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WORKSHOP ON MATHEMATICAL PHYSICS AND GEOMETRY

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Linear Superalgebra and Supermanifolds (I)

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Linear Superalgebra and Supermanifolds

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First Lecture
I Algebraic Preliminaries

Superalgebra

Terminology: "SUPER" = $\mathbb{Z}_2$-graded

Dfn. $V \in SVect \implies V \in Vect$ + Decomp. $V = V_0 \oplus V_1$
+ $\mathbb{Z}_2$-grading map $|\cdot| : V_0 - \{0\} \cup V_1 - \{0\} \to \mathbb{Z}_2$

Dfns. $v \in V$ homog. $\longleftrightarrow$ $v \in \mathrm{Dom}\, |\cdot|$
— even $\longleftrightarrow$ $|v| = 0$
— odd $\longleftrightarrow$ $|v| = 1$.

Morphisms: $V, W \in SVect \implies \mathrm{Hom}(V, W) \in SVect$

$$f \in \mathrm{Hom}(V, W)_{|f|} \longleftrightarrow f(V_\mu) \subset W_{\mu + |f|} \quad ; \ \mu \in \mathbb{Z}_2 .$$

When "graded" bases are chosen,

$$|f| = 0 \longleftrightarrow f = \begin{pmatrix} & 0 \\ 0 & \end{pmatrix} \quad \& \quad |f| = 1 \longleftrightarrow f = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$$

Tensor Product: $V, W \in SVect \implies V \otimes W \in SVect$,

$$|v \otimes w| = |v| + |w| \ ; \ v \in V, \ w \in W \text{ homog.}$$

$$\implies (V \otimes W)_\lambda = \bigoplus_{\mu + \nu = \lambda} V_\mu \otimes W_\nu .$$

Dfn. Let $\mathbb{F}$ be a field (either $\mathbb{R}$ or $\mathbb{C}$ in these notes)
An associative $\mathbb{F}$-SAlg is $A \in SVect$ +
$\mu : A \times A \to A$ bilinear, associative, distributive,
with 1_A, and $\mu(A_\mu, A_\nu) = A_\mu A_\nu \subset A_{\mu+\nu}$.

Examples: (i) $V \in SVect \implies \mathrm{End}\, V \in SAlg$ under,
$\mu(f, g) := f \circ g$ (composition).

(ii) $V \in Vect \implies \Lambda V \in SAlg$ under,
$\mu(u, v) := u \wedge v$ (exterior multipl.).

Recall: ΛV is $\mathbb{Z}$-graded $(\Lambda^j V \wedge \Lambda^k V \subset \Lambda^{j+k} V)$.

So, ΛV is $\mathbb{Z}_2$-graded $\begin{cases} (\Lambda V)_0 = \sum_j \Lambda^{2j} V \\ (\Lambda V)_1 = \sum_j \Lambda^{2j+1} V \end{cases}$

This is an example of a "Supercommutative" SAlg:

Dfn. $A \in SAlg$ is SComm $\longleftrightarrow$ $ab = (-1)^{|a||b|} ba$;
$\forall\, a, b$ homog.

(iii) Remark $\mathbb{Z}$-graded algs. which are quotients of the tensor algebra, $T(V)$, of $V \in Vect$, yield Superalgs: e.g.

Exterior alg. — Clifford alg.
Symmetric alg. — Weyl alg.
— Duffin-Kemmer algs., etc.

Tensor Product of SAlgs. $A, B \in SAlg \implies A \otimes B \in SAlg$

Dfn. $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2$

Observation (Sign rule in S-Category)

$$(\cdots a) * (b \cdots) \mapsto (-1)^{|a||b|} (\cdots b) \odot (a \cdots)$$

Typical Example: The SAlg of matrices with coeffs in a SComm SAlg. A (arising from morphisms between A-SMods.)

Recall: A-SMod is $M = M_0 \oplus M_1$ (SAbgroup)

$+$ morphism $A \times M \longrightarrow M$ $\begin{cases} A_\mu M_\nu \subset M_{\mu+\nu} \\ 1m = m \\ (a+b)m = am + bm \\ a(bm) = (ab)m \\ a(m+n) = am + an \end{cases}$

Dfn. Let $M, N \in$ A-SMod

$$f \in Hom_A(M,N) \iff f \in Hom(M,N)$$
$$+ \quad f_\mu(am) = (-1)^{\mu|a|} a f_\mu(m)$$
$\forall a \in A$ homog.

Dfn. Let $X = X_0 \cup X_1$ (disj. union) be a graded subset of an A-SMod $F = F_0 \oplus F_1$ (ie, $X_\mu \subset F_\mu$). F is free on X when,

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[dr, "\forall \varphi"'] & F \arrow[d, dashed, "\exists! \text{ SMod Morphism}"] \\ & M \end{tikzcd}$$

(graded map) $\quad M$ (any A-SMod)

If $\#(X_0)$ and $\#(X_1)$ are finite, F is the free A-SMod of rank $(\# X_0, \# X_1)$.

Lemma: The free A-SMod of rank $(\dim V_0, \dim V_1)$ is isomorphic to $A \otimes V \simeq A^{\dim V_0} \oplus A^{\dim V_1}$ with $V = V_0 \oplus V_1 \in SVect$.

(The A-SMod structure is $a(b \otimes v) := ab \otimes v$).

Proposition: Let $A \in SCommSAlg$ & $V \in SVect$. Let $\dim V < \infty$. $\exists$ Superalg Isomorphism $\Phi: A \otimes End V \longrightarrow End_A(A \otimes V)$, with,

$$\Phi(a \otimes f)(b \otimes v) = (-1)^{|f||b|} ab \otimes f(v).$$

Exercise: This alters the usual rule for matrix multiplication:

If $f \in End_A(A \otimes V) \longleftrightarrow (f_{ij}) \in Mat_{\dim V}(A)$

via, $f(1 \otimes v_j) = \sum_i f_{ij} \otimes v_i \in A \otimes V$,

Then, $g \circ f \longleftrightarrow diag(g) \circ f + antidiag(g) \circ f^*$

(see [Matrix Comp in Linear Superalg, CASV]),

where $(f_{ij})^* = ((f_{ij})_0 - (f_{ij})_1)$

Another example:

$$\text{Supersymm Alg}(V_0|V_1) = T(V) / \text{Ideal gen } \{u \otimes v - (-1)^{|u||v|} v \otimes u\}$$
$$\simeq S(V_0) \otimes \Lambda(V_1)$$

Dfn. SIdeals: $I \lhd A \iff I = I_0 \oplus I_1 = I \cap A_0 \oplus I \cap A_1$, and
$ax \in I$, $\forall a \in I$, $x \in A$.

Remark: For any SCommSAlg $A = A_0 \oplus A_1$, define

$$J = (A_1) = (A_1)_0 \oplus (A_1)_1 \ ;$$

where, $(A_1)_\mu = \{ \sum a_j \xi_j \mid a_j \in A_\mu,\ \xi_j \in A_1 \}$; $\mu = 0, 1$.

Define $\pi : A \longrightarrow A/(A_1)$, $a \longmapsto a \bmod (A_1)$.

Proposition (1) $A/(A_1)$ is a Commutative Algebra (usual sense)

(2) $z \in (A_1)$ & $z = a_1 \xi_1 + \cdots + a_r \xi_r \implies z^{r+1} = 0$ (inductic

(3) $x \in A$ is invertible $\iff \pi(x)$ is

Proof: Only non-trivial: (3) $\Leftarrow$.
If $\pi(x)\pi(y) = 1$, then $xy - 1 = z \in (A_1)$.
Hence, $z^r = 0$ and $(1+z)^{-1}$ exists and $= 1 - z + \cdots (-1)^r z^r$.
$\therefore x^{-1} = y(1+z)^{-1}$.

Corollary: The same holds for the algebra of matrices $\mathrm{Mat}_{\dim V}(A)$.

Superbilinear Forms:

Let $V \in$ SVect ; $B : V \times V \longrightarrow F$ bilinear and homogeneous with respect to the gradings

$$(V \times V)_0 = V_0 \times V_0 \cup V_1 \times V_1$$
$$(V \times V)_1 = V_0 \times V_1 \cup V_1 \times V_0$$
$$(\mathbb{R})_0 = \mathbb{R} \ ; \quad (\mathbb{R})_1 = \{0\}.$$

(i) B is Supersymmetric $\iff B(u,v) = (-1)^{|v||u|} B(v,u)$

(ii) ν is Superskew $\iff \quad = -(-1)^{|v||u|} B(v,u)$.

Non-degeneracy is as usual: $B(u,v) = 0, \forall u \implies v = 0$.

Prop (i) B even, supersymmetric & non-degenerate
$\implies B = B_0 \oplus B_1$; $B_0 : V_0 \times V_0 \to \mathbb{R}$ orthogonal
$B_1 : V_1 \times V_1 \to \mathbb{R}$ symplectic
$B(V_0, V_1) = 0$

(ii) B odd, supersymmetric (or superskew) & non deg.
$\implies B = \begin{pmatrix} 0 & G \\ \Omega & 0 \end{pmatrix}$, $\dim V_0 = \dim V_1 = 2n$

Remark: One may define "Supersesquilinear" for $H : V \times V \to \mathbb{C}$ by requiring $H(u,v) = (-1)^{|u||v|} \overline{H(v,u)}$.

(iii) H even, supersesquilinear & non-deg $\implies H = H_0 \oplus i H_1$, H_μ hermitian on V_μ and $H(V_0, V_1) = 0$.

Applications (B-adjoints).

Let $V \in SVect$ & $F \in End\, V$ & $B \in Bil\, V$, non-deg, homog.

The B-adjoint of F is F^B defined by

$$B(F^B u, v) = (-1)^{|F||u|} B(u, Fv)$$

and extended linearly so that $F = F_0 + F_1 \Rightarrow F^B = F_0^B + F_1^B$.

Proposition In terms of (graded) bases

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow F^B = \begin{cases} \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}^{-1} \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}; |B| = \\ \begin{pmatrix} 0 & G \\ \Omega & 0 \end{pmatrix}^{-1} \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & G \\ \Omega & 0 \end{pmatrix}; |B| = 1 \end{cases}$$

Remark: $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix}$ is the supertransposition map.

This, makes sense for any $F: V_0 \oplus V_1 \to W_0 \oplus W_1$ as the dual $F^*: (W_0 \oplus W_1)^* \to (V_0 \oplus V_1)^*$ defined by

$$F^*(\alpha)(v) = (-1)^{|F||\alpha|} \alpha(F(v)),$$

and extended linearly.

Exercise: Let $A \in SCommSAlg$, $V \in SVect$, $B \in Bil\, V$, non deg, homg.
Extend $B: V \times V \to \mathbb{F}$ to $(A \otimes V) \times (A \otimes V) \to A$
A-bilinearly (with good care of signs !)

$$B((f \otimes F)^B (a \otimes u), b \otimes v) = (-1)^{|f \otimes F||a \otimes u|} B(a \otimes u, (f \otimes F)(b \otimes v))$$

Then, $(f \otimes F)^B = f \otimes F^B$

Exercise Repeat the analysis for the Supersesquilinear, homog, non-degenerate case.

Lie Superalgebras.

Dfn. $L \in Lie\,SAlg \Rightarrow L \in SVect + [\,,\,]: L \times L \to L$ Superskew, satisfying, $[L_\mu, L_\nu] \subset L_{\mu+\nu}$, and

$$(-1)^{|x||z|}[[x,y],z] + (-1)^{|y||x|}[[y,z],x] + (-1)^{|z||y|}[[z,x],y] = 0$$

Extension of (Super)ring.

$L \in Lie\,SAlg$ & $A \in SCommSAlg \Rightarrow L \otimes A \in Lie\,SAlg$
under, $[x \otimes a, y \otimes b] = (-1)^{|a||y|}[x,y] \otimes ab$.

Examples: (i) $\mathfrak{gl}(V_0|V_1)$;

Let $L = End\, V$; $V \in SVect$, and define

$$[f, g] = f \circ g - (-1)^{|g||f|} g \circ f$$

extending it bilinearly. Then $\mathfrak{gl}(V_0|V_1) \in Lie\,SAlg$.

(ii) $\mathfrak{sl}(V_0|V_1) = \{ f \in \mathfrak{gl}(V_0|V_1) \mid Str\, f = 0 \}$

Dfn (SUPERTRACE) $Str: \mathfrak{gl}(V_0|V_1) \to \mathbb{F}$ is
(i) linear, and
(ii) vanishes on supercommutators.

Proposition Str is a scalar multiple of $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \mathrm{Tr}A - \mathrm{Tr}D$.

Pf: Choose basis of $\mathfrak{gl}(V_0|V_1)$:

$$F_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & 0 \end{pmatrix} . \quad \Phi_{i\nu} = \begin{pmatrix} 0 & E_{i\nu} \\ 0 & 0 \end{pmatrix}$$

$$\Theta_{\mu j} = \begin{pmatrix} 0 & 0 \\ E_{\mu j} & 0 \end{pmatrix} \quad G_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & E_{\mu\nu} \end{pmatrix}$$

The only non-vanishing supercommutators are:

$$[F_{ij}, F_{kl}] = \delta_{jk} F_{il} - \delta_{il} F_{kj} \qquad [F_{ij}, \Phi_{k\nu}] = \delta_{jk} \Phi_{i\nu}$$

$$[F_{ij}, \Theta_{\mu k}] = -\delta_{ik} \Theta_{\mu j} \qquad [G_{\mu\nu}, G_{\rho\sigma}] = \delta_{\nu\rho} G_{\mu\sigma} - \delta_{\mu\sigma} G_{\rho\nu}$$

$$[G_{\mu\nu}, \Phi_{i\rho}] = -\delta_{\mu\rho} \Phi_{i\nu} \qquad [G_{\mu\nu}, \Theta_{\rho j}] = \delta_{\nu\rho} \Theta_{\mu j}$$

$$[\Phi_{i\nu}, \Theta_{\mu j}] = \delta_{\mu\nu} F_{ij} + \delta_{ij} G_{\mu\nu}$$

Very Important sign!

Thus, if ϕ is a function vanishing on supercommutators,

$$\phi \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \phi(F_{11}) \left[\mathrm{Tr}A - \mathrm{Tr}D \right] \blacksquare$$

Exercise Compute all the ideals of $\mathfrak{gl}(V_0|V_1)$ and show that

(i) $\mathfrak{sl}(V_0|V_1)$ is simple if $\dim V_0 \neq \dim V_1$

(ii) $\mathfrak{sl}(V_0|V_1) / \{c\,\mathrm{id}\}$ is simple if $\dim V_0 = \dim V_1$

(iii) Let $V \in \mathrm{SVect}$ & $\dim V_0 = \dim V_1$.

Consider $L = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathrm{End}(V_0 \oplus V_1) \right\}$

$$\cong \mathrm{End}\, V_0 \oplus \mathrm{Hom}(V_0, V_1)$$

$$A \longleftrightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \; ; \; B \longleftrightarrow \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$$

$[A, *]$ = usual Lie commutator of A with $*$

$[B_1, B_2] = B_1 B_2 + B_2 B_1$

The only ideal of L is the set $\left\{ \begin{pmatrix} cI & 0 \\ 0 & cI \end{pmatrix} \right\}$ (this follows from Schur's lemma, because the A's act irreducibly on the B's)

Then, $\left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\} / (c\mathbb{1})$ is a simple algebra.

However, $\exists$ a non-trivial outer derivation.

$$D \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \& \quad D \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

(ie, D is a derivation that cannot be of the form $\mathrm{ad}(x)$, for some $x = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$).

Conclusion: $\exists$ simple algebras in which not every derivation is inner.

More Examples of Lie SAlgs.

(iv) $\mathfrak{osp}_B(V_0|V_1)$ & $\mathfrak{sosp}_B(V_0|V_1) = \mathfrak{osp}_B \cap \mathfrak{sl}$

(v) $\mathfrak{u}_H(V_0|V_1)$ & $\mathfrak{su}_H(V_0|V_1) = \mathfrak{u}_H \cap \mathfrak{sl}$

We shall do only $\mathfrak{u}_H(V_0|V_1)$ & $\mathfrak{su}_H(V_0|V_1)$.

Let $V \in$ SVect & $H: V\times V \to \mathbb{C}$, even, sesquilin, non-deg., so that $H = H_0 \oplus i H_1$

$$\mathfrak{u}_H(V_0|V_1) = \{\xi \in \operatorname{End} V \mid \xi = \xi_0+\xi_1 \ \& \ H(\xi_\mu u, v) + (-1)^{\mu|u|} H(u, \xi_\mu v) = 0\}$$

Prop. $\mathfrak{u}_H(V_0|V_1) \simeq \mathfrak{u}_{H_0}(V_0) \oplus \mathfrak{u}_{H_1}(V_1) \oplus \operatorname{Hom}_{\mathbb{C}}(V_0,V_1)$

Proof: $$p = \begin{pmatrix}A&0\\0&D\end{pmatrix} \in (\mathfrak{u}_H(V_0|V_1))_0 \iff \begin{cases} H_0(Au_0,v_0)+H_0(u_0,Av_0)=0\\ H_1(Du_1,v_1)+H_1(u_1,Dv_1)=0\end{cases}$$

$$q = \begin{pmatrix}0&B\\C&0\end{pmatrix} \in (\mathfrak{u}_H(V_0|V_1))_1 \iff \begin{cases} H_1(Cu_0,v_1) = iH_0(u_0,Bv_1)\\ H_0(Bu_1,v_0) = iH_1(u_1,Cv_0)\end{cases}$$

But, $H_1(Cu_0,v_1)=iH_0(u_0,Bv_1) \iff H_0(Bu_1,v_0)=iH_1(u_1,Cv_0)$

and $H_1(Cu_0,v_1)=iH_0(u_0,Bv_1) \implies C = iH_1^{-1}B^* H_0$

Application: The automorphism group of the Hermitian Lie SAlgs $\mathfrak{u}_H(V_0|V_1)$
(ref. Lect Notes in Math. 1251 (1987) 1-48.)

Dfn $$\mathfrak{A} \in \operatorname{Aut}\mathfrak{u}_H(V_0|V_1) \iff \begin{cases} \mathfrak{A} \text{ is } \mathbb{R}\text{-linear, invertible,}\\ \mathfrak{A}\,\mathfrak{u}_H(V_0|V_1)_\mu \subset \mathfrak{u}_H(V_0|V_1)_\mu\\ [\mathfrak{A}\cdot,\mathfrak{A}\cdot] = \mathfrak{A}[\cdot,\cdot]\end{cases}$$

Theorem:

1.- Assume either $\dim_{\mathbb{C}} V_0 > 1$, or $\dim_{\mathbb{C}} V_1 > 1$.
Assume $\mathfrak{u}_{H_0}(V_0) \not\simeq \mathfrak{u}_{H_1}(V_1)$
Then, $\mathfrak{A}$ is given by conjugation of the elements of $\mathfrak{u}_H(V_0|V_1)$ by an $\mathbb{R}$-linear map of the form

$$L = \left(\begin{array}{c|c} L_0 & 0\\ \hline 0 & L_1\end{array}\right)$$

where,

(i) both, L_0 & L_1 are $\mathbb{C}$-linear $\left\{\begin{array}{l}\text{in which case, both,}\\ L_0 \ \& \ L_1 \text{ are isometries, or}\\ L_0 \ \& \ L_1 \text{ are anti-isometries}\\ \text{(if sgn } H_0, \text{ sgn } H_1 \text{ allow it)}\end{array}\right.$

or

(ii) both, L_0 & L_1 are $\mathbb{C}$-antilinear $\left\{\begin{array}{l}\text{in which case,}\\ L_0 \text{ is an isometry and}\\ L_1 \text{ an anti-isometry, or}\\ \text{viceversa. (if sgns. allow it)}\end{array}\right.$

2.- Assume $\mathfrak{u}_{H_0}(V_0) \simeq \mathfrak{u}_{H_1}(V_1)$
The only additional possibility is that $\mathfrak{A}$ is given by conjugation by $L = \begin{pmatrix}0&L_0\\L_1&0\end{pmatrix}$ with L_0 and L_1 as before.

Idea of proof: By classical theory one knows Aut $\mathfrak{g}_0$
Therefore, one knows $\mathfrak{A}|_{\mathfrak{g}_0}: \mathfrak{g}_0 \to \mathfrak{g}_0$.
On the other hand, $\mathfrak{A}|_{\mathfrak{g}_1}: \mathfrak{g}_1\to\mathfrak{g}_1$ must be an $\mathbb{R}$ linear isomorphism and the eqn.

$$[\mathfrak{A}|_{\mathfrak{g}_1}\xi, \mathfrak{A}|_{\mathfrak{g}_1}\zeta] = \mathfrak{A}|_{\mathfrak{g}_0}[\xi,\zeta]$$

is used to restrict the possibilities of $\mathfrak{A}|_{\mathfrak{g}_1}$, $\mathfrak{A}|_{\mathfrak{g}_0}$, or both.

<u>Exercise</u>. Do the case $\dim_{\mathbb{C}} V_0 = \dim_{\mathbb{C}} V_1 = 1$.

Show that even though $\mathrm{Aut}\, \mathfrak{U}_H(V_0|V_1)_0$ has four connected components, any $\mathfrak{A} \in \mathrm{Aut}\, \mathfrak{U}_H(V_0|V_1)$ must have

$$\mathfrak{A}\big|_{\mathfrak{U}_H(V_0|V_1)_0} \in \left(\mathrm{Aut}\, \mathfrak{U}_H(V_0|V_1)_0\right)_e .$$

<u>Another Application</u>: The automorphism group of the Conformal Superalgebra.

<u>Dfn.</u> (Conformal SAlg)

$$\mathfrak{su}_H(V_0|V_1) \;:\; H = H_0 \oplus i H_1 \qquad \begin{cases} \mathrm{sgn}\, H_0 = (2,2) \\ \mathrm{sgn}\, H_1 = (1,0) \end{cases}$$

<u>Theorem</u> The only automorphisms of $\mathfrak{su}(2,2|1,0)$ are given by conjugation by $\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix}$ with

(a) $L_0 \in (\mathrm{Aut}\, \mathfrak{su}(2,2))_e$

or

(b) $L_0 \in (\mathrm{Aut}\, \mathfrak{su}(2,2))_P$ (antilinear anti-isometries).

<u>Proof</u>: This is actually a corollary of the previous theorem; since $\mathrm{sgn}\, H_1 = (1,0)$, L_1 cannot be an anti-isometry.

To show that the map

$$\mathfrak{su}(2,2) \ni \xi \longmapsto L_0 \xi L_0^{-1} \in \mathfrak{su}(2,2)$$
$$(L_0 \text{ antilinear antiisometry})$$

induces a transformation that reverses the orientation of space ~~[illegible]~~ ($=$ P-component of $\mathrm{Aut}\, \mathfrak{su}(2,2)$), ~~[illegible]~~

one has to look at some explicit realization of $\mathfrak{su}(2,2|1,0)$ and to study $\mathrm{Aut}\, \mathfrak{su}(2,2)$ and the effect of each of its connected components on space time [ref: OASV-Sternberg Lect. Notes in Math. <u>1251</u> (1987) 1-48].

<u>Parenthesis</u>: For the benefit of the audience, here is one convenient realization: choose a graded basis of $V_0 \oplus V_1 = \mathbb{C}^4 \oplus \mathbb{C}^1$, so that

$$H_0 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \quad ; \quad H_1 = (1_{1\times 1})$$

Then,

$$\mathfrak{su}(2,2|1,0) = \left\{ \begin{pmatrix} A & x & u \\ y & -A^* & v \\ iv^* & iu^* & 2i\,\mathrm{Im}\,\mathrm{Tr}\,A \end{pmatrix} \,\middle|\, \begin{matrix} u, v \in \mathbb{C}^2; \\ x^* = -x\,,\; y^* = -y \\ A \in \mathfrak{gl}(2,\mathbb{C}) \end{matrix} \right\}$$

A convenient representative of an antilinear antiisometry for H_0 is the 'charge conjugation' map

$$L_0 = \begin{pmatrix} \circledast & 0 \\ 0 & \circledast \end{pmatrix} : \mathbb{C}^4 \to \mathbb{C}^4$$

where, $\circledast : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{u}_2 \\ -\bar{u}_1 \end{pmatrix}$.

Thus, for example

$$L_0 \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} L_0^{-1} = \begin{pmatrix} 0 & \circledast x \circledast \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^a \\ 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^a = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is the Cramer's adjoint.

Writing $x = i \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$

the map $x \mapsto x^a$ corresponds to space inversion.