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WORKSHOP ON MATHEMATICAL PHYSICS AND GEOMETRY

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Linear superalgebra and supermanifolds (II)

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These are preliminary lecture notes, intended only for distribution to participants

Second Lecture

II Introduction to Supermanifolds

Recall: Let $A \in \mathrm{SCommSAlg}$, and $J = (A_1) = \left\{ \begin{matrix} \text{ideal gen. by} \\ \text{odd elements} \end{matrix} \right\}$

Then, A/J is a commutative algebra (in the usual sense) and one has a canonical projection defined $A \to A/J$.

Vague Question: To what extent is A determined by A/J?
(Exercise: Try an answer when $J^2 = \{0\}$)

This is where "filtrations" come in:

Note that,

$$J = \Big\{ \sum \underset{\in A_0}{f_j}\, \overset{\in A_1}{\zeta_j} + \sum \underset{\in A_1}{\xi_i}\, \overset{\in A_1}{\zeta_i} \Big\} = \Big\{ \sum f_j \zeta_j \Big\} + J^2,$$

and,

$$J^2 = \Big\{ \sum f_{ij}\, \zeta_i \zeta_j \Big\} + J^3, \dots \text{ and so on.}$$

The picture is the following:

$$\begin{tikzcd} A \arrow[d] & \supset & J \arrow[d] & \supset & J^2 \arrow[d] & \supset & \cdots \\ A/J & & J/J^2 & & J^2/J^3 & & \cdots \end{tikzcd}$$

This contains purely even elements

This contains purely odd elements

Assume that $\forall a \in A - \{0\}$ $\exists!$ integer $k(a)$ such that
$a \in J^{k(a)}$, and $a \notin J^{k(a)}$.

Then, the filtration above can be used in the following manner:

$$\begin{tikzcd} a \in A \rightsquigarrow a \in J? \arrow[r, "Y"] \arrow[d, "N"] & a \in J^2? \arrow[r, "Y"] \arrow[d, "N"] & a \in J^3? \cdots \arrow[d, phantom, "\vdots"] \\ {[a]_0 \in A/J} & {[a]_1 \in J/J^2} & {} \end{tikzcd}$$

This defines a map

$$A \xrightarrow{g_A} \bigoplus_{j \geq 0} J^j/J^{j+1} = \mathrm{Gr}\, A$$
$$a \longmapsto [a]_{k(a)}$$

Exercise: $\mathrm{Gr}\,A$ is in fact a $\mathbb{Z}$-graded algebra; the multiplication map is defined so as to yield,

$$g_A(ab) = g_A(a)\, g_A(b),$$

and

$$g_A(1_A) = 1_{\mathrm{Gr}A}$$

The correspondence $A \longrightarrow \mathrm{Gr}A$ is natural and functorial:

$$\begin{tikzcd} A \arrow[d, "\varphi"] & \rightsquigarrow & A \arrow[r, "g_A"] \arrow[d, "\varphi"] & \mathrm{Gr}A \arrow[d, "\mathrm{Gr}\varphi"] \\ B & & B \arrow[r, "g_B"] & \mathrm{Gr}B \end{tikzcd}$$

Besides, note that from the definition of g_A,

(i) $Gr^j A$ is a $Gr^0 A$-module, $\forall j$

(ii) $Gr^0 A$ consists of purely even elements,
$Gr^1 A$ consists of purely odd elements.

(iii) $Gr A$ is SComm (because A is)

(iv) $Gr A$ is generated over $Gr^0 A$ by $Gr^1 A$

(v) $Gr A \xrightarrow{\varepsilon} Gr^0 A$ (projection onto the 0^{th} component) commutes with multiplication.

The universal object with these characteristics is just $\Lambda_{Gr^0 A} Gr^1 A$; the exterior algebra of the $Gr^0 A$-module $Gr^1 A$. Thus, we have the following algebraic picture.

$$\begin{tikzcd}
 & \Lambda_{Gr^0 A} Gr^1 A \arrow[d] & \\
A \arrow[r] \arrow[dr, "\Delta"'] & Gr A \arrow[d, dashed, "\varepsilon"] \arrow[u, dashed, bend right] & \\
 & A/J &
\end{tikzcd}$$

→ Exercise:
This is an isomorph. if $Gr^1 A$ is a free $Gr^0 A$-module and $J^n = \{0\}$ for some n.

Remark: Note that the SAlg structures in $\Lambda_{Gr^0 A} Gr^1 A$ and $Gr A$ are induced by their $\mathbb{Z}$-gradation mod(2).

Questions: $Gr A \simeq Gr B \Longrightarrow A \simeq B$?

Given $\begin{matrix} A \\ \varphi \downarrow \\ B \end{matrix} \rightsquigarrow \begin{matrix} Gr A \\ \downarrow Gr\varphi \\ Gr B \end{matrix}$, can we change φ without altering $Gr\varphi$

The answers, in general, are: NO, and YES, respectively, and therefore the study of the category of Supercomm. SAlgs., cannot be subsumed into the study of their corresponding graded algebras.

Dfn. (following ref. Leites, Russ. Math. Surv. 35 (1980) 1-64 and, Manin, Gauge Field Theory and Complex Geometry, Springer-Verlag (1988)).

A real smooth (ℂ-analytic, algebraic) spmnfld is a ringed space

$(M, \mathcal{O}_M)$ $\begin{cases} M \text{ top. Hausdorff, mnfld} \\ \mathcal{O}_M \text{ sheaf of } \mathbb{R}\text{-SprAlgs} \end{cases}$

with the following conditions imposed on $\mathcal{O}_M$:

(i) $\forall x \in M$, the stalk $\mathcal{O}_{M,x}$ is a local ~~ring~~ super-ring (ie, it has a unique maximal ideal $\mathfrak{m}_x$)

(ii) The sheaf $Gr^0 \mathcal{O}_M$ is isomorphic to C^∞_M, and one has a sheaf morphism defined,

$$\mathcal{O}_M \xrightarrow[\Delta]{} C^\infty_M = \mathcal{O}_M / \mathcal{J}_M \quad ; \quad \mathcal{J}_M = (\mathcal{O}_{M_1})$$

(iii) $Gr^1 \mathcal{O}_M$ is a locally free sheaf of $Gr^0 \mathcal{O}_M$-modules of finite rank over M (and the rank is called the odd dimension of the supermanifold)

(iv) $\forall x \in M$, $\exists$ open nbd $U \ni x$, and an isomorphism of sheaves of supercommutative superalgebras over U,

$$\varphi_U : \mathcal{O}_M|_U \to Gr\, \mathcal{O}_M|_U ,$$

such that $\quad \varepsilon \circ \varphi_U = \Delta$

<u>Comments on the definition.</u>

(i) The "local model" for a supermanifold is

$$(U,\ C^\infty|_U \otimes \Lambda[\zeta^1, \ldots, \zeta^n])$$

with $U \subset \mathbb{R}^m$ an open domain, and $\{\zeta^1, \ldots, \zeta^n\}$ being a set of free generators of $Gr^1 \mathcal{O}_M$ over $C^\infty(U)$.

ie, $f \in \mathcal{O}_M(U) = C^\infty(U) \otimes \Lambda[\zeta^1, \ldots, \zeta^n]$ can be uniquely written in the form

$$f = \tilde{f} + \sum f_\mu \zeta^\mu + \sum f_{\mu\nu} \zeta^\mu \zeta^\nu + \cdots + f_{12\ldots n} \zeta^1 \cdots \zeta^n ,$$

with $\tilde{f}, f_\mu, \ldots \in C^\infty(U)$. Thus,

Superfunctions over U look exactly as sections over U of the exterior algebra bundle of a vector bundle

However, this DOES NOT mean that the category of supermanifolds is the same as the category of vectorbundles: Morphisms are different!

<u>Remark</u>: For real smooth supermanifolds (with underlying M assumed paracompact), partitions of unity exist and it is globally true that

$$\mathcal{O} \simeq \Gamma(\Lambda\, Gr^1 \mathcal{O}, \cdot),$$

but not so for complex supermanifolds.

In spite of this, there are more morphisms in the category of supermanifolds. We shall clarify this point shortly.

Dfn. A SprMnfld Morphism $\varphi : (M, \mathcal{O}_M) \longrightarrow (N, \mathcal{O}_N)$ is a pair $\varphi = (\tilde{\varphi}, \varphi^{\#})$, with

$\tilde{\varphi} : M \longrightarrow N$ continuous & $\varphi^{\#} : \mathcal{O}_N \longrightarrow \tilde{\varphi}_* \mathcal{O}_M$ sheaf morphism over N (local on each stalk)

Parenthesis. Recall how the 'direct image sheaf $\tilde{\varphi}_* \mathcal{O}_M$' is defined over N; at presheaf level:

$\forall U \subset N$ open $\quad \tilde{\varphi}_* \mathcal{O}_M (U) = \mathcal{O}_M(\tilde{\varphi}^{-1}(U))$

Dfn. A Supercoordinate system for the supermanifold $(M, \mathcal{O}_M)$ consists of an open nbd $U \subset M$, and a collection of homogeneous sections in $\mathcal{O}_M(U)$

$\{f^1, \ldots, f^m, \zeta^1, \ldots, \zeta^n\}$

with

(i) $f^i \in \mathcal{O}_M(U)_0$, $\zeta^\mu \in \mathcal{O}_M(U)_1$

(ii) $\{\tilde{f}^1, \ldots, \tilde{f}^m\}$ a coord system on U (in the usual sense)

(iii) $\{\zeta^1, \ldots, \zeta^n\}$ is maximal among all collections of odd sprfunctns on U, s.t. $\zeta^1 \cdots \zeta^n \neq 0$.

Remark: It must be clear from the definitions that supercoordinate systems always exist.

The following results, due to LEITES, combine ~~...~~ morphisms and coordinate systems as in the C^∞ theory.

(i) If $(\tilde{\varphi}, \varphi^{\#}) : (M, \mathcal{O}_M) \rightarrow (N, \mathcal{O}_N)$ is a morphism, then $\tilde{\varphi} : M \rightarrow N$ is smooth.

(ii) A Superdomain $(U, C^\infty|_U \otimes \Lambda[\zeta^1, \ldots, \zeta^n])$, $U \subset \mathbb{R}^m$, can be recovered from a knowledge of the superalgebra $\mathcal{O}(U) = C^\infty(U) \otimes \Lambda[\zeta^1, \ldots, \zeta^n]$ (In fact, one recovers the points of U and the topology).

(iii) A Superdomain Morphism

$$\varphi : (U, C^\infty|_U \otimes \Lambda[\zeta^1, \ldots, \zeta^n]) \longrightarrow (V, C^\infty_V \otimes \Lambda[\xi^1, \ldots, \xi^q])$$
$$U \subset \mathbb{R}^m \qquad V \subset \mathbb{R}^p$$

is uniquely determined by the sections

$$\varphi^{\#} g^i \in \left(C^\infty(\tilde{\varphi}^{-1}(V)) \otimes \Lambda[\zeta^1, \ldots, \zeta^n]\right)_0$$
$$\varphi^{\#} \xi^\mu \in \left(C^\infty(\tilde{\varphi}^{-1}(V)) \otimes \Lambda[\zeta^1, \ldots, \zeta^n]\right)_1$$

(iv) One may prescribe arbitrarily

p even sections $f^i \in C^\infty(\tilde{\varphi}^{-1}(V)) \otimes \Lambda[\zeta^1,\dots,\zeta^n]$,

and q odd sections $\chi^\mu \in C^\infty(\tilde{\varphi}^{-1}(V)) \otimes \Lambda[\zeta^1,\dots,\zeta^n]$,

and show that $\exists!$ Superdomain morphism $\varphi = (\tilde{\varphi}, \varphi^\#)$ with

$$\varphi^\# g^i = f^i \quad \text{and} \quad \varphi^\# \xi^\mu = \chi^\mu.$$

Back to our promise. Maps between supermanifolds are more general than maps between vector bundles.

Example: $\mathbb{R}^{1|2} = (\mathbb{R}, C^\infty|_{\mathbb{R}} \otimes \Lambda[\zeta^1,\zeta^2]) \circlearrowleft \varphi$

given on local coordinates by

$$\varphi^\# x = x + g(x)\zeta^1\zeta^2 \quad , \quad g \in C^\infty(\mathbb{R}),\ g \neq 0.$$

$$\varphi^\# \zeta^1 = \zeta^1$$

$$\varphi^\# \zeta^2 = \zeta^2$$

Consider $\varphi^\# : C^\infty(\mathbb{R}) \otimes \Lambda[\zeta^1,\zeta^2] \circlearrowleft$ defined by this morphism: it is easy to see that

$$\forall f \in C^\infty(\mathbb{R}) \qquad \varphi^\# f = f + g f' \zeta^1\zeta^2 .$$

$\Longrightarrow \varphi^\#$ is not a $C^\infty(\mathbb{R})$-module morphism!

More Examples:

1- Every supermanifold $(M, \mathcal{O}_M)$ comes equipped with the supermanifold morphism

$$\delta : (M, C^\infty_M) \longrightarrow (M, \mathcal{O}_M)$$

uniquely determined by the canonical projection

$$\mathcal{O}_M(U) \longrightarrow \left(\mathcal{O}_M/\mathcal{J}_M\right)(U) = C^\infty(U)$$

$$f \longmapsto \delta^*(f) = \tilde{f}.$$

2- (Points). The $(0,0)$-dimensional supermanifold $(\{*\}, \mathbb{R})$. For each point $p \in M$, with $(M, \mathcal{O}_M)$ a supermanifold, there is the morphism

$$\delta_p : (\{*\}, \mathbb{R}) \longrightarrow (M, \mathcal{O}_M)$$

given by $$\delta_p^\# f = \begin{cases} \tilde{f}(p) & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

for each $f \in \mathcal{O}_M(U)$.

3.- $(\{*\}, \mathbb{R})$ is a Terminal Object ; ie, $\exists$ only one constant morphism from any supermanifold into it.

$$c_{(M,\mathcal{O}_M)} : (M, \mathcal{O}_M) \longrightarrow (\{*\}, \mathbb{R})$$

defined by

$$c^{*}\lambda = \lambda \mathbb{1}_{\mathcal{O}_M(M)} \quad ; \forall \lambda \in \mathbb{R}.$$

4.- $(0,n)$-dimensional manifolds. (A remark of LEITES).

$(\{*\}, \Lambda\mathbb{R}^n)$ is a Supermanifold of dimension $(0,n)$.

One might think of $\mathrm{Aut}_{SAlg}(\Lambda\mathbb{R}^n)$ as the automorphism group of this supermanifold.

Note that $\varphi \in \mathrm{Aut}_{SAlg}(\Lambda\mathbb{R}^n)$ is always even

(ie, $\varphi : (\Lambda\mathbb{R}^n)_0 \circlearrowright$ and $\varphi : (\Lambda\mathbb{R}^n)_1 \circlearrowleft$).

On the other hand, by looking at the Lie SAlg of infinitesimal automorphisms,

$$\mathfrak{g} := \mathrm{Der}\, \Lambda\mathbb{R}^n$$

one finds $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and $\mathfrak{g}_1 \neq \{0\}$.

Thus, odd parameters must come into the description of $\mathrm{Aut}_{SAlg}(\Lambda\mathbb{R}^n)$?

Dfn. Let A be an $\mathbb{F}$-SAlg, and $\mathrm{End}_{\mathbb{F}}A$ be the (left) A-Supermodule of $\mathbb{F}$-linear maps $A \to A$.

$$\mathrm{Der}\,A = \left\{ X \in \mathrm{End}_{\mathbb{F}}A \;\middle|\; \begin{array}{l} X = X_0 + X_1 \text{ , and} \\ X_\mu(ab) = X_\mu(a)\,b + (-1)^{\mu|a|}\,a\,X_\mu(b) \;;\; a \in A \text{ homog.} \end{array} \right\}$$

Example: Determination of $\mathrm{Aut}\,\mathbb{R}^{0|1}$.

Structural Sheaf: $\{*\} \longmapsto \Lambda[\zeta]$ (Exterior algebra in one generator.)

$$\Rightarrow \mathrm{Der}\,\Lambda[\zeta] = (\mathrm{Der}\,\Lambda[\zeta])_0 \oplus (\mathrm{Der}\,\Lambda[\zeta])_1$$
$$= \{\lambda_0 \zeta \partial_\zeta \mid \lambda_0 \in \mathbb{R}\} \oplus \{\lambda_1 \partial_\zeta \mid \lambda_1 \in \mathbb{R}\}$$

Get an idea of what $\mathrm{Aut}\,\mathbb{R}^{0|1}$ by formal exponentiation:

$$\left.\begin{array}{l} (\mathrm{Exp}\,\lambda_0 \zeta\partial_\zeta)(1) = 1 \\ (\mathrm{Exp}\,\lambda_0 \zeta\partial_\zeta)(\zeta) = e^{\lambda_0}\zeta \end{array}\right\} \lambda_0 \in \mathbb{R}$$

In general, one has $\varphi_a \in \mathrm{Aut}_{SAlg}\Lambda[\zeta]$ given by,

$$1 \longmapsto 1 \quad \& \quad \zeta \longmapsto a\zeta \quad ; \quad a \in \mathbb{R}-\{0\}$$

depending on the real parameter a. However, one also finds,

$$(\mathrm{Exp}\,\lambda_1\partial_\zeta)(1) = 1 \quad \& \quad (\mathrm{Exp}\,\lambda_1\partial_\zeta)(\zeta) = \zeta + \lambda_1$$

The point is that THIS DEFINES a $\varphi_{\lambda_1} \in \mathrm{Aut}_{SAlg}\Lambda[\zeta]$, iff $\lambda_1^2 = 0$ & $\lambda_1\zeta + \zeta\lambda_1 = 0$; i.e., λ_1 is an ODD PARAMETER and $\mathrm{Aut}_{SAlg}\Lambda[\zeta]$ is $(1,1)$-dim'l.

THIRD LECTURE

III Geometric constructions based on local coordinate computations.

Dfn. A Lie Supergroup is a finite dimensional supermanifold, $(G, \mathcal{A}_G)$ with the following additional structure.

(i) A morphism $\mu : (G, \mathcal{A}_G) \times (G, \mathcal{A}_G) \longrightarrow (G, \mathcal{A}_G)$

satisfying the associativity property

$$\mu\circ(\pi_1 \times \mu\circ(\pi_2 \times \pi_3)) = \mu\circ(\mu\circ(\pi_1\times\pi_2)\times\pi_3)$$

(ii) A distinguished point $e \in G$; Hence, a distinguished morphism

$$\varepsilon_e = \delta_e \circ C : (G, \mathcal{A}_G) \longrightarrow (G, \mathcal{A}_G)$$

satisfying $\mu\circ(id \times \varepsilon_e) = id = \mu\circ(\varepsilon_e \times id)$

(iii) An involutive superdiffeomorphism

$$\sigma : (G, \mathcal{A}_G) \longrightarrow (G, \mathcal{A}_G)$$

satisfying $\mu\circ(id \times \sigma) = \varepsilon_e = \mu\circ(\sigma \times id)$.

[REF: OASV; Trans. Amer. Math. Soc. 307 (1988) 569-595.
BOYER, OASV, Trans A.M.S. 323 (1991) 151-175.]

Examples:

1] $\mathbb{R}^{m|n} = (\mathbb{R}^m, C^\infty_{\mathbb{R}^m} \otimes \Lambda[\theta^1, \dots, \theta^n])$

Linear coords: given $\{e_1, \dots, e_m\}$, basis of $\mathbb{R}^m$, the dual basis $\{x^1, \dots, x^m\}$ is the even part of the coord. system.

Define $\mu_s : \mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \longrightarrow \mathbb{R}^{m|n}$ (supersum morphism) in terms of the corresponding morphism of superalgebras

$$\mu_s^\# : C^\infty(\mathbb{R}^m) \otimes \Lambda[n] \longrightarrow C^\infty(\mathbb{R}^m \times \mathbb{R}^m) \otimes \Lambda[2n]$$

$$\left.\begin{aligned} \mu_s^\# x^i &= \pi_1^\# x^i + \pi_2^\# x^i \\ \mu_s^\# \theta^\nu &= \pi_1^\# \theta^\nu + \pi_2^\# \theta^\nu \end{aligned}\right\} \text{"Supersum morphism"}$$

where $\pi_j : \mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \longrightarrow \mathbb{R}^{m|n}$ is the projection morphism onto the j^{th} factor

Remark: Products exist in the category of supermanifolds. Locally, if $\{x^1, \dots, x^m; \theta^1, \dots, \theta^n\}$ and $\{y^1, \dots, y^p; \eta^1, \dots, \eta^q\}$ are supercoordinate systems on the domains $(U, C^\infty_{\mathbb{R}^m}|_U \otimes \Lambda[n])$, and $(V, C^\infty_{\mathbb{R}^p}|_V \otimes \Lambda[q])$, then $(U, C^\infty_{\mathbb{R}^m}|_U \otimes \Lambda[n]) \times (V, C^\infty_{\mathbb{R}^p}|_V \otimes \Lambda[q])$ is a superdomain with coordinates

$$\left\{ \pi_1^\# x^1, \dots, \pi_1^\# x^m, \pi_2^\# y^1, \dots, \pi_2^\# y^p ; \pi_1^\# \theta^1, \dots, \pi_1^\# \theta^n, \pi_2^\# \eta^1, \dots, \pi_2^\# \eta^q \right\}$$

Exercise: Taking the distinguished pt. $e = (0, \dots, 0) \in \mathbb{R}^m$ and $\sigma_s^\# : C^\infty(\mathbb{R}^m) \otimes \Lambda[n] \longrightarrow C^\infty(\mathbb{R}^m) \otimes \Lambda[n]$, as $\sigma_s^\# x^i = -x^i$; $\sigma_s^\# \theta^\mu = -\theta^\mu$

$\mathbb{R}^{m|n}$ acquires the structure of an abelian Lie supergroup.

2] $(\mathbb{R}^{1|1})^\bullet = (\mathbb{R}-\{0\}, C^\infty_{\mathbb{R}}|_{\mathbb{R}-\{0\}} \otimes \Lambda[\theta])$

Define $\mu_m : (\mathbb{R}^{1|1})^\bullet \times (\mathbb{R}^{1|1})^\bullet \longrightarrow (\mathbb{R}^{1|1})^\bullet$ by means of

$$\mu_m^\# : C^\infty(\mathbb{R}-\{0\}) \otimes \Lambda[\theta] \longrightarrow C^\infty(\mathbb{R}-\{0\} \times \mathbb{R}-\{0\}) \otimes \otimes \Lambda[\pi_1^\#\theta, \pi_2^\#\theta],$$

via,

$$\left.\begin{aligned} \mu_m^\# x &= \pi_1^\# x\, \pi_2^\# x + \pi_1^\# \theta\, \pi_2^\# \theta \\ \mu_m^\# \theta &= \pi_1^\# x\, \pi_2^\# \theta + \pi_1^\# \theta\, \pi_2^\# x \end{aligned}\right\} \quad \left(\begin{array}{c}\text{supermultiplication} \\ \text{morphism}\end{array}\right)$$

Take the distinguished point to be $1 \in \mathbb{R}-\{0\}$ and $\sigma^\# : C^\infty(\mathbb{R}-\{0\}) \otimes \Lambda[\theta] \longrightarrow C^\infty(\mathbb{R}-\{0\}) \otimes \Lambda[\theta]$, as

$$\sigma^\# x = \frac{1}{x} \quad ; \quad \sigma^\# \theta = -\frac{1}{x^2}\theta$$

<u>Exercise</u>: $(\mathbb{R}^{1|1})^\bullet$ is a Lie supergroup.

<u>Remark</u>: In the above example, the underlying manifold has to be $\mathbb{R}-\{0\}$ so that $\sigma^\#$ makes sense. However, one may consider only μ_s & μ_m for the supermanifold $\mathbb{R}^{1|1}$.

<u>Proposition</u>: The morphisms μ_m and μ_s give to $\mathbb{R}^{1|1}$ the structure of an abstract superalgebra (associativity and distributivity being stated as commutative diagrams of morphisms)

<u>First consequence</u>: View superfunctions as Supermorphisms.

Let $(U, \mathcal{O})$ be some superdomain. Then, $\exists$ a 1-1 correspondence

$$\mathcal{O}(U) \longleftrightarrow \mathrm{SMaps}((U,\mathcal{O}), \mathbb{R}^{1|1})$$

and the correspondence can be made into a map of superalgebras. The superalgebra structure in

$$\mathrm{SMaps}((U,\mathcal{O}), \mathbb{R}^{1|1})$$

is given by

$$\varphi + \psi := \mu_s \circ (\varphi \times \psi) \quad \& \quad \varphi\psi := \mu_m \circ (\varphi \times \psi).$$

The correspondence $\mathcal{O}(U) \leftrightarrow \mathrm{SMaps}((U,\mathcal{O}), \mathbb{R}^{1|1})$ is given by

$$\mathcal{O}(U) \ni f \longmapsto \varphi_f \text{ such that } \begin{cases} \varphi_f^\# t = f_0 \\ \varphi_f^\# \theta = f_1 \end{cases}$$

<u>Second consequence</u>: Typical <u>superfiber</u> of supervectorbundles.

<u>Recall</u>: in C^∞ category,

Vectorbundles and morphisms (over a fixed M)	$\longleftrightarrow$	Locally free sheaves of C^∞-modules (over M)

In fact, this is done by means of

$$\left\{\begin{array}{c}\text{Maps from a}\\ \text{trivial nbd } U \subset M\\ \text{into the fiber}\\ \text{— typically } \mathbb{R}^p.\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Sections of}\\ \text{rank } p\\ \text{vector bundle}\\ \text{over } U \subset M\end{array}\right\} \left\{\begin{array}{c}\text{free } C^\infty(U)\\ \text{module of}\\ \text{rk } p\end{array}\right.$$

Supermanifold analogue: Let $(U, \mathcal{O})$ be some superdomain.

Let $V_0 \oplus V_1 = \mathbb{R}^p \oplus \mathbb{R}^q$ be some supervector space.

Then,

$$\left\{\begin{array}{c}\text{free } \mathcal{O}(U)\\ \text{module of}\\ \text{rank } (p,q)\end{array}\right\} = \underbrace{\mathcal{O}(U) \otimes (\mathbb{R}^p \oplus \mathbb{R}^q) \longleftrightarrow S\mathrm{Maps}\big((U,\mathcal{O}), \mathbb{R}^{p+q|p+q}\big)}_{\text{correspondence of } \mathcal{O}(U)\text{-modules}}$$

$\therefore$ Typical superfiber of rk (p,q) SVect bundle $\simeq \mathbb{R}^{p+q|p+q}$

This can be applied to the following situation:

Let $(M, \mathcal{O})$ be an (m,n)-dim'l supermanifold. Then, $\mathrm{Der}\,\mathcal{O}$ is a locally free sheaf of $\mathcal{O}$-modules over M of rank (m,n) (cf. Leites Russ. Math. Surv. 1980)

$\Longrightarrow$ The corresponding supervector bundle has dimension

$$\dim(\text{base}) + \dim(\text{fiber}) = (m,n) + (m+n, m+n)$$
$$= (2m+n, 2n+m).$$

Remark: Note that the association

$$\begin{array}{c}(m,n)\text{-dim'l}\\ \text{SVect spaces}\end{array} \rightsquigarrow \begin{array}{c}(m,n)\text{-dim'l}\\ \text{Superdomains}\end{array}$$

$$V = V_0 \oplus V_1 \longmapsto SV := \big(V_0, C^\infty|_{V_0} \otimes \Lambda(V_1^*)\big)$$

can be supplemented with a corresponding assignment of morphisms, so as to get a functor

$$\mathrm{Hom}(V,W)_0 \ni F \longmapsto SF \in S\mathrm{Maps}(SV, SW)$$

This is a morphism of vector bundles:

$$(SF)^\# = \mathrm{Gr}\,(SF)^\#$$

However, the construction above suggests another correspondence:

$$V = V_0 \oplus V_1 \longmapsto V_s := \big(V_0 \oplus V_1, C^\infty|_{V_0 \oplus V_1} \otimes \Lambda(V_0^* \oplus V_1^*)\big)$$

This time we may have

$$\mathrm{Hom}(V,W) \ni F \longmapsto F_s \in S\mathrm{Maps}(V_s, W_s)$$

and F_s is "superlinear"; ie., commutes with sums and scalar multiplication defined on V_s and W_s via μ_s and μ_m (supersum and supermultipl.) componentwise.

Superlinear maps are not very interesting:

Prop. A superlinear map $V_s \to W_s$ is completely determined by its underlying continuous map $V \to W$, and the latter has to be linear

What is interesting is the posibility of defining a "superbilinear map"

$$\mathrm{Hom}(V,W)_S \times V_S \longrightarrow W_S$$

which is no longer a map of the underlying C^∞-vector bundles of the supermanifolds involved, and nevertheless, for each $F \in \mathrm{Hom}(V,W)$ (ie, each point of the underlying manifold of $\mathrm{Hom}(V,W)_S$) the restricted morphism $V_S \longrightarrow W_S$ is superlinear.

In particular, this give the possibility of looking at $\mathrm{Hom}(V,V)_S \times V_S \longrightarrow V_S$

and restrict attention to the subsupermanifold

$$GL(V_0|V_1)_S = \left(GL(V_0 \oplus V_1),\ \text{Sheaf of } \mathrm{Hom}(V,W)_S \big|_{GL(V_0\oplus V_1)}\right)$$

of dimension $((m+n)^2, (m+n)^2)$.

This brings us to the realm of "Linear supergroup actions". In general, one has the following definition

Dfn. Let $(G, \mathcal{O}_G)$ be a Lie Supergroup, and let $(M, \mathcal{O}_M)$ be a supermanifold. $(G, \mathcal{O}_G)$ acts on $(M, \mathcal{O}_M)$ from the left if $\exists$ a morphism

$$\psi : (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \longrightarrow (M, \mathcal{O}_M)$$

satisfying

$$\psi \circ (\pi_1 \times \psi \circ (\pi_2 \times \pi_3)) = \psi \circ (\mu \circ (\pi_1 \times \pi_2) \times \pi_3)$$

$$\psi \circ (\varepsilon_e \times \pi_2) = \pi_2 .$$

Dfn. Let ψ be an action. Then ψ induces a morphism

$$\psi_p : (G, \mathcal{O}_G) \longrightarrow (M, \mathcal{O}_M) \ ; \quad p \in M \text{ fixed},$$

by letting

$$\psi_p = \psi \circ (id \times \varepsilon_p)$$

where $\varepsilon_p = \delta_p \circ C : (G, \mathcal{O}_G) \longrightarrow (M, \mathcal{O}_M)$

is given by

$$\mathcal{O}_M(V) \ni f \longmapsto \varepsilon_p^* f = \begin{cases} f(p)\, 1_{\mathcal{O}_G(V)} & ; \ p \in V \\ 0 & ; \ p \notin V \end{cases}$$

The isotropy subsupergroup of the action may be "defined" ("—" because it is technically somewhat delicate; c.f. Boyer & OASV, Trans. AMS 323 (1991)) as the locus defined by the condition

$$\varepsilon_p = \psi_p$$

Example: $GL(2|2)_S$ acting on $\mathbb{R}^{2|2}$.

$$\begin{array}{cc} & (GL(2),\ C^\infty|_{GL(2)} \otimes \Lambda[4]) \times (\mathbb{R}^2,\ C^\infty|_{\mathbb{R}^2} \otimes \Lambda[2]) \\ \left(\begin{array}{c}\text{action}\\ \text{morphism}\end{array}\right) & \psi \downarrow \\ & (\mathbb{R}^2,\ C^\infty|_{\mathbb{R}^2} \otimes \Lambda[2]) \end{array}$$

Local coordinates:

$$\psi^\# \begin{pmatrix} x+\zeta \\ y+\xi \end{pmatrix} = \begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix} \begin{pmatrix} x+\zeta \\ y+\xi \end{pmatrix}$$

(ommitting references to $\pi_1^\#$ and $\pi_2^\#$)

Now fix some point $p \in \mathbb{R}^2$ and consider

$$\psi_p = \psi \circ (id \times \delta_p \circ c) : GL(2|2)_S \longrightarrow \mathbb{R}^{2|2}$$

Then,

$$\psi_p^\# \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \breve{x}(p) \\ \breve{y}(p) \end{pmatrix}$$

$$\psi_p^\# \begin{pmatrix} \zeta \\ \xi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \breve{x}(p) \\ \breve{y}(p) \end{pmatrix}$$

In particular, note that the map $\tilde{\psi}_p : GL(2) \to \mathbb{R}^2$ is,

$$\tilde{\psi}_p \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \breve{x}(p) \\ \breve{y}(p) \end{pmatrix}$$

Now, consider $\varepsilon_p : GL(2|2)_S \longrightarrow \mathbb{R}^{2|2}$:

$$\varepsilon_p^\# x = \breve{x}(p)\, 1 \qquad \varepsilon_p^\# \zeta = 0$$
$$\varepsilon_p^\# y = \breve{y}(p)\, 1 \qquad \varepsilon_p^\# \xi = 0$$

Therefore, the condition $\psi_p = \varepsilon_p$ yields

$$\psi_p^\# x = a\breve{x}(p) + b\breve{y}(p) = \breve{x}(p) = \varepsilon_p^\# x$$
$$\psi_p^\# y = c\breve{x}(p) + d\breve{y}(p) = \breve{y}(p) = \varepsilon_p^\# y$$
$$\psi_p^\# \zeta = \alpha\breve{x}(p) + \beta\breve{y}(p) = 0 = \varepsilon_p^\# \zeta$$
$$\psi_p^\# \xi = \gamma\breve{x}(p) + \delta\breve{y}(p) = 0 = \varepsilon_p^\# \xi$$

Under the assumption $\breve{x}(p) = 1$ & $\breve{y}(p) = 0$ (choice of p) these equations imply that

$$a = 1, \quad c = 0, \quad \alpha = 0, \quad \gamma = 0$$

These conditions define an embedded subsupergroup of $GL(2|2)_S$; ie, the set of matrices of the form

$$\begin{pmatrix} 1 & b+\beta \\ 0 & d+\delta \end{pmatrix}$$

which gives a $(2,2)$-dimensional subsupergroup.