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WORKSHOP ON MATHEMATICAL PHYSICS AND GEOMETRY

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An introduction to topology of 4-manifolds

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These are preliminary lecture notes, intended only for distribution to participants

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AN INTRODUCTION INTO TOPOLOGY OF 4-MANIFOLDS

(lecture notes*)

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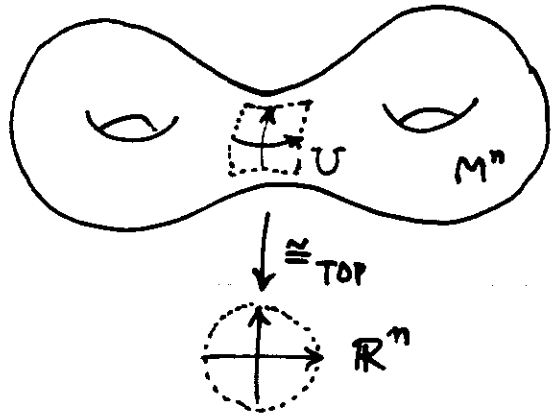
*These are informal lecture notes: the material has been selected from the books and papers listed in the Suggested literature (A) & (B).

Geometric topology : study of TOP manifolds



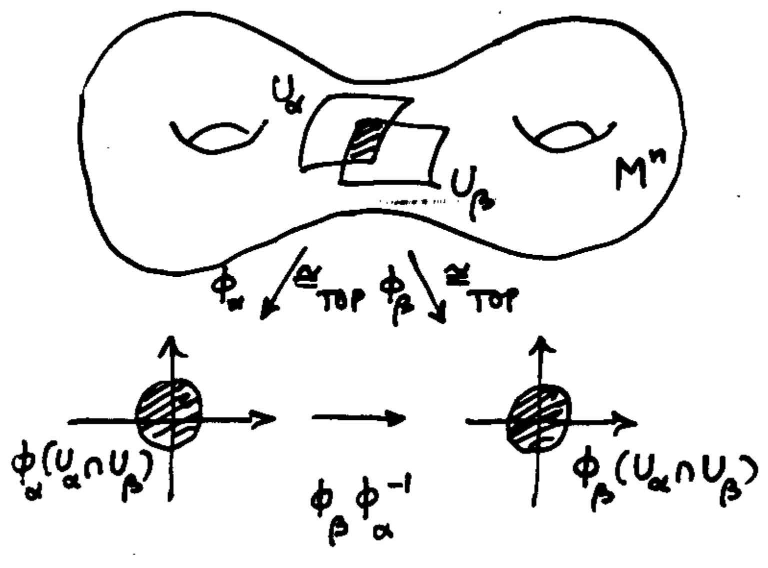
Customary goal : discover (algebraic) invariants $\rightsquigarrow$ classify all manifolds in dim n

2 separate problems : (A) existence : given an invariant
find an n-manifold w/ the inv.
(B) uniqueness : given an invariant
how many n-manifolds have it

TOP mflds too "amorphous" $\to$ must add structure $\to$ get more tools and reduce eq. classes

e.g. $M = \mathbb{R}^n$ study cont. funct. on M
better smooth funct on M

extend the idea $\to$ consider PL, DIFF atlas on M^n



PL $\overset{\cong}{\text{or}}$ DIFF

eq. relation : $\mathcal{A} = \{U_\alpha, \phi_\alpha\} \sim \mathcal{B} = \{V_\beta, \psi_\beta\} :\Leftrightarrow \mathcal{A} \cup \mathcal{B}$ atlas on M^n,
i.e. if $U_\alpha \cap V_\beta \neq \phi$ then $\phi_\alpha \phi_\beta^{-1}$ is smooth. (pl)
DIFF (PL) <u>structure</u> on M is eq. class of atlases on M^n.

<u>Basic problem</u> : When does a TOP mfld admit PL structure
if yes, is there a compatible DIFF structure, too ?

(Whitehead, 1940) DIFF $\subset$ PL
(Classica, e.g. Kerékjártó 1923) $n \leq 2$ DIFF = PL = TOP
(Moise, Bing 1950's) $n = 3$ DIFF = PL = TOP
(Milnor, 1956) $\exists$ 28 DIFF structures on S^7
(1956–1970's : Thom, Kervaire, Milnor, Munkres, Hirsch, Mazur, Poenaru, Lashof, Rothenberg, Haefliger, Smale, Novikov, Browder, Wall, Sullivan, Kirby, Siebenmann) $n \geq 5$ well understood : known obstruction to putting PL/DIFF structure on M^n : ① $k(M) \in H^4(M; \mathbb{Z}_2)$
$k(M) = 0 \Rightarrow \exists$ PL str. and there are $|H^3(M;\mathbb{Z}_2)|$ distinct ones
$k(M) = 1 \Rightarrow \nexists$ PL str.
② $n \leq 7$: Every PL n-mfld admits a compatible DIFF structure which is unique up to DIFF if $n \leq 6$. (homotopy groups of spheres) so DIFF = PL for $n \leq 6$.
③ $n \neq 4$: $\mathbb{R}^n$ has unique PL/DIFF structure whereas $\mathbb{R}^4$ has uncountably many (Taubes, Gompf)

$\boxed{\text{dimension 3}}$

<u>Poincaré Conjecture</u> : M^3 closed, $\pi_1 M = 0 \Rightarrow M^3 \cong S^3$.
GPC $(n \geq 4)$ M^n closed, $M^n \simeq S^n \Rightarrow M^n \cong S^n$.
$n \geq 5$ Smale [redacted] 1960's
$n = 4$ Freedman 1982
$n = 3$ Thurston's Geometrization Conjecture, Hyperbolic 3-manifolds theory

M^4 compact, 1-connected, DIFF 4-manifold
(Note: G fin. pres. group $\Rightarrow \exists$ 4-manifold $N^4 \ni: \pi_1 N \cong G$)
Standard alg. inv.: π_1, H_*, H^*

$$H_i(M) \cong H^{4-i}(M) \quad \text{Poincaré duality}$$

$\Rightarrow \pi_1 M = 0$ implies $H_1(M) = 0$ and $H_3(M) = 0$ hence
<u>all</u> homological information is contained <u>in $H_2(M)$.</u>
Furthermore, UCT for cohom. implies $H^2(X;\mathbb{Z}) = \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ is free ab.
so by PD $\Rightarrow$ <u>$H_2(M)$ is free abelian group.</u>

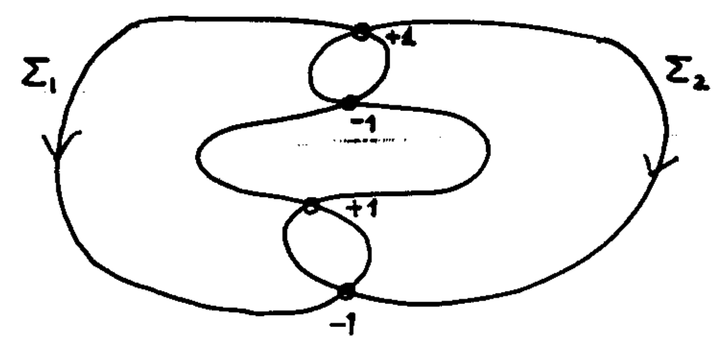
Representation of homology classes in $H_2(M)$: 3 possible ways

(1) By <u>complex</u> <u>line</u> <u>bundles</u>: $\exists$ bijection between the isomorphism classes of line bundles and H^2
(2) By <u>smoothly</u> <u>embedded</u> 2-<u>dim</u> <u>oriented</u> <u>surfaces</u> Σ in M^4:
Σ carries a fundamental homology class $[\Sigma]$ in $H_2(M)$
(3) <u>De Rham</u> <u>representation</u> of real cohomology classes by diff. forms

PD isomorphism $H^2 \cong H_1$ is equivalent (in our case) to a bilinear form

$$H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

called <u>the intersection form</u> of the manifold M^4 (unimodular and symmetric)



Geometrically:
Σ_1, Σ_2 oriented surfaces in general position:
$\Sigma_1 \cap \Sigma_2 = \{t_1, \dots, t_m\}$

To each t_i we associate ± 1 according to the matching of the orient. in the isom

$$TM \cong T\Sigma_1 \oplus T\Sigma_2$$

of the tang. bundle at t_i. $\Rightarrow$ $\Sigma_1 \cdot \Sigma_2 :=$ add the signs $\in \mathbb{Z}$

Some examples:

(1) S^4 Since $H_2(S^4)=0$ all intersection numbers vanish: $I_{S^4}=\emptyset$.

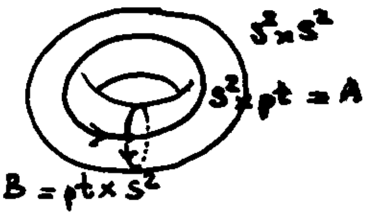
(2) $\mathbb{C}P^2$ Since $H_2(\mathbb{C}P^2;\mathbb{Z})\cong\mathbb{Z}$ and the standard generator is given by the fundamental class of the projective line $\mathbb{C}P^1\subset\mathbb{C}P^2$, the intersection form is represented by: $I_{\mathbb{C}P^2}=(1)$. (Note: $\mathbb{C}P^1=S^2$ the Riemann sphere)

(3) $\overline{\mathbb{C}P^2}:=\mathbb{C}P^2$ with the other orientation. Hence $I_{\overline{\mathbb{C}P^2}}=(-1)$.

(NB: $\nexists$ orient. reversing diffeo of $\mathbb{C}P^2$)

(4) $S^2\times S^2$ Then $H_2(S^2\times S^2;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}$ and we can represent the generators by the embedded surfaces

$$A=S^2\times pt \qquad B=pt\times S^2$$



Now $A\cap B=pt$ and each of A and B can be pushed off itself, so

$$I_{S^2\times S^2}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

i.e. $A\cdot A=B\cdot B=0$, $A\cdot B=B\cdot A=1$.

(5) $M^4\# N^4$ Then $H_2(M^4\# N^4;\mathbb{Z})\cong H_2(M^4;\mathbb{Z})\oplus H_2(N^4;\mathbb{Z})$ hence

$$I_{M\# N}=I_M\oplus I_N=\left(\begin{array}{c|c}I_M&0\\\hline 0&I_N\end{array}\right)$$

(6) The Kummer surface: $K=\{[z_0,z_1,z_2,z_3]\in\mathbb{C}P^3 \mid z_0^4+z_1^4+z_2^4+z_3^4=0\}$

Then rank $H_2(K;\mathbb{Z})$ is 22 and $I_K=E_8\oplus E_8\oplus 3\begin{pmatrix}0&1\\1&0\end{pmatrix}$ where

$$E_8=\begin{pmatrix}2&-1&0&0&0&0&0&0\\-1&2&-1&0&0&0&0&0\\0&-1&2&-1&0&0&0&0\\0&0&-1&2&-1&0&0&0\\0&0&0&-1&2&-1&0&-1\\0&0&0&0&-1&2&-1&0\\0&0&0&0&0&-1&2&0\\0&0&0&0&-1&0&0&2\end{pmatrix}$$ = the Cartan matrix for the exceptional Lie algebra e_8.

Note: For indefinite forms, the rank, signature and type form a complete set of invariants.

The classification of definite forms is more difficult: $\exists$ only one restriction on an even definite form, i.e. its signature must be divisible by 8.

$\exists$ 1 definite even form of rank 8 (namely E_8)

$\exists$ 2 posit. def. even forms of rank 16 ($E_8\oplus E_8$ and E_{16}), $\exists$ 24 of rank 24 etc. ....

In cohomology this translates into <u>cup product</u>

$$H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \xrightarrow{\cup} H^4(M;\mathbb{Z}) \cong \mathbb{Z}$$

therefore the form is an <u>invariant of the oriented homotopy type of M^4</u>.
<u>Such</u> definition of the intersection form one can extend over <u>all</u> TOP 4-manifolds:

$\alpha, \beta \in H^2(M;\mathbb{Z})$, $[M] \in H_4(M;\mathbb{Z})$ fund. class (given by choice of orient)

then $\alpha \cdot \beta := \langle \alpha \cup \beta, [M] \rangle$

(Dictum: Think with intersections, prove with cup products!)

<u>Classical</u> <u>results</u>: (A) <u>Uniqueness</u>

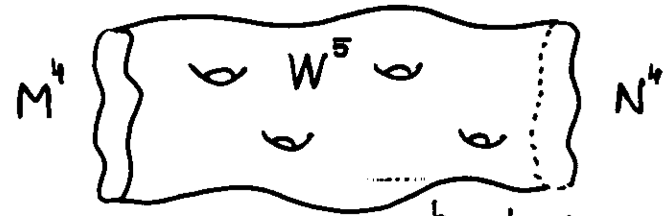
(Milnor, Whitehead, 1958) The <u>oriented</u> <u>homotopy type</u> of a simply connected, compact, oriented 4-manifold is determined by its <u>intersection</u> <u>form</u>.

(Wall, 1964) If M and N are simply connected, smooth, oriented 4-manifolds with <u>isomorphic</u> <u>intersection</u> <u>forms</u>, then for some $k \geq 0$:

$$\exists \text{ diffeomorphism } \quad M \# k(S^2 \times S^2) \cong N \# k(S^2 \times S^2)$$

(stable classif. up to diffeo)

(Wall) Two simply connected 4-manifolds with isomorphic intersection forms are h-cobordant.



h-<u>cobordism</u>

$\partial W = M \sqcup N$

$M \underset{\text{h.e.}}{\hookrightarrow} \partial W \underset{\text{h.e.}}{\hookleftarrow} N$

(Smale' h-cobordism thm) M^n, N^n simply connected n-manifolds, $n \geq 5$
h-cobordant $\Rightarrow M^n \underset{\text{DIFF}}{\cong} N^n$, i.e. $W \underset{\text{DIFF}}{\cong} M \times [0,1]$.

<u>If</u> this were true in dim 4 then <u>simply conn</u> 4-<u>manifold</u> would be <u>determined</u> <u>up to diffeo</u> by its <u>intersection form</u>.

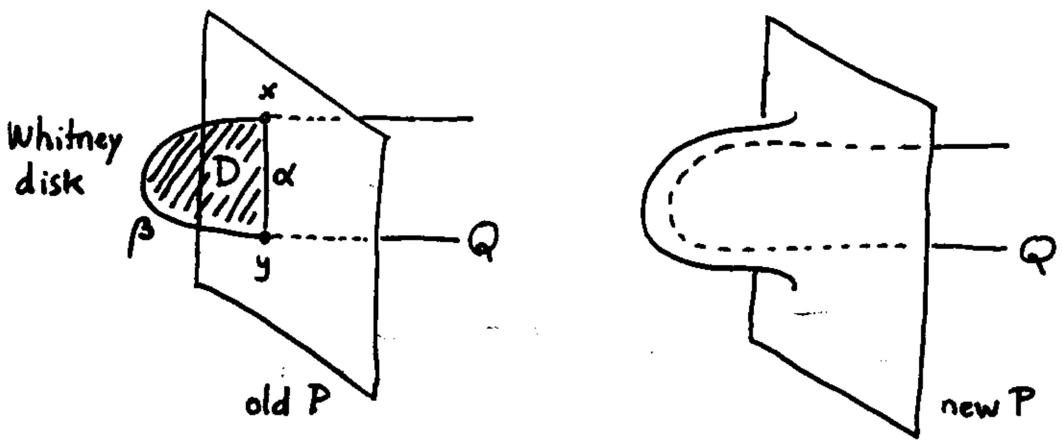
Proof of h-cobordism thm doesn't work in dim 4! Why?

The Whitney lemma: the comparison of geometric and algebraic intersection numbers

P^p, Q^q submanifolds in 1-connected ambient manifold M^m
$p+q=m$

and $P \pitchfork Q$ in finitely many points but geometric and algebraic intersections are different

$\Rightarrow \exists\, x, y \in P \cap Q \quad \operatorname{sign} x = -\operatorname{sign} y$



Choose arcs $\alpha \subset P$ and $\beta \subset Q$ from x to y.

Then $\alpha \cup \beta$ is a loop so in M it bounds a disk ≡ Whitney disk.

If embedded, can use to get an amb. isotopy of M moving P off Q hence cancell the pair (x, y). (need some extra cond on ν_D in M)

If codim $P \geq 3$ and codim $Q \geq 3$ then $\exists$ Whitney disk

More generally, if at least one codim ≥ 3.

∴ if $m \geq 5$ we have Whitney trick (lemma)!

∴ have h-cobordism theorem.

If $m = 4$ then problems occur! P and Q surfaces with possible self intersections.

Classical results: (B) existence

(Rohlin, 1952) The signature of a smooth, compact, spin 4-manifold is divisible by 16.

In our case, <u>spin</u> is just that $w_2 = 0$, hence the form is <u>even</u>.
In particular, E_8 cannot be the intersection form of a smooth 4-manifold.

1980's

(M.H. Freedman, 1982) Closed 1-connected 4-manifolds (TOP !) are completely and faithfully classified by 2 element. pieces of info:

$\boxed{\text{the intersection form}}$ <u>and</u> $\boxed{\text{the Kirby-Siebenmann obstruction } \alpha(M) \in \mathbb{Z}_2}$
i.e. $\alpha(M) = 0$ iff $M \times S^1$ admits DIFF struct.

$$\left\{\begin{matrix}\text{Compact, 1-connected} \\ \text{TOP 4-manifolds}\end{matrix}\right\} \underset{\text{corr.}}{\overset{1-1}{\longleftrightarrow}} \left\{ \langle I, \alpha \rangle , \begin{matrix} I \text{ integral unimodular symmetric bilinear form,} \\ \alpha \in \mathbb{Z}_2, \text{ and} \\ \text{if } I \text{ is even then } \frac{\sigma(I)}{8} \equiv \alpha \pmod 2 \end{matrix} \right\}$$

in particular:

existence: $\forall$ integral unimodular symmetric bilinear form I
$\exists$ TOP 4-manifold M with the intersection form $= I$

uniqueness: If I is even, then the homeom. type of M is uniq. det. by I.
If I is odd, then $\exists$ exactly 2 (homeom.) 4-manifolds realiz. I as their inters. form:
one of them with vanishing α (i.e. $M \times S^1$ admits DIFF str.)
the other one with nonvanishing α.

<u>Corollary</u> (4-dim PC) TOP 4-mfld $\simeq S^4$ is $\underset{\text{TOP}}{\cong} S^4$. (apply uniq. part for $\langle \emptyset, 0 \rangle$.)

<u>Corollary</u> (one of ancient [redacted] problems) $\exists!$ TOP 4-manifold with the inters pairing E_8.

Note: Freedman had extra hypoth. that $M \setminus pt \in$ DIFF. But:
(F. Quinn, 1982) $\forall$ noncompact TOP 4-manifold admit a DIFF structure.

Freedman's proof = surgery + Bing topology
Princeton Texas

In other words, Freedman proved

For TOP 4-manifolds all unimodular quadratic forms can occur.

Then a big surprise came when Donaldson showed

For DIFF 4-manifolds the only positive definite form which occurs is the standard form $\sum x_i^2$.

Dramatic consequences of Freedman $\oplus$ Donaldson:

$\exists$ nonstandard DIFF structure on $\mathbb{R}^4$

(Taubes, Gompf, 1988) $\exists$ uncountably many different DIFF structures on $\mathbb{R}^4$

Corollary (1963 Milnor's problem) $\nexists$ DIFF 4-mfld with int. form $E_8 \oplus E_8$.

Open problems

1. Characterization of nonsimply connected TOP 4-manifolds.
2. DIFF 4-manifolds, in particular 1-connected, in particular DIFF GPC: does S^4 admit exotic DIFF structures?

Key ideas of Freedman's proof:

Fundamental breakthrough in 1973: A. Casson's infinite construction

Does the Whitney trick work in dim 4?

Two key problems: (I) the complement is not necessarily 1-connected

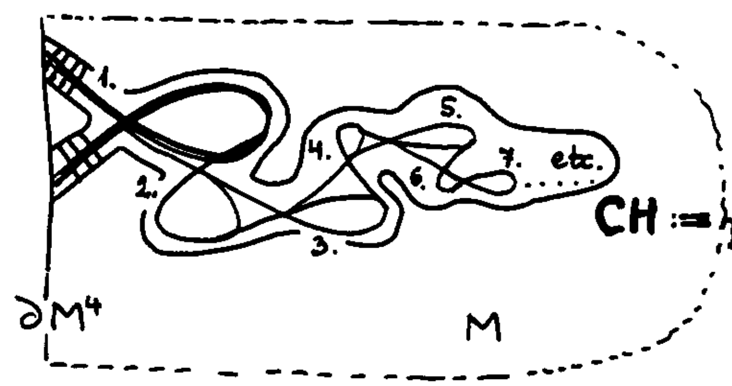
(II) the self intersections occur and can't be removed

Casson's novel idea: he added the self intersections and

he built an infinite tower of singular disks

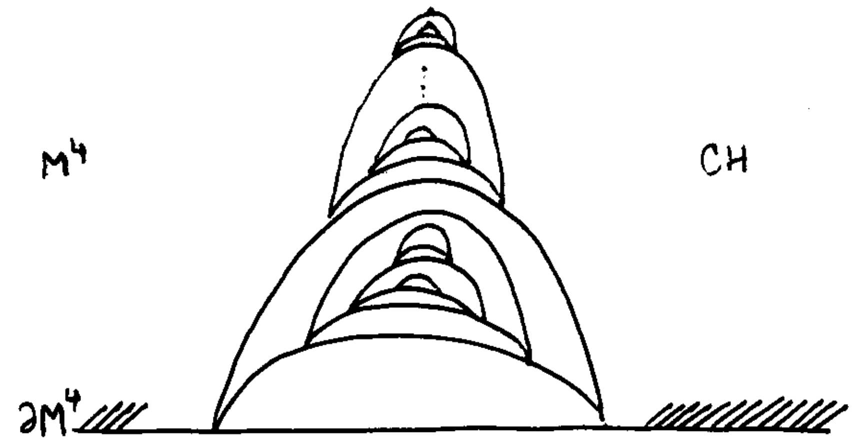
and proved $CH \underset{p.h.e.}{\simeq} D^2 \times \mathbb{R}^2 =$ open TOP 2-handle

Freedman's key discovery $CH \underset{TOP}{\cong} D^2 \times \mathbb{R}^2$!

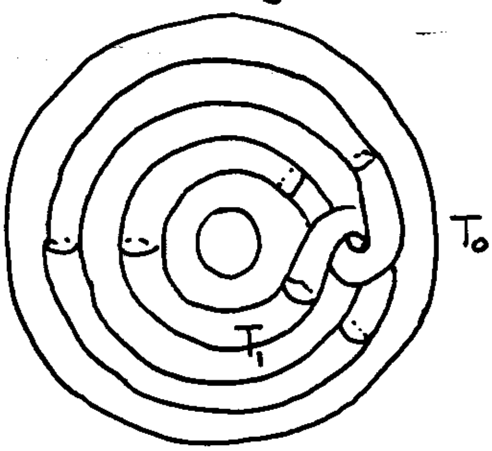


CH := reg. nbd of this tower

Freedman proves a certain reembedding theorem which enabled him to imbed Casson handle so that the closure of its frontier was just $(B^2 \times S^1)/$Whitehead continuum (or a Cantor set worth of them, in general)



Whitehead Continuum (1930's) (his counterexample to his own earlier theorem that every contractible open 3-manifold was homeom to $\mathbb{R}^3$)



$Wh := \bigcap_i T_i$

(1) Wh example is $S^3 \setminus Wh$

(2) Bing-Shapiro : $\mathbb{R}^3/_{Wh} \times \mathbb{R} \underset{TOP}{\cong} \mathbb{R}^4$

Bing-Edwards shrinking techniques for decompositions of manifolds

$$\begin{array}{ccc} CH & \overset{ABH}{\dashrightarrow} & D^2 \times \mathbb{R}^2 \\ & {}_{ABH}\searrow^{\text{cell-like maps}} \quad \swarrow_{ABH} & \\ & CH/_{\{gaps\}} & \end{array}$$

ABH := approximable by homeomorphisms

Consequently, $CH \underset{TOP}{\cong} D^2 \times \mathbb{R}^2$!

Suggested literature:

(A) survey articles

1. R. J. Stern, Instantons and the topology of 4-manifolds, Math. Intell. 5 (1983) 3, 39-44.
2. L. Lemaire, La géometrie des variétés de dimension quatre, Bull. Soc. Math. Belg. 39 (1987), 5-21.
3. L. Siebenmann, La conjecture de Poincaré topologique en dimension 4 (d'après M. Freedman), Sem. Bourbaki, No. 588.
4. M. H. Freedman, There is no room to spare in four-dimensional space, Notices Amer. Math. Soc. 31 (1984), 3-6.
5. R. Mandelbaum, Four dimensional topology: an introduction, Bull. Amer. Math. Soc. 2 (1980), 1-159.
6. R. J. Stern, Gauge theories as a tool for low dimensional topologists, Perspectives on Mat Birkhauser 1984, 497-507.

(B) books

1. S.K. Donaldson and P.B. Kronheimer, The geometry of four-manifolds, Oxford 1990
2. M. H. Freedman and F.S. Quinn, Topology of 4-manifolds, Princeton 1990
3. R. C. Kirby, The topology of 4-manifolds, Springer 1989
4. D. Freed and K. Uhlenbeck, Instantons and four-manifolds, MSRI Publ. Springer 1984
5. M.F. Atiyah, Geometry of Yang-Mills Fields, Pisa 1979
6. V. Poenaru - C. Tanasi, Palermo 198?

(C) other papers of interest:

1. N.J. Hitchin, The geometry and topology of moduli spaces, Lect. Notes Math. 1451, 1-48.
2. K. B. Marathe - G. Martucci, The geometry of gauge fields, J. Geom. Phy. 6 (1989), 1-106.
3. M. Feshbach, Topology and Physics, Ann. of Physics 192 (1989), 85-92.
4. M. Atiyah, The impact of physics on geometry, Diff. Geom. Meth. Th. Phy, Kluwer 1988, 1-9.

(D) books on quadratic forms

1. Husemoller and Milnor
2. Hirzebruch, Koh and Neumann
3. O'Meara
4. Serre