathbb{N}$ we must pick out a one dimensional subspace of $V(n,n)$ for a given b-c system. We can do this by specifying the particle type or *statistics*.

To study the possibilities, we must define an action of the symmetric group $\mathcal{S}_n$ on $V(n,n)$. We can define the action on a basis element given by Theorem 7.1 and extend by linearity. Since we have two kinds of particles, viz. the b fields labelled by $Q_1, Q_2, \ldots$ and the c fields labelled by $P_1, P_2, \ldots$ we can define a natural action of $\mathcal{S}_n$ on the labels $\{1, \ldots, n\}$ of $Q_1, \ldots, Q_n$ and of $P_1, \ldots, P_n$ separately. Thus $V(n,n)$ is an $\mathcal{S}_n \times \mathcal{S}_n$-module. Considered as an $\mathcal{S}_n$-module, $V(n,n)$ is simply the regular representation of $\mathcal{S}_n$. Then we know that $V(n,n)$ is a multiplicity free direct sum of irreducible $\mathcal{S}_n \times \mathcal{S}_n$ modules $V_\lambda \otimes V_\lambda$, where V_λ is the irreducible representation of $\mathcal{S}_n$ corresponding to the partition λ of n. The only one dimensional representations are the alternating and symmetric representations, corresponding to Fermi and Bose statistics, respectively. We thus get:

Theorem 7.2. *(a) In the case of Fermi statistics the $2n$-point function $< b(Q_1)\ldots b(Q_n)c(P_1)\ldots c(P_n) >$ is antisymmetric in the Q-variables and separately in the P-variables and, as a result,*

$$\langle b(Q_1)\ldots b(Q_n)c(P_1)\ldots c(P_n)\rangle = det\ \left(S_\alpha(Q_i,P_j)\right)\Big|_{i,j=1}^{n}, \tag{7.6}$$

where det denotes the determinant.

(b) In the case of Bose statistics, where it is conventional to write β for b and γ for c, the $2n$-point function $< \beta(Q_1)\ldots\beta(Q_n)\gamma(P_1)\ldots\gamma(P_n) >$ is symmetric in the Q-variables and separately in the P-variables and, as a result,

$$\langle \beta(Q_1)\ldots\beta(Q_n)\gamma(P_1)\ldots\gamma(P_n)\rangle = perm\ \left(S_\alpha(Q_i,P_j)\right)\Big|_{i,j=1}^{n}, \tag{7.7}$$

where perm denotes the permanent.

While Theorem 7.2 exhausts the possibilities of choosing one dimensional subspaces of $V(n,n)$ by the usual notions of the connection between permutation symmetry and particle statistics, we can investigate whether other possibilities exist under less restrictive conditions. A condition which appears quite natural is to simply demand the invariance of the $2n$-point function $< b(Q_1)\ldots b(Q_n)c(P_1)\ldots c(P_n) >$ when the Q's and P's are simultaneously permuted in like fashion, i.e.

$$\langle b(Q_1)\ldots b(Q_n)c(P_1)\ldots c(P_n)\rangle = \left\langle b(Q_{\sigma(1)})\ldots b(Q_{\sigma(n)})c(P_{\sigma(1)})\ldots c(P_{\sigma(n)})\right\rangle \tag{7.8}$$

for each $\sigma \in \mathcal{S}_n$. This clearly means that we must restrict the $\mathcal{S}_n \times \mathcal{S}_n$ representation in $V(n,n)$ to the diagonally embedded subgroup $\mathcal{S}_n$. It is obvious that the regular representation $V(n,n)$ decomposes so as to give a one dimensional representation of $\mathcal{S}_n$ for each conjugacy class of $\mathcal{S}_n$, i.e. for each partition λ of n. We thus obtain:

Theorem 7.3 *For each partition λ of n, there is a unique $2n$-point function, satisfying postulates $\mathcal{P}1, \mathcal{P}2$ and the invariance condition (7.8), given by*

$$\langle b(Q_1)\ldots b(Q_n)c(P_1)\ldots c(P_n)\rangle_\lambda = imm_\lambda\left(S_\alpha(Q_i,P_j)\right)\Big|_{i,j=1}^{n}, \tag{7.9}$$

where imm_λ denotes the immanant. This is defined for an $n \times n$ matrix (A_{ij}) by

$$imm_\lambda(A_{ij})\Big|_{i,j=1}^{n} = \sum_{\sigma\in\mathcal{S}_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} A_{\sigma(i),i} \tag{7.10}$$

where χ_λ is the character corresponding to the irreducible representation of $\mathcal{S}_n$ labelled by the partition λ of n.

Theorem 7.3 includes the case of Fermi and Bose statistics discussed in Theorem 7.2, while including other cases. Other possibilities, not covered by Theorem 7.3, abound, since the $S_n \times S_n$-module $V(n,n)$ is, of course, a $G \times G$-module for any subgroup G of S_n. In that case simply put $A_{ij} = S_\alpha(Q_i, P_j)$ and restrict the sum in (7.10) to $\sigma \in G$. For example, we could consider $G = \mathbb{Z}_n$, viewed as the group of n cyclic permutations of $\{1, \dots, n\}$. Then the coefficients in (7.10) are $1, \omega, \dots, \omega^{n-1}$ where ω is a primitive n-th root of unity and the sum in (7.10) is over the n cyclic permutations.

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Sections 2 and 3. This is taken from [9].

Section 4. The original references for the identities of Cauchy, Frobenius and Fay are [1], [6] and [4], respectively. The derivation of Cauchy's and Frobenius' identities given here is from [11]. The proof of the trisecant identity by the present method was given first in [7], while the proof of the general Fay identity was given in [9].

Section 5. The discussion of the generalised b-c system is based on [11], while the present treatment of the higher spin b-c system is from [10].

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