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Supergravity

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Analogues of the Riemannian structure for classical superspaces

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Introduction

The main object in the study of Riemannian geometry is (properties of) the Riemann tensor which, in turn, splits into Weyl tensor, traceless Ricci tensor and scalar curvature. The word "splits" above means that at every point of the Riemannian manifold the space of values of the Riemann tensor constitutes an $O(n)$-module which is the sum of three irreducible components.

More generally, let G be any group, not necessarily $O(n)$. In what follows we recall definition of *G-structure* on a manifold and (the space of) its *structure functions* (SF) which are obstructions to integrability or, in other words, to possibility of flattening the G-structure. Riemannian tensor is an example of SF. Among the most known (or popular of recent) examples of such tensors are:

- an almost *conformal structure*, $G = O(n)\times\mathbb{R}^*$, SF are called the *Weyl tensors*;
- an almost *complex structure*, $G = GL(n;\mathbb{C}) \subset GL(2n;\mathbb{R})$, SF are called the *Nijenhuis tensors*;
- an almost *symplectic structure*, $G = Sp(2n)$, (no accepted name for SF);
- Penrose' *twistor* theory, $G = U(2)\times U(2)$, SF -- *Penrose tensors* -- are called the "*α-forms*" and "*β-forms*".

Remark. The adjective "almost" should always be added until the G-structure under study is proved to be flat, i.e. integrable; by abuse of language people often omit it, we also have this bad habit.

In a very lucid paper [G] Goncharov calculated all structure functions for the classical space, i.e. an irreducible compact Hermitian symmetric space (CHSS); in his examples G is the reductive part of the stabilizer of a point of the space. He did not, however, write down the highest weights of irreducible components of SF; we do it here and interpret some of these calculations in [LPS1].

In what follows we expose some of the results of calculations of SF for classical superspaces (for definition see [S] or [L2]); the first to be served are analogues of CHSSs and we will stretch the analogy as far as we can.

The problem was raised in [L2], where some examples were indicated as being of particular interest. The theorems in the main text continue summary of about five year long tedious and labourious calculations (partly announced in [P1, P2, P3]) due to Poletaeva, who on the way corrected some statements and conjectures of [L2], [L4]. We will show that supermanifold theory naturally hints to widen the usual approach to SF in order to embrace at least the following cases:

- infinite dimensional generalizations of Riemannian geometry connected with string theories of phisicists (these infinite dimensional examples have no analogues on manifolds because they require no less then **three** odd coordinates of the superstring); our "Einstein equations" even for finite dimensional G are not what is known as *supergravity*: the corresponding G-structures are different, see [LPS1], mathematically these structures look most natural;
- the G-structures of N-extended Minkowski superspace: the tangent space to the Minkovski superspace for $N\neq 0$ is naturally endowed with a (2-step) nilpotent Lie superalgebra structure that highly resembles the contact structure on a manifold. We start studying such structures in earnest in [LPS2].

0.1.Preliminaries.

0.1.1. Structure functions. Let us retell some of Goncharov's results ([G]) and recall definitions ([St]).

Let M be a (smooth, i.e. of class C^∞) manifold of dimension n over a field $\mathbb{K}$ which in this section is either $\mathbb{R}$ or $\mathbb{C}$. Let $F(M)$ be the frame bundle over M, i.e. the canonical principal $GL(n;\mathbb{K})$-bundle. Let $G\subset GL(n;\mathbb{K})$ be a Lie group. A *G-structure* on M is reduction of the frame bundle to the principal G-bundle corresponding to the inclusion $G\subset GL(n;\mathbb{K})$, i.e. a *G-structure* is the possibility to select transition functions so that their values belong to G.

The simplest G-structure is the *flat* G-structure defined as follows. Let V be $\mathbb{K}^n$ with a fixed frame. Consider the bundle over V whose fiber over $v\in V$ consists of all frames obtained from the fixed one under the G-action, V being identified with T_vV.

Obstructions to identification of the k-th infinitesimal neighbourhood of a point $m\in M$ on a manifold M with G-structure and that of a point of the flat manifold V with the above G-structure are called *structure functions of order k*. Such an identification is possible provided all structure functions (who will be shortly refered to as SF) of lesser orders vanish. By abuse of language the space of structure functions will also be called SF.

Proposition. ([St]). *SF of order k are elements from the space of* (k,2)-*th Spencer cohomology.*

Recall the definition of the Spencer cochain complex. Let S^i denote the operator of the i-th symmetric power. Set $\mathfrak{g}_{-1} = T_mM$, $\mathfrak{g}_0 = \mathfrak{g} = Lie(G)$ and for $i > 0$ put:

$$\mathfrak{g}_i = \{X\in Hom(\mathfrak{g}_{-1},\mathfrak{g}_{i-1}):\ X(v)(w,...) = X(w)(v,...) \text{ for any } v,w \in\mathfrak{g}_{-1}\}$$
$$= S^i(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_0 \cap S^{i+1}(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_{-1}$$

and set $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \oplus_{i\geq -1}\mathfrak{g}_i$.

Suppose that

the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. (0.1.1)

Then, clearly, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*\subset \mathfrak{vect}(n) = \mathfrak{der}\ \mathbb{K}[[x_1,..., x_n]]$, where $n = \dim \mathfrak{g}_{-1}$. It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces a Lie algebra structure on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. The Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$, usually abbreviated to $\mathfrak{g}_*$, will be called *Cartan's prolong* (the result of *Cartan prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

Let E^i be the operator of the i-th exterior power; set $C^{k,s}_{\mathfrak{g}_*} = \mathfrak{g}_{k-s}\otimes E^s(\mathfrak{g}_{-1}^*)$; usually we drop the subscript or at least indicate only $\mathfrak{g}_0$. Define the differential ∂_s: $C^{k,s} \to C^{k-1,s+1}$ setting for any $v_1, ..., v_{s+1}\in V$ (as usual, the slot with the hatted variable is ignored):

$$(\partial_s f)(v_1, ..., v_{s+1}) = \Sigma(-1)^i f(v_1, ..., \hat{v}_{s+1-i}, ..., v_{s+1})(v_{s+1-i})$$

As expected, $\partial_s\partial_{s+1} = 0$, and the homology of this complex is called *Spencer cohomology* of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

0.1.2. Case of simple $\mathfrak{g}_*$ over $\mathbb{C}$. The following remarkable fact, though known to experts, is seldom formulated explicitely:

Proposition. *Let* $\mathbb{K} = \mathbb{C}$, $\mathfrak{g}_* = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ *be simple. Then only the following cases are possible:*

1) $\mathfrak{g}_2 \neq 0$ *and then* $\mathfrak{g}_*$ *is either* $\mathfrak{vect}(n)$ *or its special subalgebra* $\mathfrak{svect}(n)$ *of divergence-free vector fields, or its subalgebra* $\mathfrak{h}(2n)$ *of hamiltonian fields;*

2) $\mathfrak{g}_2 = 0$, $\mathfrak{g}_1 \neq 0$ *then* $\mathfrak{g}_*$ *is the Lie algebra of the complex Lie group of automorphisms of a CHSS* (see above).

Proposition explains the reason of imposing the restriction (0.1.1) if we wish $\mathfrak{g}_*$ to be simple. Otherwise, or on supermanifolds, where the analogue of Proposition does not imply similar restriction, we have to (and do) broaden the notion of Cartan prolong to be able to get rid of restriction (0.1.1).

When $\mathfrak{g}_*$ is a simple finite-dimensional Lie algebra over $\mathbb{C}$ computation of structure functions becomes an easy corollary of the Borel-Weyl-Bott-... (BWB) theorem, cf. [G]. Indeed, by definition $\oplus_k H^{k,2}\mathfrak{g}_* = H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ and the BWB theorem implies that, as $\mathfrak{g}$-module, $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ has as many components as $H^2(\mathfrak{g}_{-1})$ which, thanks to commutativity of $\mathfrak{g}_{-1}$, is just $E^2(\mathfrak{g}_{-1})$; their highest weights, as explained in [G], are also not difficult to deduce from the theorem, however, [G] lacks this deduction so we will give it here. Remarkably, in case 2) of Proposition SF are also of order 1 except for one case.

Let us also immediately calculate SF corresponding to case 1) of Proposition: we did not find these calculations in the literature.

In what follows $R(\Sigma a_i\pi_i)$ denotes the irreducible $\mathfrak{g}_0$-module with highest weight $\Sigma a_i\pi_i$, where π_i is the i-th fundamental weight; we will denote it sometimes by its numerical labels $R(\Sigma a_i; a)$ the highest weight with respect to the center of $\mathfrak{g}_0$ stands after semicolon, cf.[VO], Reference Chapter.

Theorem. 1)(Serre [St]). *In case 1) of Proposition structure functions can only be of order 1.*

a) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = 0$ *for* $\mathfrak{g}_* = \mathfrak{vect}(n)$ *and* $\mathfrak{svect}(m)$, m>2;

b) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = R(\pi_3) \oplus R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(2n)$, n>1;

$H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(2)$.

2)(Goncharov [G]). *SF of G-structures of classical CHSSs are of order 1 except for* $G = CO(3)$ *and their weights are* (recall that $Q_4 = Gr_2^4$):

CHSS	$\mathbb{P}^n$	Gr_m^{m+n}
weight of SF	0	$R(2, 0,...,0, -1) \oplus R(1, 0,...,0, -1, -1) \oplus R(1, 1, 0,...,0, -1) \oplus R(1, 0,...,0, -2)$
OGr_m	LGr_m	Q_n, n>4
$R(1,1, 0,...,0, -1,-1,-2)$	$R(2, 0,...,0, -1,-3)$	$R(2\pi_1+2\pi_2)$

0.1.3. SF for reduced structures. In [G] Goncharov considered (generalized) conformal structures. Structure functions for the corresponding (generalized) Riemannian structures, i.e. when $\mathfrak{g}_0$ is the semisimple part $^\wedge\mathfrak{g}$ of $\mathfrak{g}$ = Lie (G) for the cases considered by Goncharov seem to be more difficult to compute because in these cases $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}_- \oplus \mathfrak{g}_0$ and the BWB-theorem does not work. Having computed them, however, we get as a reward more SFs and, consequently, more intricate geometry.

Since $^\wedge\mathfrak{g}_1 = 0$, all we should worry about are SF of orders 1 and 2. The following statement is a direct corollary of definitions.

Proposition ([G], Th.4.7). For $^\wedge\mathfrak{g}$ and $\mathfrak{g}$ SF of order 1 are the same and SF of order 2 for $^\wedge\mathfrak{g}$ are $S^2(\mathfrak{g}_1) = S^2(\mathfrak{g}_{-1}^*)$.

Example: Riemannian geometry. Let G = O(n). In this case $\mathfrak{g}_1 = \mathfrak{g}_{-1}$ and in $S^2(\mathfrak{g}_{-1})$ a 1-dimensional subspace is distinguished; the sections through this subspace constitute a Riemannian metric g on M. (The habitual way to determine a metric on M is via a symmetric matrix, but actually this is just one scalar matrix-valued function, not n(n+1)/2-dimensional space of functions.) The values of the Riemannian tensor at a point of M constitute an O(n)-module $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ which contains a trivial component whose arbitrary section will be denoted by R. What is important, this trivial component is naturally realised as a submodule in a module isomorphic to $S^2(\mathfrak{g}_{-1})$. Thus, we have two matrix-valued functions: g and R each being a section of the trivial $\mathfrak{g}_0$-module. What is more natural than to equate them (up to a constant factor)?

$$R = \lambda\ g, \text{ where } \lambda \in \mathbb{R}. \qquad (0.1.3)$$

Let now R correspond to the Levi-Civita connection; the process of restoring R from g involves differentiations thus making (0.1.3) into *Einstein equation* (in vacuum and with cosmological term λ), a nonlinear pde.

A generalization of this example to G-structures associated with other CHSSs and to supermanifolds is considered in [LPS1].

0.1.4. SF for contact structures: Shchepochkina* prolongs. In heading a) of Proposition 0.1.2 are listed all simple Lie algebras of (polynomial or formal) vector fields except those that preserve a contact structure. Recall that a *contact structure* is a maximally nonintegrable distribution of codimension 1, cf. [A]. To consider contact structures we have to generalize slightly the notion of Cartan prolongation: the tangent space to a point of a manifold with a contact structure possesses a natural structure of a nilpotent Lie algebra (Heisenberg algebra).

This case is very attractive from supermanifold point of view because the tangent space to the N-extended Minkovski superspace is naturally endowed with a 2-step nilpotent Lie superalgebra $\mathfrak{g}_- = \oplus_{-1\geq i\geq -2}\mathfrak{g}_i$ structure with dim $\mathfrak{g}_{-1}$ = 4Nε, dim $\mathfrak{g}_{-2}$ = 4. Since for the Lie (super) algebra of contact vector fields dim $\mathfrak{g}_{-2}$ = 1, it is easier to start with contact structures.

In general, given a nilpotent Lie algebra $\mathfrak{g}_- = \oplus_{0>i\geq -d}\mathfrak{g}_i$ and a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{der}\ \mathfrak{g}_-$ which preserves $\mathbb{Z}$-grading of $\mathfrak{g}_-$, define its i-th *Shchepochkina prolong* for i > 0 to be:

* First considered by I. Shchepochkina in [Sh]

$$\mathfrak{g}_i = (S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0 \cap S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-)_i,$$

where the subscript singles out the component of degree i. Similarly to the above, define $\mathfrak{g}_*$, or rather, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, as $\oplus_{i \geq -d} \mathfrak{g}_i$; then, by the same reasons as in 0.1.1, $\mathfrak{g}_*$ is a Lie algebra (subalgebra of $\mathfrak{l}(\dim \mathfrak{g}_-)$ for d = 2) and $H^2(\mathfrak{g}_-; \mathfrak{g}_*)$ is well-defined.

The space $H^2(\mathfrak{g}_-; \mathfrak{g}_*)$ is the space of obstructions to flatness. It naturally splits into homogeneous components whose degree corresponds to what we will call the *order* of SF; in general case the minimal order of SF is 2-d. When d > 1 there is no clear correspondence between the order of SF and the number of the infinitesimal neighbourhood of a point of a supermanifold with the flat G-structure.

Example. Let $\mathfrak{g} = \mathfrak{csp}(2n)$, $\mathfrak{g}_{-1} = R(\pi_1; 1)$, $\mathfrak{g}_{-2} = R(0)$; then $\mathfrak{g}_* = \mathfrak{l}(2n+1)$ and

$$C^{k,s}_{\mathfrak{g}_*} = \mathfrak{g}_{k-2} \otimes E^s(\mathfrak{g}_{-1}^*) \oplus \mathfrak{g}_{k-3} \otimes E^{s-1}(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-2}^*.$$

The number k here is the order of SF.

Theorem. *For $\mathfrak{g}_* = \mathfrak{l}(2n+1)$ all SF vanish.*

0.1.5. SF for projective structures. It is also interesting sometimes to calculate $\oplus_k H^{k,2}(\mathfrak{g}_-; \mathfrak{h})$ for some $\mathbb{Z}$-graded subalgebras $\mathfrak{h} \subset \mathfrak{g}_*$, such that $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$. For example, for $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{g}_{-1}$ its standard (identity) representation we have $\mathfrak{g}_* = \mathfrak{vect}(n)$ and as we have seen all SF vanish; but if $\mathfrak{h} = \mathfrak{sl}(n+1) \subset \mathfrak{vect}(n)$ then the corresponding SF are nonzero and provide us with obstructions to integrability of what is called *projective connection.*

Theorem. 1) *Let $\mathfrak{g}_* = \mathfrak{vect}(n)$, $\mathfrak{h} = \mathfrak{sl}(n+1)$. Then*

SF of order 1 and 2 vanish, SF of order 3 are R(2,1, 0, ..., 0, -1)

2)*Let $\mathfrak{g}_* = \mathfrak{l}(2n+1)$, $\mathfrak{h} = \mathfrak{sp}(2n+2)$. Then SF are $R(\pi_1+\pi_2; 3)$ of order 3.*

0.1.6. Case of simple $\mathfrak{g}_*$ over $\mathbb{R}$.

Example: Nijenhuis tensor. Let $\mathfrak{g}_0 = \mathfrak{gl}(n) \subset \mathfrak{gl}(2n;\mathbb{R})$, $\mathfrak{g}_{-1}$ is the identity module. In this case $\mathfrak{g}_* = \mathfrak{vect}(n)$, however, in seeming contradiction with Theorem 0.1.2 the SF are nonzero. The reason is that now we consider not $\mathbb{C}$-linear maps but $\mathbb{R}$-linear ones.

Theorem. *Nonvanishing SF are of order 1 and constitute the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1} \otimes_{\mathbb{C}} E^2_{\mathbb{R}}(\mathfrak{g}_{-1}^*)$, where $g(cv) = \bar{c}gv$ for $c \in \mathbb{C}$, $g \in \mathfrak{gl}(n)$, $v \in V$ and a $\mathfrak{gl}(n)$-module V.*

0.2. SF on supermanifolds: Plan of campaign.

The necessary background on Lie superalgebras and supermanifolds is gathered in a condenced form in [L5]. The above definitions are generalized to Lie superalgebras via Sign Rule.

One of the slogans we are guided is "*simple $\mathbb{Z}$-graded Lie superagebras of finite growth* (SZGLSAFGs)*are as good as simple finite-dimensional Lie algebras*", there should be similar results for either. On the strength of arguments of 0.1 we shall

- list $\mathbb{Z}$-gradings of SZGLSAFGs of depth 1 and 2 (recall that a $\mathbb{Z}$-graded Lie (super)algebra of the form $\oplus_{-d \leq i \leq k} \mathfrak{g}_i$ is said to be of *depth* d and *length* k, here d, k >0.); (for the known SZGLSAFGs and d = 1 this is done in [LSV]). We should also explore the cases associated with $\mathbb{Z}$-grading of the form $\oplus_{|i| \leq 1} \mathfrak{g}_i$ of Kac-Moody (twisted loop) superalgebras.

Remarkably, there are not only "trivial" analogues of CHSS, the spaces of loops with values in a finite-dimensional CHSS, but associated with twisted loop algebras and superalgebras.

- formulate analogues of Theorems 0.1.2 and BWB for SZGLSAFGs (otherwise we will have to continue calculate everything with bare hands and there is practically nothing humainly computable left).

Can programmers help? Most part of the calculations omitted here is a par for a fast computer, especially to formulate conjectures. Now, when only cases impossible to tackle with bare hands are left, we desperately need solution to the following problem, cf. [F] and [LP]:

Problem. *Write a program for calculating (co)homology of a Lie (super)algebra $\mathfrak{g}$ with coefficients in any $\mathfrak{g}$-module .*

- calculate projective-like and reduced structures for the above and then go through the list of real forms.

Some nontrivial points in what follows are:

- Cartan prolongs of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and of $(\Pi\mathfrak{g}_{-1}, \mathfrak{g}_0)$ are essentially different;

- faithfulness of $\mathfrak{g}_0$-actions on $\mathfrak{g}_{-1}$ is violated in natural examples:

a) Grassmannians of subsuperspaces in an (n,n)-dimensional superspace when the center $\mathfrak{z}$ of $\mathfrak{g}_0$ acts trivially; retain the same definition of Cartan prolongation; the prolong is the semidirect sum $(\mathfrak{g}_{-1}, \mathfrak{g}_0/\mathfrak{z})_* \ltimes S^*(\mathfrak{g}_{-1}^*)$ with the natural $\mathbb{Z}$-grading and Lie superalgebra structure; notice that the prolong is *not* subalgebra of $\mathfrak{vect}(\dim \mathfrak{g}_{-1})$;

b) the structure preserving the exterior differential. More precisely recall, that, supermanifolds, the good counterpart of differential forms on manifolds *pseudodifferential and pseudointegrable forms. Pseudodifferential forms* on supermanifold X are functions on the supermanifold X' associated with the bundle τ obtained from the cotangent one by fiber-wise change of parity. *Differential forms* on X fiber-wise *polynomial* functions on X'. (In particular, if X is a manifold there are pseudodifferential forms.) The *exterior differential* on X is now considered as an odd ve field d on X'. Let $x = (u_1, ..., u_p, \xi_1, ..., \xi_q)$ be local coordinates on X, $x_i' = \pi(x_i)$. Then $\Sigma x_i' \partial/\partial x_i$ is the familiar coordinate expression of d. The Lie superalgebra $\mathfrak{C}(d)$ $\mathfrak{vect}(m+n/m+n)$, where (m/n) = dim X, -- the Lie superalgebra of vector fields preserving field d on X' (see definition of the Nijenhuis operator P_4 in [LKW]) -- is neither simple transitive and therefore did not draw much atention so far. Still, the corresponding structure ($\mathfrak{C}(d) = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$, where $\mathfrak{g}_0 = \mathfrak{gl}(k) \ltimes \Pi(\mathfrak{gl}(k))$ and where $\Pi(\mathfrak{gl}(k)$ is abelean a constitutes the kernel of the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ = id, the standard (identity) representation $\mathfrak{gl}(k)$) is interesting and natural. Let us call it the *exterior differential structure*; as we see, it is always integrable (like projective structure).

Theorem. *Structure functions of the exterior differential structure are 0.*

Digression. An interesting counterpart of the exterior differential structure is the odd version of the hamiltonian structure. *Pseudointegrable forms* on a supermanifold X are functions on the supermanifold 'X associated with the bundle τX obtained from the *tangent* one by fiber-wise change of parity. Fiber-wise *polynomial* functions on 'X are called polyvector fields on X. (In particular, if X is a manifold there are no pseudointegrable forms.) The *exterior differential* on X is now considered as an odd nondegenerate (as a bilinear form) bivector field div on X'. Let $x = (u_1, ..., u_p, \xi_1, ..., \xi_q)$ be local coordinates on X, $'x_i = \pi(\partial/\partial x_i)$. Then div $= \Sigma \partial^2/\partial x_i' \partial x_i$ is the coordinate expression of the Fourier transform of the exterior differential d with respect to primed variables. The Lie superalgebra $\mathfrak{aut}$(div) is isomorphic to the Lie superalgebra $\mathfrak{le}(m+n)$ which is the simple subalgebra of $\mathfrak{vect}(n+m/n+m)$ that preserves a nondegenerate odd differential 2-form $\omega = \Sigma dx_i' dx_i$; an interesting algebra is the superalgebra $\mathfrak{sle}(m+n)$ which preserves both div and ω; for both of these Lie superalgebras and their deformations the corresponding SF are calculated in [P1] and [LPS1].

Note that $\mathfrak{C}(d)$ is not even transitive; on manifolds we are accustomed to disregard such structures;

- formulation of Serre's theorem (see above) fails to be true for superalgebras; counterexamples are superalgebras of series $\mathfrak{svect}$, see below, and $\mathfrak{sle}$.

[LPS1] and [LPS2] together with this paper constitute an outcome of the first 5-year part of this plan. In these texts we deal with linear algebra: at a point; global geometry, practically not investigated, is nontrivial, cf. the review [MV].

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Terminological conventions. 1)The $\mathfrak{g}$ - module V with the highest weight ξ and *even* highest vector will be denoted by V_ξ or R(ξ).

2)Let $c\mathfrak{g}$ denote the trivial central "extent" (the result of the extention) of a Lie (super)algebra $\mathfrak{g}$, whereas $\mathfrak{p}$ stands for projectivization (as in $\mathfrak{psl}$, $\mathfrak{pq}$) and $\mathfrak{s}$ for "trace"less part (as in $\mathfrak{sl}$, $\mathfrak{sq}$, $\mathfrak{sb}$).

1.Spencer cohomology of $\mathfrak{psq}(n)$.

Before we proceed, recall that all $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{sl}(m)$ are of the form $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_1 = \mathfrak{g}_{-1}^*$, with $\mathfrak{g}_0 = c(\mathfrak{sl}(p)\oplus\mathfrak{sl}(m-p))$. As $\mathfrak{g}_0$-module, $H^2(\mathfrak{g}_{-1};\mathfrak{g}_*)$ has two components if $(p-1)(m-p-1) \neq 0$ and vanish otherwise.

The Spencer cohomology of the Lie superalgebra $\mathfrak{g} = \mathfrak{psq}(n)$, i.e. SF for the Queergrassmannians, resemble that of $\mathfrak{sl}(n)$ much more than that of $\mathfrak{sl}(m/n)$. In fact, the structure of SF for the "usual" superGrassmannian ($\mathfrak{g} = \mathfrak{sl}(m/n)$) is so complicated that just to list the answer with all particular cases takes as much space as the whole of this paper, see [P4].

Proposition ([K]). *A) All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{psq}(n)$ are of the form $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ with $\mathfrak{g}_1 = \mathfrak{g}_{-1}^*$ and $\mathfrak{g}_0 = c\mathfrak{ps}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))$, $p(n-p) \neq 0$, whereas $\mathfrak{g}_{-1}$ is one of the two irreducible $\mathfrak{g}_0$-modules in $id_p\otimes id_{n-p}^*$, where id_k denotes the standard (identity) representation of the "summand" of $\mathfrak{g}_0$ isomorphic to $\mathfrak{q}(k)$, explicitely :*

$\mathfrak{g}_{-1} = <(x\pm\pi(x))\otimes(y\pm\pi(y))$, where $x\in id_p$, $y\in id_{n-p}^*$;

B)$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$.

Theorem. *$H^{1,2}\mathfrak{g}_0 = V_{2\varepsilon_1+\varepsilon_p+\delta_1-2\delta_{n-p}}\oplus V_{\varepsilon_1+\delta_{n-p}}$; other SF vanish.*

2.Spencer cohomology of $\mathfrak{osp}(m/n)$,

2.1. $\mathbb{Z}$-gradings of depth 1. All these gradings are of the form $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1}^*$.

Proposition ([K] and [LSV]). *For $\mathfrak{osp}(m/2n)$ the following values of $\mathfrak{g}_0$ are possible for the $\mathbb{Z}$-gradings of depth 1:*

a) $\mathfrak{cosp}(m-2/2n)$ with $\mathfrak{g}_{-1} = id$;

b) $\mathfrak{gl}(r/n)$ if $m = 2r$ with $\mathfrak{g}_{-1} = E^2(id)$.

2.2. Cartan prolongs of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and $(\mathfrak{g}_{-1}, {}^\wedge\mathfrak{g}_0)$.

Proposition. *1)$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$ except for the case 2.1.b) for $r = 3$, $n = 0$ when $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(3/0)$.*

2)$(\mathfrak{g}_{-1}, {}^\wedge\mathfrak{g}_0)_* = \mathfrak{g}_{-1}\oplus {}^\wedge\mathfrak{g}_0$.

2.3. Structure functions.

Theorem. *Cases a) and b) below correspond to cases 2.2 of $\mathbb{Z}$-gradings. For cases $mn = 0$ see [G] and Introduction.*

a)As ${}^\wedge\mathfrak{g}_0$-module, $H^{2,2}{}^\wedge\mathfrak{g}_0 = S^2(\Lambda^2(\mathfrak{g}_{-1}))/\Lambda^4(\mathfrak{g}_{-1})$ and splits into the direct sum of three irreducible components whose weights are given in Table 1, where $m = 2r+2$ or $2r+3$ and $n>0$ (the case $n = 0$ is considered in [G] and Introduction).

As $\mathfrak{g}_0$-module, $H^{2,2}\mathfrak{g}_0 = H^{2,2}{}^\wedge\mathfrak{g}_0/S^2(\mathfrak{g}_{-1})$ and Table 1 also contains the highest weights of irreducible components of $H^{2,2}\mathfrak{g}_0$. For $k\neq 2$ SF vanish.

b)As $\mathfrak{g}_0$-module, $H^2(\mathfrak{g}_{-1};\mathfrak{g}_*)$ is irreducible and their highest weights are given in Table 2 for $r \neq n, n+2, n+3$.

The case $r = 4$, $n = 0$ and $r = 2$, $n = 1$ coinside, respectively, with the cases considered in a) for $\mathfrak{o}(8)$ and $\mathfrak{osp}(4/2)$.

3.Spencer cohomology of $\mathfrak{d}(\alpha)$.

Proposition ([K] and [LSV]). *All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g}$ are listed in Table 1 of [LSV]. For all these gradings $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$.*

Theorem. *For these gradings we have, respectively:*

I)$H^{1,2}\mathfrak{g}_0 = V_{(2\alpha+1)\varepsilon_1+\varepsilon_2}$; other SF vanish.

II)$H^{1,2}\mathfrak{g}_0 = V_{((\alpha+2)/\alpha)\varepsilon_1+\varepsilon_2}$; other SF vanish.

III)$H^{1,2}\mathfrak{g}_0 = V_{((\alpha-1)/(1+\alpha))\varepsilon_1+\varepsilon_2}$; other SF vanish.

4.Spencer cohomology of $\mathfrak{ab}_3$.

Proposition. *The only $\mathbb{Z}$-grading of depth 1 of $\mathfrak{g}$ is listed in Table 1 of [LSV], see also [K].*

Theorem. *For this grading $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$ and all SF vanish except $H^{1,2}\mathfrak{g}_0$ given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules*

$$0 \to X \to H^{1,2} \to V_{\varepsilon_1+2\delta_1} \to 0 \qquad (4.1)$$

where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0 \to \Pi(V_{4\varepsilon_1+2\varepsilon_2+\varepsilon_3}) \to X \to V_{3\varepsilon_1+2\delta_1} \to 0 \qquad (4.2)$$

5. Spencer cohomology of vectory Lie superalgebras in their standard grading.

Theorem (cf. Theorem 0.1.2). *1)* # *For* $\mathfrak{g}_* = \mathfrak{vect}(m/n)$, $\mathfrak{svect}(m/n)$, $\mathfrak{k}(2m+1/n)$ *and* $\mathfrak{m}(r)$ *SF vanish except for* $\mathfrak{svect}(0/n)$ *when SF are of order n and constitute the* $\mathfrak{g}_0$-*module* $\Pi^n(\mathbf{1})$.

2) For $\mathfrak{g}_* = \mathfrak{h}(0/m)$, $m > 3$, *SF are* $\Pi((R(3\phi_1)\oplus R(\phi_1))$.

3) For $\mathfrak{g}_* = \mathfrak{sh}(0/m)$, $m > 3$, *nonzero SF are same as for* $\mathfrak{h}(0/m)$ *plus additionally* $\Pi^n(R(\pi_1))$ *of order* $n-1$.

4) For $\mathfrak{g}_* = \mathfrak{sle}(n)$, $n > 1$, *nonzero SF are* $H^{1,2}\,\mathfrak{spe}(n) = S^3(\mathfrak{g}_{-1}{}^*)$, $H^{2,2}\,\mathfrak{spe}(n) = \Pi(\mathbf{1})$, $H^{n,2}\,\mathfrak{spe}(n) = \Pi^n(\mathbf{1})$.

6. Nonstandard gradings of the Lie superalgebras of hamilton and contact vector fields. For either of these superalgebras, $\mathfrak{g}(m/n)$ (= $\mathfrak{h}(2m/n)$, $\mathfrak{sh}(n)$ or $\mathfrak{k}(2m+1/n)$ for $n>1$) there is one grading of the form $\mathfrak{g} = \oplus_{1\geq i\geq -1}\,\mathfrak{g}_i$ with $\mathfrak{g}_{-1}$ = F(m/n), $\mathfrak{g}_0 = \mathfrak{g}(m/n-2)\oplus F(m/n)$, where F(m/n) is the superspace of "functions" (in our case polynomials or power series) on which $\mathfrak{g}_0$ naturally acts, and $\mathfrak{g}_1 = \mathfrak{g}_{-1}{}^*$.

By incredible effort one of us (E.P.) managed to calculate the case $\mathfrak{g} = \mathfrak{sh}(6)$. We will consider it in [LPS1]. We have no idea how to approach other, especially infinite dimensional cases: the number of irreducible components grows with n and m! The only result to this end is due to Yu. Kochetkov (1985, unpublished) who showed that SF of order 2 do contain a trivial component thus enabling us to write an analogue of Einstein equation for $\mathfrak{g}_0 = \mathfrak{h}(2m/n)$, $\mathfrak{sh}(n)$ or $\mathfrak{k}(2m+1/n)$ for $n>1$.

7. Odd analogues of Nijenhuis tensor: SF for $\mathfrak{q}(n)$. For the even and odd complex structures on supermanifolds SF are implicitely calculated in [W]. Here we calculate them explicitely for the odd structure.

Proposition. *For* $\mathfrak{g}_0 = \mathfrak{q}(m)$ *and* $\mathfrak{g}_{-1} = id$ *we have* $\mathfrak{g}_* = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0$.

Theorem. *For* $m = 1$ *SF vanish.*

For $m = 2$ *the* $\mathfrak{g}_0$-*module* $H^2(\mathfrak{g}_{-1};\mathfrak{g}_*)$ *has two components:* $E^2(id^*)\oplus E^2(id^*)$.

For $m > 2$ *the* $\mathfrak{g}_0$-*module* $H^2(\mathfrak{g}_{-1};\mathfrak{g}_*)$ *has four components:* $E^2(id^*)\oplus E^2(id^*)\oplus E^2(id^*)\oplus E^2(id^*)$.

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Classical superspaces and related structures

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Introduction

The main object in the study of Riemannian geometry is (properties of) the Riemann tensor which, in turn, splits into the Weyl tensor, Ricci tensor and scalar curvature. The word "splits" above means that at every point of the Riemannian manifold M^n the space of values of the Riemann tensor constitutes an O(n)-module which is the sum of three irreducible components (unless $n = 4$ when the Weyl tensor aditionally splits into 2 components).

More genearlly, let G be any group, not necessarily O(n). In what follows we recall definition of ***G-structure*** on a manifold and of (the space of) its *structure functions* (SFs) which are obstructions to integrability or, in other words, to possibility of flattening the G-structure. Riemannian tensor is an example of SF. Among the most known (or popular of recent) examples of G-structures are:

- an almost ***conformal structure***, $G = O(n)\times\mathbb{R}^*$, SF are called the ***Weyl tensors***;
- Penrose' *twistor* theory, $G = SU(2)\times SU(2)\times\mathbb{C}^*$, SF -- the *Penrose tensor* -- splits into 2 components whose sections are called "*α-forms*" and "*β-forms*";
- an almost ***complex structure***, $G = GL(n;\mathbb{C}) \subset GL(2n;\mathbb{R})$, SF is called the *Nijenhuis tensor*;
- an almost ***symplectic structure***, $G = Sp(2n)$, (no accepted name for SF).

The first two examples are examples of a "conformal" structure which preserves a tensor up to a scalar. In several versions of a very lucid paper [G] Goncharov calculated (among other things) all SF for all structures with a simple group of conformal transformations, whose subgroup of *linear* transformations is the reductive part of the stabilizer of a point of the space and is the "G" which determines the G-structure on the manifold. Remarkably, Goncharov's examples correspond precizely to the classical spaces, i.e. irreducible compact Hermitian symmetric spaces (CHSS). Goncharov did not, however, write down the highest weights of irreducible components of SFs; this is done in [LPS1] and some of these calculations are interpreted as leading to generalized Einstein equation.

In this talk we advertize results (mostly due to E.Poletaeva) of calculating SF (and interpretation of them) for classical superspaces who are defined and partly listed in [S] and [L2] (see also [V], containing interesting papers on supergravity and where curved supergrassmannians are introduced). The problem was raised in [L2], cf. [L4], and the above examples are now superized in [P] and [LPS]. The passage to supermanifolds naturally hints to widen the usual approach to SFs in order to embrace at least the following cases:

- 2 types of infinite dimensional generalizations of Riemannian geometry connected with: (1) string theories of physics (these infinite dimensional examples have no analogues on manifolds because they require no less then **three** odd coordinates of the superstring; the list of corresponding hermitian superspaces deduced from [S] is given in [L2]; dual pairs, etc. will be considered elsewhere) and (2) Kac-Moody (super) algebras (see Table 5);
- the G-structures of the N-extended Minkowski superspace: the tangent space to the Minkovski superspace for $N \neq 0$ is naturally endowed with a 2-step nilpotent Lie superalgebra structure that highly resembles the contact structure on a manifold. We start studying such structures in earnest in [LPS2], compare our approach with that of the GIKOS group lead by V.I. Ogievetsky. More generally, we shall calculate SF for the G-structures of the type corresponding to any "flag variety", not just Grassmannians, particular at that, see Table 1.

Elsewhere we will generalize the machinery of Jordan algebras, so useful in the study of geometry of CHSSs [Mc], to the cases we consider (this is Vinel's thesis).

Can programmers help? A good part of the calculations we need are very simple (to calculate cohomology is to solve systems of linear equations [F]). Still, though the number of papers on supergravity is counted by thousands (see reviews in our bibliography, of which [OS3], [WB], [We] are easy to understand) there is remarkably small progress in actual calculations (cf. mathematical papers [Sch], [RSh], [Me]). It is yet unclear what are all supergravities for N>1. The reason to that: the calculations *are* voluminous besides, these calculations also have to be "glued" in an answer and there are no rules for doing so, cf. [P4]. Thus the problem is a challenge for a computer scientist, our calculations, together with [LP1] and [P1-4], illustrate [LP2]. For our cohomology of our infinite dimensional Lie (super)algebras there are NO recipes at all (not even from Feigin-Fuchs nor Roger [FF]).

In this text we deal with linear algebra: at a point. The global geometry, practically not investigated, is nontrivial, cf. [M], [MV].

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Preliminaries

Terminological conventions. 1) A $\mathfrak{g}$-module V with highest weight ξ and *even* highest vector will be denoted by V_ξ or R(ξ). An irreducible module with highest weight $\Sigma a_i\pi_i$, where π_i is the i-th fundamental weight, will be denoted sometimes by its numerical labels R(Σa_i; a) the highest weight with respect to the center of $\mathfrak{g}$ stands after semicolon, cf.[OV], Reference Chapter.

2) Let $c\mathfrak{g}$ denote the trivial central "extent" (the result of the extention) of a Lie (super)algebra $\mathfrak{g}$; let p stand for projectivization (as in $\mathfrak{psl}$, $\mathfrak{pq}$) and $\mathfrak{s}$ for "trace"-less part (as in $\mathfrak{sl}$, $\mathfrak{sq}$, $\mathfrak{sh}$).

0.1. Structure functions. Let us retell some of Goncharov's results ([G]) and recall definitions ([St]).

Let M be a manifold of dimension n over a field $\mathbb{K}$; think $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$). Let F(M) be the frame bundle over M, i.e. the canonical principal GL(n; $\mathbb{K}$)-bundle. Let $G \subset GL(n; \mathbb{K})$ be a Lie group. A *G-structure* on M is reduction of the frame bundle to the principal G-bundle corresponding to inclusion $G \subset GL(n; \mathbb{K})$, i.e. a *G-structure* is the possibility to select transition functions so that their values belong to G.

The simplest G-structure is the *flat* G-structure defined as follows. Let V be $\mathbb{K}^n$ with a fixed frame. Consider the bundle over V whose fiber over $v \in V$ consists of all frames obtained from the fixed one under the G-action, V being identified with T_vV.

Obstructions to identification of the k-th infinitesimal neighbourhood of a point $m \in M$ on a manifold M with G-structure and that of a point of the flat manifold V with the above G-structure are called *structure functions of order k*. Such an identification is possible provided all structure functions of lesser orders vanish.

Proposition. ([St]). *SFs of order k are elements from the space of* (k,2)-*th Spencer cohomology.*

Recall definition of the Spencer cochain complex. Let S^i denote the operator of the i-th symmetric power. Set $\mathfrak{g}_{-1} = T_mM$, $\mathfrak{g}_0 = \mathfrak{g} = \mathrm{Lie}(G)$ and for i > 0 put:

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \oplus_{i \geq -1} \mathfrak{g}_i, \text{ where } \mathfrak{g}_i = \{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}): X(v)(w,...) = X(w)(v,...)$$

for any $v,w \in \mathfrak{g}_{-1}\} = S^i(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0 \cap S^{i+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}$.

Suppose that

the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. (0.1)

Then, clearly, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n) = \mathfrak{der}\ \mathbb{K}[[x_1,..., x_n]]$, where n = dim $\mathfrak{g}_{-1}$. It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces a Lie algebra structure on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. The Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$, usually abbreviated to $\mathfrak{g}_*$, will be called *Cartan's prolong* (the result of *Cartan prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

Let E^i be the operator of the i-th exterior power; set $C^{k,s}_{\mathfrak{g}_*} = \mathfrak{g}_{k-s} \otimes E^s(\mathfrak{g}_{-1}^*)$; usually we drop the subscript or at least indicate only $\mathfrak{g}_0$. Define the differential $\partial_s: C^{k,s} \to C^{k,s+1}$ setting for any $v_1, ..., v_{s+1} \in V$ (as always, the slot with the hatted variable is ignored):

$$(\partial_s f)(v_1, ..., v_{s+1}) = \Sigma(-1)^i [f(v_1, ..., \hat{v}_{s+1-i}, ..., v_{s+1}), v_{s+1-i}]$$

As usual, $\partial_s\partial_{s+1} = 0$, the homology of this complex is called *Spencer cohomology* of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

0.2. Case of simple $\mathfrak{g}_*$ over $\mathbb{C}$. The following remarkable fact, though known to experts, is seldom formulated explicitely:

Proposition. *Let $\mathbb{K} = \mathbb{C}$, $\mathfrak{g}_* = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ be simple. Then only the following cases are possible:*

1) $\mathfrak{g}_2 \neq 0$ and then $\mathfrak{g}_*$ is either $\mathfrak{vect}(n)$ or its special subalgebra $\mathfrak{svect}(n)$ of divergence-free vector fields, or its subalgebra $\mathfrak{h}(2n)$ of hamiltonian fields;

2) $\mathfrak{g}_2 = 0$, $\mathfrak{g}_1 \neq 0$ then $\mathfrak{g}_*$ is the Lie algebra of the complex Lie group of automorphisms of a CHSS (see above).

Proposition explains the reason of imposing the restriction (0.1) if we wish $\mathfrak{g}_*$ to be simple. Otherwise, or on supermanifolds, where the analogue of Proposition does not imply similar restriction, we have to (and do) broaden the notion of Cartan prolong to be able to get rid of restriction (0.1).

When $\mathfrak{g}_*$ is a simple finite-dimensional Lie algebra over $\mathbb{C}$ computation of structure functions becomes an easy corollary of the Borel-Weyl-Bott-... (BWB) theorem, cf. [G]. Indeed, by definition $\oplus_k H^{k,2}_{\mathfrak{g}_*} = H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ and by the BWB theorem $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$, as $\mathfrak{g}$-module, has as many components as $H^2(\mathfrak{g}_{-1})$ which, thanks to commutativity of $\mathfrak{g}_{-1}$, is just $E^2(\mathfrak{g}_{-1})$; the highest weights of these modules, as explained in [G], are also deducible from the theorem. However, [G] pityfully lacks this deduction, see [LP1] and [LPS1] where it is given with interesting interpretations.

Let us also immediately calculate SF corresponding to case 1) of Proposition: we did not find these calculations in the literature. Note that vanishing of SF for $\mathfrak{g}_* = \mathfrak{vect}$ and $\mathfrak{k}$ (see 0.5) follows from the projectivity of $\mathfrak{g}_*$ as $\mathfrak{g}_0$-modules and properties of cohomology of coinduced modules [F]. In what follows $R(\Sigma a_i\pi_i)$ denotes the irreducible $\mathfrak{g}_0$-module. The classical spaces are listed in Table 1 and some of them are bapthized for convenience of further references.

Theorem. 1)(Serre [St]). *In case 1) of Proposition structure functions can only be of order 1.*

a) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = 0$ *for* $\mathfrak{g}_* = \mathfrak{vect}(n)$ *and* $\mathfrak{svect}(m)$, m>2;

b) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = R(\pi_3) \otimes R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(2n)$, n>1;

$H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(2)$.

2)(Goncharov [G]). *SFs of Q_3 are of order 3 and constitute $R(4\pi_1)$. SF for Grassmannian Gr_m^{m+n}* (when neither m nor n is 1, i.e. Gr is not a projective space) *is the direct sum of two components whose weights and orders are as follows:*

Let A = R(2, 0, ..., 0 , -1)⊗R(1, 0, ..., 0, -1, -1), B = R(1, 1, 0, ..., 0 , -1)⊗R(1, 0, ..., 0, -2).

Then if mn ≠ 4 both A and B are of order 1;

if m = 2, n ≠ 2 A is of order 2 and B of order 1;

if n = 2, m ≠ 2 A is of order 1 and B of order 2;

if n = m = 2 both A and B are of order 2.

SF of G-structures of the rest of the classical CHSSs are the following irreducible $\mathfrak{g}_0$-modules whose order is 1 (recall that $Q_4 = Gr_2^4$)*:*

CHSS	$\mathbb{P}^n$	OGr_m	LGr_m	Q_n, n>4
weight of SF	-	$E^2(E^2(V^*))\otimes V$	$E^2(S^2(V^*))\otimes V$	$E^2(V^*)\otimes V$

$E_6/SO(10)\times U(1)$	$E_7/E_6\times U(1)$
$E^2(R(\pi_5)^*))\otimes R(\pi_5)$	$E^2(R(\pi_1)^*))\otimes R(\pi_1)$

0.3. SF for reduced structures. In [G] Goncharov considered conformal structures. SF for the corresponding generalizations of the Riemannian structure, i.e. when $\mathfrak{g}_0$ is the semisimple part $\hat{}\mathfrak{g}$ of $\mathfrak{g}$ = Lie (G), seem to be more difficult to compute because in these cases $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}_-\otimes\mathfrak{g}_0$ and the BWB-theorem does not work. Fortunately, the following statement, a direct corollary of definitions, holds.

Proposition ([G], Th.4.7). For $\mathfrak{g}_0 = \hat{}\mathfrak{g}$ and $\mathfrak{g}$ SF of order 1 are the same and SF of order 2 for $\mathfrak{g}_0 = \hat{}\mathfrak{g}$ are $S^2(\mathfrak{g}_1) = S^2(\mathfrak{g}_{-1}{}^*)$. (There are clearly no SF of order 3 for $\mathfrak{g}_0 = \hat{}\mathfrak{g}$).

Example: Riemannian geometry. Let G = O(n). In this case $\mathfrak{g}_1 = \mathfrak{g}_{-1}$ and in $S^2(\mathfrak{g}_{-1})$ a 1-dimensional subspace is distinguished; the sections through this subspace constitute a Riemannian metric g on M. (The habitual way to determine a metric on M is via a symmetric matrix, but actually this is just one scalar matrix-valued function.) The values of the Riemannian tensor at a point of M constitute an O(n)-module $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ which contains a trivial component whose arbitrary section will be denoted by R. What is important, this trivial component is realised by Proposition as a submodule in $S^2(\mathfrak{g}_{-1})$. Thus, we have two matrix-valued functions: g and R each being a section of the trivial $\mathfrak{g}_0$-module. What is more natural than to require their ratio to be a constant (rather than a function)?

$$R = \lambda\, g, \text{ where } \lambda \in \mathbf{R}. \qquad (EE_0)$$

Recall that the Levi-Civita connection is the unique symmetric affine connection compatible with the metric. Let now ι be the structure function (sum of its components belonging to the distinct irreducible O(n)-modules that constitute $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$) corresponding to the Levi-Civita connection; the process of restoring ι from g involves differentiations thus making (EE_0) into a nonlinear pde. This pde is not Einstein Equation yet. Recall that in adition to the trivial component there is another O(n)-component in $S^2(\mathfrak{g}_{-1})$, the Ricci tensor Ri. *Einstein equations* (in vacuum and with cosmological term λ) are the *two* conditions: (EE_0) and

$$Ri = 0. \qquad (EE_{ric})$$

A generalization of this example to G-structures associated with certain other CHSSs, flag varieties, and to supermanifolds is considered in [LPS1] and [LP3].

0.4. SF of flag varieties. Contact structures. In heading a) of Proposition 0.2 there are listed all simple Lie algebras of (polynomial or formal) vector fields except those that preserve a contact structure. Recall that a *contact structure* is a maximally nonintegrable distribution of codimension 1, cf. [A].

To consider contact Lie algebra we have to generalize the notion of Cartan prolongation: the tangent space to a point of a manifold with a contact structure possesses a natural structure of the Heisenberg algebra. This is a 2-step nilpotent Lie algebra. Let us consider the general case corresponding to "flag varieties" -- quotients of a simple complex Lie group modulo a parabolic subgroup. (The necessity of such a generalization was very urgent in the classification of simple Lie superalgebra, see [Shch] and [L2], where it first appeared, already superized.)

Given an arbitrary (but $\mathbf{Z}$-graded) nilpotent Lie algebra $\mathfrak{g}_- = \oplus_{0>i\geq -d}\,\mathfrak{g}_i$ and a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{der}\ \mathfrak{g}_-$ which preserves $\mathbf{Z}$-grading of $\mathfrak{g}_-$, define the i-th *prolong* of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ for i > 0 to be:

$$\mathfrak{g}_i = (S^*(\mathfrak{g}_-)^*\otimes\mathfrak{g}_0 \cap S^*(\mathfrak{g}_-)^*\otimes\mathfrak{g}_-)_i,$$

where the subscript singles out the component of degree i.Similarly to the above, define $\mathfrak{g}_*$, or rather, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, as $\oplus_{i\geq -d}\,\mathfrak{g}_i$; then, by the same reasons as in 0.1, $\mathfrak{g}_*$ is a Lie algebra (subalgebra of $\mathfrak{f}(\dim \mathfrak{g}_-)$ for d = 2 and dim $\mathfrak{g}_{-2}$= 1) and $H^i(\mathfrak{g}_-; \mathfrak{g}_*)$ is well-defined. $H^i(\mathfrak{g}_-; \mathfrak{g}_*)$ naturally splits into homogeneous components whose degree corresponds to what we will call the *order*. (For the particular case of Lie algebras of depth 2 the obtained bigraded complex was independently and much earlier defined by Tanaka [T] and used in [BS] and [O]. No cohomology was explicitely calculated, however; see calculations in [LPS2] and [LP3].)

The space $H^2(\mathfrak{g}_-; \mathfrak{g}_*)$ is the space of obstructions to flatness. In general case the minimal order of SF is 2-d. For d > 1 we did not establish correspondence between the order of SF and the number of the infinitesimal neighbourhood of a point of a supermanifold with the flat G-structure.

Examples. 1) G* is a simple Lie group, P its parabolic subgroup, G the Levi subgroup of P, $\mathfrak{g}_0$ = Lie(G), $\mathfrak{g}_-$ is the complementary subalgebra to Lie(P) in Lie(G*). The corresponding SF, calculable from the BWB-theorem if $\mathfrak{g}_*$ is finite-dimensional and simple describe for the first time the local geometry of flag varieties other than CHSSs, see [LP3] for details. Here is the simplest example.

2) Let $\mathfrak{g} = \mathfrak{csp}(2n)$, $\mathfrak{g}_{-1} = R(\pi_1; 1)$, $\mathfrak{g}_{-2} = R(0)$; then $\mathfrak{g}_* = \mathfrak{f}(2n+1)$ and

$$C^{k,s}_{\mathfrak{g}_*} = \mathfrak{g}_{k-s}\otimes E^s(\mathfrak{g}_{-1}{}^*)\oplus\mathfrak{g}_{k-s-1}\otimes E^{s-1}(\mathfrak{g}_{-1}{}^*)\otimes\mathfrak{g}_{-2}{}^*.$$

Theorem. *For* $\mathfrak{g}_* = \mathfrak{f}(2n+1)$ *all SF vanish.*

This is a reformulation of the Darboux theorem on a canonical 1-form, actually.

0.5. SF for projective structures. It is also interesting sometimes to calculate $H^2(\mathfrak{g}_-; \mathfrak{h})$ for some $\mathbf{Z}$-graded subalgebras $\mathfrak{h} \subset \mathfrak{g}_*$, such that $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$. For example, if $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{g}_{-1}$ is its standard (identity) representation we have $\mathfrak{g}_* = \mathfrak{vect}(n)$ and, as we have seen, all SF vanish; but if $\mathfrak{h} = \mathfrak{sl}(n+1) \subset \mathfrak{vect}(n)$ then the corresponding SF are nonzero and provide us with obstructions to integrability of what is called the *projective connection.*

Theorem. 1) *Let* $\mathfrak{g}_* = \mathfrak{vect}(n)$, $\mathfrak{h} = \mathfrak{sl}(n+1)$. *Then SF of order 1 and 2 vanish, SF of order 3 are* R(2,1, 0, ..., 0, -1)

2)*Let* $\mathfrak{g}_* = \mathfrak{f}(2n+1)$, $\mathfrak{h} = \mathfrak{sp}(2n+2)$. *Then SF are* $R(\pi_1+\pi_2; 3)$ *of order 3.*

0.6. Case of simple $\mathfrak{g}_*$ over $\mathbf{R}$.

Example: Nijenhuis tensor. Let $\mathfrak{g}_0 = \mathfrak{gl}(n) \subset \mathfrak{gl}(2n;\mathbf{R})$, $\mathfrak{g}_{-1}$ is the identity module. In this case $\mathfrak{g}_* = \mathfrak{vect}(n)$, however, in seeming contradiction with Theorem 0.1.2, the SF are nonzero. There is no contradiction: now we consider not $\mathbf{C}$-linear maps but $\mathbf{R}$-linear ones.

Theorem. *Nonvanishing SF are of order 1 and constitute the* $\mathfrak{g}_0$*-module* $\overline{\mathfrak{g}_{-1}}\otimes_{\mathbf{C}}E^2{}_{\mathbf{R}}(\mathfrak{g}_{-1}{}^*)$, *where* $g(cv) = \overline{c}\,\overline{v}$ *for* $c\in\mathbf{C}$, $g\in\mathfrak{gl}(n)$, $v\in\overline{V}$ *and a* $\mathfrak{gl}(n)$*-module V.*

One of our mottos is: *simple $\mathbb{Z}$-graded Lie superagebras of finite growth* (SZGLSAFGs) *are as good as simple finite-dimensional Lie algebras*; the results obtained for the latter should hold, in some form, for the former. So we calculate

SF on supermanifolds: Plan of campaign

The necessary background on Lie superalgebras and supermanifolds is gathered in a condenced form in [L5], see also [L1, L2]. The above definitions are generalized to Lie superalgebras via Sign Rule.

On the strength of the above examples we must list $\mathbb{Z}$-gradings of SZGLSAFGs of finite depth (recall that a $\mathbb{Z}$-graded Lie (super)algebra of the form $\oplus_{-d\leq i \leq k} \mathfrak{g}_i$ is said to be of *depth* d and *length* k; here d, k >0), calculate projective-like and reduced structures for the above and then go through the list of real forms.

Our theorems are cast in Tables. In Table 1 we set notations. Tables 2 and 3 complement difficult tables of [S]. Table 4 lists all symmetric superspaces of depth 1 of the form G/P with a simple finite-dimensional G. Table 5 lists all hermitian superspaces corresponding to simple loop supergroups different from the obvious examples of loops with values in a hermitian superspace. *Notice that there are 3 series of nonsuper examples.*

We compensate superfluity of exposition by wast bibliography with further results. Let us list some other points of interest in the study of SF on superspaces.

- there is no complete reducibility of the space of SF as $\mathfrak{g}_0$-module;

- Serre's theorem reformulated for superalgebras shows that there are SFs of order >1, see [LPS1];

- faithfulness of $\mathfrak{g}_0$-actions on $\mathfrak{g}_{-1}$ is violated in natural examples of: (a) supergrassmannians of subsuperspaces in an (n,n)-dimensional superspace when the center $\mathfrak{z}$ of $\mathfrak{g}_0$ acts trivially; retain the same definition of Cartan prolongation; the prolong is then the semidirect sum $(\mathfrak{g}_{-1}, \mathfrak{g}_0/\mathfrak{z})_* \ltimes S^*(\mathfrak{g}_{-1}^*)$ with the natural $\mathbb{Z}$-grading and Lie superalgebra structure; notice that the prolong is *not* subalgebra of $\mathfrak{vect}$(dim $\mathfrak{g}_{-1}$); (b) the exterior differential d preserving structure.

More precisely, recall that for supermaifolds the good counterpart of differential forms on manifolds are not differential but rather *pseudodifferential and pseudointegrable forms. Pseudodifferential forms* on a supermanifold X are functions on the supermanifold X' associated with the bundle τ^*X obtained from the cotangent one by fiber-wise change of parity. *Differential forms* on X are fiber-wise *polynomial* functions on X'. In particular, if X is a manifold there are no pseudodifferential forms. The *exterior differential* on X is now considered as an odd vector field d on X'. Let $x = (u_1, ..., u_p, \xi_1, ..., \xi_q)$ be local coordinates on X, $x_i' = \pi(x_i)$. Then $d = \Sigma x_i'\partial/\partial x_i$ is the familiar coordinate expression of d. The Lie superalgebra $\mathfrak{C}(d) \subset \mathfrak{vect}(m+n/m+n)$, where (m/n) = dim X, -- the Lie superalgebra of vector fields preserving the field d on X' (see definition of the Nijenhuis operator P_4 in [LKW]) -- is neither simple nor transitive and therefore did not draw much attention so far. Still, the corresponding G-structure ($\mathfrak{C}(d) = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$, where $\mathfrak{g}_0$ = $\mathfrak{gl}(k) \ltimes \Pi(\mathfrak{gl}(k))$ and where $\Pi(\mathfrak{gl}(k))$ is abelean and constitutes the kernel of the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ = id, the standard (identity) representation of $\mathfrak{gl}(k)$) is interesting and natural. Let us call it the *d-preserving structure.* The following theorem justifies pseudocohomology introduced in [LKW].

Theorem. *SFs of the d-preserving structure are 0.*

An interesting counterpart of the d-preserving structure is the odd version of the hamiltonian structure. In order to describe it recall that *pseudointegrable forms* on a supermanifold X are functions on the supermanifold 'X associated with the bundle τX obtained from the *tangent* one by fiber-wise change of parity. Fiber-wise *polynomial* functions on 'X are called polyvector fields on X. (In particular, if X is a manifold there are no pseudointegrable forms.) The *exterior differential* on X is now considered as an odd nondegenerate (as a bilinear form) bivector field div on X'. Let $x = (u_1, ..., u_p, \xi_1, ..., \xi_q)$ be local coordinates on X, $'x_i = \pi(\partial/\partial x_i)$. Then div $= \Sigma\partial^2/\partial x_i\, '\partial x_i$ is the coordinate expression of the Fourier transform of the exterior differential d with respect to primed variables; the operator is called "div" because it sends a polyvector field on X, i.e. a function on 'X to its divergence. The Lie superalgebra $\mathfrak{aut}$(div) is isomorphic to the Lie superalgebra $\mathfrak{le}(m+n)$ which is the simple subalgebra of $\mathfrak{vect}(n+m|n+m)$ that preserves a nondegenerate odd differential 2-form $\omega = \Sigma dx_i' dx_i$; an interesting algebra is the superalgebra $\mathfrak{sle}(m+n)$ which preserves both div and ω; for both of these Lie superalgebras and their deformations the corresponding SF are calculated in [PS] and [LPS1] .

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Notations in tables. Everywhere we assume the notational conventions of [S] and definitions adopted there without mentioning this specifically. In Table 1 $\mathfrak{g} = (\mathrm{Lie}(S_c))\otimes\mathbb{C}$, NCHSS is an abbreviation for noncompact hermitian symmetric space, in the diagram of $\mathfrak{g}$ the vertex defining the minimal parabolic subalgebra $\mathfrak{p} = \mathrm{Lie}(P)$, such that X can be presented as $(S_c)^{\mathbb{C}}/P$, is shaded. In Table 4 we call a homogeneous space G/P, where G is a simple Lie supergroup P its parabolic subsupergroup corresponding to several omitted generators of a Borel subalgebra (description of these generators can be found in [L3, # 31]), of *depth* d and *length* l if such are the depth and length of Lie (G) in the $\mathbb{Z}$-grading compatible with that of Lie(P). Note that all superspaces of Table 4 possess an hermitian structure (hence are of depth 1) except PeGr (no hermitian structure), PeQ (no structure, length 2), $CGr_{0,k}^{0,n}$ and $SCGr_{0,k}^{0,n}$ (no structure, lengths n-k and, resp. n-k-1)

Table 1. Hermitian symmetric spaces

Name of CHSS X	$X=S_c/G_c$	$\mathfrak{g}_0=(\mathfrak{g}_c)^{\mathbb{C}}$	The diagram of $\mathfrak{g}$	$\mathfrak{g}_{-1}\cong T_0X$	$(S_c)^*$	names of NCHSS
$\mathbb{C}P^n$	$SU(n+1)/U(n)$	$\mathfrak{gl}(n)$		id	$SU(1, n)$	$\mathbb{C}P^n$
Gr_p^{p+q}	$SU(p+q)/S(U(p)\times U(q))$	$\mathfrak{s}(\mathfrak{gl}(p)\oplus\mathfrak{gl}(q))$		$id\otimes id^*$	$SU(p, q)$	$^*Gr_p^{p+q}$
OGr_n	$SO(2n)/U(n)$	$\mathfrak{gl}(n)$		$\Lambda^2 id$	$SO(n, n)$	$^*OGr_n$
Q_n	$SO(n+2)/SO(2)\times SO(n)$	$\mathfrak{co}(n)$		id	$SO(n, 2)$	$^*Q_n$
LGr_n	$Sp(2n)/U(n)$	$\mathfrak{gl}(n)$		$S^2 id$	$Sp(2n;\mathbb{R})$	$^*LGr_n$
$(\mathbb{O}P^2)$	$E_6/SO(10)\times U(1)$	$\mathfrak{co}(10)$			E_6^*	
	$E_7/E_6\times U(1)$	$\mathfrak{ce}_6$			E_7^*	

Occasional isomorphisms: $Gr_p^{p+q}\cong Gr_q^{p+q}$, $Q_1\cong\mathbb{C}P^1$, $Q_3\cong LGr_2$, $Q_2\cong S^2\times S^2$, $OGr_2\cong LGr_1\cong\mathbb{C}P^1$, $OGr_3\cong Gr_3^4$, $Q_4\cong Gr_2^4$.

Table 2. Dual pairs of homogeneous symmetric superspaces

$(\mathfrak{p})\mathfrak{sl}_r(m\|2n)/\mathfrak{osp}(m,p\|2n)$	$(\mathfrak{p})\mathfrak{su}(m,p\|2n,n)/\mathfrak{osp}(m,p\|2n)$
$(\mathfrak{p})\mathfrak{sl}_r(2m\|2n)/(\mathfrak{p}_r)\mathfrak{s}_r\mathfrak{l}(m\|n)$	$(\mathfrak{p})\mathfrak{su}^*(2m\|2n)/(\mathfrak{p}_r)\mathfrak{s}_r\mathfrak{l}(m\|n)$
$\mathfrak{psl}_r(n\|n)/\mathfrak{pq}_r(n)$	$^o\mathfrak{pq}(n)/\mathfrak{pq}_r(n)$
$\mathfrak{psl}_r(n\|n)/\mathfrak{spe}_r(n)$	$\mathfrak{supe}(n)/\mathfrak{spe}_r(n)$
$(\mathfrak{p})\mathfrak{su}(m,p\|n,q)/$	$(\mathfrak{p})\mathfrak{su}(m,p+s-r\|n,v+q)/$
$/(\mathfrak{p})\mathfrak{s}(\mathfrak{u}(r+s,r\|t+v,v)\bullet$	$/(\mathfrak{p})\mathfrak{s}(\mathfrak{u}(r+s,r\|t+v,v)\bullet$
$\bullet\,\mathfrak{u}(m-r-s,p-r\|n-t-v,q-t)$	$\bullet\,\mathfrak{u}(m-r-s,p-r\|n-t-v,q-t)$
$(\mathfrak{p})\mathfrak{su}(2n,m\|2n,2q)/\mathfrak{osp}^*(2m\|2n,2q)$	$(\mathfrak{p})\mathfrak{su}^*(2m\|2n)/\mathfrak{osp}^*(2m\|2n,2q)$
$\mathfrak{psu}(m,p\|n,q)/\mathfrak{puq}(n,p)$	$\mathfrak{supe}(n)/\mathfrak{puq}(n,p)$
$\mathfrak{psu}^*(2n\|2n)/\mathfrak{pq}^*(2n)$	$^o\mathfrak{pq}(n)/\mathfrak{pq}^*(2n)$
$\mathfrak{psu}^*(2n\|2n)/\mathfrak{su}^*(2n)$	$\mathfrak{supe}(2n)/\mathfrak{spe}^*(2n)$
$\mathfrak{osp}(m,p\|2n)/\mathfrak{osp}(s+r,r\|2q)\bullet$	$\mathfrak{osp}(m,p+s-r\|2n)/\mathfrak{osp}(s+r,r\|2q)\bullet$
$\bullet\,\mathfrak{osp}(m-r-s,p-s\|2n-2q)$	$\bullet\,\mathfrak{osp}(m-r-s,p-s\|2n-2q)$
$\mathfrak{osp}(m,p\|2n)/\mathfrak{u}(m/2,p/2\|n,q)$	$\mathfrak{osp}^*(m\|2n,2q)/\mathfrak{u}(m/2,p/2\|n,q)$
$\mathfrak{osp}^*(2m\|2n,2q)/\mathfrak{osp}^*(2p\|2s+2r,2r)\bullet$	$\mathfrak{osp}^*(2m\|2n,2q+2s-2r)/$
$\bullet\,\mathfrak{osp}^*(2m-2p\|2n-2r-2s,2q-2r)$	$/\mathfrak{osp}^*(2p\|2s+2r,2r)\bullet$
	$\bullet\,\mathfrak{osp}^*(2m-2p\|2n-2r-2s,2q-2r)$
$\mathfrak{osp}^*(2m\|2n,n)/\mathfrak{osp}_{\mathbb{C}}(m\|n)$	$\mathfrak{osp}(2m\|2n,n)/\mathfrak{osp}_{\mathbb{C}}(m\|n)$
$\mathfrak{psq}_r(2n)/\mathfrak{p}_r\mathfrak{s}_r\mathfrak{q}(n)$	$\mathfrak{psq}^*(2n)/\mathfrak{p}_r\mathfrak{s}_r\mathfrak{q}(n)$
$\mathfrak{psq}_r(2n)/^0\mathfrak{p}_r\mathfrak{q}(n)$	$\mathfrak{psq}^*(2n)/^0\mathfrak{p}_r\mathfrak{q}(n)$
$\mathfrak{psuq}(m,p)/\mathfrak{ps}(\mathfrak{uq}(r+s,r)\bullet$	$\mathfrak{psuq}(m,p+s-r)/\mathfrak{ps}(\mathfrak{uq}(r+s,r)\bullet$
$\bullet\,\mathfrak{uq}(m-r-s,p-r))$	$\bullet\,\mathfrak{uq}(m-r-s,p-r))$
$\mathfrak{psuq}(m,p)/\mathfrak{pu}(r+s,r\|m-r-s,p-r)$	$\mathfrak{psuq}(m,p+s-r)/\mathfrak{pu}(r+s,r\|m-r-s,p-r)$
$\mathfrak{spe}_r(2n)/\mathfrak{upe}(n)$	$\mathfrak{spe}^*(2n)/\mathfrak{upe}(n)$
$\mathfrak{spe}_r(2n)/\mathfrak{s}_r\mathfrak{pe}(n)$	$\mathfrak{spe}^*(2n)/\mathfrak{s}_r\mathfrak{pe}(n)$
$\mathfrak{sh}(n,p)/Ш\,(k,m,p,n)$	$\mathfrak{sh}(n,p+l-k)/Ш\,(k,m,p,n)$

Table 3. Selfdual homogeneous symmetric superspaces

$(\mathfrak{p})\mathfrak{su}^*(2m|2n)/(\mathfrak{p})\mathfrak{s}(\mathfrak{u}^*(2p|2q)\bullet\mathfrak{u}^*(2m-2p|2n-2q))$;
$(\mathfrak{p})\mathfrak{sl}_r(m|2n)/(\mathfrak{p})\mathfrak{s}(\mathfrak{gl}_r(p|q)\bullet\mathfrak{gl}_r(n-p|n-q))$;
$(\mathfrak{p})\mathfrak{su}(2m,\ m|2n,\ n)/\mathfrak{p}_{im}\mathfrak{s}_{im}\mathfrak{l}(m|n)$; $^o\mathfrak{pg}(n)/\mathfrak{p}(^o\mathfrak{g}(p)\bullet{}^o\mathfrak{g}(n-p))$;
$^o\mathfrak{pg}(n)/\mathfrak{p}_r\mathfrak{s}_{im}\mathfrak{l}(p|n-p)$; $\mathfrak{supe}(n)/\mathfrak{s}(\mathfrak{upe}(p)\bullet\mathfrak{upe}(n-p))$; $\mathfrak{supe}(n)/\mathfrak{p}_{im}\mathfrak{s}_r\mathfrak{l}(p|n-p)$
$\mathfrak{osp}(2m,\ m|2n)/\mathfrak{gl}_r(m|n)$; $\mathfrak{osp}^*(2m|2n,\ n)/\mathfrak{u}^*(m|n)$; $\mathfrak{psg}_r(n)/\mathfrak{ps}(\mathfrak{g}_r(p)\bullet\mathfrak{g}_r(n-p))$
$\mathfrak{psg}_r(n)/\mathfrak{pgl}_r(p|-p)$; $\mathfrak{psg}^*(2n)/\mathfrak{ps}(\mathfrak{g}^*(2p)\bullet\mathfrak{g}^*(2n-2p))$; $\mathfrak{psug}(2m,\ m)/\mathfrak{p}_{im}\mathfrak{s}_{im}\mathfrak{g}(m)$;
$\mathfrak{psug}(2m,\ m)/^o\mathfrak{p}_{im}\mathfrak{g}(m)$; $\mathfrak{spe}_r(n)/\mathfrak{s}(\mathfrak{pe}_r(n-p)\bullet\mathfrak{pe}_r(p))$; $\mathfrak{spe}_r(n)/\mathfrak{sl}_r(p|n-p)$
$\mathfrak{spe}^*(2n)/\mathfrak{s}(\mathfrak{pe}^*(2p)\bullet\mathfrak{pe}^*(2n-2p))$; $\mathfrak{spe}^*(2n)/\mathfrak{su}^*(2p|2n-2p)$; $\mathfrak{sh}(2n,\ n)/Ц_r(n)$

Table 4. Classical superspaces of depth 1

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	Interpretation	Underlying domain	Name of the superdomain
$\mathfrak{sl}(m\|n)$	$\mathfrak{s}(\mathfrak{gl}(p\|q)\bullet\mathfrak{gl}(m-p\|n-q))$	$id\bullet id^*$	Supergrassmannian of the $(p\|q)$-dimensional subsuperspaces in $\mathbb{C}^{m\|n}$-dimensional one	$Gr_p^m\times Gr_q^n$	$Gr_{p,q}^{m,n}$
$\mathfrak{psl}(m\|n)$	$\mathfrak{ps}(\mathfrak{gl}(p\|p)\bullet\mathfrak{gl}(m-p\|n-q))$	$id\bullet id^*$	Same for $m=n$, $p=q$	$Gr_p^m\times Gr_p^m$	$Gr_{p,p}^{m,m}$
$\mathfrak{osp}(m\|2n)$	$\mathfrak{cosp}(m-2\|2n)$	id	Superquadric of $(1\|0)$-dimensional isotropic with respect to the nondege-nerate even form lines in $\mathbb{C}^{m\|n}$	Q_{m-2}	$Q_{m-2,n}$
$\mathfrak{osp}(2m\|2n)$	$\mathfrak{gl}(m\|n)$	$E^2\ id$	Ortholagrangean supergrassmannian of $(m\|m)$-dimensional isotropic with respect to the nondegenerate even form subsuperspaces in $\mathbb{C}^{2m\|n}$	$OGr_m\times{}^*LGr_n$ $^*OGr_m\times LGr_n$	$OLGr_{m,n}$
$\mathfrak{sq}(n)$ $\mathfrak{psq}(n)$	$\mathfrak{s}(\mathfrak{q}(p)\bullet\mathfrak{q}(n-p))$ $\mathfrak{ps}(\mathfrak{q}(p)\bullet\mathfrak{q}(n-p))$	$irr(id\bullet id^*)$	Queergrassmannian of q-symmetric $(p\|p)$-dimensional subsuperspace in $\mathbb{C}^{n\|n}$	Gr_p^n	QGr_p^n
$\mathfrak{pe}(n)$ $\mathfrak{spe}(n)$	$\mathfrak{cpe}(n-1)$ $\mathfrak{cspe}(n-1)$	id	Odd superquadric of $(1\|0)$-dimensional isotropic with respect to the nondegenerate odd form lines in $\mathbb{C}^{n\|n}$	$\mathbb{CP}^{n-1}$	PeQ_{n-1}
$\mathfrak{pe}(n)$ $(\mathfrak{spe}(n))$	$\mathfrak{gl}(p\|n-p)$ $(\mathfrak{sl}(p\|n-p))$	$\pi(S^2\ (id))$ or $\pi(E^2\ (id))$	Odd lagrangean supergrassmannian of $(p\|n-p)$-dimensional (and with a fixed volume for $\mathfrak{spe}$) subsuperspaces in $\mathbb{C}^{n\|n}$ isotropic with respect to the odd symmetric or skewsymmetric form	Gr_p^n	$PeGr_p^n$
$\mathfrak{vect}(0\|n)$	$\mathfrak{vect}(0\|n-k)\bullet\mathfrak{gl}(k;\Lambda(n-k))$	$\Lambda(k)\bullet\pi(id)$	Curved supergrassmanian of $(0\|1)$-dimensional subsupermanifolds in $\mathbb{C}^{0\|n}$		$CGr_{0,k}^{0,n}$
$\mathfrak{svect}(0\|n)$	$\mathfrak{vect}(0\|n-k)\bullet\mathfrak{sl}(k;\Lambda(n-k))$	$\pi(Vol)$if $k=1$	Same with volume elements preserved in the sub- and ambient supermanifolds		$SCGr_{0,k}^{0,n}$
$\mathfrak{h}(0\|m)$ $\mathfrak{sh}(m)$	$\mathfrak{h}(0\|m-2)\bullet\Lambda(m-2)\cdot z$ $\mathfrak{sh}(m-2)\bullet\Lambda(m-2)\cdot z$	$\pi(id)$	Curved superquadeic of $(0\|1)$-dimensional isotropic with respect to the (partly) split symmetric form subsupermanifolds in $\mathbb{C}^{0\|m}$		$CQ_{m-2,0}$

$\delta(\alpha)= =osp(4\|2)_\alpha$	$cosp(2\|2)\cong(gl(2\|1))$	id	$\mathbb{C}P^1\times\mathbb{C}P^1$
$ab(3)$	$cosp(2\|4)$ $c\delta(2)$	$L_{3\varepsilon_1}$	$\mathbb{C}P^1\times Q_5$

Table 5. Gradings of twisted loop (super)algebras corresponding to hermitian superdomains

$g_\varphi^{(m)}$	φ	*grading elements from* $\mathfrak{h}$	$(g_\varphi^{(m)})_0$
$sl(2m/2n)^{(2)}$	$(-st)\circ Ad\ diag(\pi_{2m},J_{2n})$	$diag(1_m,-1_m,1_n,-1_n)$	$sl(m/n)^{(1)}$
$sl(2m)^{(2)}$	$(-t)\circ Ad\ (\pi_{2m})$		$sl(m)^{(1)}$
$sl(2n)^{(2)}$	$(t)\circ Ad\ (J_{2n})$		$sl(n)^{(1)}$
$sl(n/n)^{(2)}$	π	$diag(1_p,0_{n-p},1_p,0_{n-p})$	$s(gl(p/p)^{(2)}{}_\pi \oplus gl(n-p/n-p)^{(2)}{}_\pi)$
$sl(n/n)^{(2)}$	$\pi\circ(-st)$	$diag(1_p,-1_{n-p},-1_p,1_{n-p})$	$s(gl(p/p)^{(2)}{}_{\pi\circ(-st)} \oplus gl(n-p/n-p)^{(2)}{}_{\pi\circ(-st)})$
$osp(2m/2n)^{(2)}$	$\varphi_{m,n}Ad\ diag(1_{2m-1},1,1_{2n})$	$diag(2J_2,\ O_{2(m+n-1)})$	$(cosp(2m-2/2n))^{(1)}{}_{\varphi_{m-1,n}}$
$o(2m)^{(2)}$			$(co(2m-2))^{(1)}$
$psq(2n)^{(4)}$	$(-st)\circ\sigma_i$	$diag(J_{2n},J_{2n})$	$psq(n)^{(2)}{}_{\delta_{-1}}$
$s\mathfrak{h}(2n)^{(2)}$	A	$H_{\xi_2\xi_3}$	$(s\mathfrak{h}(2n-2)\oplus\Lambda(2n-2))^{(2)}{}_A$
$psq(n)^{(2)}$	σ_{-1}	$diag(1_p,0_{n-p},1_p,0_{n-p})$	$ps(q(p)^{(2)}{}_{\delta_{-1}}\oplus q(n-p)^{(2)}{}_{\delta_{-1}})$

On Einstein equations on manifolds and supermanifolds

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Abstract. Analogues of Einstein equations (EE) are written for certain non-Riemannian manifolds, who locally are as certain compact Hermitian symmetric (super)spaces, e.g. as the Grassmannian Gr_{2n}^{4n}. Similar analogues are indicated for supermanifolds, in particular, for certain infinite dimensional ones. Some of these infinite dimensional analogues of EE equations are realized on the total spaces of Fock bundles over supermanifolds with no less than 3 odd coordinates and their invariance group is the N-extended Neveu-Schwarz superalgebra for N>2. Our EE are *not* supergravity equations; supergravity equations require a contact-like structure and are discussed elsewhere.

Introduction

The main object in the study of Riemannian geometry is (properties of) the Riemann tensor which, in turn, splits into the Weyl tensor, the traceless Ricci tensor and the scalar curvature. All these tensors are obstructions to the possibility of "flattening" the manifold on which they are considered. The word "splits" above means that at every point of the Riemannian manifold the space of values of the Riemann tensor constitutes an O(n)-module which consists of the three (if n ≠ 4) irreducible components (for n = 4 there are 4 components because the Weyl tensor splits additionally in this case).

More genearlly, let G be any group, not necessarily O(n). In what follows we will recall definition of *G-structure* on a manifold and (the space of) its *structure functions* (shortly referred to as SFs). SF are obstructions to integrability or, in other words, to possibility of flattening the G-structure. The Riemannian tensor is an example of SF. Among the most known (or popular of recent) other examples of such tensors are:

- an almost *conformal structure*, $G = O(n)\times\mathbb{R}^*$, SF are called the *Weyl tensor*;
- Penrose' *twistor* theory, $G = SU(2)\times SU(2)\times\mathbb{C}^*$, SF are called the "*α-forms*" and "*β-forms*";
- an almost *complex structure*, $G = GL(n;\mathbb{C})\subset GL(2n;\mathbb{R})$, SF are called the *Nijenhuis tensor*;
- an almost *symplectic structure*, G = Sp(2n), (no accepted name for SF).

Remark. The adjective "almost" should be always added until the G-structure under study is proved flat, i.e. integrable; by abuse of language people often omit it.

In several versions of a very lucid paper [G] Goncharov calculated all structure functions for analogues of conformal structure. In other words, his model manifold is a classical space, i.e. an irreducible compact Hermitian symmetric space (CHSS); and therefore in his examples G is the reductive part of the stabilizer of a point of the space.

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0.1. Structure functions. Let us retell some of Goncharov's results ([G]) and recall definitions ([St]).

Let M be a manifold of dimension n over a field $\mathbb{K}$. Let F(M) be the frame bundle over M, i.e. the principal $GL(n;\mathbb{K})$-bundle. Let $G\subset GL(n;\mathbb{K})$ be a Lie group. The *G-structure* on M is the reduction of the principal $GL(n;\mathbb{K})$-bundle to the principal G-bundle, i.e. the possibility to select transition functions so that their values belong to G.

The simplest G-structure is the *flat* G-structure defined as follows. Let V be $\mathbb{K}^n$ with a fixed frame. The flat structure is the bundle over V whose fiber over $v\in V$ consists of all frames obtained from the fixed one under the G-action, V being identified with T_vV.

In textbooks on differential geometry (e.g. in [St]) it is explained that obstructions to identification of the k-th infinitesimal neighbourhood of a point $m\in M$ on a manifold M with G-structure and that of a point of the flat manifold V with the above G-structure are called *structure functions of order k*.

Such an identification is possible *provided all structure functions of lesser orders vanish.*

Proposition. ([St]). *SFs constitute the space of* the (k,2)*-th Spencer cohomology.*

The Spencer cochain complex whose cohomology are mentioned in Proposition is defined as follows. Let S^i denote the operator of the i-th symmetric power. Set $\mathfrak{g}_{-1} = T_mM$, $\mathfrak{g}_0 = \mathfrak{g} = \mathrm{Lie}(G)$ and for $i > 0$ put:

$$\mathfrak{g}_i = \{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) : X(v)(w,\dots) = X(w)(v,\dots) \text{ for any } v,w \in \mathfrak{g}_{-1}\}$$
$$= S^i(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0 \cap S^{i+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}.$$

Now set $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \oplus_{i \geq -1} \mathfrak{g}_i$.

Suppose that

the $\mathfrak{g}_0$*-module* $\mathfrak{g}_{-1}$ *is faithful.* (1)

Then, clearly, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n) = \mathfrak{der}\ \mathbb{K}[[x_1,\dots, x_n]]$, where $n = \dim \mathfrak{g}_{-1}$. It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces same on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. The Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$, usually abbreviated to $\mathfrak{g}_*$, will be called *Cartan's prolong* (the result of the *Cartan prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

Let E^i be the operator of the i-th exterior power; set

$C^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} = \mathfrak{g}_{k-s} \otimes E^s(\mathfrak{g}_{-1}^*)$;

we usually drop the subscript of $C^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$ or at least indicate only $\mathfrak{g}_0$.

Define the differential $\partial_s : C^{k,s} \dashrightarrow C^{k,s+1}$ setting for any $v_1, \dots, v_{s+1} \in V$ (as usual, the slot with the hatted variable is ignored):

$$(\partial_s f)(v_1, \dots, v_{s+1}) = \Sigma(-1)^i [f(v_1, \dots, \hat{v}_{s+1-i}, \dots, v_{s+1}), v_{s+1-i}]$$

As expected, $\partial_s \partial_{s+1} = 0$, and the homology of this complex is called *Spencer cohomology* of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

0.2. Case of simple $\mathfrak{g}_*$ over $\mathbb{C}$. The following remarkable fact, though known to experts, is seldom formulated explicitely:

Proposition. *Let* $\mathbb{K} = \mathbb{C}$, $\mathfrak{g}_* = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ *be simple. Then only the following cases are possible:*

1) $\mathfrak{g}_2 \neq 0$ *and then* $\mathfrak{g}_*$ *is either* $\mathfrak{vect}(n)$ *or its special subalgebra* $\mathfrak{svect}(n)$ *of divergence-free vector fields, or its subalgebra* $\mathfrak{h}(2n)$ *of hamiltonian fields;*

2) $\mathfrak{g}_2 = 0$, $\mathfrak{g}_1 \neq 0$ *then* $\mathfrak{g}_*$ *is the Lie algebra of the complex Lie group of automorphisms of a CHSS* (see above).

Proposition explains the reason of imposing the restriction (0.1) if we wish $\mathfrak{g}_*$ to be simple. Otherwise, or on supermanifolds, where the analogue of Proposition does not imply similar restriction, we have to (and do) broaden the notion of Cartan prolong to be able to get rid of restriction (0.1).

When $\mathfrak{g}_*$ is a simple finite-dimensional Lie algebra over $\mathbb{C}$ computation of structure functions becomes an easy corollary of the Borel-Weil-Bott-... (BWB) theorem, cf. [G]. Indeed, by definition $\oplus_k H^{k,2}_{\mathfrak{g}_*} = H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ and by the BWB theorem $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$, as $\mathfrak{g}$-module, has as many components as $H^2(\mathfrak{g}_{-1})$ which, thanks to commutativity of $\mathfrak{g}_{-1}$, is just $E^2(\mathfrak{g}_{-1})$; the highest weights of these modules, as explained in [G], are also

deducible from the theorem. However, [G] pityfully lacks this deduction, see [LP1] and [LPS1] where it is given with interesting interpretations.

Let us also immediately calculate SF corresponding to case 1) of Proposition: we did not find these calculations in the literature. Note that vanishing of SF for $\mathfrak{g}_* = \mathfrak{vect}$ and $\mathfrak{f}$ (see 0.5) follows from the projectivity of $\mathfrak{g}_*$ as $\mathfrak{g}_0$-modules and properties of cohomology of coinduced modules [F]. In what follows $R(\Sigma a_i \pi_i)$ denotes the irreducible $\mathfrak{g}_0$-module. The classical spaces are listed in Table 1 and some of them are bapthized for convenience of further references.

Theorem. 1)(Serre [St]). *In case 1) of Proposition structure functions can only be of order 1.*

a) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = 0$ *for* $\mathfrak{g}_* = \mathfrak{vect}(n)$ *and* $\mathfrak{svect}(m)$, $m>2$;

b) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = R(\pi_3) \oplus R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(2n)$, $n>2$;

$H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(4)$.

2)(Goncharov [G]). *SFs of* Q_3 *are of order 3 and constitute* $R(4\pi_1)$. *SF for Grassmannian* Gr_m^{m+n} (when neither m nor n is 1, i.e. Gr is not a projective space) *is the direct sum of two components whose weights and orders are as follows:*

Let $A = R(2, 0, \dots, 0, -1) \oplus R(1, 0, \dots, 0, -1, -1)$, $B = R(1, 1, 0, \dots, 0, -1) \oplus R(1, 0, \dots, 0, -2)$.

Then *if* $mn \neq 4$ *both A and B are of order 1;*
if $m = 2$, $n \neq 2$ *A is of order 2 and B of order 1;*
if $n = 2$, $m \neq 2$ *A is of order 1 and B of order 2;*
if $n = m = 2$ *both A and B are of order 2.*

SF of G-structures of the rest of the classical CHSSs are the following irreducible $\mathfrak{g}_0$*-modules whose order is 1* (recall that $Q_4 = Gr_2^4$):

CHSS	$\mathbb{P}^n$	OGr_m	LGr_m	Q_n, $n>4$
weight of SF	-	$E^2(E^2(V^*)) \oplus V$	$E^2(S^2(V^*)) \oplus V$	$E^2(V^*) \oplus V$
		$E_6/SO(10) \times U(1)$	$E_7/E_6 \times U(1)$	
		$E^2(R(\pi_5)^*)) \oplus R(\pi_5)$	$E^2(R(\pi_1)^*)) \oplus R(\pi_1)$	

0.3. SF for reduced structures. In [G] Goncharov considered conformal structures. SFs for the corresponding generalizations of the Riemannian structure, i.e. when $\mathfrak{g}_0$ is the semisimple part $\hat{\mathfrak{g}}$ of $\mathfrak{g} = \mathrm{Lie}(G)$, seem to be more difficult to compute because in these cases $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}_- \oplus \mathfrak{g}_0$ and the BWB-theorem does not work. Fortunately, the following statement, a direct corollary of definitions, holds.

Proposition ([G], Th.4.7). *For* $\mathfrak{g}_0 = \hat{\mathfrak{g}}$ *and* $\mathfrak{g}$ *SF of order 1 are the same and SF of order 2 for* $\mathfrak{g}_0 = \hat{\mathfrak{g}}$ *are* $S^2(\mathfrak{g}_1) = S^2(\mathfrak{g}_{-1}^*)$. (There are clearly no SF sof order >2 for $\mathfrak{g}_0 = \hat{\mathfrak{g}}$).

Example: Riemannian geometry. Let $G = O(n)$. In this case $\mathfrak{g}_1 = \mathfrak{g}_{-1}$ and in $S^2(\mathfrak{g}_{-1})$ a 1-dimensional subspace is distinguished; the sections through this subspace constitute a Riemannian metric g on M. (The habitual way to determine a metric on M is via a symmetric matrix, but actually this is just one scalar matrix-valued function.) The values of the Riemannian tensor at a point of M constitute an O(n)-module $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ which contains a trivial component whose arbitrary section will be denoted by R. What is important, this trivial component is realised by Proposition as a submodule in $S^2(\mathfrak{g}_{-1})$. Thus, we have two matrix-valued functions: g and R each being a section of the trivial $\mathfrak{g}_0$-module. What is more natural than to require their ratio to be a constant (rather than a function)?

$$R = \lambda\, g, \text{ where } \lambda \in \mathbb{R}. \qquad (EE_0)$$

Recall that the Levi-Civita connection is the unique symmetric affine connection compatible with the metric. Let now t be the structure function (sum of its components belonging to the distinct irreducible O(n)-modules that constitute $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$) corresponding to the Levi-Civita connection; the process of restoring t from g involves differentiations thus making (EE_0) into a nonlinear pde. This pde is not Einstein Equation yet. Recall that in adition to the trivial component there is another O(n)-component in $S^2(\mathfrak{g}_{-1})$, the Ricci tensor Ri. *Einstein equations* (in vacuum and with cosmological term λ) are the *two* conditions: (EE_0) and

$$Ri = 0. \qquad (EE_{ric})$$

Notice that we have no SF of order 1 to think about. This is not so for superspaces or flag manifolds.

A generalization of this example to G-structures associated with certain other CHSSs, flag varieties, and to supermanifolds is considered in [LPS1] and [LP3].

The prerequisites on symmetric spaces see in [H]; on symmetric superspaces in [S].

1. SF for reduced structures -- analogues of EE on manifolds

In [G] Goncharov did not explicitely calculate SFs for G-structures corresponding to the reduction of the generalized conformal structure. Let us fill in this gap: let us explicify Proposition G for the classical CHSSs. (The excedptions do not give analogues of EE and are considered in [LP3].)

Proposition. *Let* $\mathfrak{g}_0$ *be the semisimple part* $\hat{\mathfrak{g}}$ *of* $\mathfrak{g}$ = Lie (G) *corresponding to a CHSS. Then SF of order 2 are:*

CHSS	$\mathbb{P}^n$	Gr_m^{m+n}	OGr_m	LGr_m
weight of SF	$R(\pi_2)$	$R(2\pi_1) \oplus R(2\pi_1^*) \oplus R(\pi_2) \oplus R(\pi_2^*)$	$R(0, \dots, 0, -2, -2) \oplus R(0, \dots, 0, -1, -1, -1, -1)$	$R(0, \dots, 0, -2, -2) \oplus R(0, \dots, 0, -4)$

Let us show what, in our opinion, plays the role of EE on some CHSS different from the quadric. Let R be a section of the vector bundle with the above SF as the fiber; if SF consists of several components denote them $R = R_1 + R_2$ in accordance with the decomposition of the module of SFs as indicated above or in what follows. Consider SF corresponding to the canonical connection (the restoring of this connection involves differentiations).

An analogue of EE_0 :

$v = \lambda R_2^n$ (or $v = \lambda R^n$ if R has just one irreducible component) (EE_0)

where v is a fixed volume element in the following cases:

1) $\mathbb{P}^{2n}$ or Gr_{2n}^{4n} (the conventional EE_0 is just it for n = 1);
2) $\mathbb{P}^{2n}$;
3) OGr_{4n}, we set $v = \lambda R_2^n$ (the conventional EE_0 is just it for n = 1).

Analogues of (EE_{ric}) are equations

$R_1 = 0$ (if there is such a component) (EE_{ric})

All these equations are meaningful provided SFs of order 1, $T = \oplus T_i$, vanish:

$T = 0$ (if there is such a component) (EE_{tor})

Notice that if the space of SFs is irreducible there is no EE_{ric}.

2. EE on supermanifolds

The necessary background on Lie superalgebras and supermanifolds is gathered in a condenced form in [L]. The above definitions are generalized to Lie superalgebras via Sign Rule.

Let us try to list all possible analogues of the above EE on supermanifolds.

1) The first idea is to replace $\mathfrak{o}(m)$ with $\mathfrak{osp}(m|2n)$ for a $\mathbb{Z}$-grading of the form

$$\mathfrak{osp}(m|2n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \text{ with } \mathfrak{g}_0 = \mathfrak{cosp}(m-2|2n) \text{ and } m > 2.$$

2) The next step is to replace $\mathfrak{osp}(m|2n)$ with its odd (periplectic) analogues: $\mathfrak{pe}(n)$ and $\mathfrak{spe}(n)$ and the "mixture" of these: $\mathfrak{pe}(n) \oplus \mathbb{C}(az+bd)$, where a, b ∈ ℂ, d is the outer derivative of $\mathfrak{spe}(n)$, i.e. $\mathfrak{pe}(n) = \mathfrak{spe}(n) \oplus \mathbb{C}d$, z the central element. In matrix realization we can take $d = \mathrm{diag}(1_n, -1_n)$, $z = 1_{2n}$, definitions see in [L4].

Why is m ≠ 0 in 1)? Might it be that an analogue of EE is connected not with $\mathfrak{sp}(2n)$, the Lie algebra of *linear* symplectic transformations, but with the infinite dimensional Lie algebra of *all* symplectic transformations, i.e. the Lie algebra $\mathfrak{h}(2n|0)$ of Hamiltonian vector fields? As Theorem 0.1.2 states, the answer to the above suggestions is NO: SFs are only of order 1 (and are investigated in [P4]).

Let us not give up: the algebra $\mathfrak{o}(m)$ has one more analogue -- Lie superalgebra $\mathfrak{h}(0|m)$ of Hamiltonian vector fields on (0|m)-dimensional supermanifold. So another possibility is to

3) replace $\mathfrak{osp}(m|2n)$ with $\mathfrak{h}(2n|m)$, where $m \neq 0$;

4) replace $\mathfrak{osp}(m|2n)$ with $\mathfrak{t}(2n+1|m)$;

and consider odd analogues of 3) and 4):

5) replace $\mathfrak{pe}(n)$ and $\mathfrak{spe}(n)$ in 2) with $\mathfrak{le}(n)$, and $\mathfrak{sle}(n)$;

6) replace $\mathfrak{pe}(n)$ with $\mathfrak{m}(n)$, and $\mathfrak{sm}_\lambda(n)$.

We should also explore the cases associated with $\mathbb{Z}$-grading (if any) of Kac-Moody (twisted loop) superalgebras of the form $\oplus_{|i| \leq 1} \mathfrak{g}_i$. Remarkably, there are not only "trivial" analogues of CHSS, the spaces of loops with values in a finite-dimensional CHSS! There are CHSSs associated with *twisted* loop algebras and superalgebras, cf. [LSV].

3. Spencer cohomology of $\mathfrak{osp}(m|n)$

3.1. $\mathbb{Z}$-gradings of depth 1. All these gradings are of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1}^*$.

Proposition ([K] and [LSV]). *For $\mathfrak{osp}(m/2n)$ the following values of $\mathfrak{g}_0$ are possible for the $\mathbb{Z}$-gradings of depth 1:*

a) $\mathfrak{cosp}(m-2/2n)$;

b) $\mathfrak{gl}(r/n)$ if $m = 2r$.

3.2. Cartan prolongs of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and $(\mathfrak{g}_{-1}, {}^\wedge\mathfrak{g}_0)$.

Proposition. *a) $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$ except for the case 2.1.b) for $r = 3$, $n = 0$ when*

$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(3/0)$.

b) $(\mathfrak{g}_{-1}, {}^\wedge\mathfrak{g}_0)_* = \mathfrak{g}_{-1} \oplus {}^\wedge\mathfrak{g}_0$.

3.3. Structure functions.

Theorem. *Cases a) and b) below correspond to cases 2.2 of $\mathbb{Z}$-gradings. The cases $mn = 0$ see in [G] and Introduction.*

a) As ${}^\wedge\mathfrak{g}_0$-module, $H^{2,2}{}^\wedge\mathfrak{g}_0 = S^2(\Lambda^2(\mathfrak{g}_{-1}))/\Lambda^4(\mathfrak{g}_{-1})$ and splits into the direct sum of three irreducible components whose weights are given in Table 1.

As $\mathfrak{g}_0$-module, $H^{2,2}\mathfrak{g}_0 = H^{2,2}{}^\wedge\mathfrak{g}_0/S^2(\mathfrak{g}_{-1})$ and Table 1 also contains the highest weights of irreducible components of $H^{2,2}\mathfrak{g}_0$. For $k \neq 2$ SF vanish.

b) As $\mathfrak{g}_0$-module, $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ is irreducible and their highest weights are given in Table 2 for $r \neq n, n+2, n+3$.

The case $r = 4$, $n = 0$ and $r = 2$, $n = 1$ coinside, respectively, with the cases considered in a) for $\mathfrak{o}(8)$ and $\mathfrak{osp}(4/2)$.

4. Spencer cohomology of $\mathfrak{spe}(n)$

Proposition (cf. [K] with [LSV]). *All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g}$ are listed in Table 1 of [LSV]. They are :*

a^{sy}) $\mathfrak{g}_0 = \mathfrak{sl}(n\text{-}p/p)$, $\mathfrak{g}_{-1} = S^2(id)$, $\mathfrak{g}_1 = E^2(id^*)$;

a^{sk}) $\mathfrak{g}_0 = \mathfrak{sl}(n\text{-}p/p)$, $\mathfrak{g}_1 = S^2(id)$, $\mathfrak{g}_{-1} = E^2(id^*)$

b) $\mathfrak{g}_0 = \mathfrak{pe}(n\text{-}1)$, $\mathfrak{g}_{-1} = id$ (considered endowed with a symmetric form). In these cases $\mathfrak{g}_* = \mathfrak{g}$.

Theorem. *a^{sk}) Nonvanishig SF are of order 1 and constitute a completely reducible $\mathfrak{g}_0$-module described in Table 3.*

b) For $\mathfrak{g}_0 = \mathfrak{pe}(n\text{-}1)$, $\mathfrak{spe}(n\text{-}1)$, $\mathfrak{cpe}(n\text{-}1)$, and $\mathfrak{cspe}(n\text{-}1)$ and the above $\mathfrak{g}_{-1}$ All SF vanish except for $H^{1,2}\mathfrak{spe}(n\text{-}1) = \Pi(\mathfrak{g}_{-1}) = \Pi(V_{\varepsilon_1})$ and there are following nonsplit exact sequences of $\mathfrak{spe}(n\text{-}1)$-modules:

$0 \to V_{\varepsilon_1+\varepsilon_2} \to H^{2,2}\mathfrak{spe}(n\text{-}1) \to \Pi(V_{2\varepsilon_1+2\varepsilon_2}) \to 0$ for $n>4$

$0 \to X \to H^{2,2}\mathfrak{spe}(3) \to \Pi(V_{3\varepsilon_1}) \to 0$, where X is determined from the following nonsplit exact sequences of $\mathfrak{spe}(3)$-modules:

$0 \to V_{\varepsilon_1+\varepsilon_2} \to X \to \Pi(V_{2\varepsilon_1+2\varepsilon_2}) \to 0$

and $0 \to X \to H^{2,2}\mathfrak{cpe}(3) \to V_{2\varepsilon_1} \to 0$, where X is determined from the following nonsplit exact sequences of $\mathfrak{spe}(3)$-modules:

$0 \to \Pi(V_{2\varepsilon_1+2\varepsilon_2}) \to X \to \Pi(V_{3\varepsilon_1}) \to 0$

Besides, there are exact sequences:

$0 \to H^{2,2}\mathfrak{spe}(n\text{-}1) \to H^{2,2}\mathfrak{pe}(n\text{-}1) \to V_{2\varepsilon_1} \to 0$

and

$0 \to H^{2,2}\mathfrak{spe}(n\text{-}1) \to H^{2,2}\mathfrak{cspe}(n\text{-}1) \to V_{2\varepsilon_1} \to 0$

both for $n>3$; and exact sequence

$0 \to \Pi(V_{2\varepsilon_1+2\varepsilon_2}) \to H^{2,2}\mathfrak{cpe}(n\text{-}1) \to V_{2\varepsilon_1} \to 0$ for $n>4$

Moreover, $H^{2,2}\mathfrak{cpe}(n\text{-}1) = \Pi(S^2((E^2(V_{\varepsilon_1}))/<(\Pi(\mathbb{1}))/E^4(V_{\varepsilon_1}))$

5. An analogue of a theorem by Serre for Lie superalgebras: concequences of involutivity

The theorem we have ascribed above to Serre is actually a corollary of his initial statement that $\mathbb{Z}$-graded Lie algebra of the form $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ is *involutive* if and only if its Spencer cohomology $H^{k,s}\mathfrak{g}_*$ vanishes for $s \geq 0$ ([St]). For superalgebras we only need the *only if* part for the time being. To formulate it we have to superize the notion of involutivity. Let us do so and recall the classical definition of involutivity for Lie algebras as well.

Let $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie superalgebra, $\{a_1, \dots, a_n\}$ a (homogeneous) basis of $\mathfrak{g}_{-1}$. Clearly, the map

$a_i: \mathfrak{g} \to \mathfrak{g}$, $x \to [x, a_i]$

is a homomorphism of $\mathfrak{g}_{-1}$-modules. A $\mathbb{Z}$-graded Lie algebra of the form $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ is called *involutive* if the maps a_i are onto. To superize it we have to require the same of the even maps a_i. Additionally we must demand vanising of the homology of the odd maps a_i (well-defined thanks to the Jacoby identity).

In scientific terms this is formulated as follows. For a Lie superalgebra $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ set:

$\mathfrak{g}^r = \ker \mathfrak{g}_1 \cap \ker \mathfrak{g}_2 \cap \dots \cap \ker \mathfrak{g}_r$, $\mathfrak{g}^r = \oplus_{i \geq -1} \mathfrak{g}^r_i$.

Notice that $a_r(\mathfrak{g}^{r-1}_i) \subset \mathfrak{g}^{r-1}_{i-1}$. The Lie superalgebra $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ will be called *involutive* if the following conditions are fulfilled:

(1) $\mathfrak{g}^n = \mathfrak{g}_{-1}$;

(2) $a_r(\mathfrak{g}^{r-1}) = \mathfrak{g}^{r-1}$ if a_r is even;

(3) $a_r(\mathfrak{g}^{r-1}) = \mathfrak{g}^r$ if a_r is odd.

Theorem. *If* $\mathfrak{g}$ *is involutive then* $H^{k,s}\mathfrak{g}_* = 0$ *for* $s \geq 0$.

6. Spencer cohomology of vectory Lie superalgebras in the standard grading (Definitions see in [L4].)

Theorem (cf. Theorem 0.1.2). *1) For* $\mathfrak{g}_* = \mathfrak{vect}(m|n)$, $\mathfrak{svect}(m|n)$, $\mathfrak{k}(2m+1|n)$ *and* $\mathfrak{m}(n)$ *SF vanish except for* $\mathfrak{svect}(0|n)$ *when SF are of order n and constitute the* $\mathfrak{g}_0$*-module* $\Pi^n(\mathbf{1})$.

2) For $\mathfrak{g}_* = \mathfrak{h}(0|m)$, $m > 3$, *SF are* $\Pi(R(3\phi_1) \oplus R(\phi_1))$.

3) For $\mathfrak{g}_* = \mathfrak{sh}(0|m)$, $m > 3$, *nonzero SF are same as for* $\mathfrak{h}(0|m)$ *and an additional direct summand* $\Pi^n(R(\pi_1))$ *of order n-1.*

4) For $\mathfrak{g}_* = \mathfrak{sle}(n)$, $n > 1$, *nonzero SF are* $H^{1,2}\mathfrak{spe}(n) = S^3(\mathfrak{g}_{-1}^*)$, $H^{2,2}\mathfrak{spe}(n) = \Pi(\mathbf{1})$, $H^{n,2}\mathfrak{spe}(n) = \Pi^n(\mathbf{1})$.

Let $\lambda \in \mathbb{C}$, ω be the canonical odd 2-form and R a section through $H^{2,2}\mathfrak{spe}(n) = \Pi(\mathbf{1})$. Thus, there is a possibility to write two analogues of EE_0 for $\mathfrak{sle}(n)$:

$$\omega = \lambda R, \qquad (EE_0(\text{2-form}))$$

and

$$v = \lambda R, \qquad (EE_0(\text{volume}))$$

where v is the volume form (of parity congruent to n mod 2).

7. Nonstandard grading of the Lie superalgebras of hamilton or contact vector fields

There is one grading of either of these superalgebras (that we denote by $\mathfrak{g}(m|n) = \mathfrak{h}(2m|n)$, $\mathfrak{sh}(n)$ or $\mathfrak{k}(2m+1|n)$) of the form

$\mathfrak{g}_{-1} = F(m|n)$, $\mathfrak{g}_0 = \mathfrak{g}(m|n-2) \oplus F(m|n-2)$ for $n > 1$,

where $F(m|n-2)$ is the superspace of "functions" (polynomials or power series in our case) on which $\mathfrak{g}_0$ naturally acts, and $\mathfrak{g}_1 = \mathfrak{g}_{-1}^*$. For $n > 2$ $\mathfrak{g}_{-1}$ is not purely odd and is isomorphic to the total space of the Fock bundle over a (2m, n-2)-dimensional symplectic supermanifold.

By an incredible effort one of us (E.P.) managed to calculate SFs of order 1 for $\mathfrak{g} = \mathfrak{sh}(6)$. The space of these SFs is not completely reducible, some of the indecomposable components look as complicated as follows:

$$\begin{array}{ccccc} x & \rightarrow & o & \leftarrow & x \\ \downarrow & & \uparrow & & \downarrow \\ o & \leftarrow & x & \rightarrow & o \\ \uparrow & & \downarrow & & \uparrow \\ x & \rightarrow & o & \leftarrow & x \end{array}$$

According to sec. 2 all these SF constitute constraints similar to the Wess-Zumino constraints in supergravity and must vanish; the lack of complete reducibility implies that only part of these relations are relevant (thick dots). We have no idea how to approach other, especially infinite dimensional, cases: the number of SF grows with m and n! Yu. Kochetkov showed (unpublished) that for $\mathfrak{g}(m|n) = \mathfrak{h}(2m|n)$ or $\mathfrak{h}(n)$ there is always a trivial component (perhaps, there are several) in the space of 2nd order SFs. An answeer might come from programmers: cohomology is a computerizable problem [LP1] .

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Contact type structures on supermanifolds and supergravities

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Abstract. A language suitable to describe nonholonomic mechanics on (super)manifolds is applied to derive supergravity equations on an N-extedded Minkowski superspace for any N.

Introduction

This paper is a continuation of [LPS], where the necessary background is presented. In [LPS] we have written certain analogues of Einstein equations on (super)manifolds. These analogues , however, are not supergravity equations. The reason is that they naively superise the technique of differential geometry developed only for the case when the tangent space is considered endowed with the trivial (zero) Lie bracket (this technique appeared under the impression that partial derivatives commute).

As shown in [L2] (see also [L4], [LSV], [LP]), the supermanifold theory naturally hints to devise a new language sutable to describe the structure of an N-extended Minkowski superspace. The tangent space to the Minkovski superspace for $N \neq 0$ is naturally endowed with a (2-step) nilpotent Lie superalgebra structure that highly resembles the contact structure on a manifold, cf [A].

(The hasty reader might think that this can never happen, by the parenthetical remark above the tangent space can only be endowed with the trivial bracket. This, however, does happen. The simplest example: let $\alpha = dt - \Sigma_{i \leq n} p_i dq_i$ be the contact form on a (2n+1)-dimensional manifold M. Then a canonical basis of the tangent space to every point of M is constituted by vector fields $\partial/\partial t$, $\partial/\partial q_i$, and $\partial/\partial p_i + q_i \partial/\partial t$. The fields $\partial/\partial p_i$ won't do: they are not invariant under contact transformations. Thus the tangent space is naturally endowed with a Heisenberg algebra structure.)

Here we give the definitions that allow one to calculate structure functions (analogues of the Riemann tensor) for various contact-like structures: of the "naive" even one; another, odd one, with plenty of its interesting satellite structures, and of the complexified N-extended Minkowski superspace.

Our theorems have an interesting counterpart in classical mechanics: they enable us to study nonholonomic mechanics in parallel with the holonomic one and verify integrability of differential equations whose symmetries are induced not from point transformations but from contact one: the two possible cases [ALV] (only for the first one there were means of description -- Spencer cohomology). In the last century Hertz noticed that some of nonholonomic problems are not variational ones [VG]. In our context his remark indicates the source of difficulties (discused by V.Ogievetsky and E.Sokachev) in expressing some of supergravity equations in a Lagrangean form: this might be just impossible.

Since the local geometry is given by a G-structure, it was natural to investigate Minkovsky superspaces from that point of view in order to write SUGRA equations, but all the previous attempts tried to adjust the text-book technique which does not treat contact like cases, see a moving account in [VG], and therefore does not lead to SUGRA, with the possible exception of N=1; this unlucky coincidence delayed our substantiation of the hypothesis from [L] on the correct description of SUGRA.

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0.1. SF for contact structures: Shchepochkina prolongs[1]. Proposition 0.2 [LPS] lists all simple $\mathbb{Z}$-graded Lie algebras of finite growth (SZGLAFGs) admitting a $\mathbb{Z}$-grading of depth 1 (i.e. of the form $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$). Among such we find all simple Lie algebras of vector fields (with polynomial coefficients) except those that preserve a contact structure whose canonical $\mathbb{Z}$-grading is of depth 2. Recall that a *contact structure* is a maximally nonintegrable distribution of codimension 1, cf. [A]. To embrace contact structures we have to slightly generalize the notion of Cartan prolongation: the tangent space to a point of a manifold with a contact structure possesses a natural structure of a nilpotent Lie algebra (Heisenberg algebra).

Given a $\mathbb{Z}$-graded nilpotent Lie algebra $\mathfrak{g}_- = \oplus_{0 > i \geq -d} \mathfrak{g}_i$ and a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{der}\ \mathfrak{g}_-$ which preserves $\mathbb{Z}$-grading of $\mathfrak{g}_-$, define its i-th *Shchepochkina*[2] *prolong* for i > 0 to be:

$$\mathfrak{g}_i = (S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0 \cap S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-)_i,$$

where the subscript singles out the component of degree i.Similarly to the above, define $\mathfrak{g}_*$, or rather, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, as $\oplus_{i \geq -d} \mathfrak{g}_i$; then, by the same reasons as in [LPS], $\mathfrak{g}_*$ is a Lie algebra and $H^2(\mathfrak{g}_-; \mathfrak{g}_*)$ is well-defined. The space $H^2(\mathfrak{g}_-; \mathfrak{g}_*)$ is the space of obstructions to flatness. It naturally splits into homogeneous components whose degree corresponds to the order of SF; in general case the minimal order of SF is 2-d.

Example. Let $\mathfrak{g} = \mathfrak{csp}(2n)$, $\mathfrak{g}_{-1} = R(\pi_1; 1)$, $\mathfrak{g}_{-2} = R(0)$; then $\mathfrak{g}_* = \mathfrak{k}(2n+1)$ and

$$C^{k,s}{}_{\mathfrak{g}_*} = \mathfrak{g}_{k-s} \otimes E^s(\mathfrak{g}_{-1}{}^*) \oplus \mathfrak{g}_{k-s-1} \otimes E^{s-1}(\mathfrak{g}_{-1}{}^*) \otimes \mathfrak{g}_{-2}{}^*.$$

The number k here is the order of SF.

Theorem. *For* $\mathfrak{g}_* = \mathfrak{k}(2n+1)$ *all SF vanish.*

Remark. This is a conceptual reformulation of Darboux's theorem on the lack of parameters for the contact form.

Proof of this theorem illustrates the might of science: since $\mathfrak{k}(2n+1)$, as $\mathfrak{k}(2n+1)$-module, is induced from a character of $\mathfrak{g}_0$ nontrivial on the center, $H^*(\mathfrak{g}_-; \mathfrak{g}_*) = 0$ with Poincaré's Lemma, see [F] .

0.2. SF for projective structures. It is also interesting sometimes to calculate $\oplus_k H^{k,2}(\mathfrak{g}_-; \mathfrak{h})$ for some $\mathbb{Z}$-graded subalgebras $\mathfrak{h} \subset \mathfrak{g}_*$, such that $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$. For example, let $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{g}_{-1}$ its standard (identity) representation. Then $\mathfrak{g}_* = \mathfrak{vect}(n)$ and all SF vanish ([LPS]); but if $\mathfrak{h} = \mathfrak{sl}(n + 1) \subset \mathfrak{vect}(n)$ then the corresponding SF are nonzero and provide with obstructions to integrability of what is called *projective connection.*

Theorem. 1) *Let* $\mathfrak{g}_* = \mathfrak{vect}(n)$, $\mathfrak{h} = \mathfrak{sl}(n+1)$. *Then*

SF of order 1 and 2 vanish, SF of order 3 are R(2,1, 0, ..., 0, -1)

---

[1]For depth 2 this construction was developed in [T] but nobody, the author included, understood its importance. We thank S.Shnider, who indicated [T] to us.

[2]This construction was first described in [Sh].

2)*Let* $\mathfrak{g}_* = \mathfrak{k}(2n+1)$, $\mathfrak{h} = \mathfrak{sp}(2n+2)$. *Then SF are* $R(\pi_1 + \pi_2; 3)$ *of order 3.*

0.3. SF on supermanifolds. Our motto is

"simple $\mathbb{Z}$-graded Lie superagebras of finite growth (SZGLSAFGs)*are as good as simple finite-dimensional Lie algebras"*

There should be similar results for either. On the strength of arguments of sec. 0 we shall

- list $\mathbb{Z}$-gradings of SZGLSAFGs of depth 2 *similar to that of* $\mathfrak{k}(2n+1)$(this is deducible from [K], [L1], [S2]) and in what follows we will explain which of all $\mathbb{Z}$-gradings of depth 2 we have in mind;
- calculate projective-like and reduced structures for the above.

0.3.1. Darboux theorem on supermanifolds. ([L1]). *Let* ω *be a homogeneous (with respect to parity) nondegenerate (as a bilinear functional) closed differential 2-form on a supermanifold M over* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *In a neighbourhood of any point there is a coordinate system such that*

$$\omega = \Sigma dp_i dq_i + \Sigma \varepsilon_j (d\xi_j)^2, \text{ where } \varepsilon_j = \pm 1 \text{ for } \mathbb{K} = \mathbb{R} \text{ and } \varepsilon_j = 1 \text{ for } \mathbb{K} = \mathbb{C},$$
$$1 \leq i \leq n,\ 1 \leq j \leq m \text{ and } (2n, m) = \dim M, \text{ if } p(\omega) = 0;$$

and

$$\omega = \Sigma d\pi_i dq_i,\ 1 \leq i \leq n, \text{ where } (n, n) = \dim M, \text{ if } p(\omega) = 1.$$

Proof of this theorem for the *family* of forms depending on a parameter running a supermanifold is given in [SH]. Same arguments as in the case of manifolds (cf. [A], App. 4), derive from the above theorem the classification of 1-forms:

Corollary. *Let* α *be a differential 1-form which determines a distribution of codimention* $p(\alpha)$ *such that* $d\alpha$ *is nondegenerate (i.e.* α *is maximally nonintegrable). Then either*

$$\alpha = dt + \Sigma(p_i dq_i - q_i dp_i) + \Sigma \varepsilon_j \xi_\varphi d\xi_\varphi \text{ where } \varepsilon_j = \pm 1 \text{ for } \mathbb{K} = \mathbb{R} \text{ and } \varepsilon_j = 1 \text{ for } \mathbb{K} = \mathbb{C},$$
$$1 \leq i \leq n,\ 1 \leq j \leq m \text{ and } \dim M = (2n+1, m), \text{ if } p(\omega) = 0;$$

or

$$\alpha = d\tau + \Sigma(\pi_i dq_i + q_i d\pi_i),\ 1 \leq i \leq n, \text{ where } \dim M = (n, n+1), \text{ if } p(\omega) = 1.$$

We have often heard that "Riemannian geometry has parameters whereas the symplectic one does not". It is our aim to elucidate this phrase: we have shown (Th. 0.2 in [LPS]) that the symplectic geometry does have parameters, the torsion, which being of order 1 does not prevent one to reduce a 2-form to a canonical form. The curvature, alias an SF of order 2, might have been the problem; whereas contact structures have no SFs at all.

0.3.2. SFs on the N-extended complexified Minkowski supermanifold. In this case $\mathfrak{g}_0 = \mathfrak{g} = \mathfrak{o}(4) \oplus \mathfrak{o}(N) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{o}(N)$, $\mathfrak{g}_- = \oplus_{-1 \geq i \geq -2} \mathfrak{g}_i$ with $\mathfrak{g}_{-1} = id_1 \otimes Id \oplus id_2 \otimes Id^*$, dim $\mathfrak{g}_{-2} = id_1 \otimes id_2{}^*$, where id_j is the space of the standard (identity) representation of the j-th summand $\mathfrak{sl}(2)$, Id the space of the standard representation of $\mathfrak{o}(N)$. The corresponding matrix representaion was first found for a particular real form of $\mathfrak{g}_* = \mathfrak{g}_- \oplus \mathfrak{g}_0$ by Golfand and Likhtman

Terminological conventions. 1)The $\mathfrak{g}$-module V with the highest weight ξ and *even* highest vector will be denoted by V_ξ or R(ξ).

2)Let $\mathfrak{cg}$ denote the trivial central "extent" (the result of the extention) of a Lie (super)algebra $\mathfrak{g}$.

1. SF for contact structures.

Theorem. *For* $\mathfrak{k}(2n+1|m)$ *and* $\mathfrak{m}(n)$ *all SF vanish.*

Proof: same as of Th. 0.1.

1. SF for N = 1 superMinkowski structures

The G-structure of the Minkowski space can be viewed as either (pseudo) Riemannian or, equivalently, twistor one. Their "straightforward" superizations are considered in [LPS] and [P], respectively. Neither of these superisations are what is accepted as supergravity nor supertwistors. The reason is that Minkowski superspace is naturally endowed with a contact-type structure.

"Recall" first of all, what is the complexified and a compactified Minkowski superspace $\mathcal{M}(N)$, cf.[M].

An account of physical reasons for the restrictions $N \le 4$ for the Yang-Mills and $N \le 8$ for the supergravity theories can be found in [OS].

Consider the Lie supergroup SL(N|4) and its parabolic subsupergroup P corresponding to the two odd simple roots in the base (system of simple roots) of the form

```
       +           +
0---●---0--...--0---●---0   (there are N-1 white nodes in the middle)
```

Let $G = SL(N|4)_{red} = SL(N)\times SL(2)\times SL(2)\times\mathbb{C}^\times$. Then $\mathcal{M}(N) = SL(N|4)/P$ endowed with the natural G-structure. The conventional versions of the Minkovsky superspace correspond to a certain real form of the (complex) superspace $\hat{}\mathcal{M}(N)$ with the $\hat{}G$-structure, i.e. the reduced G-structure. Clearly,

$\hat{}\mathcal{M}(N) = P/\hat{}G$, where $\hat{}G = SL(N)\times SL(2)\times SL(2)$.

Of interest are also SFs of an enlargement of $\hat{}\mathcal{M}(N)$ obtained by *dimensional reduction*, physicists' name for the passage from $\hat{}\mathcal{M}(N)$ to

$\hat{}\mathfrak{K}(N) = P/\hat{}G$, where $\hat{}G = Q\times SL(2)\times SL(2)$ and Q is a parabolic subgroup of SL(N),

i.e. the passage to a smaller parabolic P, the one with the diagram

```
      +          +       +              +
0---●---0--...  --0--  ...--0--....--0---●---0
```

Theorem. *All the orders and weights of all the SF for N = 1 Minkowski superspace* $\hat{}\mathcal{M}(N)$ and the Minkowski space for comparison *are as follows* (dash means that there are no SF of this order; notice that the orders of SF for N = 0 can only equal to 1 or 2):

order of SF	weights for N = 0	weights for N = 1
0		$3\varepsilon_1 + \delta_1$, $2\varepsilon_1 + 2\delta_1$, $\varepsilon_1 + 3\delta_1$
1	-	$3\varepsilon_1 + 2\delta_1$, $2\varepsilon_1 + 3\delta_1$, ε_1, δ_1
2	$4\varepsilon_1$, $4\delta_1$, $2\varepsilon_1+2\delta_1$, 0	$\varepsilon_1+\delta_1$, 0, 0
3		$3\varepsilon_1$, $3\delta_1$
4		-

The cocycles corresponding to these weights are:

for N = 0:

weights	cocycles
$4\varepsilon_1$	$\mathbf{Wy_1} = e_1{}^2\bullet((f_1'\wedge f_2'\bullet f_2{}^2)$
$4\delta_1$	$\mathbf{Wy_2} = e_{1'}{}^2\bullet((f_1\wedge f_2\bullet f_{2'}{}^2)$
$2\varepsilon_1+2\delta_1$	$\mathbf{Ri} = e_{1'}{}^2\bullet((f_{1'}\wedge f_{2'}\bullet f_2{}^2) + e_1{}^2\bullet((f_1\wedge f_2\bullet f_{2'}{}^2)$
0	$\mathbf{R}=$ $[e_1{}^2\bullet((f_1'\wedge f_2'\bullet f_1{}^2)+2e_1e_2\bullet((f_{1'}\wedge f_{2'}\bullet f_1f_2)+e_2{}^2\bullet((f_1'\wedge f_2'\bullet f_2{}^2)]+$ $[e_{1'}{}^2\bullet((f_1\wedge f_2\bullet f_{1'}{}^2)+2e_{1'}e_{2'}\bullet((f_1\wedge f_2\bullet f_1'f_2')+e_{2'}{}^2\bullet((f_1\wedge f_2\bullet f_{2'}{}^2)]$

for N = 1:

weights	cocycles
$3\varepsilon_1 + \delta_1$	$T_1 = (e_1'\bullet e_1')f_2{}^2$
$2\varepsilon_1 + 2\delta_1$	$T_2 = (e_1\bullet e_1)\bullet f_2f_2'$
$\varepsilon_1 + 3\delta_1$	$T_3 = (e_1'\bullet e_1)\bullet f_2'{}^2$
$3\varepsilon_1 + 2\delta_1$	$\mathbf{To_1} = (e_1\bullet e_1')\bullet(f_2\wedge(f_2'\bullet f_2)$
$2\varepsilon_1 + 3\delta_1$	$\mathbf{To_2} = (e_1'\bullet e_1)\bullet(f_2'\wedge(f_2'\bullet f_2)$
ε_1	$\mathbf{To_3} = \Sigma_j\Sigma_i(e_j'\bullet e_i)\bullet(f_i\wedge(f_j'\bullet f_2))-$ $\Sigma_i(e_i'\bullet e_1)\bullet(f_1\wedge(f_i'\bullet f_2)+f_2\wedge(f_i'\bullet f_1))$
δ_1	$\mathbf{To_4} = \Sigma_j\Sigma_i(e_j'\bullet e_i)\bullet(f_i\wedge(f_2'\bullet f_j))-$ $\Sigma_i(e_i'\bullet e_1)\bullet(f_1\wedge(f_2'\bullet f_i)+f_2\wedge(f_1'\bullet f_2))$
$\varepsilon_1+\delta_1$	$\mathbf{Wy} = \Sigma_i (e_1e_i)\bullet f_if_2'+\Sigma_i e_{1'}e_i'\bullet f_if_2' +$ $\Sigma_i e_1e_i\bullet f_if_2'+\Sigma_i e_1e_i\bullet f_{i'}f_2 -$ $\Sigma_ie_if_i\bullet(f_2\bullet f_2')/2-e_1f_1\bullet(f_2'\bullet f_2)+e_1f_2\bullet(f_2'\bullet f_1)+$ $\Sigma_ie_i'f_i'\bullet(f_2'\bullet f_2)/2+e_1'f_1'\bullet(f_2'\bullet f_2)-e_1'f_2'\bullet(f_1'\bullet f_2)$
0	$\mathbf{R} = a[e_1'\bullet f_1\bullet(f_1'\bullet f_2)+e_{2'}\bullet f_1\bullet(f_2'\bullet f_2)-e_1'\bullet f_2\bullet(f'\bullet f_1)-$ $e_2'\bullet f_2\bullet(f_2'\bullet f_1)-$ $e_1\bullet e_1\bullet(f_1\bullet f_1)-2e_1\bullet e_2\bullet(f_1\bullet f_2)- e_2\bullet e_2\bullet(f_2\bullet f_2)] +$ $b[e_1\bullet f_1'\bullet(f_2'\bullet f_1)+e_2\bullet f_1'\bullet(f_2'\bullet f_2)- e_1\bullet f_2'\bullet(f_1'\bullet f_1)-$ $e_2\bullet f_2'\bullet(f_1'\bullet f_2)-$ $e_1'\bullet e_{1'}\bullet(f_1'\bullet f_1')-2e_1'\bullet e_2'\bullet(f_1'\bullet f_2')- e_{2'}\bullet e_{2'}\bullet(f_2'\bullet f_2')]$
$3\varepsilon_1$	$\mathbf{Ri_1} = e_1\bullet(f_1'\wedge f_2')\bullet f_2{}^2+e_1{}^2\bullet f_1'\bullet(f_2'\bullet f_2)-e_1{}^2\bullet f'\bullet(f_1'\bullet f_2)$

δ_1	$\mathbf{Ri}_2 = c_1' \otimes (f_1 \wedge f_2) \otimes f_2^2 + c_1'^2 \otimes f_1 \otimes (f_2' \otimes f_2) - c_1^2 \otimes f_2' \otimes (f_1' \otimes f_2)$

To interpret the supergravity in the same way as we have treated the Einstein equations [LPS], define the supergravity equation as follows. On $^\wedge\mathcal{M}(N)$, the stationary subgroup of a point (which coincides with $^\wedge G$) preserves

$\varepsilon_L \otimes \mathrm{vol} \otimes \varepsilon_R$, where ε_L and ε_R are spinorial metrics on the left and right chiral superspaces [OS] and vol is the volume element preserved by SL(N).

Now set similarly to (EE_0)

$$\mathbf{R} = \lambda \varepsilon_L \otimes \mathrm{vol} \otimes \varepsilon_R. \qquad (EE_0(\lambda))$$

The tensor **R** depends on a parameter, the ratio a/b which runs the projective line $\mathbf{P}^1$. Physicists call this parameter the *Gates-Sigel parameter*.

Notice immediately, that before considering $(EE_0(\lambda))$ we must vanish all SFs of lesser orders. This gives us the constraints:

$$\mathbf{T}_i = 0 \text{ and } \mathbf{T}_{0j} = 0.$$

Analogues of (EE_{ric}) seem to be [to write a differential equation from the above data is a separate problem that will be dealt with in a separate publication] any or the both of the equations

$$\mathbf{Ri}_j = 0,$$

which are well-defined as integrability condition provided the constraint

$$\mathbf{W}_y = 0$$

takes place.

Different choices correspond to different supergravities (minimal, flexible, etc.). The calculations are pretty bothersome and will be published elsewhere; in the continuation of this paper we will list SFs for $\mathcal{M}(N)$ with N<9.

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