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Algebraic Geometry of Instantons

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These are preliminary lecture notes, intended only for distribution to participants

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Introduction. Instantons are (anti-)self-dual $SU(2)$-connections on the 4-dimensional sphere S^4. The set of guage equivalence classes of instantons with Pontrjagin number k carries a natural structure $I(k)$ of $(8|k|-3)$-dimensional real analytic manifold. The Penrose transformation enables us to exploit algebraic geometry for the study of instantons, especially the moduli space $I(k)$. In fact, pulling back an instanton to $\mathbf{P}^3_{\mathbf{C}}$ by the twistor space structure $\nu : \mathbf{P}^3_{\mathbf{C}} \to S^4$, we obtain an algebraic vector bundle of rank 2. This procedure gives rise to a bijective correspondence between the set of guage equivalence classes of instantons with Pontrjagin number k and the set of isomorphism classes of algebraic vector bundles E with the following properties (see Theorem 4.5)

(1) E is quaternionic

(2) The restriction of E to any fiber of ν is trivial

(3) $c_2(E) = |k|$.

An algebraic vector bundle of rank 2 on $\mathbf{P}^3_{\mathbf{C}}$ with the properties (1) and (2) is called an instanton bundle. An instanton bundle is μ-stable except for a trivial case where $k = 0$. On the other hand, we have the theory of moduli space of stable sheaves. In terms of the moduli space of stable sheaves, the condition (1) in the above can be understood that the moduli space $I(k)$ is contained in the *real* part of the moduli space $\overline{M}(|k|, 2)$ of semi-stable sheaves of rank 2 on $\mathbf{P}^3_{\mathbf{C}}$with $c_1 = 0$, $c_2 = |k|$ and $c_3 = 0$. The above condition (2) shows that $I(k)$ is an open set in classical topology in the real part. Thus the condition (2) does not fit in seemingly with algebraic geometry. In fact, $I(k)$ cannot be, by nature, an algebraic set and we can hope, at best, that it is semi-algebraic. For example, it is well-known that $I(1)$ is inside the 5-dimensional ball. Verdiera's description ([10]) of instanton bundles, however, shows that the conditions are translated into linear and quadratic equations

if we use real coordinates and monads. All the difficulties are reduced to the positive definiteness of a Hermitian form.

We start from algebraic equations given by Verdier's description and get a semi-algebraic set, taking the image by a real rational map. $I(k)$ is the quotient of this semi-algebraic set by an action of a real algebraic group. We obtain a compactification of $I(k)$ as the closure of $I(k)$ in an ambient projective variety where $I(k)$ is embedded semi-algebraically. We see then that this compactification is nothing but Donaldson's compactification ([4]).

This note is divided into two parts. The first part deals with the differential geometric viewpoint of instantons and ends by reducing them to algebraic vector bundles on $\mathbf{P}^3_{\mathbf{C}}$ with the above properties (1) and (2). I extracted ideas of the explanation from the beautiful lecture note by M. F. Atiyah ([1]) and added some explicit computations. The second part is essentially the same as the latter half of my joint work with G. Trautmann ([6]). Here we depend on Verdier's description of instantons and interpret it in term of a monad other than that used by Verdier. Then we combine our interpretation with the geometric invariant theory to get our main result.

1 Connections on C^r-bundles

Throughout this section K denotes the real number field $\mathbf{R}$ or the complex number field $\mathbf{C}$. Let X be a differentiable manifold and E a K^r-bundle or a vector bundle of rank r. Locally E is a direct product of K^r and X, more precisely there is an open covering $\{U_\alpha\}$ of X and a collection of differentiable maps $g_{\alpha\beta}$ of $U_\alpha \cap U_\beta$ to $GL(r, K)$ such that $g_{\alpha\alpha}$ is the constant map to the identity matrix I_r and $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. $U_\alpha \times K^r$ and $U_\beta \times K^r$ are identified on $U_\alpha \cap U_\beta$ via

$$U_\alpha \times K^r|_{U_\alpha \cap U_\beta} \ni (x, \xi) \longmapsto (x, g_{\beta\alpha}(x)\xi) \in U_\beta \times K^r|_{U_\alpha \cap U_\beta}$$

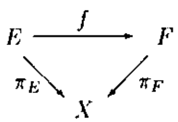
By this identification we can glue $U_\alpha \times K^r$ together to get a K-vector bundle E. $\{g_{\alpha\beta}\}$ is called the transition matrices of E and r is, by definition, the rank of E. If X is a complex manifold, K is the complex number field $\mathbf{C}$ and if we can choose $\{U_\alpha\}$ and $\{g_{\alpha\beta}\}$ so that $g_{\alpha\beta}$'s are holomorphic, then E is said to be a holomorphic vector bundle. A differentiable (or holomorphic) vector bundle is a differentiable (or complex, resp.) manifold and the natural projection $\pi : E \to X$ is a differentiable (or holomorphic, resp.) map. If we take the dual vector space of K^r, then $\{{}^t g_{\alpha\beta}^{-1}\}$ defines another vector bundle which is called the dual vector bundle of E and denoted by $E^\vee$.

Example 1.1. Let $\{U_\alpha\}$ be an open covering by charts of a differentiable manifold X and $(x_1^\alpha, \ldots, x_n^\alpha)$ be a local coordinate system on U_α. If we define $g_{\alpha\beta}$ to be the $(n \times n)$-matrix of differentiable functions on $U_\alpha \cap U_\beta$

$$\left(\frac{\partial x_i^\alpha}{\partial x_j^\beta}\right)$$

then the collection $\{g_{\alpha\beta}\}$ defines an $\mathbf{R}$-vector bundle of rank n on X. This is called the tangent bundle of X and we denote it by T_X. The dual bundle T_X^* of T_X is called the cotangent bundle of X. Assuming that X is a complex manifold and taking a holomorphic coordinate system $(x_1^\alpha, \ldots, x_n^\alpha)$ on each chart, $g_{\alpha\beta}$ is a matrix of holomorphic functions on $U_\alpha \cap U_\beta$. Then we obtain a holomorphic vector bundle of rank n on X. This is the holomorphic tangent bundle of X and denoted also by T_X. The dual bundle Ω_X is the holomorphic cotangent bundle on X.

Let E and F be differentiable (or holomorphic) vector bundle on X. A differential (or holomorphic, resp.) map $f : E \to F$ is called a morphism of vector bundles if f is compatible with the projections and the map $f_x : E_x \to F_x$ of fibers is a linear map of vector spaces for every $x \in X$, where $E_x = \pi_E^{-1}(x)$ and $F_x = \pi_F^{-1}(x)$ with π_E and π_F the projections of E and F, respectively.



Let r and s be the ranks of E and F, respectively. In terms of the transition matrices $\{g_{\alpha\beta}^E\}$ and $\{g_{\alpha\beta}^F\}$ of E and F, the morphism f is a collection $\{f_\alpha\}$ of differrential (or holomorphic, resp.) maps of U_α to $\mathrm{Hom}_K(K^r, K^s) \cong K^{rs}$ such that $f_\alpha g_{\alpha\beta}^E = g_{\alpha\beta}^F f_\beta$ on $U_\alpha \cap U_\beta$

If a morphism $f : E \to F$ is an embedding, then E is said to be a subbundle of F. If f is a surjection, then we call F a quotient bundle of E. An isomorphism of vector bundles is a morphism which induces an

isomorphism on each fiber. In the case where E is a subbundle of F, by choosing suitable local bases of fibers, $g_{\alpha\beta}^F$ is written in the form

$$\begin{pmatrix} g_{\alpha\beta}^E & A_{\alpha\beta} \\ 0 & g'_{\alpha\beta} \end{pmatrix}$$

and then $\{g'_{\alpha\beta}\}$ gives rise to a quotient bundle of F which is denoted by F/E. It is not hard to see that if F is a quotient bundle of E, then there is a subbundle S of E such that F is isomorphic to E/S.

For K-vector bundles E and F on X, the tensor products of the transition matrices of E and F define another vector bundle $E \otimes_K F$ whose fibers are the tensor products of those of E and F. The vector bundle is called the tensor product of E and F. Similarly we can define a vector bundle whose fiber over a point x is $\mathrm{Hom}_K(E_x, F_x)$. This is denoted by $\mathrm{Hom}_K(E, F)$. As in the case of vector spaces we have a canonical isomorphism of $E^\vee \otimes_K F$ to $\mathrm{Hom}_K(E, F)$.

We denote the sheaf of $\mathbf{C}$-valued C^∞ p-forms on X by $\mathcal{A}^p$. Let E be a differentiable $\mathbf{C}$-vector bundle of rank r on X. For an open set U of X, we set

$$\mathcal{A}^p(E)(U) = \{E\text{-valued } \mathbf{C}^\infty\ p\text{—forms on } U\}.$$

An element ζ of $\mathcal{A}^p(E)(U)$ can be written in the form

$$\zeta_\alpha = \begin{pmatrix} \omega_{1\alpha} \\ \vdots \\ \omega_{r\alpha} \end{pmatrix}$$

on $U \cap U_\alpha$, where $\omega_{i\alpha}$'s are p-forms on $U \cap U_\alpha$. The collection $\{\zeta_\alpha\}$ subjects to the following transformation rule:

$$g_{\alpha\beta} \begin{pmatrix} \omega_{1\beta} \\ \vdots \\ \omega_{r\beta} \end{pmatrix} = \begin{pmatrix} \omega_{1\alpha} \\ \vdots \\ \omega_{r\alpha} \end{pmatrix}$$

on $U \cap U\alpha \cap U_\beta$. The correspondence $U \to \mathcal{A}^p(E)(U)$ defines a sheaf $\mathcal{A}^p(E)$ on X.

A connection ∇ is a $\mathbf{C}$-linear map of sheaf $\mathcal{A}^0(E)$ to $\mathcal{A}^1(E)$ with the following property

(1.2) for every open $U \subset X$, every $f \in \mathcal{A}^0(U)$ and every $\zeta \in \mathcal{A}^0(E)(U)$, we have

$$\nabla(U)(f\zeta) = df \cdot \zeta + f\nabla(U)(\zeta).$$

This is the same as defining a $\mathbf{C}$-linear map $\widetilde{\nabla}$ of $\mathcal{A}^0(E)(X)$ to $\mathcal{A}^1(E)(X)$ such that for every $f \in \mathcal{A}^0(X)$ and every $\zeta \in \mathcal{A}^0(E)(X)$, we have

$$\widetilde{\nabla}(f\zeta) = df \cdot \zeta + f\widetilde{\nabla}(\zeta).$$

For a p-form ω and a local section ζ of E, we define

$$\nabla(\omega\zeta) = d\omega \cdot \zeta + (-1)^p\omega \wedge \nabla(\zeta).$$

Then we get a $\mathbf{C}$-linaer map of $\mathcal{A}^p(E)$ to $\mathcal{A}^{p+1}(E)$ because each element of $\mathcal{A}^p(E)(X)$ is locally written in the form $\omega_1\zeta_1 + \cdots + \omega_s\zeta_s$ with ω_i a p-form and ζ_i a local section of E. Let us look at a special case

$$\nabla^2 : \mathcal{A}^0(E) \xrightarrow{\nabla} \mathcal{A}^1(E) \xrightarrow{\nabla} \mathcal{A}^2(E).$$

For an $f \in \mathcal{A}^0(U)$ and a $\zeta \in \mathcal{A}^0(E)(U)$, we have

$$\begin{aligned} \nabla^2(f\zeta) &= \nabla(df \cdot \zeta + f\nabla(\zeta)) \\ &= d^2f \cdot \zeta - df \wedge \nabla(\zeta) + df \wedge \nabla(\zeta) + f\nabla^2(\zeta) \\ &= f\nabla^2(\zeta). \end{aligned}$$

Thus ∇^2 is compatible with the multiplication by C^∞-functions, that is, ∇^2 can be regarded as a C^∞-section of $\bigwedge^2 T_X^* \otimes_{\mathbf{R}} E^\vee \otimes_{\mathbf{C}} E$.

Definition 1.3. The global section $F(\nabla)$ of $\bigwedge^2 T_X^* \otimes_{\mathbf{R}} E^\vee \otimes_{\mathbf{C}} E$ defined by ∇^2 is called the curvature of ∇.

Let us examine the meaning of connections and their curvatures by using local bases of vector bundles. Take an open set U of X where E is trivial, that is, there is an isomorphism $\varphi : E|_U \to U \times \mathbf{C}^r$. Let $e_1, \ldots, e_r$ be the sections of E over U such that $\varphi e_1, \ldots, \varphi e_r$ are the sections defined by the standard basis of $\mathbf{C}^r$. This $(e_1, \ldots, e_r)$ is called a frame of E over U. If ∇ is a connection on E, then $\nabla(e_i)$ can be written in a form

$$\nabla(e_i) = \sum_{j=1}^{r} \theta_{ji} e_j$$

with $\theta_{ji} \in \mathcal{A}^1(U)$. For $\zeta = \sum_{i=1}^r a_i e_i$ in $\mathcal{A}^0(E)(U)$, we have that $a_i \in \mathcal{A}^0(U)$ and

$$\begin{aligned} \nabla(\zeta) &= \sum_{i=1}^r da_i e_i + \sum_{i,j} a_i\theta_{ji}e_j \\ &= \sum_{j=1}^r (da_j + \sum_{i=1}^r a_i\theta_{ji})e_j. \end{aligned}$$

Taking another frame $(e'_1, \ldots, e'_r)$, we write $e'_i = \sum_{j=1}^r g_{ji}e_j$ and $e_i = \sum_{j=1}^r g'_{ji}e'_j$ with $g_{ji}, g'_{ji} \in \mathcal{A}^0(U)$. Then, for $g = (g_{ij})$, we have $g^{-1} = \left(g'_{ij}\right)$ and we see

$$\begin{aligned} \nabla(e'_i) &= \sum_{j=1}^r dg_{ji} \cdot e_j + g_{ji}\nabla(e_j) \\ &= \sum_{j=1}^r \left(dg_{ji} \cdot e_j + g_{ji} \sum_{k=1}^r \theta_{kj}e_k\right) \\ &= \sum_{j=1}^r \left\{dg_{ji}\left(\sum_{k=1}^r g'_{kj}e'_k\right) + g_{ji}\sum_{\ell,k}\theta_{\ell j}g'_{k\ell}e'_k\right\} \\ &= \sum_{k=1}^r \left(\sum_{j=1}^r g'_{kj}dg_{ji} + \sum_{1\le j,\ell\le r} g'_{k\ell}\theta_{\ell j}g_{ji}\right) e'_k \end{aligned}$$

The $r \times r$-matrix $\theta_e = (\theta_{ij})$ is called the connection matrix of ∇ withe respect to the frame $e = (e_1, \ldots, e_r)$. Then what we saw is

(1.4) $\theta_{e'} = g^{-1}dg + g^{-1}\theta_e g.$

Taking an open covering $\{U_\alpha\}$ of X such that $E|_{U_\alpha} \cong U_\alpha \times \mathbf{C}^r$ and fixing a frame e_α of $\{e_1^\alpha, \ldots, e_r^\alpha\}$ of E over U_α, we have $e_i^\beta = \sum_{j=1}^r g_{ji}e_j^\alpha$, where $(g_{ij}) = g_{\alpha\beta}$. By (1.4) we see the following.

Proposition 1.5. *A collection $\{\theta_\alpha\}$ of $r \times r$-matrices θ_α whose entries are 1-forms on U_α gives rise to a connection on E if and only if it subjects to the following rule of transformations:*

$$\theta_\beta = g_{\beta\alpha}dg_{\alpha\beta} + g_{\beta\alpha}\theta_\alpha g_{\alpha\beta} \qquad \text{on } U_\alpha \cap U_\beta,$$

where $\{g_{\alpha\beta}\}$ is the collection of the transition matrices of E.

Since the curvature $F(\nabla)$ is a global morphism of E to $\bigwedge^2 T_X^* \otimes_{\mathbf{R}} E$, the representation of $F(\nabla)_\alpha$ by the frame e_α has the following transformation rule:

(1.6) $F(\nabla)_\beta = g_{\beta\alpha} F(\nabla)_\alpha g_{\alpha\beta}$.

Moreover, we see

$$\begin{aligned}\nabla^2(e_i^\alpha) &= \nabla\left(\sum_{j=1}^r \theta_{ji} e_j^\alpha\right) \\ &= \sum_{j=1}^r \left(d\theta_{ji} e_j^\alpha - \sum_{k=1}^r (\theta_{ji} \wedge \theta_{kj})\, e_k^\alpha\right) \\ &= \sum_{j=1}^r \left(d\theta_{ji} + \sum_{k=1}^r \theta_{jk} \wedge \theta_{ki}\right) e_j^\alpha.\end{aligned}$$

We obtain therefore

Proposition 1.7. $F(\nabla)_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha$.

2 Hermitian bundles and integrable connections

Let E be a differentiable $\mathbf{C}$-vector bundle of rank r and let G be a Lie subgroup of $GL(r, \mathbf{C})$. When we can choose a system of transition matrices $\{g_{\alpha\beta}\}$ such that for every $x \in U_\alpha \cap U_\beta$, we have $g_{\alpha\beta}(x) \in G$, we say that E has a reduction to G or that E is a G-bundle. These transition matrices define isomorphisms

$$U_\beta \times G|_{U_\alpha \cap U_\beta} \ni (x, a) \longmapsto (x, g_{\alpha\beta}(x)a) \in U_\alpha \times G|_{U_\alpha \cap U_\beta}.$$

Gluing $\{U_\alpha \times G\}$ by using these isomorphisms, we obtain a G-principal fiber bundle P on X. P is a differentiable manifold with a G-action from right.

G acts on G by conjugations:

$$\text{for } g \in G, ad(g) : G \ni x \longmapsto gxg^{-1} \in G.$$

Obviously $ad(gg') = ad(g)ad(g')$. Then we have the diagonal action of G on $P \times G$:

$$P \times G \ni (z, x) \longmapsto (zg, ad(g^{-1})x) \in P \times G.$$

The quotient of $P \times G$ by this action is denoted by $P \times_{ad} G$ or G_P. G_P is a fiber bundle whose fiber is G. Set $\mathcal{G}_E$ to be the set of sections $\Gamma(X, G_P)$ of the fiber bundle G_P. $G_P|_{U_\alpha}$ is isomorphic to $U_\alpha \times G$ and for $x \in U_\alpha \cap U_\beta$, (x, g) in $U_\alpha \times G$ should be identified with $(x, g_{\beta\alpha}(x) g g_{\alpha\beta}(x))$ of $U_\beta \times G$. Thus a member t of $\mathcal{G}_E$ is a collection $\{t_\alpha\}$ of C^∞-maps t_α of U_α to G such that

$$g_{\beta\alpha} t_\alpha g_{\alpha\beta} = t_\beta \quad \text{or} \quad t_\alpha g_{\alpha\beta} = g_{\alpha\beta} t_\beta \quad \text{on} \quad U_\alpha \cap U_\beta.$$

Taking another member $s = \{s_\alpha\}$ of $\mathcal{G}_E$, let us consider $ts = \{t_\alpha s_\alpha\}$, where $(t_\alpha s_\alpha)(x) = t_\alpha(x) s_\alpha(x)$ in G for $x \in U_\alpha$. Then we see

$$t_\alpha s_\alpha g_{\alpha\beta} = t_\alpha g_{\alpha\beta} s_\beta = g_{\alpha\beta} t_\beta s_\beta \text{ on } U_\alpha \cap U_\beta$$

and hence $\mathcal{G}_E$ is a group with respect to the multiplication $(t, s) \mapsto ts$.

Definiton 2.1. The group $\mathcal{G}_E$ is called the guage transformation group of P (or, E).

A system $\{e^\alpha\}$ of frames e^α of a G-bundle E is called a system of G-frames if the transformation matrices with respect to the system are in G. Fix a system of G-frames $\{e^\alpha\}$ of E and take an element $t = \{t_\alpha\}$ of the guage transformation group $\mathcal{G}_E$. Since for every $x \in U_\alpha$, $t_\alpha(x)$ is an element of G, $(t_\alpha(e_1^\alpha), \ldots, t_\alpha(e_r^\alpha))$ is a frame of E over U_α. Moreover, if we write $t_\alpha(e_i^\alpha) = \sum_{j=1}^r \tau_{ji}^\alpha e_j^\alpha$, then we have

$$\begin{aligned} t_\alpha(e_i^\alpha) &= \sum_{j=1}^r \tau_{ji}^\alpha e_j^\alpha = \sum_{j=1}^r \tau_{ji}^\alpha \sum_{k=1}^r g_{kj}^{\beta\alpha} e_k^\beta \\ &= \sum_{k=1}^r \left(\sum_{j=1}^r g_{kj}^{\beta\alpha} \tau_{ji}^\alpha\right) e_k^\beta = \sum_{k=1}^r \left(\sum_{j=1}^r \tau_{kj}^\beta g_{ji}^{\beta\alpha}\right) e_k^\beta \\ &= \sum_{j=1}^r g_{ji}^{\beta\alpha} \sum_{k=1}^r \tau_{kj}^\beta e_k^\beta = \sum_{j=1}^r g_{ji}^{\beta\alpha} t_\beta(e_k^\beta) \end{aligned}$$

Therefore, $t_\alpha e^\alpha = \{t_\alpha(e_1^\alpha), \ldots, t_\alpha(e_r^\alpha)\}$ forms a G-frame of E, in other words, $\mathcal{G}_E$ is contained in the automorphism group of the G-bundle E.

Definition 2.2. A connection ∇ of a G-bundle E is called a G-connection if for a system of G-frames of E, the connection matrix is contained in $\mathfrak{g}$-valued 1-forms, where $\mathfrak{g}$ is the Lie algebra of G.

Let $\{e^\alpha\}$ be asystem of G-frames of E such that the connection matrices θ_α of ∇ with respect this system are $\mathfrak{g}$-valued 1-froms; θ_α is contained in $\mathcal{A}^1(U_\alpha)\otimes_{\mathbf{R}}\mathfrak{g}\subset\mathcal{A}^1(U_\alpha)\otimes_{\mathbf{R}}\mathfrak{gl}(r,\mathbf{C})$. Pick an element $t=\{t_\alpha\}$ of $\mathcal{G}_E$ and set $f^\alpha=t_\alpha e^\alpha$. Then, by (1.4) the connection matrix θ'_α of ∇ with respect to f^α is written as follows

$$\theta'_\alpha=t_\alpha^{-1}dt_\alpha+t_\alpha^{-1}\theta_\alpha t_\alpha=t_\alpha^{-1}dt_\alpha+ad(t_\alpha)\theta_\alpha$$

Thus θ'_α is $\mathfrak{g}$-valued 1-froms, too.

Proposition 2.3. *A G-connection is transformed to another G-connection by an element of the guage transformation group of E.*

The group G acts on $\mathfrak{g}$ by the adojoint representation $ad: G\to GL(\mathfrak{g})$. Then G acts on $P\times\mathfrak{g}$ as in the construction of G_P and we have an $\mathbf{R}$-vector bundle $P\times\mathfrak{g}/G$ which is denoted by $ad_E(\mathfrak{g})$.

If ∇ is a G-connection and if θ_α is the connection matrix of ∇ over U_α with respect to a system of G-frames $\{e^\alpha\}$, then the curvature matrix has the form

$$F(\nabla)_\alpha=d\theta_\alpha+\theta_\alpha\wedge\theta_\alpha$$

Shrinking U_α and taking local coordinates $x_1,\ldots,x_n$ in U_α, θ_α can be written in the form

$$\theta_\alpha=A_1dx_1+\cdots+A_ndx_n$$

with $A_i\in\mathcal{A}^0(ad_E(\mathfrak{g}))(U_\alpha)$. Then we see easily that dA_i is in $\mathcal{A}^1(ad_E(\mathfrak{g}))(U_\alpha)$ and $\theta_\alpha\wedge\theta_\alpha=\sum_{i<j}[A_i,A_j]dx_i\wedge dx_j$. Thus $F(\nabla)_\alpha$ is actually in $\mathcal{A}^2(ad_E(\mathfrak{g}))(U_\alpha)$.

Proposition 2.4. *If ∇ is a G-connection, then the curvature $F(\nabla)$ of ∇ is contained in $\Gamma(X,\bigwedge^2T_X^*\otimes_{\mathbf{R}}ad_E(\mathfrak{g}))$.*

Now we shall study a special case.

Definition 2.5. A $\mathbf{C}$-vector bundle E on X is called a Hermitian bundle if there is a C^∞-map h of $E\otimes_{\mathbf{C}}E$ to $\mathbf{C}$ such that at each point x of X, $h(x): E_x\otimes_{\mathbf{C}}E_x\to\mathbf{C}$ gives rise to a positive definite Hermitian form.

If E is a Hermitian bundle and if h is the Hermitian form on E, then at each point x of X, we can find an open neighborhood U of x and a frame

$(e_1,\ldots,e_r)$ of E over U which is orthonormal with respect to $h(y)$ for all $y\in U$. Using these frames, we see that E has a reduction to the unitary group $U(r)$ or E is a $U(r)$-bundle. This system of frames is called a unitary frame of E.

Remark 2.6. If X is paracompact,then every differentiable $\mathbf{C}$-bundle on X carries a Hermitian form.

Let ∇ be a connection on a Hermitian bundle E. Taking a unitary frame $\{e^\alpha\}$ of E, we denote the connection matrices by θ_α. Our Hermitian form h on E defines a C^∞-map $T_X^*\otimes_{\mathbf{R}}E\otimes_{\mathbf{C}}E\to T_X^*\otimes_{\mathbf{R}}\mathbf{C}$. We use the notation (ξ,η) instead of $h(\xi,\eta)$. Pick ξ, η in $\mathcal{A}^0(E)(U_\alpha)$ and then we can write

$$\xi=\sum_{i=1}^r a_ie_i^\alpha,\qquad \eta=\sum_{i=1}^r b_ie_i^\alpha$$

with a_i, b_i differentiable functions on U_α. Since

$$\nabla(\xi)=(e_1^\alpha,\ldots,e_r^\alpha)\left(\begin{pmatrix}da_1\\ \vdots\\ da_r\end{pmatrix}+\theta_\alpha\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix}\right)$$

we have

$$(\nabla(\xi),\eta)=\sum_{i=1}^r da_i\bar b_i+(a_1,\ldots,a_r)^t\theta_\alpha\begin{pmatrix}\bar b_1\\ \vdots\\ \bar b_r\end{pmatrix}.$$

Similarily,

$$(\xi,\nabla(\eta))=\sum_{i=1}^r a_id\bar b_i+(a_1,\ldots,a_r)\bar\theta_\alpha\begin{pmatrix}\bar b_1\\ \vdots\\ \bar b_r\end{pmatrix}.$$

Thus $d(\xi,\eta)=(\nabla(\xi),\eta)+(\xi,\nabla(\eta))$ for all ξ and η if and only if ${}^t\theta_\alpha=-\bar\theta_\alpha$ or θ_α ia an element of $\mathcal{A}^1(ad_E(\mathfrak{u}(r)))$, where $\mathfrak{u}(r)$ is the Lie algebra of $U(r)$.

Definition 2.7. A connection ∇ of a Hermitian bundle E is called a Hermitian connection if for all $\xi,\eta\in\mathcal{A}^0(E)(U)$, we have

$$d(\xi,\eta)=(\nabla(\xi),\eta)+(\xi,\nabla(\eta)).$$

What we have showed right before the definition is the following.

Proposition 2.8. *∇ is a $U(r)$-connection with respect to a unitary frame if and only if it is a Hermitian connection.*

Assume now that X is a complex manifold and take a local (holomorphic) coordinates $z_1, \dots, z_n$ in an open set U. Z_j can be written in the form $x_j + \sqrt{-1}y_j$ with $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ a system of real coordinates. The 1-forms $dx_j + \sqrt{-1}dy_j$ and $dx_j - \sqrt{-1}dy_j$ are denoted by dz_j and $d\bar{z}_j$, respectively. $dz_1, d\bar{z}_1, dz_2, d\bar{z}_2, \dots, dz_n, d\bar{z}_n$ form a basis of $\mathcal{A}^1(U) = (T_X^* \otimes_{\mathbf{R}} \mathbf{C})(U)$ over $\mathcal{A}^0(U)$. The forms spanned by $\{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \mid i_1 < \cdots < i_p, j_1 < \cdots < j_q\}$ over $\mathcal{A}^0(U)$ are called (p,q)-forms or forms of type (p,q). For a vector bundle E on X, $\mathcal{A}^{p,q}(E)(U)$ is the set of E-valued (p,q)-forms on U and $\mathcal{A}^{p,q}(E)$ denotes the sheaf of E-valued (p,q)-forms.

For a C^∞-function $f \in \mathcal{A}^0(U)$ on U, we get

$$\begin{aligned} df &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i \right) \\ &= \sum_{i=1}^{n} \left\{ \frac{1}{2} \left(\frac{\partial f}{\partial x_i} - \sqrt{-1}\frac{\partial f}{\partial y_i} \right) dz_i + \frac{1}{2} \left(\frac{\partial f}{\partial x_i} + \sqrt{-1}\frac{\partial f}{\partial y_i} \right) d\bar{z}_i \right\}. \end{aligned}$$

By setting

$$d'f = \sum_{i=1}^{n} \frac{1}{2} \left(\frac{\partial f}{\partial x_i} - \sqrt{-1}\frac{\partial f}{\partial y_i} \right) dz_i = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i$$

and $$d''f = \sum_{i=1}^{n} \frac{1}{2} \left(\frac{\partial f}{\partial x_i} + \sqrt{-1}\frac{\partial f}{\partial y_i} \right) d\bar{z}_i = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

we see that $df = d'f + d''f$ and f is holomorphic if and only if $d''f = 0$. We can extend d, d' and d'' to operations on (p,q)-forms so that for $\omega = f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, we have

$$\begin{aligned} d\omega &= df \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \\ d'\omega &= d'f \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \\ d''\omega &= d''f \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}. \end{aligned}$$

Note that $d'\omega$ is a $(p+1,q)$-form and $d''\omega$ is a $(p,q+1)$-form.

If E is a holomorphic vector bundle on X, then we can find a system of frames $\{e^\alpha = (e_1^\alpha, \dots, e_r^\alpha)\}$ such that the isomorphism of $E|_{U_\alpha}$ to $U_\alpha \times \mathbf{C}^r$ with

respect to e^α is holomorphic. This system of frames is called a holomorphic frame.

If both $(e_1, \dots, e_r)$ and $(e'_1, \dots, e'_r)$ are holomorphic frames of E over U, then we can write $e_i = \sum_{j=1}^r g_{ji} e'_j$ with g_{ji} holomorphic functions on U, in particular, the transition matrices with respect to a system of holomorphic frames are holomorphic. Taking a holomorphic frame $(e_1, \dots, e_r)$ of E over U, an element ζ in $\mathcal{A}^{p,q}(E)(U)$ can be written in a form

$$\zeta = \sum_{i=1}^{r} a_i e_i \quad \text{with } a_i \in \mathcal{A}^{p,q}(U).$$

Let us define

$$d''\zeta = \sum_{i=1}^{r} (d''a_i) e_i.$$

Then we see that $d''\zeta$ is in $\mathcal{A}^{p,q+1}(E)(U)$. If $\{e'_1, \dots, e'_r\}$ is another holomorphic frame and if we write $e_i = \sum_{j=1}^r g_{ji} e'_j$, then

$$\zeta = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} g_{ij} a_j \right) e'_i.$$

Since $d''(g_{ij} a_j) = d'' g_{ij} \wedge a_j + g_{ij} d'' a_j = g_{ij} d'' a_j$, one sees

$$\sum_{i=1}^{r} d'' \left(\sum_{j=1}^{r} g_{ij} a_j \right) e'_i = \sum_{i=1}^{r} d'' a_i \sum_{j=1}^{r} g_{ji} e'_j = \sum_{i=1}^{r} d'' a_i e_i$$

and hence the definition of $d''\zeta$ is independent of the choice of holomorphic frames. This means that the local d'' are glued together to get a $\mathbf{C}$-linear map

$$d'' : \mathcal{A}^{p,q}(E) \longrightarrow \mathcal{A}^{p,q+1}(E).$$

Since $\mathcal{A}^1(E)$ is a direct sum of $\mathcal{A}^{1,0}$ and $\mathcal{A}^{0,1}$, a connection ∇ on E is the sum of two maps ∇' and ∇'':

$$\begin{aligned} \nabla' &: \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{1,0}(E) \\ \nabla'' &: \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{0,1}(E) \end{aligned}$$

Definition 2.9. Let E be a holomorphic vector bundle on X with a Hermitian structure. A connection ∇ on E is said to be *compatible* with the holomorphic and Hermitian structure or an *integrable* Hermitian connection if

(1) $d(\xi,\eta) = (\nabla(\xi),\eta) + (\xi,\nabla(\eta))$ and

(2) $\nabla'' = d''$.

We have the following basic result.

Proposition 2.10. *Every holomorphic, Hermitian vector bundle carries a unique integrable connection.*

Proof. Let E be a holomorphic, Hermitian vector bundle and ∇ be an integrable connection. Fixing a holomorphic frame of E on an open set U, the connection matrices of ∇' and ∇'' are denoted by θ' and θ'', respectively. θ' is an $r \times r$-matrix whose entries are $(1,0)$-forms and we see by (2) of the definition that $\theta'' = 0$. Choosing a $U(r)$-frame of E over U and denoting the transformation of the holomorphic frame to the $U(r)$-frame by g, we can write the connection matrix τ with respect to the $U(r)$-frame in the form

$$\tau = g^{-1}dg + g^{-1}(\theta' + \theta'')g$$

If we set τ' (or, τ'') to be the $(1,0)$-part (or, $(0,1)$-part, resp.) of τ, then we see

$$\tau'' = g^{-1}d''g + g^{-1}\theta''g = g^{-1}d''g$$

which is determined by g. Moreover, since τ is in $\mathcal{A}^1(ad_E(\mathfrak{u}(r)))$, one sees that ${}^t\bar{\tau} = -\tau$. Hence, by comparing the types, one gets $\tau' = -{}^t\bar{\tau}'' = -{}^t(\overline{g^{-1}d''g})$, which is also determined by g. Now the connection thus obtained from g is independent of the choice of the holomorphic frames and the $U(r)$-frames (left to the reader). This means that the connection is unique. Taking a holomorphic frame $(e_1,\dots,e_r)$ of E over U and set $h = (h_{ij})$ with $h_{ij} = (e_i, e_j)$. Then, a connection with the connection matrix θ is Hermitian if and only if we have the equality

$$dh = {}^t\theta h + h\bar{\theta}.$$

If we set $\theta = {}^t(d'h \cdot h^{-1})$, then it is easily seen that this θ satisfies the above equality, of type $(1,0)$ and subjects to the transformation rule of connection matrices. Q.E.D.

For an integrable Hermitian connection ∇, $\nabla''^2 = 0$ because $\nabla'' = d''$. Thus, if we decompose the curvature $F(\nabla)$ into the sum $F(\nabla)^{2,0} + F(\nabla)^{1,1} +$

$F(\nabla)^{0,2}$ along types, then $F(\nabla)^{0,2}$ must be 0. On the other hand, the matrix $F(\nabla)$ is skew-Hermitian. Hence $F(\nabla)^{2,0} = -{}^t\overline{F(\nabla)^{0,2}} = 0$. This shows that the curvature of an integrable Hermitian connection is of type $(1,1)$. The converse of this is also true.

Theorem 2.11. *Let E be a Hermitian vector bundle over a complex manifold and let ∇ be a Hermitian connection on E. E carries a complex structure with respect to which ∇ is an integrable Hermitian connection if and only if the curvature of ∇ is of type $(1,1)$.*

3 Hodge's $*$-operator

An orientation of a real vector space V is an ordered system of basis $(x_1,\dots,x_n)$ of V. Two ordered bases of V give rise to the same orientation if the determinant of the transformation of one to the other is positive. In particular, for a permutation σ of the set $\{1,\dots,n\}$, $(x_{\sigma(1)},\dots,x_{\sigma(n)})$ determines the same orientation as $(x_1,\dots,x_n)$ if and only if the signature of σ is 1.

Let us fix an inner product $(\ ,\ )$ on the dual space $V^\vee$ and an orientation of V. Giving an inner product on $V^\vee$ is equivalent to giving an element $\sum_{i,j} h_{ij}x_i \otimes x_j$ in $S^2(V)$ such that the matrix $h = (h_{ij})$ is positive definite. We can find an ordered basis $(x_1,\dots,x_n)$ of V such that it gives our fixed orientation and the h with respect to this basis is the identity matrix. For an even permutation $(i_1,\dots,i_p,j_1,\dots,j_{n-p})$ of $(1,\dots,n)$, we define

$$*(x_{i_1} \wedge \cdots \wedge x_{i_p}) = x_{j_1} \wedge \cdots \wedge x_{j_{n-p}}$$

and then extend this to a $\mathbf{C}$-linear transformation of the exterior algebra $\bigwedge V \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus \bigwedge^p V \otimes_{\mathbf{R}} \mathbf{C}$. This linear map $*$ is called Hodge's $*$-operator. It is easy to see that $*$ is compatible with the representations of the special ortogonal group $SO(n)$ on $\bigwedge^p V$ and $\bigwedge^{n-p} V$ and hence it is independent of the choice of oriented orthonormal bases of V.

A differentiable manifold is said to be orientable if there is a chart $\{U_\alpha\}$ and an ordered system of local coordinates $(x_1^\alpha,\dots,x_n^\alpha)$ in U_α such that $\det(\frac{\partial x_i^\alpha}{\partial x_j^\beta})$ is positive at each point of $U_\alpha \cap U_\beta$.

Assume that X is an orientable Riemannian manifold. Riemannian structure on X is a differentiable, symmetric bilinear map h of $T_X \otimes_{\mathbf{R}} T_X$ to $\mathbf{R}$ such that at each point x of X, h induces a positive definite bilinear form $h(x)$ on the tangent space of X at x. Then we can define a $*$-operator on the exterior algebra $\bigwedge T^*_{X,x} \otimes_{\mathbf{R}} \mathbf{C}$ over the complexified cotangent space $T^*_{X,x} \otimes_{\mathbf{R}} \mathbf{C}$ of X at x.

A complex manifold is canonically orientable. In fact, if $z_1^\alpha = x_1^\alpha + \sqrt{-1}y_1^\alpha, \dots, z_n^\alpha = x_1^\alpha + \sqrt{-1}y_n^\alpha$ are complex coodinates in a chart U^α, then $(x_1^\alpha, y_1^\alpha, \dots, x_1^\alpha, y_n^\alpha)$ gives rise to a system of real coordinates in U^α which provides us with a global orientation. Let h be a Hermitian structure on the complex tangent bundle of X. h induces a unitary structure $h(x)$ on the complex tangent space $T_{X,x}$ of X at x. $h(x)$ is represented by $H = \sum_{i,j} h_{i,j} dz_i \otimes d\bar{z}_j$ with $h(x) = (h_{ij})$. After changing coordinates differetiably we may assume that $h(x)$ is the identity matrix and then $H = \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i)$. Thus, applying the above procedure to the $\mathbf{R}$-vector space V spanned by $\{dx_1, dy_1, \dots, dx_n, dy_n\}$ in $T^*_{\mathbf{C}} = T^*_{X,x} \otimes_{\mathbf{R}} \mathbf{C}$, we can define a $*$-operator on the algebra $\bigwedge T^*_{\mathbf{C}}$.

Since a $*$-operator is independent of the choice of oriented orthonormal basis, we have a global $*$-operator on the sheaf $\oplus \mathcal{A}^p$ and the space of sections $\oplus \mathcal{A}^p(U)$ on an open set U.

Let A, B and M be mutually disjoint subsets of $\{1, \dots, n\}$. Set dz_A to be $dz_{i_1} \wedge \dots \wedge dz_{i_a}$, where $A = \{i_1, \dots, i_a\}$ and $i_1 < \dots < i_a$. Similarily, we define $d\bar{z}_B$. Let w_M be the form $\prod_{j \in M}(dz_j \wedge d\bar{z}_j)$. Then we have

$$*(dz_A \wedge d\bar{z}_B \wedge w_M) = (-1)^{m + \frac{p(p+1)}{2}} (-2\sqrt{-1})^{p-n} \sqrt{-1}^{a-b} dz_A \wedge d\bar{z}_B \wedge w_N,$$

where $m = \sharp M$, $p = \sharp A + \sharp B + 2m$ and $N = \{1, \dots, n\} \setminus (A \cup B \cup M)$.

Now assume that X is a 4-dimensional, orientable Riemanian manifold. Then the $*$-operator induces an involution on $\mathcal{A}^2(U)$ and hence $\mathcal{A}^2(U)$ is the direct sum of two subspaces

$$\begin{aligned} \mathcal{A}^2(U)_+ &= \{\omega \mid \omega \text{ is a } 2-\text{form with } *\omega = \omega\} \\ \mathcal{A}^2(U)_- &= \{\omega \mid \omega \text{ is a } 2-\text{form with } *\omega = -\omega\} \end{aligned}$$

A member of $\mathcal{A}^2(U)_+$ (or, $\mathcal{A}^2(U)_-$) is called a *self-dual* (or, *antiself-dual, resp.*) form. Let $(x_1, \dots, x_4)$ be a oriented system of coodinates in U such that the Riemannian metric is given by $\sum_{i=1}^4 dx_i \otimes dx_i$. We see then that $\mathcal{A}^2(U)_+$ is spanned by

$$dx_1 \wedge dx_2 + dx_3 \wedge dx_4, dx_1 \wedge dx_3 - dx_2 \wedge dx_4, dx_1 \wedge dx_4 + dx_2 \wedge dx_3$$

and $\mathcal{A}^2(U)_-$ is spanned by

$$dx_1 \wedge dx_2 - dx_3 \wedge dx_4, dx_1 \wedge dx_3 + dx_2 \wedge dx_4, dx_1 \wedge dx_4 - dx_2 \wedge dx_3.$$

If one sets $z_1 = x_1 + \sqrt{-1}x_2$ and $z_2 = x_3 + \sqrt{-1}x_4$, then one has

$$dx_1 = \frac{1}{2}(dz_1 + d\bar{z}_1) \quad dx_2 = -\frac{\sqrt{-1}}{2}(dz_1 - d\bar{z}_1)$$

$$dx_3 = \frac{1}{2}(dz_2 + d\bar{z}_2) \quad dx_4 = -\frac{\sqrt{-1}}{2}(dz_2 - d\bar{z}_2).$$

The above computations show that on $\mathcal{A}^2(U)_+$

$$\begin{aligned} dx_1 \wedge dx_2 + dx_3 \wedge dx_4 &= \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \\ dx_1 \wedge dx_3 - dx_2 \wedge dx_4 &= \frac{1}{2}(dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2) \\ dx_1 \wedge dx_4 + dx_2 \wedge dx_3 &= -\frac{\sqrt{-1}}{2}(dz_1 \wedge dz_2 - dz_1 \wedge dz_2) \end{aligned}$$

and that on $\mathcal{A}^2(U)_-$

$$\begin{aligned} dx_1 \wedge dx_2 - dx_3 \wedge dx_4 &= \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2) \\ dx_1 \wedge dx_3 + dx_2 \wedge dx_4 &= \frac{1}{2}(dz_1 \wedge d\bar{z}_2 - dz_2 \wedge d\bar{z}_1) \\ dx_1 \wedge dx_4 - dx_2 \wedge dx_3 &= \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_2 + dz_2 \wedge dz_1). \end{aligned}$$

Let us introduce the notion of primitive elements in this case.

Definition 3.1. Under the above situation, let u be the $(1,1)$-form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4 = \frac{\sqrt{-1}}{2}(dz_1 \wedge dz_1 + dz_2 \wedge dz_2)$. An element ω is said to be primitive if we have $u \wedge *\omega = 0$.

By the above computation we see that the space $\mathcal{A}^{1,1}(U)$ of $(1,1)$-forms on U is the direct sum of $\mathcal{A}^2(U)_-$ and the space generated by the form u and that $\mathcal{A}^2(U)_-$ is contained in the space of primitive elements. Since $u \wedge *u = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ we get the following.

Lemma 3.2. *Under the above situation the space $\mathcal{A}^2(U)_-$ is exactly the space of primitive $(1,1)$-forms.*

Two Riemannian metrics h and h' on X are said to be confomally equivalent if there is a $\mathbf{R}$-valued differentiable function f on X such that the value of f is positive at every point and $h = fh'$. Note that on a 4-dimentional manifold $*$-operator is conformally invariant on $\mathcal{A}^2(U)$, that is, conformally equivalent Riemannian metrics give rise to the same $*$-operator.

4 Penrose transformation

We shall first recall a real structure on $X = \mathbf{P}^3_{\mathbf{C}}$. Let $\mathbf{H}$ be the quaternions $\mathbf{R}+\mathbf{R}i+\mathbf{R}j+\mathbf{R}k$ over $\mathbf{R}$. Identifying $\mathbf{C}$ with the subfield $\mathbf{R}+\mathbf{R}i$ of $\mathbf{H}$, the multiplication by complex numbers to $\mathbf{H}$ from left defines an action of $\mathbf{C}$ on $\mathbf{H}^2$ and then $\mathbf{H}^2$ is isomorphic to $\mathbf{C}^4$ as $\mathbf{C}$-vector spaces. The multiplication by j to $\mathbf{H}$ provides us with an antilinear automorphism of $\mathbf{C}^4$ which induces a continuous map $|\sigma|$ of X to itself. Since $j^2 = -1$, $|\sigma|^2 = \mathrm{id}$. For a section a of $\mathcal{O}_X$ over an open set, we define $\sigma^*(a)$ to be $\overline{a \cdot |\sigma|}$. The couple $\sigma = (|\sigma|, \sigma^*)$ is now a real form of X, in other words, there is an $\mathbf{R}$-scheme $X_{\mathbf{R}}$ such that $X_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ is isomorphic to X over $\mathbf{C}$.

Note that X has no real or σ-fixed points. A complex line in X is said to be a real line if it is stable under the action of σ. It is easy to see that a line is real if and only if it is the line joining a point x and $\sigma(x)$. On the other hand, the above identification of $\mathbf{C}^4$ with $\mathbf{H}^2$ gives rise to a real analytic morphism π of X to $\mathbf{P}^1_{\mathbf{H}}$. Since $\mathbf{P}^1_{\mathbf{H}} = \mathbf{H} \cup \{\infty\}$, it is not hard to see that $\mathbf{P}^1_{\mathbf{H}}$ is real anlytically isomorphic to S^4. Every fiber of ν is a line and moreover a line of X is real if and only if it is a fiber of ν.

Let x_∞ be the north pole $(0,1)$ in $\mathbf{P}^1_{\mathbf{H}} \cong S^4$ and let ℓ_∞ be the real line $\nu^{-1}(x_\infty)$. Hyperplanes of $X = \mathbf{P}^3_{\mathbf{C}}$ which contain the line ℓ_∞ are parametrized by another real line ℓ_0 over the south pole $x_0 = (1,0)$. If a hyperplane P^μ corresponds to a point $(1,\mu)$ of ℓ_0, then $P^\mu \setminus \ell_\infty$ is $\{(1,\mu,u,v) \mid u,v \in \mathbf{C}\}$ as a subset. ν maps a point $(1,\mu,u,v) \in P^\mu \setminus \ell_\infty$ to the point $y = (1+\mu j)^{-1}(u+vj) \in \mathbf{H} = \mathbf{P}^1_{\mathbf{H}} \setminus \{x_\infty\}$. If we write $y = y_1 + y_2 i + y_3 j + y_4 k$, then

$$\begin{aligned} y_1 &= u_1 + \mu_1 v_1 + \mu_2 v_2 \\ y_2 &= u_2 - \mu_1 v_2 + \mu_2 v_1 \\ y_3 &= v_1 + \mu_1 u_1 - \mu_2 u_2 \end{aligned}$$

$$y_2 = v_2 + \mu_1 u_2 - \mu_2 u_1,$$

where $\mu = \mu_1 + \sqrt{-1}\mu_2$, $u = u_1 + \sqrt{-1}u_2$ and $v = v_1 + \sqrt{-1}v_2$. Set

$$\tau = \begin{pmatrix} 1 & 0 & \mu_1 & \mu_2 \\ 0 & 1 & \mu_2 & -\mu_1 \\ \mu_1 & -\mu_2 & 1 & 0 \\ -\mu_2 & \mu_1 & 0 & 1 \end{pmatrix}$$

One finds then

$$\det \tau = (1 + {\mu_1}^2 + {\mu_2}^2) > 0.$$

Setting $(x_1, \ldots, x_4)$ to be the standard basis of $\mathbf{R}^4$, we define $x^\mu = (x_1^\mu, \ldots, x_4^\mu)$ by the equation

$$\begin{pmatrix} x_1^\mu \\ x_2^\mu \\ x_3^\mu \\ x_4^\mu \end{pmatrix} = \tau^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Since $\det\tau > 0$, the orientation of x^μ is independent of μ. If we introduce a complex structure to $\mathbf{H} \cong \mathbf{R}^4$ by $z_1^\mu = x_1^\mu + \sqrt{-1}x_2^\mu$, $z_2^\mu = x_3^\mu + \sqrt{-1}x_4^\mu$, then ν induces a biholomorphic map of $P^\mu \setminus \ell_\infty$ to $\mathbf{R}^4$ with this complex structure. The Riemannian metrics h^μ determined by $\sum_{i=1}^4 dx_i^\mu \otimes dx_i^\mu$ are conformally equivalent with each other because

$$\sum_{i=1}^4 dx_i \otimes dx_i = (1 + {\mu_1}^2 + {\mu_2}^2) \sum_{i=1}^4 dx_i^\mu \otimes dx_i^\mu.$$

The complex structure given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

instead of τ corresponds to the point $(0,1)$ of ℓ_0 and the system of coordinates of this structure is denoted by $x^\infty = (x_1^\infty, \ldots, x_4^\infty)$. As remarked at the end of the preceding section, the above oriented Riemanian structures provide us with the same $*$-operator. If we denote the space of $(1,1)$-formes on $\mathbf{R}^4$ with respect to the (z_1^μ, z_2^μ) by $\mathcal{A}_\mu^{1,1}$ and the space of antiself-daul 2-forms on

$\mathbf{R}^4$ with respect to the above $*$-operator by $\mathcal{A}_-$. By Lemma 3.2 we see that $\mathcal{A}_- \subset \bigcap_{\mu \in \ell_0} \mathcal{A}_\mu^{1,1}$. Now the following is almost obvious.

Proposition 4.1. $\mathcal{A}_- = \bigcap_{\mu \in \ell_0} \mathcal{A}_\mu^{1,1}$

Pick a point p on S^4 and embed S^4 into $\mathbf{R}^5$ by the equation $\sum_{i=1}^4 x_i^2 + (x_5 - 1)^2 = 1$ with $p = (0,0,0,0,2)$. Taking a point q of S^4 other than p, the line in $\mathbf{R}^5$ joining p and q meets at one point t with $\mathbf{R}^4 = \{x_5 = 0\}$. By sending q to t we get a real analytic map ρ_p of $S^4 \setminus \{p\}$ to $\mathbf{R}^4$ which is called the streographic projection from p. We can endow S^4 with an oriented Riemannian structure such that for every point p of S^4, ρ_p gives rise to a conformal equivalence between $S^4 \setminus \{p\}$ and the standard oriented Riemannian structure of $\mathbf{R}^4$.

Definition 4.2. Let E be a $SU(2)$-bundle on S^4 and let ∇ be a $SU(2)$-connection on E. ∇ is called an *inststanton* if the curvature $F(\nabla)$ satisfies

$$*F(\nabla) = -F(\nabla)$$

for the $*$-operator with respect to the above oriented conformal structure.

Let $(\tilde{E}, \tilde{\nabla})$ be the pull-back of an instanton (E, ∇) by the map $\nu : \mathbf{P}^3_{\mathbf{C}} \to S^4$. Assigning a point p of S^4 for the north pole, we can apply the above consideration to the entries of the curvature $F(\nabla)$ and see that for every hyperplane P containing the line $\nu^{-1}(p)$, all the entries of $F(\tilde{\nabla})$ are of type $(1,1)$ on $P \setminus \nu^{-1}(p)$. This implies that the curvature $F(\tilde{\nabla})$ is of type $(1,1)$. Then, by Theorem 2.11 $\tilde{E}$ carries a holomorphic structure with respect to which $\tilde{\nabla}$ is an integrable Hermitian connection. We denote this holomorphic vector bundle on $\mathbf{P}^3_{\mathbf{C}}$ by $E(\nabla)$. Since $F(\tilde{\nabla})$ is trivial on each real line which is simply connected, $E(\nabla)$ is trivial on every real line.

The Hermitian structure of E is equivalent to an antilinear isomorphism of E to the dual bundle $E^\vee$ and hence an isomorphism λ_0 to $\bar{E}^\vee$. If $\{\theta_\alpha\}$ is a system of connection matrices of ∇, then $\{-{}^t\theta_\alpha\}$ defines a connection on $E^\vee$. Since ∇ is Hermitian, $-{}^t\theta_\alpha = \bar{\theta}_\alpha$ and hence $\{\theta_\alpha\}$ is a system of connection matrices on $\bar{E}^\vee$. Since the pull-back of this connection to E by λ_0 is the connection ∇, we see that λ_0 induces an isomorphism $\lambda : E(\nabla) \to \sigma^*(E(\nabla)^\vee)) \cong \sigma^*(E(\nabla))$ of holomorphic vector bundles.

In order to study holomorphic vector bundle in a wider category, let us understand vector bundles in terms of coherent sheaves. A holomorphic vector bundle on a complex manifold Y is given by an open covering $\{U_\alpha\}$

of Y and a collection of holomorphic transition matrices $\{g_{\alpha\beta}\}$. By patching holomorphic free sheaves $\mathcal{O}_{U_\alpha}^{\oplus r}$ by using $\{g_{\alpha\beta}\}$, we get a coherent locally free sheaf. Conversely, a coherent locally free sheaf on Y is also given by an open covering and transition matrices, which provide us with a holomorphic vector bundle. Thus the category of holomorphic vector bundles on a complex manifold is equivalent to the category of a coherent locally free sheaves.

On the other hand, if Y is a projective variety over $\mathbf{C}$, then the category $\mathcal{C}^h$ of anlytic coherent sheaves on Y is equivalent to the catergory $\mathcal{C}^{al}$ of algebraic coherent sheaves on Y (GAGA-prinsiple), more precisely, if we denote the holomorphic (or, algebraic) structure sheaf of Y by $\mathcal{O}_Y^h$ (or, $\mathcal{O}_Y$, resp.), then the functor

$$\mathcal{C}^{al} \ni F \longmapsto F \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^h \in \mathcal{C}^h$$

is an equivalence of categories. Thus, from now on, we undestand a vector bundle on a projective variety to be an algebraic, coherent locally free sheaf. Our $E(\nabla)$ is then a coherent locally free sheaf on $P^3_{\mathbf{C}}$.

Definition 4.3. Let F be a coherent sheaf on $X = \mathbf{P}^3_{\mathbf{C}}$. Since σ is an involution on X, we can naturally identify F with $\sigma^*(\sigma^*(F))$. F is said to be *quaternionic* (or, *real*) if there is an isomorphism ψ of F to $\sigma^*(F)$ such that $\sigma^*(\psi) \cdot \psi = -\mathrm{id}$ (or,$\sigma^*(\psi) \cdot \psi = \mathrm{id}$, resp.)

It is easy to see that the above λ makes $E(\nabla)$ a quaternionic sheaf. For our instanton ∇, we have a non-positive integer k such that

$$8\pi^2 k = -\int_{S^4} \mathrm{Trace}(F(\nabla) \wedge F(\nabla)).$$

This k is called the Pontrjagin number of E. The second Chern class $c_2(E(\nabla))$ is represented by the $(2,2)$-form

$$-\frac{1}{4\pi^2} \det \nu^*(F(\nabla))$$

which is equal to

$$\frac{1}{8\pi^2} \mathrm{Trace}(F(\nabla) \wedge F(\nabla)).$$

Thus we see that $c_2(E(\nabla)) = -k$.

So far we showed that if ∇ is an instanton with Pontrjagin number k, then we can construct a vector bundle $F = E(\nabla)$ of rank 2 on X with the following properties:

(4.4)
(a) F is quaternionic
(b) for every real line ℓ, the restriction $F|_\ell$ is trivial
(c) $c_2(F) = -k$.

Let $G(3,1)$ be the Grassmann variety of lines in X. Then $Z = \{(x,\ell) \in X \times G(3,1) \mid x \in \ell\}$ is a subvariety of $X \times G(3,1)$ and the natural projections $p_1 : Z \to X$ and $p_2 : Z \to G(3,1)$ are flat, proper morphism. In fact, p_1 is a $\mathbf{P}^2$-bundle whose fiber over a point x is $\mathbf{P}^2$ formed by lines passing through x and p_2 is a $\mathbf{P}^1$-budle whose fiber over y is $\mathbf{P}^1$ fromed by points on the line corresponding to y.

Let V be the 4-dimensional $\mathbf{C}$-vector space $\mathbf{H}^2$. A line ℓ in X corresponds to a 2-dimensional qutient space W_ℓ of $V^\vee$. We have a surjection $\bigwedge^2 V^\vee$ to the 1-dimensinal space $\bigwedge^2 W_\ell$ which gives rise to a point y_ℓ of $\mathbf{P}(\bigwedge^2 V^\vee) \cong \mathbf{P}^5_{\mathbf{C}}$. This map $\ell \mapsto y_\ell$ is an immersion of $G(3,1)$ into $\mathbf{P}^5_{\mathbf{C}}$ which is called the Plücker embedding. On the other hand, the involution σ on X introduced in the beginning of this section provides us with a real structure on the vector space $\bigwedge^2 V^\vee$ and hence we have a natural real part $\mathbf{P}^5_{\mathbf{R}}$ in $\mathbf{P}^5_{\mathbf{C}}$. Our S^4 is nothing but the intersection $G(3,1) \cap \mathbf{P}^5_{\mathbf{R}}$ in $\mathbf{P}^5_{\mathbf{C}}$.

Pick an algebraic vector bundle F on X with the property (4.4). By the property (b) there is a Zariski open set U in $G(3,1)$ such that $S^4 \subset U$ and that for every point y of U, the restriction of $\tilde{F} = p_1^*(F)$ to the fiber $p_2^{-1}(y)$ is a trivial bundle. Thus $(p_2)_*\tilde{F}|_U$ can be regarded as a holomorphic vector bundle of rank 2 on U. Ristricting $(p_2)_*\tilde{F}|_U$ to S^4, we obtain a real analytic, a fortiori, differntiable $\mathbf{C}$-vector bundle E. Note that the fiber E_y at each point y of S^4 is the space of holomorphic sections of the trivial bundle $F|_{\nu^{-1}(y)}$. By tracing the way of construction of the quaternionic structure on $E(\nabla)$ in opposite direction, we see that the quaternionic structure of F gives us a Hermitian form on E. The quaternionicity implies that this Hermitian form is positive definite. Pulling back this Hermitian structure to F, we obtain the unique integrable Hermitian connection $\tilde{\nabla}$ on F.

Let J be the defining ideal of a real line ℓ in X. We see that the conormal bundle J/J^2 of ℓ in X is isomorphic to $\mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(-1)$. Then $F \otimes J/J^2$ is isomorphic to $\mathcal{O}_\ell(-1)^{\oplus 4}$ because F is trivial on the line ℓ. Consider the exact sequence

$$0 \longrightarrow F \otimes J/J^2 \longrightarrow F \otimes \mathcal{O}_X/J^2 \longrightarrow F \otimes \mathcal{O}_\ell \longrightarrow 0.$$

The above result shows that $H^1(X, F \otimes J/J^2) = 0$ and hence the holomorphic sections of $F \otimes \mathcal{O}_\ell$ which provide a trivialization lift to holomorphic sections of $F \otimes \mathcal{O}_X/J^2$. This means that F is trivial along the first infinitesimal neighbouhood of ℓ and hence the curvature of $\tilde{\nabla}$ contains no 2-forms involving any direction along fibers of ν. We see therefore that $\tilde{\nabla}$ descends to a connection ∇ on E. Since $F(\tilde{\nabla})$ is of type $(1,1)$, ∇ must be an instanton by virtue of Proposition 4.1.

Summarizing the above results, we have the following theorem.

Theorem 4.5. *The set of isomorphism (i. e. guage equivalence) classes of instantons with Pontrjagin number k is in bijective correspondence with the set of isomorphism classes of algebraic vector bundles of rank 2 on X with the property* (4.4).

5 Moduli spaces of algebraic vector bundles

Let $(Y, \mathcal{O}_Y(1))$ be a couple of a non-singular projective variety Y and an ample line bundle $\mathcal{O}_Y(1)$ on Y, that is, there is a positive integer a and an embedding j of Y to a projective space $\mathbf{P}^N$ such that for the line bundle $\mathcal{O}_{\mathbf{P}^N}(1)$ corresponding to hyperplanes, we have an isomorphism of $\mathcal{O}_Y(a) = \mathcal{O}_Y(1)^{\otimes a}$ to $j^*(\mathcal{O}_{\mathbf{P}^N}(1))$. For a coherent sheaf E on Y, there is a non-empty open set U of Y such that $E|_U$ is isomorphic to a free sheaf $\mathcal{O}_U^{\oplus r}$. This r is called the rank of E and denoted by $r(E)$. Since Y is non-singular and projective, we can define the Chern classes of E by using resolusion by coherent, locally free resolusion of E. We denote the degree of the first Chern class $c_1(E)$ of E with respect to $\mathcal{O}_Y(1)$ by $d(E, \mathcal{O}_Y(1))$ or simply $d(E)$. The alternating sum $\sum(-1)^i \dim H^i(Y, E \otimes \mathcal{O}_Y(m))$ is a polynomial in m. We denote the polynomial by $\chi(E(m))$ and call it the Hilbert polynomial of E. If $r(E) \neq 0$, then we set

$$\mu(E) = d(E, \mathcal{O}_Y(1))/r(E)$$
$$P_E(m) = \chi(E(m))/r(E)$$

A coherent locally free sheaf on Y is called a vector bundle. A coherent sheaf on Y is said to be torsion free if it is a subsheaf of a vector bundle. A coherent, torsion free sheaf is of rank 0 if and only if it is 0 as a sheaf.

Definition 5.1. (1) A coherent sheaf E on Y is said to be μ-stable (or μ-semi-stable) if (a) E is torsion free and (b) for every coherent subsheaf F of E with $0 < r(F) < r(E)$, we have $\mu(F) < \mu(E)$ (or, $\mu(F) \leq \mu(E)$, resp.).

(2) A coherent sheaf E on Y is said to be stable (or semi-stable) if (a) E is torsion free and (b) for every coherent subsheaf F of E with $0 < r(F) < r(E)$, we have $P_F(m) < P_E(m)$ (or, $P_F(m) \leq P_E(m)$, resp.) for all sufficiently large m.

By Riemann-Roch Theorem we have the following implications:

μ-stable $\Longrightarrow$ stable $\Longrightarrow$ semi-stable $\Longrightarrow$ μ-semi-stable.

The following is not hard to prove.

Proposition-Definition 5.2. *If E is a semi-stable sheaf on Y, then there is a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{\alpha-1} \subset E_\alpha = E$ by coherent subsheaves of E with the following properties:*
(a) for $1 \leq i \leq \alpha$, we have $P_{E_i}(m) = P_E(m)$
(b) for $1 \leq i \leq \alpha$, E_i/E_{i-1} is stable.
Moreover, if we have another filtration $0 = E'_0 \subset E'_1 \subset \cdots \subset E'_{\beta-1} \subset E'_\beta = E$ with the properties (a) and (b), then we have $\alpha = \beta$ and there is a permutation γ of the set $\{1, \ldots, \alpha$ such that E'_i/E'_{i-1} is isomorphic to $E_{\gamma(i)}/E_{\gamma(i)-1}$.
$\mathrm{gr}(E) = \bigotimes_{i=1}^{\alpha} E_i/E_{i-1}$ is independent of the choice of the filtration. Two semi-stable sheaves E and F are saide to be S-equivalent if $\mathrm{gr}(E) \cong \mathrm{gr}(F)$. The S-equivalence is an equivalence relation ande denoted by $E \underset{S}{\sim} F$. Note that if one of E and F are S-equivalent and one of them is stable, then they are isomorphic.

Let $H(m)$ be a polynomial in m. We set

$$\bar{\Sigma}_Y(H) = \{E \mid E \text{ is semi-stable and } \chi(E(m)) = H(m)\}/\underset{S}{\sim}$$

and

$$\Sigma_Y(H) = \{E \mid E \text{ is stable and } \chi(E(m)) = H(m)\}/\cong.$$

Obviously $\Sigma_Y(H)$ is a subset of $\bar{\Sigma}_Y(H)$.

Theorem 5.3. *$\bar{\Sigma}_Y(H)$ carries a natural structure of a scheme $\overline{M}_Y(H)$ and $\Sigma_Y(H)$ forms an open subscheme $M_Y(H)$ in $\overline{M}_Y(H)$. $\overline{M}_Y(H)$ is locally of finite type and moreover if the characteristic of the ground field is 0, then it is projective.*

Though we shall here avoid explaining the real meaning of the *natural* structure, let us note that it implies the following universal property:

> Let T be an algebraic scheme and let $\tilde{E}$ be a T-flat coherent sheaf on $Y \times T$ such that for every geometric point t of T, $\tilde{E}_t$ is semi-stable and its S-equivalence class $[\tilde{E}_t]$ is contained in $\overline{\Sigma}_Y(H)$. Then the map $t \mapsto [\tilde{E}_t]$ gives rise to a morphism of T to $\overline{\Sigma}_Y(H)$.

6 The universal extension on $\mathbf{P}^3$

First of all we shall recall the Riemann-Roch formula on $\mathbf{P}^3$. For a coherent sheaf F on $\mathbf{P}^3$, we have the following equality:

$$\text{(R-R)} \qquad \begin{aligned} \chi(F(m)) = {} & \frac{rm^3}{6} + \left(\frac{c_1}{2} + r\right)m^2 + \\ & \left(\frac{c_1^2}{2} + 2c_1 - c_2 + \frac{11}{6}r\right)m + \\ & \frac{c_1^3}{6} - \frac{c_1c_2}{2} + \frac{c_3}{2} + c_1^2 - 2c_2 + \frac{11}{6}c_1 + r, \end{aligned}$$

where c_i is the i-th Chern class of F and r is the rank of F.

Let E be a coherent sheaf on $\mathbf{P}^3_k$ with the following properties

$$\text{(UV)} \begin{cases} (1) & E \text{ is } \mu\text{-semi-stable}, \\ (2) & c_1(E) = c_3(E) = 0, \\ (3) & H^0(\mathbf{P}^3, E) = H^2(\mathbf{P}^3, E) = 0. \end{cases}$$

The properties (1) and (2) imply that $H^3(\mathbf{P}^3, E) = \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^3}}(E, \mathcal{O}_{\mathbf{P}^3}(-4))^\vee = 0$. Then (3) and (R-R) show that $\dim H^1(\mathbf{P}^3, E) = 2c_2(E) - r(E)$. On the other hand, we have a canonical isomorphisms

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}^3}}(H^1(E) \otimes \mathcal{O}_{\mathbf{P}^3}, E) & \xleftarrow{\sim} H^1(\mathbf{P}^3, \mathcal{H}om_{\mathcal{O}_{\mathbf{P}^3}}(H^1(E) \otimes \mathcal{O}_{\mathbf{P}^3}, E)) \\ & \cong \mathrm{Hom}_k(H^1(E), H^1(E)). \end{aligned}$$

Let ξ be the element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}^3}}(H^1(E) \otimes \mathcal{O}_{\mathbf{P}^3}, E)$ which corresponds to the identity map of $H^1(E)$ by the above isomorphisms. The ξ defines an extension

$$0 \longrightarrow E \longrightarrow U(E) \longrightarrow H^1(E) \otimes \mathcal{O}_{\mathbf{P}^3} \longrightarrow 0.$$

Definition 6.1. $U(E)$ is called the universal extension of E.

The following are very basic properties of the universal extensions.

Lemma 6.2. *Let E be a coherent sheaf on $\mathbf{P}^3$ with the property* (UV).

(1) $c_1(U(E)) = 0$, $c_2(U(E)) = c_2(E)$, $c_3(U(E)) = c_3(E)$ *and* $r(U(E)) = 2c_2(E)$ *and hence we have*

$$P_{U(E)}(m) = \frac{m^3}{6} + m^2 + \frac{4}{3}m.$$

(2) $H^i(\mathbf{P}^3, U(E)) = 0$ *for all i.*

(3) *If a coherent sheaf F fits in an exact sequence*

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_{\mathbf{P}^3}^{\oplus s} \longrightarrow 0$$

and if $H^i(\mathbf{P}^3, F) = 0$ for $i = 0$, 1, then $s = b(E)$ and F is isomorphic to $U(E)$.

The universal extension $U(E)$ of E is not necesarily semi-stable even if E is semi-stable. To state an affirmative result for the semi-stability of the universal extensions, let us introduce the following notion.

Definition 6.3. For a coherent sheaf F on a variety Y, we denote the dual $\mathcal{H}om_{\mathcal{O}_Y}(E, \mathcal{O}_Y)$ by $E^\vee$. There is a natural homomorphism γ of E to its double dual $(E^\vee)^\vee$. E is said to be reflexive if γ is an isomorphism.

Note that a vector bundle on a variety is reflexive.

Proposition 6.4. *Let E be a reflexive, μ-stable sheaf with the property* (UV) *and S an extension of $\mathcal{O}_{\mathbf{P}^3}^{\oplus u}$ by E:*

$$0 \longrightarrow E \longrightarrow S \longrightarrow \mathcal{O}_{\mathbf{P}^3}^{\oplus u} \longrightarrow 0.$$

If $H^0(\mathcal{O}_{\mathbf{P}^3}, S) = 0$, then S is stable. In particular, the universal extension $U(E)$ is stable if $c_2(E) \geq 0$.

Proof. Since $c_2(E) \geq 0$, the coefficient of m in $P_S(m)$ is less than $\frac{11}{6}$. Let F be a coherent subsheaf of S. We have to prove that $P_F(m) < P_S(m)$ and hence we may assume that S/F is torsion free and $c_1(F) = 0$. Since S/F is μ-semi-stable, we see that $G = H^0(\mathbf{P}^3, S/F) \otimes \mathcal{O}_{\mathbf{P}^3}$ is naturally a subsheaf

of S/F. Let F' be the inverse image of G by the surjection $S \to S/F$. If $s = r(F)$ and $t = \dim H^0(\mathbf{P}^3, S/F)$, then for large m,

$$\begin{aligned} P_{F'}(m) &= \frac{1}{s+t}\{\chi(F(m)) + \chi(\mathcal{O}_{\mathbf{P}^3}(m)^{\oplus t})\} \\ &= \frac{1}{s+t}\left\{sP_F(m) + \left(\frac{m^3}{6} + m^2 + \frac{11}{6}m + 1\right)t\right\} \\ &> \frac{1}{s+t}\{sP_F(m) + tP_S(m)\} \end{aligned}$$

If $P_{F'}(m) < P_S(m)$, then above inequality means that $P_F(m) < P_S(m)$. Thus we may assume that $H^0(\mathbf{P}^3, S/F) = 0$. Setting $E' = E \cap F$, we have three cases.

Case I. Assume that $E' \neq 0$, E. Since E/E' is a subsheaf of the torsion free S/F and $E/E' \neq 0$, we see that $0 < r(E') < r(E)$. Then the μ-stability of E implies that $d(E') < 0$ and hence $d(F/E') > 0$. On the other hand, F/E' can be regarded as a subsheaf of S/E which is a trivial vector bundle. This means that $d(F/E') \leq 0$. Thus we come to a contradiction.

Case II. Let us next treat the case where $E' = 0$. As in the case I, $F = F/E'$ is a subsheaf of a trivial vector bundle. Since E is reflexive, so is S. Then F is reflexive because S/F is torsion free. These and the assumption that $c_1(F) = 0$ imply that F is a trivial vector bundle. We assumed that $H^0(\mathbf{P}^3, S) = 0$ and hence $H^0(\mathbf{P}^3, F) = 0$. Therefore, $F = 0$ in this case.

Case III. Finally assume that $E' = E$. We have the following exact commutative diagram.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & S/F & = & S/F & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E & \longrightarrow & S & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}^{\oplus u} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & F/E & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since S/F is torsion free, the rightmost column of the above diagram shows that F/E is reflexive. This and the fact that $c_1(F/E) = 0$ show that

$F/E \cong \mathcal{O}_{\mathbf{P}^3}^{\oplus\alpha}$ and hence $S/F \cong \mathcal{O}_{\mathbf{P}^3}^{\oplus(u-\alpha)}$. This means that $S/F = 0$ because $H^0(\mathbf{P}^3, S/F) = 0$. Thus we see that $F = S$. Q. E. D.

Let $\overline{M}(n,r)$ be the moduli space of semi-stable sheaves E of rank r on $\mathbf{P}^3$ with $c_1(E) = 0$, $c_2(E) = n$ and $c_3(E) = 0$. There is a Zariski open set $V(n,r)$ of $\overline{M}(n,r)$ formed by the S-equivalence classes $[E]$ of semi-stable E whose $\mathrm{gr}(E)$ have the property (UV) and whose universal extensions $U(E)$ are semi-stable. Note that by Proposition 6.4 $V(n,r)$ contains all μ-stable vector bundles with the property (UV) in $\overline{M}(n,r)$. We get a map $\varphi(n,r)$ of $V(n,r)$ to $\overline{M}(n,2n)$ by sending $E \in V(n,r)$ to $[U(E)] \in \overline{M}(n,2n)$.

Theorem 6.5. (1) *φ is a morphism.*

(2) *Let $V(n,r)^\mu$ be the open set of $\overline{M}(n,r)$ formed by the μ-stable, reflexive sheaves E such that $H^p(\mathbf{P}^3, E(q)) = 0$ if $(p,q) = (1,-2)$, $(2,0)$ or $(2,-2)$. Then $V(n,r)^\mu$ is an open subschem of $V(n,r)$ and the morphism $\varphi(n,r)$ induces an immersion of $V(n,r)^\mu$ to $\overline{M}(n,2n)$.*

To show a nature of $\varphi(n,r)$ we shall give an interesting example.

Example 6.6. Let ℓ be a line in $\mathbf{P}^3$ and let L be the line bundle $\mathcal{O}_\ell(1)$ on ℓ. Pick a μ-stable vector bundle E of rank 2 on $\mathbf{P}^3$ with the property (UV) and $c_2(E) = n-1$. Assume that E is trivial on the line ℓ (if the characteristic of the ground field is 0, then E is trivial on almost all lines). Since L is generated by two global sectiions, we have a surjection

$$u : E \longrightarrow E|_\ell \longrightarrow L.$$

Then $E' = \ker(u)$ has the property (UV) and is μ-stable, too. By an easy computation we see that $c_2(E') = n$. Look at the exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^3, L) \longrightarrow H^1(\mathbf{P}^3, E') \longrightarrow H^1(\mathbf{P}^3, E) \longrightarrow 0.$$

Take a basis $\{\xi_1, \xi_2\}$ of $H^0(\mathbf{P}^3, L)$ and extend it to a basis $\{\xi_1, \dots, \xi_{2n-2}\}$ of $H^1(\mathbf{P}^3, E')$. $\xi = \{\xi_1, \dots, \xi_{2n-2}\}$ can be regarded as an element of

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}^3}}(H^1(E') \otimes \mathcal{O}_{\mathbf{P}^3}, E') \cong H^1(\mathbf{P}^3, E')^{\oplus(2n-2)}$$

and it defines an extension of $H^1(\mathbf{P}^3, E') \otimes \mathcal{O}_{\mathbf{P}^3}$ by E' which is isomorphic to the universal extension $U(E')$. The natural map

$$\delta : \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}^3}}(H^1(E') \otimes \mathcal{O}_{\mathbf{P}^3}, E') \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}^3}}(H^1(E') \otimes \mathcal{O}_{\mathbf{P}^3}, E)$$

gives rise to a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & L & = & L & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & E & \longrightarrow & F & \xrightarrow{\gamma} & H^1(E') \otimes \mathcal{O}_{\mathbf{P}^3} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & E' & \longrightarrow & U(E') & \longrightarrow & H^1(E') \otimes \mathcal{O}_{\mathbf{P}^3} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where F is the extension defined by $\delta(\xi)$. Since ξ_1 and ξ_2 are sent to zero in $H^1(\mathbf{P}^3, E)$, F contains $G = \mathcal{O}_{\mathbf{P}^3}^{\oplus 2}$ which is mapped to a direct factor of $H^1(\mathbf{P}^3, E') \otimes \mathcal{O}_{\mathbf{P}^3}$ by γ. The fact that $H^0(\mathbf{P}^3, U(E')) = 0$ and $\dim H^0(\mathbf{P}^3, L) = 2$ implies that the map of G to L is nothing but the natural surjective map $\zeta : H^0(\mathbf{P}^3, L) \otimes \mathcal{O}_{\mathbf{P}^3} \to L$. Thus $U(E')$ contains $G' = \ker(\zeta)$.

By the construction of G, the natural homomorphism of $U(E')/G'$ to F/G is an isomorphism and we have an exact sequence

$$0 \longrightarrow E \longrightarrow F/G \longrightarrow \mathcal{O}_{\mathbf{P}^3}^{\oplus(\beta-\alpha)} \longrightarrow 0.$$

Since $H^0(\mathbf{P}^3, G) \cong H^0(\mathbf{P}^3, F)$ and since $H^1(\mathbf{P}^3, G) = 0$, $H^1(\mathbf{P}^3, F/G)$ vanishes. By the exactness of the middle column of the above diagram and the fact that $H^1(\mathbf{P}^3, U(E')) = 0$ we have that $H^1(\mathbf{P}^3, F) = 0$ and hence $H^1(\mathbf{P}^3, F/G) = 0$. Thus, by Lemma 6.2, F/G is isomorphic to $U(E)$. It is not hard to see that $Q_\ell = G'$ is stable, independent of the choice of ζ and $P_{Q_\ell}(m) = P_{U(E)}(m)$. Therefore, $\mathrm{gr}(U(E'))$ depends only on the location of the line ℓ and is independent of the choice of the surjection u. The surjections u's are parametrized by $\mathbf{P}^3$ and different points in $\mathbf{P}^3$ give us different E'. Thus $\varphi(n,2)$ contracts the $\mathbf{P}^3$.

7 Verdier's description of instantons

Now we shall go back to the situation of §4, for example, X denotes $\mathbf{P}^3_{\mathbf{C}}$ with a strange involution σ and $X_{\mathbf{R}}$ is the real form of X defined by the σ.

Definition 7.1. A vector bundle E of rank 2 on $X = \mathbf{P}^3_{\mathbf{C}}$ is called an instanton bundle if it is isomorphic to the $E(\nabla)$ obtained from an instanton ∇.

If the second Chern class is not zero, then an instanton bundle is μ-stable. In fact, if $E = E(\nabla)$ is not μ-stable, then it contains $\mathcal{O}_X$. Since $c_2(E) > 0$, $J = E/\mathcal{O}_X$ is torsion free but not locally free. Pick a point x of X where J is not locally free. Then, on the real line joining x and $\sigma(x)$, E is not trivial, which violate (b) of (4.4).

It is known that for an instanton bundle E, $H^1(X, E(-2))$ vanishes. Forgetting the real structure of instanton bundles and taking this vanishing of the cohomology we come to the notion of mathematical instantons.

Definition 7.2. A vector bundle E of rank 2 on $X = \mathbf{P}^3_{\mathbf{C}}$ is called a mathematical instanton bundle if it is stable, $c_1(E) = 0$ and $H^1(X, E(-2)) = 0$.

The set of mathemetical instanton bundles E with $c_2(E) = n$ forms an open subscheme of the moduli space $\overline{M}(n, 2)$ which we denote $MI(n)$. As is well-known, if E is a mathematical instanton bundle, then $H^1(X, E(-a)) = 0$ for every $a \geq 2$ and hence, by Serre duality, $H^2(X, E(b)) = 0$ for every $b \geq -2$. Therefore, a mathematical instanton bundle has the property (UV). Since mathematical instanton bundles are μ-stable, the vanishing of the cohomologies show that $\varphi(n, 2)$ induces an immersion $\varphi(n)$ of $MI(n)$ to $\overline{M}(n, 2n)$.

The map sending a mathematical instanton bundle E to $\sigma^*(E)$ induces an automorphism of $MI(n)$ which is also denoted by σ. A *real* point, that is, a fixed point by σ corresponds to a real or quaternionic bundle. If a points of a connected component of the *real* part $MI(n)(\mathbf{R}) = MI(n)^\sigma$ corresponds to a quaternionic bundle, then so do all the points of the component. Since the set of bundles with the property (b) of (4.4) forms an open set of $MI(n)(\mathbf{R})$, the set $I(n)$ of instanton bundles is open in $MI(n)(\mathbf{R})$ in the classical topology (that is, the topology by the absolute value).

Let us examine the above viewpoint more precisely. Since X has a real form $X_{\mathbf{R}}$ and $\mathcal{O}_X(2)$ descends to a line bundle on $X_{\mathbf{R}}$, every moduli space $\overline{M}$ of semi-stable sheaves has a real form, that is, there is an $\mathbf{R}$-scheme $\overline{M}_{\mathbf{R}}$ such that $\overline{M} \cong \overline{M}_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. If E is an instanton bundle, then the quaternionic structure of E gives rise to an isomorphism $\zeta : H^1(X, E) \to H^1(X, \sigma^*(E))$ such that $\sigma^*(\zeta) \cdot \zeta = -\mathrm{id}$ and hence it induces a quaterninic structure on $H = H^1(X, E) \otimes_{\mathbf{C}} \mathcal{O}_X$. Thus $E \otimes_{\mathcal{O}_X} H^\vee$ is a real sheaf. Then $\mathrm{Ext}^1_{\mathcal{O}_X}(H, E) \cong$

$H^1(X, E \otimes_{\mathcal{O}_X} H^\vee) \cong \mathrm{Hom}_{\mathcal{O}_X}(H, H)$ carries a real structure and moreover id in the last space is a real vector. This means that the universal extension $U(E)$ is quaternionic.

$\varphi(n, 2)$ is defined over $\mathbf{R}$ and hence there is an $\mathbf{R}$-morphism $\varphi(n)_{\mathbf{R}}$ of an open subscheme of $\overline{M}(n, 2)_{\mathbf{R}}$ containing $I(n)$ to $\overline{M}(n, 2n)_{\mathbf{R}}$. $\varphi(n)_{\mathbf{R}}$ is an immersion over $I(n)$. On the other hand, it is known that for an instanton bundle E, $\mathrm{Ext}^2_{\mathcal{O}_X}(E, E) = 0$ and then it is easy to see that $\mathrm{Ext}^2_{\mathcal{O}_X}(U(E), U(E)) = 0$. This means that $\overline{M}(n, 2)_{\mathbf{R}}$ and $\overline{M}(n, 2n)_{\mathbf{R}}$ are smooth at E and $U(E)$, respectively. Thus we have the following.

Proposition 7.3. *The moduli space $I(n)$ of instanton bundles with $c_2 = n$ is an open submanifold (in the classical topology) of the set of real points of the projective algebraic scheme $\overline{M}(n, 2)_{\mathbf{R}}$ and it can be regarded as a submanifold of the set of real points of another projective algebraic scheme $\overline{M}(n, 2n)_{\mathbf{R}}$.*

Our main interest in the sequel is compactification of $I(n)$.

Definition 7.4. The closure (in classical topology) of $I(n)$ in $M(n, 2)_{\mathbf{R}}(\mathbf{R})$ is denoted by $\overline{I}(n)$. We denote the closure (in the classical topology) of $I(n)$ in $M(n, 2n)_{\mathbf{R}}(\mathbf{R})$ by $\overline{\overline{I}}(n)$.

Pick an instanton bundle E with $c_2(E) = n - 1$ and a real line ℓ. By the definition of instanton bundle, $E|_\ell$ is quaternionic and isomorphic to $\mathcal{O}_\ell^{\oplus 2}$ whose quaternionic structure is given by a symplectic basis $\{e_1, e_2\}$. Since $\mathcal{O}_\ell(1)$ is quaternionic, $\Xi = \mathrm{Hom}_{\mathcal{O}_\ell}(E|_\ell, \mathcal{O}_\ell(1))$ is a four dimensional $\mathbf{C}$-vector space with natural $\mathbf{R}$-structure. An element θ in Ξ is real if and only if $\theta(e_1) = s$ and $\theta(e_2) = -\sigma^*(s)$. This implies that a real element θ is always surjective because the common zero of s and $-\sigma^*(s)$ should be a real point of ℓ and ℓ has no real points. On the other hand, for a real θ, $E(\theta) = \ker(\theta)$ inherits a quaternionic structure from that of E. Fixing E and ℓ, the set of $E(\theta)$'s is parametrized by $\mathbf{P}^3_{\mathbf{R}}(\mathbf{R})$ in $\overline{M}(n, 2)_{\mathbf{R}}(\mathbf{R})$.

As we have seen in the previous section, all the points of $\mathbf{P}^3_{\mathbf{R}}(\mathbf{R})$ in the above are sent to one point of $\overline{M}(n, 2n)_{\mathbf{R}}(\mathbf{R})$ by $\varphi(n)_{\mathbf{R}}$. We shall see later that the image point is in $\overline{\overline{I}}(n)$.

Now let us recall Verdier's description of instanton bundles and then write down the conditions explicitely by using systems of coordinates. In the first place we shall fix notation. Let n be a positive integer, $\mathcal{W}$ an n-dimensional $\mathbf{C}$-vector space with real structure. Fixing a real basis $\{e_1, \ldots, e_n\}$ of $\mathcal{W}$, we can write

$$\mathcal{W} = (e_1\mathbf{R} + \cdots + e_n\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}.$$

ι denotes the conjugation on $\mathcal{W}$ which defines the real structure. Let $\mathcal{U}$ be a $(2n+2)$-dimensional $\mathbf{C}$-vector space and fix a basis $\{u_1, \ldots, u_{2n+2}\}$ of $\mathcal{U}$. For a four dimensional $\mathbf{C}$-vector space $\mathcal{V} = \mathbf{C}Z_0 + \mathbf{C}Z_1 + \mathbf{C}Z_2 + \mathbf{C}Z_3$, σ is the antilinear automorphism of $\mathcal{V}$ such that

$$\sigma(Z_0) = Z_1, \sigma(Z_1) = -Z_0, \sigma(Z_2) = Z_3, \sigma(Z_3) = -Z_2.$$

If we let the row vector (a_0, a_1, a_2, a_3) represent an element $f = \sum_{i=0}^{3} a_i Z_i$ in $\mathcal{V}$, then $\sigma(f)$ is represented by

$$(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3)\Sigma,$$

where

$$\Sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Denoting the identity matrix of degree $n+1$ by I_{n+1}, set

$$J_1 = \begin{pmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{pmatrix}$$

and $J_2 = -J_1$. By using J_1, we define an altenating form $j_1 : \mathcal{U} \otimes_{\mathbf{C}} \mathcal{U} \to \mathbf{C}$ as follows:

$$j_1(\mathbf{a} \otimes \mathbf{b}) = (a_1, \ldots, a_{2n+2})J_1 \begin{pmatrix} b_1 \\ \vdots \\ b_{2n+2} \end{pmatrix},$$

where $\mathbf{a} = \sum a_i u_i$ and $\mathbf{b} = \sum b_i u_i$. We use the notation of inner product $(\mathbf{a}, \mathbf{b})$ instead of $j_1(\mathbf{a} \otimes \mathbf{b})$. Let $j_2 : \mathcal{U} \to \mathcal{U}$ be the antilinear map defined by

$$j_2(\mathbf{a}) = (u_1, \ldots, u_{2n+2})J_2 \begin{pmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_{2n+2} \end{pmatrix},$$

where $\mathbf{a} = \sum a_i u_i$.

For elements $\mathbf{a} = \sum a_i u_i$ and $\mathbf{b} = \sum b_i u_i$ in $\mathcal{U}$, we have the following equalities:

$$\begin{aligned} (j_2(\mathbf{a}), j_2(\mathbf{b})) &= (\bar{a}_1, \ldots, \bar{a}_{2n+2})^t J_2 J_1 J_2 \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_{2n+2} \end{pmatrix} \\ &= (\bar{a}_1, \ldots, \bar{a}_{2n+2}) J_1 \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_{2n+2} \end{pmatrix} = \overline{(\mathbf{a}, \mathbf{b})} \\ (\mathbf{a}, j_2(\mathbf{b})) &= (a_1, \ldots, a_{2n+2}) J_1 J_2 \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_{2n+2} \end{pmatrix} = \sum_{i=1}^{2n+2} a_i \bar{b}_i \end{aligned}$$

Thus $\langle \mathbf{a}, \mathbf{b} \rangle = (\mathbf{a}, j_2(\mathbf{b}))$ is a positive definite Hermitian form.

Let ψ be a $\mathbf{C}$-linear map of $\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}$ to $\mathcal{U}$. Define $\rho(\psi)$ to be

$$\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V} \xrightarrow{\iota \otimes \sigma} \mathcal{W} \otimes_{\mathbf{C}} \mathcal{V} \xrightarrow{\psi} \mathcal{U} \xrightarrow{j_2^{-1}} \mathcal{U}.$$

Then $\rho(\psi)$ is a $\mathbf{C}$-linear map. Let us consider the following two conditions on ψ.

(INT 1) For every $v \in \mathcal{V}$, $\psi(\mathcal{W} \otimes v)$ is contained in an isotropic subspace of j_1.

(INT 2) $\psi = \rho(\psi)$.

For a given $\psi \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}, \mathcal{U})$ satisfying the above conditions (INT 1) and (INT 2), let $\tilde{\psi}$ be the homomorphism

$$\mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_X(-1) \xrightarrow{1 \otimes \alpha} \mathcal{W} \otimes_{\mathbf{C}} \mathcal{V} \otimes_{\mathbf{C}} \mathcal{O}_X \xrightarrow{\psi \otimes 1} \mathcal{U} \otimes_{\mathbf{C}} \mathcal{O}_X,$$

where we identify X with $\mathbf{P}(\mathcal{V}^\vee)$ and $\alpha : \mathcal{O}_X(-1) \to \mathcal{V} \otimes_{\mathbf{C}} \mathcal{O}_X$ is a natural map. If we assume that $\tilde{\psi}$ is injective and $\mathrm{im}(\tilde{\psi})$ is a subbundle of $\mathcal{U} \otimes_{\mathbf{C}} \mathcal{O}_X$, then the condition (INT1) implies that

$$\mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_X(-1) \xrightarrow{\tilde{\psi}} \mathcal{U} \otimes_{\mathbf{C}} \mathcal{O}_X \xrightarrow{{}^t\tilde{\psi}} \mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{O}_X(1)$$

is a monad whose cohomology sheaf $E(\psi)$ is a vector bundle of rank 2 on X with $c_1(E(\psi)) = 0$ and $c_2(E(\psi)) = n$. The condition (INT 2) means that $E(\psi)$ is quaternionic.

Theorem 7.5([10, Theorem 6.6]). *$E(\psi)$ is an instanton bundle. Conversely, if E is an instanton bundle, then there is a $\psi \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}, \mathcal{U})$ satisfying the conditions* (INT 1) *and* (INT 2) *such that E is isomorphic to $E(\psi)$.*

The latter part of the theorem is rather complicated. The main point of the first part of the theorem is that the restriction of $E(\psi)$ to a real line is always trivial. The triviality comes from the fact that $\langle \mathbf{a}, \mathbf{b} \rangle$ is positive definite. We shall give another proof of the first part in §6 (see Remark 9.9).

If $\{Z_0^\vee, Z_1^\vee, Z_2^\vee, Z_3^\vee\}$ is the dual basis of $\{Z_0, Z_1, Z_2, Z_3\}$, then the above ψ can be written down in the following way:

$$\psi(e_i) = \sum_{\substack{1 \le k \le 2n+2 \\ 0 \le \ell \le 3}} a_{k\ell}^{(i)} u_k \otimes Z_\ell^\vee .$$

Thus ψ is represented by an n-ple $(A_1, \ldots, A_n)$ of $(2n+2) \times 4$-matrices, where $A_i = (a_{k\ell}^{(i)})$. The condition (INT 1) means that $(\psi(e_i), \psi(e_j)) = 0$ as polynomials in $Z_0^\vee, \ldots, Z_3^\vee$ for all i, j. Since we have

$$(\psi(e_i), \psi(e_j)) = \sum_{0 \le k, \ell \le 3} (a_{0k}^{(i)}, \ldots, a_{2n+2\,k}^{(i)}) J_1 \begin{pmatrix} a_{0\ell}^{(j)} \\ \vdots \\ a_{2n+2\,\ell}^{(j)} \end{pmatrix} Z_k^\vee Z_\ell^\vee ,$$

The condition is equivalent to

(INT 1′) The 4×4-matrix ${}^tA_i J_1 A_j$ is alternating for all i, j.

Using the n-ple $(A_1, \ldots, A_n)$, $\rho(\psi)$ is represented by

$$(J_2^{-1} A_1 {}^t\Sigma, \ldots, J_2^{-1} A_n {}^t\Sigma) = (J_1 \overline{A}_1 {}^t\Sigma, \ldots, J_1 \overline{A}_n {}^t\Sigma),$$

Thus the condition (INT 2) means that

(INT 2′) $A_i = J_1 \overline{A}_i {}^t\Sigma$ for all i.

8 A compact semi-algebraic set

In the preceding section we exploited a monad to get an instanton bundle. We have another type of monad to represent an instanton bundle. In fact, an instanton bundle E with $c_2(E) = n$ is the cohomology sheaf of a monad

$$\mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow \mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X^1(1) \longrightarrow \mathcal{W}' \otimes_{\mathbf{C}} \mathcal{O}_X$$

If E is represented by $\psi \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}, \mathcal{U})$, then we can construct a commutative diagram by using two monads.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_X(-1) & \xrightarrow{\omega(\tilde{\psi})} & \mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X^1(1) & \to & \mathcal{W}' \otimes_{\mathbf{C}} \mathcal{O}_X & \to & 0 \\
 & & \tilde{\psi}\downarrow & & \alpha\downarrow & & \| & & \\
0 & \to & \mathcal{U} \otimes_{\mathbf{C}} \mathcal{O}_X & \xrightarrow{\beta} & \mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{V}^\vee \otimes_{\mathbf{C}} \mathcal{O}_X & \to & \mathcal{W}' \otimes_{\mathbf{C}} \mathcal{O}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{O}_X(1) & = & \mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{O}_X(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the middle column and middle row are exact. If one looks into Beilinson's spectral sequence, then one finds that β in the above diagram is ${}^t\psi \cdot {}^tj_1$. Thus we have

$$\alpha \cdot \omega(\tilde{\psi}) = {}^t\psi \cdot {}^tj_1 \cdot \tilde{\psi}.$$

The isomorphism classes of monads of two types are in bijective correspondence. It is easy to see that $\mathrm{coker}(\omega(\tilde{\psi})) = U(E)$ and the map $\varphi(n, 2)$ is nothing but forgetting the third term of the monad in the first row of the above diagram. Thus, to study $\overline{\overline{I}}(n)$ we have to ignore the condition that $\mathrm{im}(\tilde{\psi})$ is a subbundle and have to use the first row instead of the leftmost column. For simplicity sake we use $-\omega(\tilde{\psi})$ rather than $\omega(\tilde{\psi})$. Let $\omega(\psi)$ be the $\mathbf{C}$-linear map ${}^t\psi \cdot j_1 \cdot \psi$ of $\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}$ to $\mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{V}^\vee$. If we regard $\omega(\psi)$ as a linear map of $\mathcal{W}$ to $\mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{V}^\vee \otimes_{\mathbf{C}} \mathcal{V}^\vee$, then it is represented by an $n \times n$-matrix $\Omega(\psi)$ whose entries are in $\mathcal{V}^\vee \otimes_{\mathbf{C}} \mathcal{V}^\vee$. By using the basis we fixed in the preceding section, we can write down $\Omega(\psi)$ in the form

$$\Omega(\psi) = \begin{pmatrix} {}^tA_1J_1A_1 & \cdots & {}^tA_1J_1A_n \\ {}^tA_2J_1A_1 & \cdots & {}^tA_2J_1A_n \\ \cdots & \cdots & \cdots \\ {}^tA_nJ_1A_1 & \cdots & {}^tA_nJ_1A_n \end{pmatrix}.$$

Since ${}^tA_iJ_1A_j$ is alternating by (INT 1′), it is in $\bigwedge^2 \mathcal{V}^\vee$, which we denote by s_{ij}. Moreover, we see that ${}^tA_jJ_1A_i = {}^t({}^tA_i{}^tJ_1A_j) = -{}^t({}^tA_iJ_1A_j) = {}^tA_iJ_1A_j$. Thus we obtain a symmetric matrix

$$S(\psi) = (s_{ij}).$$

By using the condition (INT 2′), we get

$${}^tA_iJ_1A_j = {}^tA_iJ_1(J_1\overline{A}_j{}^t\Sigma) = -{}^tA_i\overline{A}_j{}^t\Sigma = {}^tA_i\overline{A}_j\Sigma$$

If we write ${}^tA_iJ_1A_j$ in the form $\left(m^{ij}_{k\ell}\right)$, then this equation means

$$\textbf{(INT 3)} \left\{ \begin{array}{ll} m^{ij}_{kk} = 0,\ m^{ij}_{k\ell} = -m^{ij}_{\ell k} & \text{and} \\ m^{ij}_{k\ell} = (-1)^{\ell-1}\sum_{t=1}^{2n+2} a^{(i)}_{tk}\bar{a}^{(j)}_{t\ell'}, & \text{where } \ell' = \ell + (-1)^{\ell}. \end{array} \right.$$

$H_0(n) = \mathrm{Hom}_{\mathbf{C}}(\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}, \mathcal{U})$ is an $\mathbf{R}$-vector space. Let $Q_0(n)$ be the algebraic set in $H_0(n)$ defined by the equations (INT 1′) and (INT 2′) which are real polynomials. Since the equations (INT 1′) and (INT 2′) are homogeneous over $\mathbf{R}$ (quadratic and linear, respectively), $Q_0(n)$ gives rise to an algebric (a fotiori, closed) set $Q(n)$ in the real projective space $H(n) = H_0(n) - \{0\}/\mathbf{R}^*$.

Put $\mathcal{W}_0$ be the real vector space $\sum_{i=1}^n \mathbf{R}e_i$. σ induces a real structure on $\bigwedge^2 \mathcal{V}$. In fact, setting $Z_{ij} = Z_i \wedge Z_j$, we see that Z_{01}, $Z_{13} + Z_{02}$, $\sqrt{-1}\,(Z_{13} - Z_{02})$, $Z_{03} - Z_{12}$, $\sqrt{-1}\,(Z_{03} + Z_{12})$ and Z_{23} form a real basis. We denote the $\mathbf{R}$-vector space spanned by this basis by $(\bigwedge^2 \mathcal{V})_0$. Then $\bigwedge^2 \mathcal{V} = (\bigwedge^2 \mathcal{V})_0 \otimes_{\mathbf{R}} \mathbf{C}$. The equation (INT 3) and the symmetricity of $S(\psi)$ show

$$\begin{array}{l} \sum_{t=1}^{2n+2} a^{(i)}_{t0}\bar{a}^{(j)}_{t0} = m^{ij}_{01} = m^{ji}_{01} = \sum_{t=1}^{2n+2} a^{(j)}_{t0}\bar{a}^{(i)}_{t0} \\ m^{ij}_{13} = \sum_{t=1}^{2n+2} a^{(i)}_{t1}\bar{a}^{(j)}_{t2} \\ m^{ij}_{02} = -m^{ij}_{20} = -m^{ji}_{20} = \sum_{t=1}^{2n+2} a^{(j)}_{t2}\bar{a}^{(i)}_{t1} \\ m^{ij}_{12} = -\sum_{t=1}^{2n+2} a^{(i)}_{t1}\bar{a}^{(j)}_{t3} \\ m^{ij}_{03} = -m^{ij}_{30} = -m^{ji}_{30} = \sum_{t=1}^{2n+2} a^{(j)}_{t3}\bar{a}^{(i)}_{t1} \\ \sum_{t=1}^{2n+2} a^{(i)}_{t2}\bar{a}^{(j)}_{t2} = m^{ij}_{23} = m^{ji}_{23} = \sum_{t=1}^{2n+2} a^{(j)}_{t2}\bar{a}^{(i)}_{t2} \end{array}$$

Therefore, m^{ij}_{01} and m^{ij}_{23} are real, $\bar{m}^{ij}_{13} = m^{ij}_{02}$ and $m^{ij}_{03} = -m^{ij}_{12}$. This implies that

$$\begin{array}{rcl} s_{ij} & = & m^{ij}_{01}Z^\vee_{01} + m^{ij}_{13}Z^\vee_{13} + m^{ij}_{02}Z^\vee_{02} + m^{ij}_{12}Z^\vee_{12} + \\ & & m^{ij}_{03}Z^\vee_{03} + m^{ij}_{23}Z^\vee_{23} \\ & = & m^{ij}_{01}Z^\vee_{01} + \Re(m^{ij}_{13})(Z^\vee_{13} + Z^\vee_{02}) + \\ & & \Im(m^{ij}_{13})\sqrt{-1}\,(Z^\vee_{13} - Z^\vee_{02}) + \Re(m^{ij}_{03})(Z^\vee_{03} - Z^\vee_{12}) + \\ & & \Im(m^{ij}_{03})\sqrt{-1}\,(Z^\vee_{03} + Z^\vee_{12}) + m^{ij}_{23}Z^\vee_{23}, \end{array}$$

where for a complex number z, $\Re(z)$ and $\Im(z)$ are the real part and the imaginary part of z, respectively. These equalities show that $S(\psi)$ is a member of $K_0(n) = \mathrm{Hom}_{\mathbf{R}}(\mathcal{W}_0, \mathcal{W}_0^\vee \otimes_{\mathbf{R}} (\bigwedge^2 \mathcal{V})_0^\vee)$. The map $Q_0(n) \ni \psi \xrightarrow{f_0} S(\psi) \in K_0(n)$ is defined by real homogeneous forms. Furthermore, if $S(\psi) = 0$, then $m^{ii}_{10} = m^{ii}_{01} = m^{ii}_{23} = m^{ii}_{32} = 0$ for all i. The above computation shows that this is equivalent to vanishing of $a^{(i)}_{k\ell}$ for all k, ℓ, i, which means that $\psi = 0$. Thus the map f_0 induces a real algebraic morphism f of $Q(n)$ to the real projective space $K(n) = K_0(n) - \{0\}/\mathbf{R}^*$. To state our result, we need the concept of semi-slgebraic sets.

Definition 8.1. Let Y be a subset of a real algebriac variety Z. Y is said to be semi-algebraic at a point z of Z if there exists an open neighborhood (in the classical topology) V of z in Z and a finite number of elements f_{ij} and g_{ik} of $\Gamma(V, \mathcal{O}_Z)$ such that

$$Y \cap V = \bigcup_i \{\, y \in V \mid f_{ij} = 0, g_{ik} > 0, \text{for all } j, k \}$$

If Y is semi-algebraic at every point of Z, it is said that Y is semi-algebraic in Z.

The following due to Seidenberg is one of basic results on semi-algebraic sets.

Theorem 8.2 ([8]). *If $f : Z \to Z'$ is a morphism of real algebraic manifolds and if Y is semi-algebraic in Z, then $f(Y)$ is semi-algebraic in Z'.*

What we have seen in the above is

Proposition 8.3. *Put $K_0(n) = \mathrm{Hom}_{\mathbf{R}}(\mathcal{W}_0, \mathcal{W}_0^\vee \otimes_{\mathbf{R}} (\bigwedge^2 \mathcal{V})_0^\vee)$ and $K(n) = K_0(n) - \{0\}/\mathbf{R}^*$. $f(Q(n)) = P(n)$ is compact (a fortiori, closed in $K(n)$)*

and semi-algebraic set in $K(n)$. A point of $K_0(n)$ gives rise to a point of $P(n)$ if and only if it is represented by the matrix $S(\psi)$ with ψ satisfies the conditions (INT 1) and (INT 2).

We shall now study the effect of changing basis of $\mathcal{W}$ on the set $P(n)$. Pick another real basis $\{e'_1, \ldots, e'_n\}$ of $\mathcal{W}$ and write $e'_i = \sum_{j=1}^n b_{ji} e_j$ with $b_{ji} \in \mathbf{R}$. Then ψ is represented by matrices

$$(\sum_{j=1}^n A_j b_{j1}, \ldots, \sum_{j=1}^n A_j b_{jn}) = (A'_1, \ldots, A'_n).$$

Let us examin two conditions (INT 1′) and (INT 2′) for $(A'_1, \ldots, A'_n)$. Since

$$\begin{aligned} {}^t({}^tA'_i J_i A'_j) &= {}^t(\sum_{1\le k,\ell\le n} ({}^tA_k b_{ki}) J_1 (A_\ell b_{\ell j})) \\ &= \sum_{1\le k,\ell \le n} b_{ki} {}^t({}^tA_k J_1 A_\ell) b_{\ell j} = - \sum_{1\le k,\ell\le n} b_{ki} {}^tA_k J_1 A_\ell b_{\ell j} \\ &= -{}^tA'_i J_i A'_j, \end{aligned}$$

(INT 1′) for $(A'_1, \ldots, A'_n)$ is satisfied. $(A'_1, \ldots, A'_n)$ obviously satifies the condition (INT 2′) because b_{ji} are real. The above computation shows that if ψ' denotes the map represented by $(A'_1, \ldots, A'_n)$, then

$$S(\psi') = {}^tBS(\psi)B,$$

where $B = (b_{ij}) \in GL(n, \mathbf{R})$. We get therefore,

Lemma 8.4. *Let $GL(n, \mathbf{R})$ acts on $K_0(n)$ as follows:*

$$\text{for } B \in GL(n, \mathbf{R}), S \in K_0(n), S \longmapsto {}^tBSB.$$

Then this induces an action of $GL(n, \mathbf{R})$ on $K(n)$ and $P(n)$ is stable under this action.

Let $p = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$ be a point of $(\bigwedge^2 \mathcal{V})^\vee_0$ which satisfies the equation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$. The equation means that the Pfaffian of the matirix (p_{ij}) is zero, where we set $p_{ij} = -p_{ji}$. Thus $\sum_{i<j} p_{ij} Z_{ij}$ is pure or there are linear forms $L_i = \sum_{j=0}^3 \alpha_{ij} Z_j$, $(i = 0, 1)$ such that $\sum_{i<j} p_{ij} Z_{ij} = L_0 \wedge L_1$. Since p is real, we may assume that $L_1 = \sigma(L_0)$. Pick a point $q = (q_{01}, \ldots, q_{23})$ in $(\bigwedge^2 \mathcal{V})^\vee_0$ which is different from p and satisfies $q_{01}q_{23} -$

$q_{02}q_{13} + q_{03}q_{12} = 0$. And write $\sum_{i<j} q_{ij} Z_{ij}$ in the form $L_2 \wedge L_3$ with $L_3 = \sigma(L_2)$. Then $Z'_0 = L_0$, $Z'_1 = L_1$, $Z'_2 = L_2$, $Z'_3 = L_3$ form a basis of $\mathcal{V}$ and the action of σ with respect to this basis is represented by Σ, too. By writing A_i for this basis, we obtain the representation of $S(\psi)$ in terms of $Z'_{ij} = Z'_i \wedge Z'_j$. Note that if we choose a suitable q, then the transformation from Z_i's to Z'_i's is represented by a unitary matrix.

9 Boundary of P

We shall denote $\mathbf{P}^3_{\mathbf{C}}$ by X in this section, too and identify it with $\mathbf{P}(\mathcal{V}^\vee)$. We also denote the polynomial $Z^\vee_{01}Z^\vee_{23} - Z^\vee_{02}Z^\vee_{13} + Z^\vee_{03}Z^\vee_{13}$ by g. Note that g is the equation of the Grassmann of lines in X in $\mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$. Let us start with an easy lemma.

Lemma 9.1. *Let S be a member of $\mathrm{Hom}_{\mathbf{C}}(\mathcal{W}, \mathcal{W}^\vee \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}^\vee)$ and let $u(S)$ be the homomorphism of $\mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_X(-1)$ to $\mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X(1)$ which is defined by the multiplication by S. Assume that $\det S \not\equiv 0 \bmod g$ as a polynomial in $Z^\vee_{ij}$, where S is regarded as a matrix whose entries are in $\bigwedge^2 \mathcal{V}^\vee$. Then*

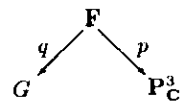
(1) *$u(S)$ is injective,*

(2) *$E(S) = \mathrm{coker}(u(S))$ is torsion free,*

(3) *the set $\{\det S = 0\} \cap G$ is exactly the set of lines ℓ such that $E(S)|_\ell$ is not isomorphic to the trivial bundle of rank $2n$,*

where G is the Grassmann of the lines in X.

Proof. Let K (or, I) be the kernerl (or, image, resp.) of $u(S)$. Since both are torsion free, for sufficiently general line ℓ in X, we see that $K|_\ell$ and $I|_\ell$ are subsheaves of $\mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_\ell(-1)$ and $\mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X(1)|_\ell$, respectively. Thus $H^0(\ell, I(-1)|_\ell) = 0$ and if $K \neq 0$, then $H^1(\ell, K(-1)|_\ell) \neq 0$. Look at the following diagram



where $\mathbf{F}$ is the incidence correspondence. Since p is flat, we have exact sequences

$$0 \longrightarrow q_*p^*(I(-1)) \longrightarrow R^1q_*p^*(K(-1)) \longrightarrow$$
$$R^1q_*p^*(\mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_X(-2)) \xrightarrow{u} R^1q_*p^*(I(-1)) \longrightarrow 0,$$
$$R^1q_*p^*(I(-1)) \xrightarrow{v} R^1q_*p^*(\mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X) \longrightarrow R^1q_*p^*(E(S)).$$

It is easy to see that $R^1q_*p^*(\mathcal{W}\otimes_{\mathbf{C}}\mathcal{O}_X(-2)) \cong \mathcal{W}\otimes_{\mathbf{C}}\mathcal{O}_G(-1)$, $R^1q_*p^*(\mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X) \cong \mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{O}_G$ and the composition $v \cdot u$ is nothing but the multiplication by S. Since $p^*(I(-1))$ is torsion free, so is $q_*p^*(I(-1))$. On the other hand, we know that $q_*p^*(I(-1)) = 0$ on a non-empty open set. Thus $q_*p^*(I(-1)) = 0$ on the whole space. We have seen in the above that if $K \neq 0$, then $R^1q_*p^*(K(-1)) \neq 0$. This means that if $u(S)$ is not injective, then $\mathrm{rank}(\mathrm{im}(v \cdot u)) < n$ and hence $\det S = 0$ on G. This completes the proof of (1).

Now we have an exact sequence

$$\begin{array}{ccccc} 0 \to q_*p^*(E(S)) \longrightarrow & R^1q_*p^*(\mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_X(-2)) & & & \\ & \wr\Vert & & & \\ & \mathcal{O}_G(-1)^{\oplus n} & & & \\ \xrightarrow{w} & R^1q_*p^*(\mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X) & \longrightarrow & R^1q_*p^*(E(S)) \to 0, & \\ & \wr\Vert & & & \\ & \mathcal{O}_G^{\oplus n} & & & \end{array}$$

where w is the homomorphism of the multiplication by S. This sequence shows that if $\det S \not\equiv 0 \pmod g$, then $q_*p^*(E(S))$ must be 0. On the other hand, since $K = 0$, $\mathrm{depth}_{\mathcal{O}_{X,x}} E(S)_x \geq \dim \mathcal{O}_{X,x} - 1$. Hence if the torsion part T of $E(S)$ is non-trivial, then it is supported by a closed subset of codimension one. Then the subsheaf $q_*p^*(T)$ of $q_*p^*(E(S))$ is of positive rank because general lines in X meet $\mathrm{Supp}(T)$ at a finite set of points. This contradicts our assumption. Therefore, we see that $E(S)$ is torsion free.

To prove (3), let B be the set of pinch points of $E(S)$. B is of codimension, at least, two in X. If a line ℓ is contained in B, then the kernel of $u(S)|_\ell$ contains an invertible sheaf $\mathcal{O}_\ell(a)$ with $a \leq -2$. Since the map w in the above sequence is compatible with base change, the non-zero vector space $H^1(\ell, \mathcal{O}_\ell(a))$ is a subspace of $\ker(w(\ell))$. Thus $\ell \in \{\det S = 0\} \cap G$. Assume next that ℓ is not in B. Then we can find a hyperplane Y containing ℓ such

that $Y \cap B$ is a finite set of points. Pick a line ℓ' in Y. At a general point of $q^{-1}(\ell')$, $p^*(u(S))|_{q^{-1}(\ell')}$ is injective and hence injective on $q^{-1}(\ell')$. This implies that $p^*(E(S))$ is flat over G at ℓ'. There is an exact sequence

$$0 \longrightarrow \mathcal{W} \otimes_{\mathbf{C}} \mathcal{O}_Y(-2) \longrightarrow \mathcal{W}^\vee \otimes_{\mathbf{C}} \Omega_X|_Y \longrightarrow E(S)|_Y \longrightarrow 0.$$

Thus if the torsion part of $E(S)|_Y$ is non-trivial, it must be supported by curves and contained in $Y \cap B$. This is impossible and hence $E(S)|_Y$ is torsion free. The dual plain $Y^\vee$ is contained in G and the diagram in the beginning of this proof induces

$$\begin{array}{ccc} & \mathbf{F}_{Y^\vee} & \\ {}^{q'}\swarrow & & \searrow^{p'} \\ Y^\vee & & Y \end{array}$$

Using the flatness of $p^*(E(S))$ over G in a neighborhood of $Y^\vee$ and the base change theorem we see that $\{\det S = 0\} \cap Y^\vee = \mathrm{Supp}(R^1q'_*p'^*(E(S)|_Y))$. Therefore, our assertion follows from a well-known result on $\mathbf{P}^2$ [5, p 373]. Q.E.D.

As a corollary to the above lemma we obtain a basic result.

Proposition 9.2. *Let S and $u(S)$ be as in Lemma 9.1. Assume that $\{\det S = 0\}$ does not meet the real part of the Grassmann $\{g = 0\} \cap \mathbf{P}((\wedge^2 \mathcal{V})_0^\vee)$. Then $E(S) = \mathrm{coker}(u(S))$ is a vector bundle and there is an exact sequence*

$$0 \longrightarrow F \longrightarrow E(S) \longrightarrow \mathcal{O}_X^{\oplus \alpha} \longrightarrow 0,$$

where $\alpha = \dim H^0(X, E(S)^\vee)$.

Proof. In the first place, note that $c_1(E(S)) = 0$. By our assumption on $\det S$ we can apply Lemma 9.1 to this case and know that $E(S)$ is a μ-semi-stable sheaf. Assume that $E(S)$ is not locally free and pick a pinch point x of $E(S)$. By (3) of Lemma 9.1 all the lines passing through x are "jumping lines" or contained in $\{\det S = 0\} \cap G$. Thus $\det S = 0$ at the real line $x\sigma(x)$ which is in the real part of the Grassmann. This violates our assumption and hence $E(S)$ must be locally free. Since $E(S)^\vee$ is a μ-semi-stable of degree 0, we have an injection

$$0 \longrightarrow H^0(X, E(S)^\vee) \otimes_{\mathbf{C}} \mathcal{O}_X \xrightarrow{h} E(S)^\vee.$$

Setting $F = \operatorname{coker}(h)^\vee$, there is an exact sequence

$$0 \longrightarrow F \longrightarrow E(S) \stackrel{f}{\longrightarrow} \mathcal{O}_X^{\oplus\alpha}.$$

f is generically surjective and hence $A = \operatorname{im}(f)$ is a torsion free sheaf of rank α. Now let us look at the diagram in the proof of Lemma 9.1. On one hand, $q_*p^*(A(-1))$ is torsion free. On the other hand, it is zero at a general point of G. Thus we get an exact sequence

$$0 \longrightarrow R^1q_*p^*(F(-1)) \longrightarrow R^1q_*p^*(E(S)(-1)) \longrightarrow R^1q_*p^*(A(-1)) \longrightarrow 0.$$

As we have seen in the proof of Lemma 9.1,

$$\{\det S = 0\} \cap G = \operatorname{Supp}(R^1q_*p^*(E(S)(-1))),$$

which is, by the above exact sequence, equal to $\operatorname{Supp}(R^1q_*p^*(F(-1))) \cup \operatorname{Supp}(R^1q_*p^*(A(-1)))$. Let T be $\mathcal{O}_X^{\oplus\alpha}/A$. Then T is a torsion sheaf and $R^1q_*p^*(A(-1)) \cong q_*p^*(T(-1))$. It is easy to see that a line which meets $\operatorname{Supp}(T)$ at a finite set of points contained in $\operatorname{Supp}(q_*p^*(T(-1)))$ and then we see that a line in X is contained in $\operatorname{Supp}(R^1q_*p^*(A(-1)))$ if and only if it meets $\operatorname{Supp}(T)$. Thus, for a point x in $\operatorname{Supp}(T)$, the real line $\overline{x\sigma(x)}$ is in $\{\det S = 0\}$. This contradicts our assumption. Therefore, $T = 0$, in other words, f is surjective as required. Q.E.D.

Pick an S in $K_0(n)$ which represents a point of $P(n)$. Since $S = S(\psi)$ for some $\psi \in Q_0(n)$ and since $\operatorname{rank}(\psi) \le 2n+2$, we see that if we regard $\omega(\psi)$ a linear map of $\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}$ to $\mathcal{W}^\vee \otimes_{\mathbf{C}} \mathcal{V}^\vee$, then $\operatorname{rank}(\omega(\psi)) \le 2n+2$ (see the beginning of §5). Let us consider the following exact commutative diagram which is dual to a part of the diagram in the beginning of the preceding section:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{W}\otimes_{\mathbf{C}}\mathcal{O}_X(-1) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{W}\otimes_{\mathbf{C}}\mathcal{V}\otimes_{\mathbf{C}}\mathcal{O}_X & \stackrel{\gamma}{\longrightarrow} & \mathcal{U}^\vee\otimes_{\mathbf{C}}\mathcal{O}_X & & \\
 & & & & \downarrow & & \downarrow\delta & & \\
0 & \longrightarrow & E(S)^\vee & \longrightarrow & \mathcal{W}\otimes_{\mathbf{C}}\Omega_X^\vee(-1) & \longrightarrow & \mathcal{W}^\vee\otimes_{\mathbf{C}}\mathcal{O}_X(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

This shows us that $H^0(X, \mathcal{W}\otimes_{\mathbf{C}}\Omega_X^\vee(-1)) \cong \mathcal{W}\otimes_{\mathbf{C}}\mathcal{V}$. Since ${}^t\omega(\psi) = H^0(\delta\cdot\gamma)$, we obtain that

$$\dim H^0(X, E(S)^\vee) = \dim \mathcal{W}\otimes_{\mathbf{C}}\mathcal{V} - \operatorname{rank}({}^t\omega(\psi)) \ge 2n-2.$$

Assume that $\{\det S = 0\}$ contains no real lines. Then, by the above proposition, there is an exact sequence

$$0 \longrightarrow F \longrightarrow E(S) \longrightarrow H^0(X, E(S)^\vee)^\vee \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow 0.$$

If $\operatorname{rank}\omega(\psi) < 2n+2$, then $\operatorname{rank}\omega(\psi) \le 2n$ because $\omega(\psi)$ is alternating. In this case, $\dim H^0(X, E(S)^\vee) = 2n$ and hence $E(S)$ is trivial. This is impossible because $c_2(E(S)) = n > 0$. Thus $\operatorname{rank}\omega(\psi) = 2n+2$ and then F is of rank 2 and locally free. Since $c_1(F) = 0$ and $H^0(X, F) = 0$, F is μ-stable vector bundle. By the definition of $E(S)$, we have that $H^i(X, E(S)) = 0$ for all i. Hence F satisfies the condition (UV) and $E(S)$ is the universal extension of F (see Lemma 1.3). Moreover, by Verdier's Theorem (Theorem 7.5) F is an instanton bundle. We have proved the following.

Proposition 9.3. *Let S be an element in $K_0(n)$ which represents a point of $P(n)$ such that $\{\det S = 0\}$ contains no real lines. Then there is an exact sequence*

$$0 \longrightarrow F \longrightarrow E(S) \longrightarrow \mathcal{O}_X^{\oplus 2n-2} \longrightarrow 0$$

such that F is an instanton bundle and $E(S)$ is the universal extension of F.

Next assume that $\{\det S = 0\}$ contains a real line $p = (p_{01}, \ldots, p_{23}) \in G \cap \mathbf{P}((\bigwedge^2 \mathcal{V})_0^\vee)$. Then $\det S(p) = \det(s_{ij}(p)) = 0$. Since $S(p)$ is symmetric, real matrix, we can find a B in the orthogonal group $O(n, \mathbf{R})$ (a fortiori, in $GL(n, \mathbf{R})$) such that $BS(p){}^tB = (t_{ij})$ with $t_{11} = 0$. By the computation at the end of the preceding section, after changing the real basis of $\mathcal{W}_0$ and a suitable choice of generators of $(\bigwedge^2 \mathcal{V})_0^\vee$, we may assume that $s_{11}(1, 0, \ldots, 0) = 0$ or $m_{01}^{11} = 0$. Since by (INT 3) $m_{01}^{11} = \sum_{t=1}^{2n+2} |a_{t0}^{(1)}|^2 = \sum_{t=1}^{2n+2} |a_{t1}^{(1)}|^2$, the condition provides us with

$$a_{ti}^{(1)} = 0 \text{ for } i = 0, 1 \text{ and all } t.$$

Substituting $a_{ti}^{(1)}$, $(i = 0, 1)$ in (INT 3) by 0, we get

$$\begin{aligned} s_{1j} &= m_{23}^{1j} Z_{23}^\vee \\ s_{j1} &= m_{23}^{j1} Z_{23}^\vee. \end{aligned}$$

If $m^{11}_{23} = 0$ in addition to the above, then $A_1 = 0$ and hence $s_{1j} = s_{j1} = 0$ for all j. Suppose that $m^{11}_{23} \neq 0$. Define B' to be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\frac{m^{21}_{23}}{m^{11}_{23}} & 1 & 0 & 0 & \cdots & 0 \\ -\frac{m^{31}_{23}}{m^{11}_{23}} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ -\frac{m^{n1}_{23}}{m^{11}_{23}} & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Then we have

$$B'S{}^tB' = \begin{pmatrix} m^{11}_{23}Z^{\vee}_{23} & 0 \cdots 0 \\ 0 & \\ \vdots & S_1 \\ 0 & \end{pmatrix}$$

If $\det S_1(q) = 0$ with q a real line, then for a suitable $B_1 \in GL(n-1, \mathbf{R})$, the $(2,2)$ element of

$$\begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix} B'S{}^tB' \begin{pmatrix} 1 & 0 \\ 0 & {}^tB_1 \end{pmatrix}$$

vanishes at q. The same argument as above shows that if we choose $B_1' \in GL(n-1, \mathbf{R})$ suitably, then $B_1'S_1{}^tB_1'$ has a form

$$\begin{pmatrix} M_2 & 0 \\ 0 & S_2 \end{pmatrix}$$

where $M_2 = a_2L_2 \wedge \sigma L_2$ with $a_2 \in \mathbf{R}$ and $L_2 \in \mathcal{V}^{\vee}$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & B_1' \end{pmatrix} B'S{}^tB' \begin{pmatrix} 1 & 0 \\ 0 & {}^tB_1' \end{pmatrix} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & S_2 \end{pmatrix}$$

with $M_i = a_iL_i \wedge \sigma L_i$ with $a_i \in \mathbf{R}$ $L_i \in \mathcal{V}^{\vee}$. Continuing this procedure, we can find a $C \in G$ such that

$$CS{}^tC = \begin{pmatrix} M_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & M_r \\ & & & & S_r \end{pmatrix}$$

where $M_i = a_iL_i \wedge \sigma L_i$ with $a_i \in \mathbf{R}$ and $L_i \in \mathcal{V}^{\vee}$ and where $\{\det S_r = 0\}$ contains no real lines or $r = n$, that is, S_r does not appear.

Lemma 9.4. *Let L, L' be linearly indepndent elements of $\mathcal{V}^{\vee}$ and let $M = L \wedge L'$. If we regard M as a global section of $\Omega_X(2)$, then the quotinet sheaf $C = \Omega_X(1)/M\mathcal{O}_X(-1)$ is isomorphic to the sheaf Q_ℓ, where ℓ is the line in X defined by $L = L' = 0$ and where Q_ℓ is the kernel of a surjection of $\mathcal{O}_X^{\oplus 2}$ to $\mathcal{O}_\ell(1)$.*

Proof. We have the Koszul cmplex defined by L and L' and the Euler sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus 2} \longrightarrow I_\ell(2) \longrightarrow 0,$$
$$0 \longrightarrow \Omega_X(2) \longrightarrow \mathcal{V}^{\vee} \otimes_{\mathbf{C}} \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X(2) \longrightarrow 0,$$

where I_ℓ is the defining ideal of ℓ. Since both sequence are exact and since the map of $\mathcal{O}_X(1)^{\oplus 2}$ to $\mathcal{V}^{\vee} \otimes_{\mathbf{C}} \mathcal{O}_X(1)$ defined by the multiplication by (L, L') induces the map of $\mathcal{O}_X$ to $\Omega_X(2)$ of multiplication by M, we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_X(2) & \longrightarrow & C(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(1)^{\oplus 2} & \longrightarrow & \mathcal{V}^{\vee} \otimes_{\mathbf{C}} \mathcal{O}_X(1) & \longrightarrow & \mathcal{O}_X(1)^{\oplus 2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_\ell(2) & \longrightarrow & \mathcal{O}_X(2) & \longrightarrow & \mathcal{O}_\ell(2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The rightmost column shows our assertion. Q.E.D.

If $a_1 a_2 \cdots a_r \neq 0$, then $E(S)$ is isomorphic to $Q_{\ell_1} \oplus \cdots \oplus Q_{\ell_r} \oplus E(S_r)$, where ℓ_i is the real line defined by $L_i = \sigma(L_i) = 0$. Note that $Q_{\ell_i} \cong \Omega_X(1)/M_i\mathcal{O}_X(-1)$ is quaternionic. By Proposition 9.2, $E(S_r)$ is locally free, of rank $2n - 2r$, trivial on every real line and there is an exact sequence

$$0 \longrightarrow F(S_r) \longrightarrow E(S_r) \longrightarrow \mathcal{O}_X^{\oplus\alpha} \longrightarrow 0,$$

where $\alpha = \dim H^0(X, E(S_r)^\vee)$. Since $\operatorname{rank}\omega(\psi) \leq 2n - 2$ and M_i gives rise to a rank 2 (4×4)-block in $\Omega(\psi)$, we see that $\dim H^0(X, E(S_r)^\vee) \geq 4(n-r) - (2n + 2 - 2r) = 2n - 2r - 2$. Thus, as we have seen in the proof of Proposition 9.3, $\alpha = 2n - 2r - 2$ or $2n - 2r$. The latter case is absurd because $H^0(X, E(S_r)) = 0$. $F(S_r)$ is therefore of rank 2, which means that $E(S_r)$ is the universal extension of the instanton bundle $F(S_r)$

Remark 9.5. Since $F(S_r)$ is an instanton bundle, Verdier's Theorem shows that S_r must be represented by a point of $P(n - r)$. We can directly see this by applying repeatedly Lemma 10.2 in the next section.

Let $SK_0(n)$ be the set $\{T \in K_0(n) \mid T$ is represented by a symmetric $n \times n$-matrix whose entries are in $(\bigwedge^2 V^\vee)_0\}$ and let $SK(n)$ be $SK_0(n) - \{0\}/\mathbf{R}^*$. Clearly $SK(n)$ is a linear suspace of $K(n)$. $GL(n, \mathbf{R})$ acts on $SK(n)$ as in Lemma 8.4 and its center acts trivially on $SK(n)$. Thus $PGL(n, \mathbf{R})$ acts on $SK(n)$.

Proposition 9.6. *For the above action of $PGL(n, \mathbf{R})$ on $SK(n)$, a geometric point A of $SK(n)$ is not semi-stable (or, not stable) if and only if there exists a B in $GL(n, k(A))$ and a positive integer α such that $\alpha \leq n - \alpha + 1$ (or, $\alpha \leq n - \alpha$, resp.) and that for $BA{}^tB = (a_{ij})$, $a_{ij} = 0$ if (1) $i \leq n - \alpha + 1$ (or, $\leq n - \alpha$, resp.) and $j \leq \alpha$ or (2) $i \leq \alpha$ and $j \leq n - \alpha + 1$ (or, $\leq n - \alpha$, resp.). Thus if A is not semi-stable, then $\det A = 0$.*

Proof. We may extend the base field to $\mathbf{C}$. Then, since we have an isogeny of $SL(n, \mathbf{C})$ to $PGL(n, \mathbf{C})$, we may prove our assertion with respect to the action of $SL(n, \mathbf{C})$. Assume that a geometric point A of $SK(n)$ is not semi-stable (or, not stable). Then there is a one parameter subgroup λ of $SL(n, \mathbf{C})$ such that $\mu(A, \lambda) < 0$ (or, ≤ 0, resp.). If we choose a suitable basis of W,

then the action of λ on W is the multiplication of the matrix

$$\lambda(t) = \begin{pmatrix} t^{r_1} & & & 0 \\ & t^{r_2} & & \\ & 0 & \ddots & \\ & & & t^{r_n} \end{pmatrix}$$

where $r_1, r_2, \ldots, r_n$ are integers such that $r_1 \leq r_2 \leq \cdots \leq r_n$ and $r_1 + r_2 + \cdots + r_n = 0$. Changing a basis of W, A is transformed to $BA{}^tB$. If we denote the matrix $BA{}^tB$ by (a_{ij}), then the action of $\lambda(t)$ is represented by

$$a_{ij} \longmapsto t^{r_i + r_j} a_{ij}$$

We need a simple numerical lemma.

Lemma 9.7. *Let $r_1, r_2, \ldots, r_n$ be a sequence of integers such that $r_1 \leq r_2 \leq \cdots \leq r_n$, $r_1 + r_2 + \cdots + r_n = 0$ and not all of r_i's are zero. Then there is a positive integers α and β such that $\alpha \leq n - \alpha + 1$, $\beta \leq n - \beta$ and $r_\alpha + r_{n-\alpha+1} \leq 0$ and $r_\beta + r_{n-\beta} < 0$.*

Proof. If $r_1 + r_n \leq 0$, then we may put $\alpha = 1$. Assume that $r_1 + r_n > 0$. We have

$$\begin{aligned} 0 < r_1 + r_n &= -(r_2 + \cdots + r_{n-1}) \\ &= -\textstyle\sum_{i=1}^{m-1}(r_{i+1} + r_{n-i}) - \frac{1}{2}\{1 + (-1)^{n-1}\}\, r_{m+1}, \end{aligned}$$

where m is the biggest integer with $m \leq n/2$. Thus one of $r_{i+1} + r_{n-i}$ is negative, which proves the existence of α. If $r_1 + r_{n-1} < 0$, then we may choose 1 to be β. Assume that $r_1 + r_{n-1} \geq 0$. We then obtain

$$\begin{aligned} 0 \leq r_1 + r_{n-1} &= -(r_2 + \cdots + r_{n-2} + r_n) \\ &= -\textstyle\sum_{i=2}^{m-1}(r_i + r_{n-i}) - \frac{1}{2}\{1 + (-1)^n\}\, r_m - r_n, \end{aligned}$$

where m is the least integer with $m \geq n/2$. Since r_n must be positive, one of $r_i + r_{n-i}$ is negative. This completes the proof of our lemma.

Let us go back to the proof of Proposition 9.6. Pick an α in the above lemma. Then we see that $r_i + r_j \leq 0$ if (1) $i \leq n - \alpha + 1$ and $j \leq \alpha$ or (2) $i \leq \alpha$ and $j \leq n - \alpha + 1$. Since $\mu(\lambda, A) < 0$, a_{ij} should be 0 if (i,j) is in this range. If we pick β instead of α, then the proof of the condition for A to be not stable is similar to the above. Conversely, suppose that we have B and α which satisfy our condition for A to be not semi-stable in our proposition. Let $\{e'_1, \ldots, e'_n\}$ be the basis of $\mathcal{W}$ after the linear transformation by B. Take such a one parameter subgroup λ of $SL(n, k(A))$ that

$$\lambda(t)(e'_i) = \begin{cases} t^{\alpha-n}e'_i & \text{if } 1 \le i \le \alpha \\ te'_i & \text{if } \alpha < i \le n+1-\alpha \\ t^{n+1-\alpha}e'_i & \text{if } n+1-\alpha < i \le n \end{cases}$$

Then, for this λ, we have that $\mu(\lambda, A) < 0$. If B and α satisfy our condition for A to be not stable, then we pick the one parameter group λ:

$$\lambda(t)(e'_i) = \begin{cases} t^{-1}e'_i & \text{if } 1 \le i \le \alpha \\ e'_i & \text{if } \alpha < i \le n-\alpha \\ te'_i & \text{if } n-\alpha < i \le n \end{cases}$$

It is clear that for this λ, $\mu(\lambda, A) \le 0$. Q.E.D.

Now we come to the main result of this section.

Theorem 9.8. *Let S be a member of $SK_0(n)$ which represents a point of $P(n)$.*

(1) There is a B in $GL(n, \mathbf{R})$, members $L_1, \ldots, L_r$ of $\mathcal{V}$, real numbers $a_1, \ldots, a_r$ and a symmetric S_r with $\{\det S_r = 0\}$ containing no real lines such that

$$BS{}^tB = \begin{pmatrix} M_1 & & & 0 & \\ & \ddots & & & \\ & & \ddots & & \\ 0 & & & M_r & \\ & & & & S_r \end{pmatrix}$$

where $M_i = a_iL_i \wedge \sigma L_i$ and where if $r = n$, then we understand that S_r does not appear.

(2) S is semi-stable with respect to the action of $PGL(n, \mathbf{R})$ on $SK(n)$ if and only if $\prod a_i \ne 0$, that is, $\det S \not\equiv 0$.

(3) S is stable if and only if $r = 0$.

(4) If S is semi-stable, then $E(S) \cong Q_{\ell_1} \oplus \cdots Q_{\ell_r} \oplus E(S_r)$, where ℓ_i is the real line defined by $L_i = \sigma(L_i) = 0$ (for the definitions of $E(S)$ and Q_{ℓ_i}, see Lemma 9.1 and just after Lemma 9.4, respectively). Moreover, $E(S_r)$ is the universal extension of an instanton bundle with $c_2(E) = n - r$.

Proof. (1) was proved in the above. (2) is obvious by Proposition 9.6. If $r > 0$, then it is easy to see that there is a B' in $GL(n, \mathbf{C})$ such that $B'S{}^tB'$ has the 0-matrix of degree r at the upper-left corner. Then, by Proposition 9.6, S is not stable. Assume that $r = 0$. If S is not stable, then there is a positive integer α such that $\alpha \le n - \alpha$ and that by choosing a suitable basis over $\mathbf{C}$, S is transformed to a matrix

$$\begin{pmatrix} 0 & 0 & C \\ 0 & D_{11} & D_{12} \\ C & D_{21} & D_{22} \end{pmatrix}$$

where C and D_{22} are square matrices of degree α, D_{11} is $(n-2\alpha) \times (n-2\alpha)$ and D_{12} and where D_{21} are $(n-2\alpha) \times \alpha$ and $\alpha \times (n-2\alpha)$, respectively. Changing the order of rows, we come to the matrix

$$\begin{pmatrix} C & D_{21} & D_{22} \\ 0 & D_{11} & D_{12} \\ 0 & 0 & C \end{pmatrix}.$$

This means that there is an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E(S) \longrightarrow E_2 \longrightarrow 0,$$

where $E_1 = E(C)$ and $E_2 = E(T)$ with

$$\begin{pmatrix} D_{11} & D_{12} \\ 0 & C \end{pmatrix}.$$

Since both $\det C$ and $\det T$ are not zero at any real line, Proposition 9.2 tells us that E_1 and E_2 are locally free and we have exact sequences

$$0 \longrightarrow F_i \longrightarrow E_i \longrightarrow H^0(X, E_i^\vee)^\vee \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow 0.$$

If $\dim H^0(X, E_i^\vee) = r(E_i)$ or $r(E_i) - 1$, then E_i is a trivial bundle because $c_1(E_i) = 0$. Since neither E_1 nor E_2 is trivial bundle, $\dim H^0(X, E_i^\vee) \le r(E_i) - 2$ and hence $\dim H^0(X, E(S)^\vee) \le r(E(S)) - 4$. On the other hand, we know that $\dim H^0(X, E(S)^\vee) = r(E(S)) - 2$. This is a contradiction and we see that S is stable. (4) was proved right after Lemma 9.4. Q.E.D.

Remark 9.9. If $E(S)$ is locally free and of rank $2n$, then the above theorem shows that $\{\det S = 0\}$ contains no real lines. Then, by Lemma 9.1

$E(S)$ is trivial on every real line and hence it is the universal extension of a vector bundle of rank 2 whose restriction to any real line is trivial (see the proof of Proposition 9.3). This is another proof of the first part of Verdier's theorem (Theorem 7.5).

Let $SM(n)$ be the good quotient of $SK(n)^{ss}$ by $PGL(n,\mathbf{R})$. The above theorem shows that $P(n)_0 = \{x \in P(n) \mid x$ is represented by a matrix S with $\det S \not\equiv 0\}$ is $PGL(n,\mathbf{R})$-stable and closed (in the classical topology) in $SK(n)^{ss}$. Thus the image $\overline{P}(n)$ of $P(n)_0$ in $SM(n)$ is closed [7, Theorem 3] and semi-algebriac. Since $SM(n)$ is projective, $\overline{P}(n)$ is compact.

There exists a $PGL(n,\mathbf{R})$-stable, Zariski closed set R in $SK(n)^{ss}$ such that for a geometric point S in $SK_0(n)$, $u(S)$ is injective and $E(S)$ is a semi-stable sheaf if and only if it does not give rise to a geometric point of R. The image $\overline{R}$ of R to $SM(n)$ is closed set and does not meet $\overline{P}(n)$. There is an $\mathbf{R}$-morphism θ of $SM(n) - \overline{R}$ to $\overline{M}(n,2n)_{\mathbf{R}}$ (see Proposition 7.3) which sends each point of $SM(n) - \overline{R}$ to the S-equivalence class of the corresponding semi-stable sheaf. Since $\overline{P}(n)$ is compact and semi-slgebric, so is $\theta(\overline{P}(n))$ in $\overline{M}(n,2n)_{\mathbf{R}}(\mathbf{R})$.

Proposition 9.10. *The set $\coprod_{a=0}^{n} I(n-a) \times \Sigma^a(S^4)$ is compact and semi-algebraic in $\overline{M}(n,2n)_{\mathbf{R}}(\mathbf{R})$, where S^4 parametrizes the real lines in X, Σ^b stands for b-th symmetric product and where we understand that $I(0)$ is one point. Moreover, $I(n)$ is open in the compact set.*

Proof. Our assertion is obvious because Theorem 9.8 shows us that $\theta(\overline{P}(n))$ is exactly the set in the proposition.

10 A compactification of the moduli space of instantons

We have proved so far that $\coprod_{a=0}^{n} I(n-a) \times \Sigma^a(S^4)$ is compact and semi-algebraic in $\overline{M}(n,2n)_{\mathbf{R}}(\mathbf{R})$ and hence $\overline{I}(n)$ is contained in the set. In this section we are going to show that these two sets coincide. As in §5, $Q(n)$ denotes the real, closed subscheme of $H(n) = \mathrm{Hom}_{\mathbf{C}}(\mathcal{W} \otimes_{\mathbf{C}} \mathcal{V}, \mathcal{U}) - \{0\}/\mathbf{R}^*$ defined by the equations (INT 1′) and (INT 2′). Pick a point x in $Q(n)$. x is represented by an n-ple $\mathbf{A} = (A_1, \ldots, A_n)$ of $(2n+2) \times 4$-matrices. For

$A_i = (a^{(i)}_{k\ell})$, the equation (INT 2′) is written in the form

$$a^{(i)}_{n+1+r\,0} = \bar{a}^{(i)}_{r1}, \quad a^{(i)}_{n+1+r\,1} = -\bar{a}^{(i)}_{r0},$$
$$a^{(i)}_{n+1+r\,2} = \bar{a}^{(i)}_{r3}, \quad a^{(i)}_{n+1+r\,3} = -\bar{a}^{(i)}_{r2}.$$

Thus $Q(n)$ is a real subscheme in the linear subspace $T(n)$ of $H(n)$ defined by the above equations. For an n-ple $\mathbf{A} = (A_1, \ldots, A_n)$ which gives rise to a point of $T(n)$, set

$$(q)_n \begin{cases} F^{ij}_m(n,\mathbf{A}) = (-1)^{m-1}\left\{\sum_{t=1}^{n+1} a^{(i)}_{tm}\bar{a}^{(j)}_{tm'} - \sum_{t=1}^{n+1} a^{(i)}_{tm'}a^{(j)}_{tm}\right\} \\ F^{ij}_{k\ell}(n,\mathbf{A}) = (-1)^{\ell-1}\sum_{t=1}^{n+1} a^{(i)}_{tk}\bar{a}^{(j)}_{t\ell'} + (-1)^{k}\sum_{t=1}^{n+1} \bar{a}^{(i)}_{tk'}a^{(j)}_{t\ell} + \\ \qquad (-1)^{k-1}\sum_{t=1}^{n+1} a^{(i)}_{t\ell}\bar{a}^{(j)}_{tk'} + (-1)^{\ell}\sum_{t=1}^{n+1} \bar{a}^{(i)}_{t\ell'}a^{(j)}_{tk} \end{cases}$$

where for an integer α, α' is the integer $\alpha + (-1)^{\alpha}$. $Q(n)$ is defined by the equations $\{F^{ij}_m(n,\mathbf{A}) = 0, F^{ij}_{k\ell}(n,\mathbf{A}) = 0 \mid 1 \le i < j \le n, 0 \le m \le 3, 0 \le k < \ell \le 3\}$ in $T(n)$.

Take a point y in $Q(n-1)$ and let $\mathbf{B} = (B_2, \ldots, B_n)$ represent y. This satifies the equations $F^{ij}_m(n-1,\mathbf{B}) = 0$, $F^{ij}_{k\ell}(n-1,\mathbf{B}) = 0$ $(2 \le i < j \le n)$. Writing

$$B_i = \begin{pmatrix} B'_i \\ B''_i \end{pmatrix}$$

with B'_i and B''_i $(n \times 4)$-matrices, we set

$$\tilde{B}_1 = \begin{pmatrix} 1\ 1\,0\,0 \\ 0 \\ 1\ {-1}\,0\,0 \\ 0 \end{pmatrix}, \quad \tilde{B}_i = \begin{pmatrix} 0\,0\,0\,0 \\ B'_i \\ 0\,0\,0\,0 \\ B''_i \end{pmatrix} \quad (2 \le i < j \le n)$$

Then, for $\delta(y) = \delta(\mathbf{B}) = \tilde{\mathbf{B}} = (\tilde{B}_1, \ldots, \tilde{B}_n)$, we have that $F^{1j}_m(n,\tilde{\mathbf{B}}) = 0$, $F^{1j}_{k\ell}(n,\tilde{\mathbf{B}}) = 0$ and that if $i \ge 2$, then $F^{ij}_m(n,\tilde{\mathbf{B}}) = F^{ij}_m(n-1,\mathbf{B}) = 0$ and $F^{ij}_{k\ell}(n,\tilde{\mathbf{B}}) = F^{1j}_{k\ell}(n-1,\mathbf{B}) = 0$. Thus $\delta(y)$ is a point of $Q(n)$.

Let $R(n)$ be the open subscheme of $Q(n)$ consisting of the points which define instanton bundles and $Q(n)^{ss}$ be the set of points which give rise to semi-stable sheaves. Note that $R(n)$ is a smooth real algebraic manifold [10, §5, §6]. Our main result in this section is stated as follows.

Theorem 10.1. *$Q(n)^{ss}$ is the closure of $R(n)$ in $Q(n)^{ss}$ and* $\dim_{\mathbf{R}} R(n) = 3n^2 + 13n - 1$.

Proof. First note that the latter assertion is contained in [10] and that codimension of $R(n)$ in $T(n)$ is equal to the number of the equations to define $Q(n)$. Our proof is by induction on n. When $n = 1$, our assertion is obvious. In fact, $Q(1)^{ss} = T(1)$ and $Q(1)^{ss} - R(1)$ is a thin set. Assume that our theorem is true up to $n-1$. We claim the following.

Claim: There is a dense Zariski open set $R'(n-1)$ of $R(n-1)$ such that for every y in $R'(n-1)$, $\delta(y)$ is a smooth point of $Q(n)^{ss}$ and $\dim_{\delta(y)} Q(n)^{ss} = \dim R(n)$.

Let $\mathbf{a}_i = (a_{1i}, \ldots, a_{n+1\,i})$ $(0 \le i \le 3)$ and let A be the matrix

$$A = \begin{pmatrix} {}^t\mathbf{a}_0 & {}^t\mathbf{a}_1 & {}^t\mathbf{a}_2 & {}^t\mathbf{a}_3 \\ {}^t\bar{\mathbf{a}}_1 & -{}^t\bar{\mathbf{a}}_0 & {}^t\bar{\mathbf{a}}_3 & -{}^t\bar{\mathbf{a}}_2 \end{pmatrix}.$$

We define $D(n, A)$ to be the $10 \times (8n+8)$-matrix

$$\begin{pmatrix} -\bar{\mathbf{a}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_0 & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{a}}_0 & -\bar{\mathbf{a}}_1 & \mathbf{0} & \mathbf{0} & -\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{0} & \mathbf{0} \\ -\bar{\mathbf{a}}_3 & \mathbf{0} & -\bar{\mathbf{a}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{a}_2 & \mathbf{0} & \mathbf{a}_0 \\ \bar{\mathbf{a}}_2 & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{a}}_1 & \mathbf{0} & \mathbf{a}_3 & -\mathbf{a}_0 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{a}}_0 & \mathbf{0} & \mathbf{0} & -\mathbf{a}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\bar{\mathbf{a}}_3 & \bar{\mathbf{a}}_0 & \mathbf{0} & -\mathbf{a}_2 & \mathbf{0} & \mathbf{0} & \mathbf{a}_1 \\ \mathbf{0} & \bar{\mathbf{a}}_2 & \mathbf{0} & \bar{\mathbf{a}}_0 & -\mathbf{a}_3 & \mathbf{0} & -\mathbf{a}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\bar{\mathbf{a}}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_2 \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{a}}_2 & -\bar{\mathbf{a}}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{a}_2 & \mathbf{a}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{a}}_2 & \mathbf{0} & \mathbf{0} & -\mathbf{a}_3 & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ is the 0-row vector of degree $n+1$. Pick a point y in $Q(n-1)^{ss}$ and let it be represented by $\mathbf{B} = (B_2, \ldots, B_n)$ with $B_i = (b^{(i)}_{k\ell})$. Let us set

$$J(\mathbf{B}) = \begin{pmatrix} D(n, \tilde{B}_2) & -D(n, \tilde{B}_1) & 0 & & & \\ D(n, \tilde{B}_3) & 0 & -D(n, \tilde{B}_1) & & 0 & \\ \vdots & \vdots & 0 & \ddots & & \\ \vdots & \vdots & \vdots & & \ddots & \\ D(n, \tilde{B}_n) & 0 & 0 & \ldots & 0 & -D(n, \tilde{B}_1) \end{pmatrix}.$$

Let $\alpha_{ri} = in + r + i + 2$ and $\beta_{rj} = (r+1)(8n+8) + (n+1)(j-1) + 1$. We denote by $\widehat{J(\mathbf{B})}$ the minor matrix of degree $10n - 10$ of $J(\mathbf{B})$ made by choosing columns $(\ldots, \alpha_{r1}, \ldots, \alpha_{r4}, \alpha_{r6}, \beta_{r1}, \ldots, \beta_{r5}, \ldots)_{0 \le r \le n-2}$. The points y with $\det \widehat{J(\mathbf{B})} \ne 0$ form a Zariski open set Z of $Q(n-1)^{ss}$. Let us look at a special point y_0 in $Q(n-1)^{ss}$ which is represented by $\mathbf{B}_0 = (B'_2, \ldots, B'_n)$ with

$$B'_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & \cdots & \cdots & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ & \cdots & \cdots & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & \cdots & \cdots & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ & \cdots & \cdots & \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ i-1 \\ \\ \\ n \\ n+1 \\ \\ \\ n+i-1 \\ \\ \\ \\ \end{matrix}$$

Then we see that

$$\widehat{J(\mathbf{B}_0)} = \begin{pmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & \ddots \\ & & & T_{n-1} \end{pmatrix},$$

where T_i is a square matrix of degree 10 with the following form

$$\left(\begin{array}{ccccc|ccccc} & & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & -1 & 1 & 0 & 0 & 1 \\ & & \mathbf{0} & & & 0 & 0 & 1 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 1 & 0 \\ & & & & & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & & & & & \\ 1 & 0 & 0 & -1 & 0 & & & & & \\ 0 & -1 & 0 & 0 & 0 & & & * & & \\ 0 & 1 & -1 & 0 & -1 & & & & & \\ 0 & 0 & 1 & 0 & -1 & & & & & \end{array}\right).$$

As is easily seen, the determinant of this matrix is not zero and hence $\det J(\widehat{\mathbf{B}_0}) \neq 0$. Thus Z is not empty.

By induction hypothesis, every point of Z is the limit of a sequence of points in $R(n-1)$. This and the fact that Z is a non-empty open set imply that $Z' = R(n-1) \cap Z$ is not empty. Let y be a point of Z'. The Jacobian $J(\delta(y))$ of the equation $(q)_n$ at $\delta(y)$ is

$$\begin{pmatrix} & & & J(\mathbf{B}) & & & \\ 0 & D(n,\tilde{B}_3) & -D(n,\tilde{B}_2) & 0 & \cdots & \cdots & 0 \\ 0 & D(n,\tilde{B}_4) & 0 & -D(n,\tilde{B}_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & 0 \\ 0 & D(n,\tilde{B}_n) & 0 & \cdots & \cdots & 0 & -D(n,\tilde{B}_2) \\ 0 & 0 & D(n,\tilde{B}_4) & -D(n,\tilde{B}_3) & 0 & \cdots & 0 \\ \vdots & \vdots & & \cdots & & \cdots & \\ 0 & 0 & \cdots & \cdots & 0 & D(n,\tilde{B}_{n-1}) & -D(n,\tilde{B}_n) \end{pmatrix}$$

(Note that $\overline{F}^{ij}_{k\ell}(n,\mathbf{A}) = F^{ij}_{uv}(n,\mathbf{A})$ for some u, v and $\overline{F}^{ij}_{m}(n,\mathbf{A}) = F^{ij}_{m'}(n,\mathbf{A})$). Since y is in $R(n-1)$ the lower block of the above matrix is of rank $5(n-1)(n-2)$. On the other hand, on the columns we picked up to make the minor $\widehat{J(\mathbf{B})}$ the lower block has only 0. Thus we see that $\operatorname{rank}(J(\delta(y))) = 5n(n-1)$, which means that $Q(n)^{ss}$ is smooth at $\delta(y)$.

The actions of $GL(n-1,\mathbf{R})$ on $\mathcal{W}_0$ and $Sp(n) = Sp_{\mathbf{C}}(n) \cap U(2n)$ on $\mathcal{U}$ (here $\dim \mathcal{W}_0 = n-1$ and $\dim \mathcal{U} = 2n$) induce an action of

$$G = (PGL(n-1,\mathbf{R}) \times Sp_{\mathbf{R}}(n))/\{-1,+1\}$$

on $R(n-1)$ (see [10, §6]). As Verdier proved, $R(n-1)$ is a principal fiber bundle over the moduli space of instanton bundles with group G. Since the moduli space of instanton bundles with fixed instanton number is connected ([9,3]) and since G has at most two connected components, $R(n-1)$ has at most two connected components and the action of a suitable element g of G transforms one component to another. Now put $R'(n-1) = Z' \cup gZ'$. Since $R(n-1)$ is smooth, the complement of $R'(n-1)$ in $R(n-1)$ is a thin set and then $R'(n-1)$ is dense in $R(n-1)$. This completes the proof of our claim.

Lemma 10.2. *Let ψ be an element of $Q_0(n)$ which gives rise to a point of $Q(n)^{ss} - R(n)$. Then, after changing frames of $\mathcal{V}$, $\mathcal{W}_0$ and $\mathcal{U}$, there is a $\mathbf{B} = (B_2,\ldots,B_n)$ which represents a point of $Q(n-1)^{ss}$ such that ψ is represented by $\delta(\mathbf{B})$.*

Proof. If we choose a suitable frames of $\mathcal{V}$ and $\mathcal{W}_0$, then ψ is represented by $\mathbf{A} = (A_1,\ldots,A_n)$ such that for all $2 \le i \le n$, ${}^tA_1J_1A_i = 0$ and that

$$A_1 = \begin{pmatrix} {}^t\mathbf{a}_0 & {}^t\mathbf{a}_1 & 0 & 0 \\ {}^t\mathbf{a}_1 & -{}^t\bar{\mathbf{a}}_0 & 0 & 0 \end{pmatrix},$$

where the notation is the same as in the proof of the above claim. Set

$$\begin{aligned} x_1 &= \sum_{i=1}^{n+1} a_{0i}u_i + \sum_{i=1}^{n+1} \bar{a}_{1i}u_{n+1+i} \\ x_2 &= \sum_{i=1}^{n+1} a_{1i}u_i - \sum_{i=1}^{n+1} \bar{a}_{0i}u_{n+1+i}. \end{aligned}$$

Then we have that $x_2 = j_2(x_1)$. Since $A_1 \neq 0$, $x_i \neq 0$ and hence $(x_1,x_2) = \langle x_1,x_1\rangle \neq 0$. Moreover, we see that $\langle x_1,x_2\rangle = (x_1,j_2(x_2)) = (x_1,-x_1) = 0$. Thus, putting

$$\begin{aligned} y_1 &= \frac{x_1 + x_2}{\sqrt{2\langle x_1,x_1\rangle}} \\ y_{n+2} &= \frac{x_1 - x_2}{\sqrt{2\langle x_1,x_1\rangle}}, \end{aligned}$$

we have that $\langle y_1, y_{n+1}\rangle = -1$, $\langle y_1, y_1\rangle = 1$, $\langle y_{n+1}, y_{n+1}\rangle = 1$ and $\langle y_1, y_{n+1}\rangle = 0$. Now let $\mathcal{U}'$ be the orthogonal complement of the vector subspace spanned by x_1 and x_2 with respect to $(*,*)$. Then it is clear that $j_2(\mathcal{U}') = \mathcal{U}'$ and j_1 induces a non-degenerate alternating form on $\mathcal{U}'$. It is easy to see that we can find a basis $\{y_2, \ldots, y_{n+1}, y_{n+3}, \ldots y_{2n+2}\}$ of $\mathcal{U}'$ which is symplectic with respect to $j_1|_{\mathcal{U}'}$ and unitary with respect to $(*,*)$. Then $\{y_1, \ldots, y_{2n+2}\}$ is a basis of $\mathcal{U}$ which is symplectic and unitary. If we write

$$u_i = \sum_{j=1}^{2n+2} g_{ji} y_j$$

then $g = (g_{ij})$ is an element of $Sp(n+1) = Sp_{\mathbf{C}}(n+1) \cap U(2n+2)$ and $g\mathbf{A} = (gA_1, \ldots, gA_n)$ represents ψ with respect to the new basis. If we replace the first member e_1 of the basis of W_0 by $\sqrt{\frac{2}{(x_1,x_1)}}\, e_1$, then

$$gA_1 = \begin{pmatrix} 1\ 1\ 0\ 0 \\ 0 \\ 1\ -1\ 0\ 0 \\ 0 \end{pmatrix}.$$

Since ${}^t(gA_1)J_1 gA_i = {}^tA_1 J_1 A_i = 0$, we have that for all $2 \le i \le n$, the first and $(n+2)$-th rows of gA_i are 0. By the fact that g is in $Sp(n)$, $g\mathbf{A}$ satisfies the equations (INT 1′) and (INT 2′). This completes the proof of our lemma.

Let us go back to the proof of Theorem 10.1. Pick an $\mathbf{A} = (A_1, \ldots, A_n)$ which represents a point of $Q(n)^{ss} - R(n)$. Then, by the above lemma, we may assume that $\mathbf{A} = \delta(\mathbf{B})$. If $\mathbf{B}$ is not contained in $R'(n-1)$, then it is the limit of a sequence of points of $R'(n-1)$ by induction and by the fact that $R'(n-1)$ is dense in $R(n-1)$. Thus we have only to prove that for $\mathbf{B} \in R'(n-1)$, $\delta(\mathbf{B}) = \mathbf{A}$ is the limit of a sequence of points of $R(n-1)$. In this case, $\mathbf{A}$ represents a smooth point in $Q(n)^{ss}$. If we note that the dimension of $Q(n)^{ss} - R(n)$ is less than $\dim R(n)$ and that $\dim_{\mathbf{A}} Q(n)^{ss} = \dim R(n)$, we see that our assertion is obvious. Q.E.D.

Combining the above theorem and Proposition 9.10, we get the following.

Theorem 10.3. *$\overline{\overline{I}}(n) = \coprod_{a=0}^{n} I(n-a) \times \Sigma^a(S^4)$ and it is semi-algebraic in $\overline{M}(n, 2n)_{\mathbf{R}}(\mathbf{R})$.*

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