INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.545/10

WORKSHOP ON MATHEMATICAL PHYSICS AND GEOMETRY

(4 - 15 March 1991)

Representations of Virasoro and Kac-Moody algebras: An algebraic geometrical viewpoint

Cesare Reina
Scuola Internazionale Superiore di Studi Avanzati
Via Beirut 2-4
34014 Trieste
Italy

These are preliminary lecture notes, intended only for distribution to participants

REPRESENTATIONS OF VIRASORO AND KAC-MOODY ALGEBRAS: AN ALGEBRAIC GEOMETRICAL VIEWPOINT

Cesare Reina
SISSA - Trieste

Contents:

First Draft, March 1991

INTRODUCTION

Representation theory of Kac-Moody and Virasoro algebras plays a central rôle in Conformal Field Theory (C.F.T.). Indeed the local fields forming the operator algebra can be classified according to the irreducible representations of the Virasoro algebra, and their correlation functions are build up out of certain basic "conformal blocks" which in turn are entirely determined by the conformal invariance. For those theories where gauge symmetry is present (such as the Wess-Zumino-Witten models) the current algebra of the theory is actually an (affine) Kac-Moody algebra whose representation theory plays a fundamental rôle as well.

Most of the known results concerning representation theory of these algebras come from the realm of algebra, where one builds up Verma modules, or in physical terms (sectors of) the Fock space of the theory. Recently however some of these results have been given a geometrical description, much in the spirit of Borel-Weil-Bott approach to representation theory of compact Lie groups.

The basic example is the case of loop-groups, for which at least the fundamental representation of their central extensions has been realized as the space of holomorphic sections of a line bundle on the infinite grassmannian $Gr(H)$ of an Hilbert space (see Pressley and Segal).

Another basic fact is that there is a homomorphism of (a completion) of the classical Virasoro algebra (i.e. with central extension $c=0$) into the Lie algebra of vector fields over the moduli spaces $\hat{\mathcal{M}}_g$ of triples (C, p, z) where C is an algebraic curve, $p \in C$ a point and z a local coordinate at p. Over $\hat{\mathcal{M}}_g$ we have the "universal curve" $\mathcal{C} \xrightarrow{\pi} \hat{\mathcal{M}}_g$, together with its pluricanonical relative bundles $K^j_{\mathcal{C}/\hat{\mathcal{M}}}$. The direct images $\lambda_j = \det \pi_* K^j_{\mathcal{C}/\hat{\mathcal{M}}}$ are non trivial line bundles over $\hat{\mathcal{M}}_g$, whose chern class is given by the Mumford formula

$$c_1(\lambda_j) = (6j^2 - 6j + 1)\, c_1(\lambda_0)\,.$$

This coincides precisely with the central extension one gets in physics (see e.g. AG-G-R), and the mathematical reason for this coincidence has been completely described in A-D-K-P.

The third basic fact has to do with the Segal-Sugawara construction of the central extension of the Virasoro algebra acting on a highest weight representation of a Kac-Moody algebra of level k. Here again geometry plays a role, as such a central extension comes from a heat equation, generalizing to the non abelian case the classical heat equation for the Θ-functions. The geometrical meaning of such a heat equation is that it gives a projectively flat connection on certain vector bundles over the moduli spaces of curves (Hitchin).

In these lectures I will concentrate on the first two facts, giving an account of the main constructions and results.

1-. THE ALGEBRA $\mathcal{D}^1_r$

At the classical level (i.e. with all the central extensions set to zero) the infinite dimensional algebras(1) we shall be dealing with have nice realizations in terms of geometric objects. Recall that

1) The oscillator algebra is generated by elements $\{e_m\}$, $m \in \mathbb{Z}$ with the commutation relations

$$[e_m, e_n] = m\,\delta_{m,-n}\, c_o$$
$$[e_m, c_o] = 0$$

For $c_0 = 0$, this is an abelian algebra, whose generators can be identified with multiplication operators $e_m = z^m$, $m \in \mathbb{Z}$ for $z \in \mathbb{C} - \{0\}$.

2) The $sl(r,\mathbb{C})$ - Kac Moody algebra is generated by elements $\{\tau^a_m\}$ $m \in \mathbb{Z}$, $(a = 1, \dots r^2-1)$ with commutation relations

$$[\tau^a_m, \tau^b_n] = f^{ab}_c\, \tau^c_{m+n} + m\,\delta^{ab}\delta_{m,-n}\, c_k$$
$$[\tau^a_m, c_k] = 0$$

where f^{ab}_c are the structure constants of $sl(r,\mathbb{C})$. As well known, for $c_k = 0$, one can identify $\tau^a_m = \tau^a z^m$, where τ^a is a basis for $sl(r,\mathbb{C})$

3) The Virasoro algebra is generated by $\{\ell_m\}$ $m \in \mathbb{Z}$ with the relations

$$[\ell_m, \ell_n] = (m-n)\,\ell_{m+n} + (m^3 - m)\,\delta_{m,-n}\,\frac{c_v}{12}$$
$$[\ell_m, c_v] = 0$$

and when $c_v = 0$, one can set $\ell_m = -z^{m+1}\frac{d}{dz}$.

Of course these three algebras are not unrelated

---

(1) For the time being, we will not introduce topology, and we will consider formal linear combinations of the generators listed below.

structures, because, as their realizations show, all together they form a semidirect sum

$$[e_m, \tau^a_n] = 0$$
$$[e_m, e_n] = m\, e_{m+n} - m^2 \delta_{m,-n}\, c$$
$$[\tau^a_m, e_m] = n\, \tau^a_{m+n}$$

with all the other bracketts vanishing.

This semidirect sum has a classical realization in terms of differential operators. Consider the Lie algebra $\mathcal{D}^1_r$ of regular differential operators of order ≤ 1 acting on holomorphic functions $f: \mathbb{C}^* \to \mathbb{C}^r$, such that $\mathcal{D}^1_r / \mathcal{D}^0_r$ is a vector field on $\mathbb{C}^*$. (Here $\mathcal{D}^0_r$ is the algebra of differential operators of order 0). Then we have an extension

$$0 \longrightarrow \mathcal{D}^0_r \longrightarrow \mathcal{D}^1_r \longrightarrow \underline{d} \longrightarrow 0$$

where $\mathcal{D}^0_r$ ($\underline{d}$) is isomorphic to (suitable completion) of the direct sum of the oscillator algebra and the Kac-Moody algebra (the Virasoro algebra) with all the central terms equal to zero. So the semidirect sum of the three algebras above with no central extensions is actually $\mathcal{D}^1_r$.

One can show that $H^2(\mathcal{D}^1_r, \mathbb{C}) = \mathbb{C}^4$, and therefore with all the c's $\neq 0$ the algebra above is the universal central extension $\hat{\mathcal{D}}^1_r$ of $\mathcal{D}^1_r$. Our main goal is to explain how these central extensions arise in geometrical terms. The basic idea comes from geometric quantization; Suppose we have a homomorphism $\varphi: \mathcal{D}^1_r \longrightarrow \mathcal{X}(\mathcal{V})$ of $\mathcal{D}^1_r$ into the Lie algebra of vector fields on some manifold $\mathcal{V}$. Then, given a line bundle $L \to \mathcal{V}$ with connection ∇, we can lift vector fields on $\mathcal{V}$ to act on sections of L via the covariant derivatives. Obviously $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + R(X,Y)$, where R is the curvature of ∇; pulling this back to $\mathcal{D}^1_r$ via φ one would like to get the desired extension $\hat{\mathcal{D}}^1_r$. Incidentally one would get in this way a representation of $\hat{\mathcal{D}}^1_r$ on the space of sections of L.

As for the manifold $\mathcal{V}$, a good candidate is given by the following reasoning. The algebra $\mathcal{D}^1_r$ (with yet another topology) has a direct geometrical meaning, since it can be identified with an algebra of 1-cocycles on an algebraic curve. To see how this goes, consider the datum of

i) a (stable) rank r vector bundle E over a (smooth) pointed curve $C \ni p$. Set $d = \deg E$.

ii) a local coordinate z on a neighborhood $U_0 \ni p$, vanishing at p.

iii) a local trivialization $\psi : E|_{U_0} \to U_0 \times \mathbb{C}^r$ of E on U_0.

Now, together with i) one has the sheaf $\mathcal{D}^1(E)$ of differential operators of order ≤ 1 with scalar symbol acting on sections of E, and this comes with exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{End}_0 E \arrow[r] \arrow[d] & \mathcal{D}^1_0(E) \arrow[r] \arrow[d] & K^{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{End}\, E \arrow[r] \arrow[d, "tr"'] & \mathcal{D}^1(E) \arrow[r] \arrow[d, "tr"'] & K^{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{D}^1(L) \arrow[r] \arrow[d] & K^{-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where

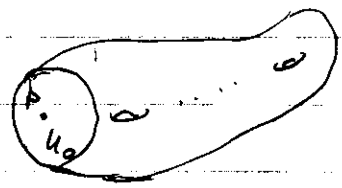
$\mathrm{End}\, E$ (resp. $\mathrm{End}_0 E$) is the sheaf of (resp. traceless) endomorphisms of E, and $tr : \mathrm{End}\, E \to \mathcal{O}$ is the trace homomorphism.

$\mathcal{D}^1(L)$ is the sheaf of differential operators of order ≤ 1 acting on sections of $L = \det E$

Notice that for E stable, $\mathcal{D}^1_0(E) \simeq \mathrm{End}_0 E \oplus K^{-1}$ is a

trivial extension since $H^1(C, k \otimes End_0 E) = H^0(C, End_0 E) = 0$.
Also $End\, E = End_0 E \oplus End_c E$ is a trivial central extension of $End_0 E$ by the sheaf of central endomorphisms $End_c E \simeq \mathcal{O}$.
Instead the second and third lines are not trivial extensions, the obstruction class being directly related to the chern class of E.

The datum of ii) defines for us a Stein covering of the curve C given by $\{U_0, U_1\}$, with $U_1 = C - \{p\}$.



Now the algebra $\mathcal{C}^1(\mathcal{U}, \mathcal{D}^1(E))$ of one cocycles with values in $\mathcal{D}^1(E)$ is isomorphic to $\Gamma(U_0 - \{p\}, \mathcal{D}^1(E))$, and using the local coordinate z and the trivialization ϕ this in turn is isomorphic to (a suitable closure) of the algebra on a punctured disk $\Delta^* = \Delta - \{0\} \subset \mathbb{C}$ spanned by

$$\left(z^m \mathbb{1},\ \tau^a z^m,\ z^{m+1}\frac{d}{dz}\right) \qquad m \in \mathbb{Z}.$$

In particular, we have the following decompositions of linear spaces

$$\mathcal{C}^1(\mathcal{O}) = \Gamma(U_1, \mathcal{O}) \oplus \Gamma(U_0, \mathcal{O}) \oplus H^1(C, \mathcal{O})$$
$$\mathcal{C}^1(End\, E) = \Gamma(U_1, End\, E) \oplus \Gamma(U_0, End\, E) \oplus H^1(C, End\, E)$$
$$\mathcal{C}^1(k^{-1}) = \Gamma(U_1, k^{-1}) \oplus \Gamma(U_0, k^{-1}) \oplus H^1(C, k^{-1}).$$

Notice that the first two terms of each line are separately Lie algebras, while their direct sum is not. The third summand of each line has a direct geometrical meaning, being identified with the tangent space at (smooth) points of suitable moduli spaces (see appendix).

Summing up we have

<u>Proposition</u> Given a quintuple (C, p, z, E, ϕ) as above, we have an isomorphism

$$\chi : \mathcal{D}^1_r \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{D}^1(E))$$

Remark. Notice again that $\mathcal{D}_r^1$ should be given the appropriate topology. For more details see [ADKP], where the case $r = 1$ is discussed in details. The extension of their proofs to $r > 1$ is obvious.

2. b-c SYSTEMS AND THE MUMFORD FORMULA

2.1. A b-c system on a curve C is a pair of local fields $b = b(z)\,dz^{1-j}$, $c = c(z)\,dz^{j}$, (i.e. of local sections of the pluricanonical bundles K^{1-j} and K^{j} respectively) with action functional

$$S(b,c) = \int_C b\,\bar{\partial}_j c .$$

The path integral, for fermi statistics, by definition reads

$$\int d\mu_b\, d\mu_c\, e^{-S(b,c)} = \det{}'\bar{\partial}_j \quad \Lambda^{max}\ker\bar{\partial} \otimes \Lambda^{max}\operatorname{coker}\bar{\partial}$$

showing that $\det{}'\bar{\partial}_j$ is a section of the determinant index bundle L_j of the family of $\bar{\partial}_j$ operators on K^j parametrized by the complex structure on C. While the absolute value of $\det{}'\bar{\partial}_j$ is well defined by the Quillen metric

$$\| \det{}'\bar{\partial}_j \|_Q^2 = e^{-\zeta'(0)}$$

where $\zeta(s)$ is the ζ-function of the Laplacian on K^j, its phase is not well defined (i.e. we have an anomaly) because on the determinant index bundle L_j there is no global angular form unless its Chern class vanishes.

The first geometric picture we get can be described as follows. Let $\mathcal{M}_g$ be the moduli space of curves of genus g, and let $\mathcal{C} \xrightarrow{\pi} \mathcal{M}_g$ be the "universal-curve" over $\mathcal{M}_g$ (1), i.e. a fibered manifold s.t. $\pi^{-1}(m) = \mathcal{C}_m$ is in the equivalence class parametrized by m. Over $\mathcal{C}$ we have the pluricanonical relative bundle $K^j = K^j_{\mathcal{C}/\mathcal{M}_g}$.

$$\begin{array}{ccl} \mathcal{C} & \leftarrow & K^j \\ \pi \big\downarrow & & \\ \mathcal{M}_g & \leftarrow & \det \pi_! K^j = L_j \end{array}$$

and on $\mathcal{M}_g$ we get the determinant index bundle. Its chern class is easily computed via the Grothendieck-Riemann-Roch formula, giving

$$ch(L_j) = \int ch K^j \cdot Td\, K^{-1} =$$

$$= \int \left(1 + j c_1(K) + \frac{j^2}{2} c_1(K)^2 + \dots\right)\left(1 - \frac{1}{2} c_1(K) + \frac{1}{12} c_1(K)^2 + \dots\right)$$

So

$$c_1(L_j) = (6j^2 - 6j + 1) \frac{1}{12} \int c_1(K)^2 = (6j^2 - 6j + 1)\, c_1(L_0).$$

2.2.- Let us go back to algebra. The algebra d^F of finite linear combinations of $\ell_n = z^{n+1} \frac{d}{dz}$, $n \in \mathbb{Z}$ naturally acts on j-th differentials $c = \sum_k c_k z^{-k} (dz)^j$, giving us a matrix representation

$$\phi_j : d^F \longrightarrow a_\infty^F$$

where a_∞^F is the Lie algebra of matrices (a_{ij}) $(i, j \in \mathbb{Z})$ such that $a_{ij} = 0$ for $|i-j| \gg 0$. This has a universal central extension, given by the cocycle

$$\psi(E_{ij}, E_{ji}) = -\psi(E_{ji}, E_{ij}) = 1 \qquad i \leq 0,\ j > 0$$
$$\psi(E_{ij}, E_{ji}) = 0 \qquad \text{otherwise}$$

---

(1) This does not actually exist. So either one takes $\mathcal{M}_g$ to be the moduli space of curves without automorphisms, or one thinks of $\mathcal{C}$ as the collection of all deformations of curves, i.e. the stack of moduli.

By pull-back under ϕ_j, we get a 2-cocycle on d^F.
An easy computation gives

$$(\phi_j^* \psi)(e_n, e_m) = -\delta_{m,-n}\, \frac{m^3 - m}{6}(6j^2 - 6j + 1) =$$

i.e.

$$(\phi_j^* \psi) = (6j^2 - 6j + 1)(\phi_0^* \psi)$$

A comparison with Mumford's formula for $c_1(L_j)$ rises the problem of explaining why we have such a coincidence. This problem has been completely solved by [ADKP]. In the rest of this section we will recall some of their ideas and results.

2.3. The first step is to enlarge a bit the moduli space of curves, in order to realize a subspace of $\underline{d}$ larger than $H^1(C, k^{-1})$ in terms of vector fields. Indeed one introduces an infinite dimensional manifold $\hat{\mathcal{M}}_g$ as the moduli spaces of triples (C, p, z). This manifold is homotopically equivalent to $\mathcal{M}_g''$, this latter being the moduli space of triples (C, p, v), where v is a tangent vector to C at p. Moreover for $g \gg 1$ the surjection $\mathcal{M}_g'' \to \mathcal{M}_g$ induces an isomorphism in second cohomology.

From sect. 1 we know that (a suitable completion of) $\underline{d}$ is actually isomorphic to $\mathcal{E}_x^1(k^{-1})$, once the datum of $x = (C, p, z)$ is given. Now the splitting

$$\mathcal{E}_x^1(k^{-1}) = \Gamma(U_1, k^{-1}) \oplus \Gamma(U_0, k^{-1}) \oplus H^1(C, k^{-1})$$

shows that there is a sequence

$$\begin{array}{ccccccccc}
0 & \to & \Gamma(U_1, k^{-1}) & \longrightarrow & \mathcal{E}_x^1 & \longrightarrow & \Gamma(U_0, k^{-1}) \oplus H^1(C, k^{-1}) & \longrightarrow & 0 \\
 & & \| & & \wr\| & & \wr\| & & \\
0 & \to & d_x & \longrightarrow & d & \xrightarrow{ev_x} & T_x \hat{\mathcal{M}}_g & \longrightarrow & 0
\end{array}$$

d_x playing the rôle of the isotropy subalgebra at $x \in \hat{\mathcal{M}}_g$.

The isomorphism $T_x \hat{\mathcal{M}}_g \simeq \Gamma(U_0, k^{-1}) \oplus H^1(C, k^{-1})$ gives actually a splitting of the tangent space to $\hat{\mathcal{M}} \to \mathcal{M}$ into the vertical part and $T_C \mathcal{M}_g \simeq H^1(C, k^{-1})$.

Now given a line bundle $L \to \hat{\mathcal{M}}_g$, the pull back under ev_x of its first chern class gives us a central extension $\hat{\underline{d}}_L$ of $\underline{d}$. This is nothing but a form of geometric quantization, as explained above.

2.4. To make contact between the geometrical extensions constructed in sect 2.3 and the algebraic one of 2.2, one exploits the Krichever maps given by pluricanonical bundles into the infinite Grassmannian $Gr(H)$. We will not describe such a manifold nor the krichever maps here, and we will refer to [PS] [SW] and [ADKP] for detailed accounts. We simply recall that

i) $Gr(H)$ is a homogeneous space of $Gl_{res}(H)$, a subgroup of bounded invertible operators on a suitable topological linear space X.

ii) a suitable completion $gl_{res}(H)$ of d_∞^F acts holomorphically on $Gr(H)$, i.e. we have a homomorphism

$$\chi : gl_{res}(H) \longrightarrow \mathcal{H}Gr(H)$$

into the Lie algebra of holomorphic vector fields on $Gr(H)$.

iii) the universal central extension of $gl_{res}(H)$ or of $Gl_{res}(H)$ induces a line bundle Det^{-1} on $Gr(H)$

iv) there is a map $k_j : \hat{\mathcal{M}}_g \longrightarrow Gr(H)$ such that
$k_j^* Det^{-1} = \det \pi_! \partial_j = L_j$

For a suitable model of $Gr(H)$, its first chern class $c_1(Det^{-1})$ exists as a form on $Gr(H)$, and its evaluation at the base point on pairs of fundamental vector fields of $gl_{res}(H)$ (obviously) coincides with the universal central extension of $gl_{res}(H)$ (and hence of a_∞^F).

Now the basic idea is to look at the diagram

$$\begin{array}{ccc} \underline{d}/\underline{d}_x & \xrightarrow{\phi_j} & g/g_{k_j(x)} \\ \| & & \| \\ T_x\mathcal{M} & \xrightarrow{dk_j} & T_{k_j(x)}Gr(H) \end{array}$$

where $g = gl_{res}(H)$ for short and $g_{k_j(x)}$ is the isotropy subalgebra of $k_j(x) \in Gr(H)$. From the commutativity of this diagram one gets

$$\phi_j^* \psi \simeq k_j^* c_1(\mathrm{Det}^{-1})$$

which explains the coincidence between 2.1 and 2.2.

<u>Remark</u>. This is a very short and intuitive account of what is going on. Filling in all the details requires much more work for which we refer to [ADKP].

3-. THE ABELIAN SUGAWARA FORMULA AND θ-FUNCTIONS.

3.1-. Up to now we have been working with systems without gauge symmetry. Suppose instead we take a C.F.T with an abelian gauge symmetry, i.e. with a classical gauge algebra given by the oscillator algebra 1) of section 1 (and with $c_0=0$). As we know this algebra, which we denote by $\mathcal{D}_1^0$ (i.e. differential operators of degree zero and rank 1), can be made isomorphic to $\mathcal{E}_X^1(\mathcal{O})$, once the datum of $X=(C,p,z)$ is given. From the decomposition

$$\mathcal{E}_X^1(\mathcal{O}) = \Gamma(U_1,\mathcal{O}) \oplus \Gamma(U_0,\mathcal{O}) \oplus H^1(C,\mathcal{O})$$

it's clear that the candidate moduli space will be such that $H^1(C,\mathcal{O})$ is its tangent space: namely the Jacobian $J(C)$ of C. Or, more generally, a component $\mathrm{Pic}^d(C)$ of the Picard group of C. For the time being, we will fix $d=g-1$.

The appearing of abelian varieties in this instance is by no means a surprise, as it is well known (at least in the finite dimensional case) that the central extension of $\mathcal{D}_1^0$ with $c_0=k\in\mathbb{Z}$ (also called the Heisemberg algebra) has nice finite dimensional representations concretely realized in terms of θ-functions of level k, for which we refer to Mumford [M] and Igusa [I]. In this section we will recall some simple facts as a warm up exercice in view of the non abelian generalization. The problem is to understand how the Virasoro algebra $\underline{d}$ acts on these representation space

3.2 -. A first answer comes from algebra (see e.g. [K.P])
Infact the Virasoro algebra can be represented in terms of certain quadratic elements of (a completed) universal enveloping algebra $\hat{\mathcal{U}}(a_\infty)$ of a_∞ - In physics such a construction is known as the (abelian) Sugawara formula.
We think of e_m as the multiplication operator by z^m acting on the algebra $\mathbb{C}[z,z^{-1}]$; in this way we get a homomorphism $\mathcal{D}_1^0 \to a_\infty$ given by $z^m \mapsto \Lambda_m$ where

$$\Lambda_m = \sum_{i\in\mathbb{Z}} E_{i,i+m}$$

in terms of the basis $(E_{i,j})_{k\ell} = \delta^i_k \delta^j_\ell$ of a_∞.
We set

$$L_m = \frac{1}{\beta}\sum_{j\in\mathbb{Z}} :\Lambda_{-j}\Lambda_{m+j}: \qquad m\in\mathbb{Z}$$

where the normal ordering $: \;:$ is defined by

$$\begin{aligned} :\Lambda_i\Lambda_j: &= \Lambda_i\Lambda_j \qquad i\le j \\ &= \Lambda_j\Lambda_i \qquad i>j \end{aligned}$$

It is easy to check that the L_m's satisfy the Virasoro commutation relations for $\beta = 2K = 2c_0$, $c_v = 1$, i.e. we have

$$\begin{aligned} [\Lambda_m, \Lambda_n] &= m\,\delta_{m,-n}\,k \\ [\Lambda_m, L_n] &= m\, e_{m+n} \\ [L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\,\delta_{m,-n} \end{aligned}$$

<u>Exercise.</u> Let $V(\lambda)$ be a highest weight repr. of the oscillator algebra, with highest weight $|0\rangle$, i.e.

$$\Lambda_0|0\rangle = \lambda|0\rangle, \qquad e_m|0\rangle = 0 \quad (m>0)$$

Let us show that the commutation relations above between Λ_m, L_n imply $\beta = 2k$. It is enough to compute

$$\begin{aligned} [\Lambda_1, L_{-1}]|0\rangle &= \Lambda_1 L_{-1}|0\rangle = \Lambda_1\left(\frac{2}{\beta}\Lambda_{-1}\Lambda_0\right)|0\rangle = \frac{2}{\beta}\lambda\Lambda_1\Lambda_{-1}|0\rangle \\ &= \frac{2}{\beta}\lambda(\Lambda_{-1}\Lambda_1 + k)|0\rangle = \frac{2\lambda k}{\beta}|0\rangle \end{aligned}$$

on the other hand $[\Lambda_1, L_{-1}] = \Lambda_0$ imply $\beta = 2k$.

3.3 —. Notice that this realization is different from that given e.g. in [K, P] for which $c_V = k$. In particular for $c_0 = c_V = k$, one can recover the central extensions by pulling back via the Krichever map the first Chern class of $(\mathrm{Det}^{-1})^k$ on $Gr(H)$. We will not dwell in this direction any further, but refer to [ADKP] [FR1] [FR2] for more details.

3.4 —. The geometrical counterpart of the construction 3.2 has been given by Hitchin [H], already in the non abelian case (see sect. 4). We will first sketch the classical construction underlying the present abelian situation. The basic idea is to set up a "relative" formulation of the representations of the oscillator algebra on Θ-functions, recalled at the end of sect. 3.1.

To this purpose, let us consider a universal deformation $\mathcal{C} \xrightarrow{\pi} \Delta \subset \mathbb{C}^{3g-3}$ of a smooth algebraic curve C. Associated to this we have the corresponding family of moduli spaces $J^d \xrightarrow{f} \Delta$ of degree d line bundles, i.e. a fibered manifold such that $f^{-1}(t) = \mathrm{Pic}^d(\pi^{-1}(t))$. This is actually a deformation of $\mathrm{Pic}^d(C)$. Summing up we have a diagram

$$\begin{tikzcd} & & L \arrow[dl] \\ J^d \arrow[d, "f"'] & \tilde{\mathcal{C}} \arrow[l, "\tilde{\pi}"'] \arrow[d, "\tilde{f}"] & \\ \Delta & \mathcal{C} \arrow[l, "\pi"'] & \end{tikzcd}$$

where $\tilde{\mathcal{C}} = f^*\mathcal{C}$, and a Poincaré line bundle on $\tilde{\mathcal{C}}$ such that $L|_{\tilde{\pi}^{-1}(p)}$ is in the equivalence class of p on the curve parametrized by $f(p)$. For $d = g-1$ one can define on J^d the relative Θ-line bundle i.e. the determinant index bundle

$$L_\Theta = \det \pi_! L$$

Now we have a surjection $\mathcal{D}_1^0 \longmapsto T_p^v J^d$ given by $z_\cdot^m \mapsto [z_\cdot^m] \in H^1(f(p), \operatorname{End} L|_{\tilde{\pi}^{-1}(p)}) = H^1(f(p), \mathcal{O}_{f(p)})$, where $T^v J^d$ is the relative (i.e. vertical) tangent sheaf. This extends to a Lie algebra homomorphism.

$$\mathcal{D}_1^0 \longmapsto \Gamma(T^v J^d)$$

The datum of L_Θ^k ($k \in \mathbb{N}$) on J^d defines a central extension of the relative tangent sheaf $T^v J^d$, and by pull-back a central extension of $\mathcal{D}_1^0$, as usual. In other words we get a projective representation of $\mathcal{D}_1^0$ on the space of level k Θ-functions with a central extension given by the two-cocycle corresponding to the first chern class $c_1(L_\Theta^k) = k\, c_1(L_\Theta)$.

The action of the Virasoro algebra on such representations can be now given in terms of infinitesimal deformations of triples $(J^d, L_\Theta, \theta^{(k)})$ (see appendix), or which is the same as an action on the space of sections of the vector bundle $F = f_* L_\Theta^k$ on Δ, with fibres $F_t = H^0(J_t^d, L_\Theta^k)$. This entails again an extension of the tangent sheaf to Δ by $\operatorname{End} F$ to get

$$0 \to \operatorname{End} F \to \mathcal{D}^1(F) \to T\Delta \to 1$$

which can be represented by the curvature of a connection on $F \to \Delta$.

The Chern character of F can be computed via the Groethendrek - Riemann - Roch theorem as

$$ch(F) = f_*\left(ch\, L_\theta^k \cdot Td(T_1^v J^\alpha)\right) = f_* ch\, L_\theta^k$$

where π_* is the integral along the fibres of $f : J^\alpha \to \Delta$.

Now

$$ch_n\, L_\theta^k = \frac{k^n}{n!}\, \omega^n$$

where $\omega = c_1(L_\theta)$. In particular since $\dim H^0(J_C, L_\theta^k) = k^g$ for a curve of genus g, we have

$$\text{rank}\, F = k^g = ch_0(F) = \pi_*(ch_g\, L_\theta^k) = \frac{k^g}{g!}\, \pi_*\, \omega^g .$$

Next

$$c_1(F) = f_*\, ch_{g+1}(F) = \frac{k^{g+1}}{(g+1)!}\, f_*\, \omega^{g+1}$$

One can choose ω as $\omega_\theta + \alpha$ where α is the pull back of the chern class λ of the line bundle $\pi_* K$ on $\hat{\Delta}$ (i.e. the restriction to Δ of the Hodge class). Accordingly

$$c_1(F) = \frac{k^{g+1}}{(g+1)!}\, f_*[(g+1)\, \omega_\theta^g\, \alpha] = k\, rkF \cdot \lambda$$

and more generally

$$ch(F) = rkF \sum_{n=0}^{\infty} k^n \lambda^n$$

showing that

$$F \sim k\lambda \otimes \mathbb{1}$$

where $\mathbb{1}$ is the trivial bundle of rank equal to rkF on Δ.

Actually all this local computation on Δ makes sense on the moduli stack of curves. The formula

$$c_1(F) = k\, rkF \cdot \lambda$$

is the analogue of the Mumford formula for the abelian Sugawara construction, and tells us that the central extension of the Virasoro algebra in this representation, actually comes from the line bundle $k\cdot\lambda$. This matches with the computation 4.2, because the operators $\sum :\Lambda_{-j}\Lambda_{m+j}:$ can be now identified with the covariant derivatives w.r.t. a connection kA on the line bundle with class $k\lambda$. The normalization $\beta = k$ scales the connection to A and the central extension to λ.

3.5 -. The scaling by k makes contact with the heat operator

$$\frac{\partial}{\partial T_{ij}} + \frac{1}{k}\frac{\partial^2}{\partial x_i \partial x_j}$$

whose global solutions are the level k Θ-functions. For more details see [H], Remark 4.12.1 -

3.6 -. Here we have another coincidence between algebraic rep. theory and a geometric construction. One can easily generalize the Grassmannian construction of sect 2 to get representations of $\mathcal{S}_1^1$ with central extensions $c_0 = k$, $c_V = k$. The effect of the scaling to $c_V = 1$ is however unclear in this set up, and deserves a closer inspection.

4-. NON ABELIAN SUGAWARA FORMULA AND MODULI SPACES OF VECTOR BUNDLES.

4.1-. The results of sect. 3 have been recently generalized by Hitchin [H] to non abelian Kac-Moody algebras. The geometric picture is actually the generalization of sect 4.4 where moduli spaces of vector bundles replace J^d. As in the abelian case one would start with

$$\begin{tikzcd} & & \mathcal{E} \arrow[dl] \\ \mathcal{V} \arrow[d, "f"'] & \tilde{\mathcal{C}} \arrow[l, "\pi"'] \arrow[d, "\tilde{f}"] & \\ \Delta & \mathcal{C} \arrow[l, "\pi"'] & \end{tikzcd}$$

where $\mathcal{V}$ is the moduli space of couples (E, C), where E is a stable vector bundle of rank r and degree d and C a curve occurring as a fibre of $\pi : \mathcal{C} \to \Delta$. $\mathcal{E}$ is the analogue of the Poincare bundle L, which however exists only for r, d coprime. In this case one constructs the analogue of the Θ-bundle on $\mathcal{V}$ by

$$L_\theta = \det \bar{\pi}_! E$$

Actually L_θ can be defined also for generic r, d (see ~~Drezet~~-Narashiman [DN] by a different construction.

By a theorem of Narashiman and Ramanan [NR] the family $\mathcal{V} \to \Delta \subset \mathbb{C}^{3g-3}$ is actually a universal family (for $g \geq 3$), i.e. there is no Shottchy problem for rank higher than 1. So we can forget the curves $\mathcal{C}, \tilde{\mathcal{C}}$ and directly work with the problem of studying deformations of $(\mathcal{V}_0, L_0^k, s_0)$ for $\mathcal{V}_0 = f^{-1}(0)$, $L_0^k = L_\theta^k|_{\mathcal{V}_0}$, and s_0 a section of L_0^k. (see the appendix for an introduction to the deformation theory of such triples). The rationale for this is that $H^0(\mathcal{V}_0, \mathcal{L}_0^k)$ carries a natural representation of a Kac-Moody algebra with values in $gl(r, \mathbb{C})$ and central extension k, and, lifting vector fields on Δ to derivations on sections of the vector bundle $F = f_* \mathcal{L}_\theta^k$, a representation of the Virasoro algebra as well.

4.2. The obvious idea of computing the extension one gets in this way by means of G-R-R theorem, as we did in sect. 3.4, is hard to follow. Indeed one has little control on the Todd class of the relative tangent bundle to $\mathcal{V}$. A way out, found by Hitchin, is to directly construct a connection on F as follows. Let $\{U_\alpha\}$ be a covering for $\mathcal{V}$ with local coordinates (t_α^A, v_α^i) $(A = 1, \dots 3g-3 \quad i = 1, \dots r^2(g-1)+1)$ over which we can trivialize $\mathcal{L}_\theta$. The family of vector fields $\frac{\partial}{\partial t_\alpha^A}$ are such that on each intersection $U_\alpha \cap U_\beta$ the vector field

$$X_{\alpha\beta}^A = \frac{\partial}{\partial t_\alpha^A} - \frac{\partial}{\partial t_\beta^A}$$

projects to zero on Δ, and therefore defines a class in $H^1(\mathcal{V}_t, T)$ for each $t \in \Delta$, which is the Kodaira Spencer class for the infinitesimal deformation of $\mathcal{V}_t$ in the direction t^A.

Now trivializing $\mathcal{L}_\theta$ we can interpret $\frac{\partial}{\partial t_\alpha^A}$ as a first order differential operator on $L = \mathcal{L}_\theta^k$ and the difference

$$Y_{\alpha\beta}^A = \frac{\partial}{\partial t_\alpha^A} - \frac{\partial}{\partial t_\beta^A}$$

defines now a class in $H^1(\mathcal{V}_t, \mathcal{D}^1(L))$ whose

symbol class is the Kodaira Spencer class $[X_{d_B}]$.
As we know from the appendix deforming sections of $L = L_\theta^k$ as well is the same as giving a cohomology class in $\mathbb{H}^1(d^1s)$. Now there is a canonical way of associating a holomorphic symmetric tensor to such a class. Consider the short exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{D}^1(L) & \to & \mathcal{D}^2(L) & \xrightarrow{\sigma} & S^2T & \to & 0 \\ & & \downarrow s & & \downarrow s & & \downarrow & & \\ 0 & \to & L & \longrightarrow & L & \longrightarrow & 0 & \to & 0 \end{array}$$

where $\mathcal{D}^i(L)$ are the sheaf of differential operators of order $\leq i$ on L, and σ is the top-symbol map. The vertical arrows are just the evaluations on the section s.
Associated to the first line we have a long cohomology sequence,

$$0 \to H^0(\mathcal{V}, \mathcal{D}^1(L)) \to H^0(\mathcal{V}, \mathcal{D}^2(L)) \to H^0(\mathcal{V}, S^2T) \xrightarrow{\delta} H^1(\mathcal{V}, \mathcal{D}^1(L)) \to \cdots$$

and to the entire diagram a long sequence of hypercohomology groups

$$0 \to \mathbb{H}^0(\mathcal{V}, d^1_s) \to \mathbb{H}^0(\mathcal{V}, d^2_s) \to H^0(\mathcal{V}, S^2T) \xrightarrow{\delta_s} \mathbb{H}^1(\mathcal{V}, d^1 s) \to \cdots$$

Now given a global section G of S^2T, we choose on each open set $U_\alpha \subset \mathcal{V}$ a holomorphic section Δ_α of $\mathcal{D}^2(L)$, i.e. a second order local holomorphic differential operator such that $\sigma(\Delta_\alpha) = G$. On each intersection $U_\alpha \cap U_\beta$, $\Delta_\alpha - \Delta_\beta$ is a section of $\mathcal{D}^1(L)$ because their top symbols coincide. Actually $[\Delta_\alpha - \Delta_\beta] = \delta G \in H^1(\mathcal{V}, \mathcal{D}^1(L))$.
To get $\delta_s G$, we consider a pair of cocycles

$$(\Delta_\alpha - \Delta_\beta, -\Delta_\alpha s)$$

in $\mathcal{C}^1(\mathcal{V}, \mathcal{D}^1(L)) \oplus \mathcal{C}^0(\mathcal{V}, L)$ and set $\delta_s G = [(\Delta_\alpha - \Delta_\beta, -\Delta_\alpha s)]$.
Now $\sigma \delta_s G \in H^1(\mathcal{V}, T)$, and Hitchin shows that for any $A = 1, \ldots, 3g-3$ one can choose $G^A \in H^0(\mathcal{V}, S^2T)$

such that $\frac{\nabla\delta(G^A)}{2k+\lambda} = [X^A_{\alpha\beta}] \in H^1(\mathcal{V},T)$. Here k is the level and λ an integer such that the canonical line bundle $K_{\mathcal{V}_0} = L_0^{-\lambda}$. (One knows that $\lambda = 2m$, with m the maximum common divisor of r, d). Accordingly the 1-cocycle $\delta G^A = \frac{1}{2k+\lambda}(\Delta^A_\alpha - \Delta^A_\beta) \in \mathcal{C}^1(\mathcal{V}, \mathcal{D}^1(L))$ is cohomologous to $Y^A_{\alpha\beta}$, i.e.

$$\frac{\partial}{\partial t^A_\alpha} - \frac{\partial}{\partial t^A_\beta} = \frac{1}{2k+\lambda}(\Delta^A_\alpha - \Delta^A_\beta) + D^A_\beta - D^A_\alpha$$

for some $D^A_\alpha \in \mathcal{C}^0(\mathcal{V}, \mathcal{D}^1(L))$. Hence the operator

$$\frac{\partial}{\partial t^A_\alpha} - \frac{1}{2k+\lambda}\Delta^A_\alpha + D^A_\alpha = \frac{\partial}{\partial t^A} + P_A\Big|_{U_\alpha}$$

is a globally defined holomorphic "heat" operator.

Hitchin shows that the operator $\frac{\partial}{\partial t^A} + P_A$ defines a connection of F, and for our concern a representation of Virasoro algebra. Moreover he shows that this connection has central curvature, i.e. it is projectively flat.

4.3. As is well known there is a non abelian analogue of the Sugawara construction 3.2 (see e.g. Goddard - Olive [GO] or Pressley and Segal [PS]) which for simple Lie algebras $\mathfrak{g}$ yield a Virasoro algebra with central extension

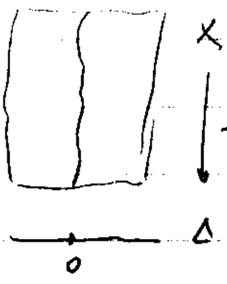
$$c_V = \frac{k \dim \mathfrak{g}}{k+h}$$

with h the dual Coxeter number. Here we have possibly another coincidence, between algebraic and geometrical results; a fact which requires some explanation.

APPENDIX: SOME DEFORMATION THEORY

a) Deformations of manifolds

Let M be a compact complex manifold, $\Delta \subset \mathbb{C}^n$ a ball.
A *deformation* of M over Δ is a manifold X with a proper surjective holomorphic map $f : X \to \Delta$ and an embedding of $M \hookrightarrow X$ which induces an isomorphism of M and the central fibre $f^{-1}(0)$.



One covers X with open sets $\mathcal{U}_\alpha$ and coordinates $\varphi_\alpha : \mathcal{U}_\alpha \to U_\alpha \times \Delta$ and sets $\varphi_\alpha(p) = (z^i_\alpha, t)$ with $t = \pi(p)$, $i = 1, \dots, \dim M$.
Then on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ one has transition functions

$$z^i_\alpha = f^i_{\alpha\beta}(z^j_\beta, t)$$

which can be taken to coincide with those of M at $t=0$. If $v = v^r \frac{\partial}{\partial t^r}$ is a tangent vector to Δ at $t=0$, one defines

$$b_{\alpha\beta} = v(f^i_{\alpha\beta}) \frac{\partial}{\partial z^i_\alpha}$$

which is clearly a holomorphic tangent vector field on $M_0 \cap \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. From the identity $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ one gets that the family of vector fields $\{b_{\alpha\beta}\}$ form a 1-cocycle on M_0, and hence it determines a class $[b_{\alpha\beta}] \in H^1(M_0, TM_0)$. The map $KS : T_0\Delta \to H^1(M_0, TM_0)$ is called the *Kodaira Spencer map*. One knows that if $H^0(M_0, TM_0) = 0$, a deformation for which KS is an isomorphism is a universal deformation of M, in the sense that any other deformation of M is isomorphic to the pull-back of X under a base change.

In particular, if $M = C$ is a smooth algebraic curve of genus $g \geq 2$, we have $\dim_{\mathbb{C}} H^1(C, K^{-1}) = 3g-3$, where K is the canonical sheaf, and $TC = K^{-1}$. One can easily construct universal deformations, thanks to the following

Lemma. Let C be a smooth algebraic curve and $p \in C$

a generic point. Then there is an isomorphism

$$H^1(C, K^{-1}) \simeq H^0(C, K^{-1}((3g-3)p)/K^{-1})$$
$$\simeq \mathbb{C}\{e_{-2}, e_{-3}, \dots, e_{-(3g-3)-1}\}$$

So we see that there is a subspace of the Virasoro algebra parametrizing infinitesimal deformations of C.

b) Deformations of triples (C, E, s)

We want next to sketch how one can construct deformations of a manifold C, a vector bundle $E \to C$ and a section $\sigma : C \to E$. We construct a family X above, together with a vector bundle $\mathcal{E} \to X$ and an isomorphism between E and the central vector bundle.

$$\begin{tikzcd} E \arrow[d, dash] & & \mathcal{E} \arrow[dl] \\ C \arrow[r, hook] \arrow[d] & X \arrow[d] & \\ (*) \arrow[r] & \Delta_t \times \Delta_v & \end{tikzcd}$$

We cover X with open sets of the form $U_\alpha \times \Delta_t \times \Delta_v$ (with local coordinates (z_α, t, v)) over which $\mathcal{E}$ trivializes, and we denote by ξ_α the fibre coordinates of $\mathcal{E}|_{U_\alpha \times \Delta_t \times \Delta_v}$. On overlaps, the clutching functions will have the form

$$z_\alpha = f_{\alpha\beta}(z_\beta, t)$$
$$\xi_\alpha = g_{\alpha\beta}(z_\beta, t, v)\, \xi_\beta$$

and a section s of $\mathcal{E}$ will be given by local $\mathbb{C}^r$ valued functions $s_\alpha(z_\alpha, t, v)$ such that

$$s_\alpha(z_\alpha, t, v) = g_{\alpha\beta}(z_\beta, t, v)\, s_\beta(z_\beta, t, v).$$

Intuitively v is a (multi)-parameter along which the vector bundle E is deformed, while varying t both C and $E \to C$ will be deformed.

We will restrict ourselves to study infinitesimal deformations by setting $t^2 = v^2 = 0$. Expanding the clutching functions above one gets

$$z_\alpha = f_{\alpha\beta}(z_\beta, 0) + t\, b_{\alpha\beta}$$

$$s_\alpha(z_\alpha) + t\left(b_{\alpha\beta}\frac{\partial s_\alpha}{\partial z_\alpha} + h_\alpha(z_\alpha)\right) + v\, m_\alpha(z_\alpha) =$$

$$= \left(g_{\alpha\beta}(z_\beta, 0, 0) + t\, c_{\alpha\beta} + v\, a_{\alpha\beta}\right)\left(s_\beta(z_\beta) + t\, h_\beta(z_\beta) + v\, m_\beta(z_\beta)\right)$$

where $f_{\alpha\beta}(z_\beta, 0)$, $g_{\alpha\beta}(z_\beta, 0, 0)$ are clutching functions for $E \to C$, and

$$b_{\alpha\beta} = \frac{\partial f_{\alpha\beta}}{\partial t}\Big|_{t=0}, \quad c_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial t}\Big|_{t=0}, \quad a_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial v}\Big|_{v=0}$$

$$h_i = \frac{\partial s_i}{\partial t}\Big|_{t=0}, \quad m_i = \frac{\partial s_i}{\partial v}\Big|_{v=0} \qquad (i = \alpha, \beta).$$

Dropping zeroes, one gets at the lowest order

$$z_\alpha = f_{\alpha\beta}(z_\beta) \qquad s_\alpha(z_\alpha) = g_{\alpha\beta}(z_\beta)\, s_\beta(z_\beta)$$

telling us that we are dealing with a triple (C, E, s), $s : C \to E$. At first order in t we have

$$b_{\alpha\beta}\frac{\partial s_\alpha}{\partial z_\alpha} + h_\alpha = g_{\alpha\beta}\, h_\beta + c_{\alpha\beta}\, s_\beta$$

i.e.

$$(t) \qquad h_\alpha - g_{\alpha\beta}\, h_\beta = X_{\alpha\beta}\, s_\alpha$$

where

$$X_{\alpha\beta} = b_{\alpha\beta}\frac{\partial}{\partial z_\alpha} + c_{\alpha\beta}\, g_{\alpha\beta}^{-1}$$

And at the first order in v

$$m_\alpha = g_{\alpha\beta}\, m_\beta + a_{\alpha\beta}\, s_\beta$$

i.e.

$$(v) \qquad m_\alpha - g_{\alpha\beta}\, m_\beta = A_{\alpha\beta}\, s_\alpha$$

where

$$A_{\alpha\beta} = a_{\alpha\beta}\, g_{\alpha\beta}^{-1}.$$

Now the cocycle conditions $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ at first order in t yield that the family $\left\{ b_{\alpha\beta}\frac{\partial}{\partial z_\alpha} \right\}$ is actually a one cocycle with values in K^{-1}. In the same way the conditions

$g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma}$ at the first order in v tell us that $\{ A_{\alpha\beta}\, e_\alpha \otimes e^*_\beta \}$ is a 1-cocycle with values in $\operatorname{End} E$ (here $e_\alpha \otimes e^*_\alpha$ are the generators of $\operatorname{End} E \simeq E \otimes E^*$). At the first order in v one gets that $\{ b_{\alpha\beta} \frac{\partial}{\partial t_\alpha} + X_{\alpha\beta}\, e_\alpha \otimes e^*_\alpha \}$ is a one cocycle with values in $\mathcal{D}^1(E)$.

Accordingly the infinitesimal deformation of $E \to C$ gives us maps

$$\frac{\partial}{\partial v} \longmapsto [A_{\alpha\beta}\, e_\alpha \otimes e^*_\alpha] \in H^1(C, \operatorname{End} E)$$

$$\downarrow$$

$$\frac{\partial}{\partial t} \longmapsto [b_{\alpha\beta} \frac{\partial}{\partial t_\alpha} + X_{\alpha\beta}\, e_\alpha \otimes e^*_\alpha] \in H^1(C, \mathcal{D}^1(E)).$$

$$\downarrow \pi \qquad\qquad \downarrow$$

$$[b_{\alpha\beta} \frac{\partial}{\partial t_\alpha}] \in H^1(C, K^{-1}).$$

The relations (t) and (v) above which tell us how the section s is deformed, have a cohomological meaning as well. Consider the complexes

$$0 \longrightarrow \mathcal{D}^i(E) \xrightarrow{\cdot s} E \longrightarrow 0$$

$i = 0, 1$, where $\cdot s$ is the evaluation of the differential operator $D \in \mathcal{D}^i(E)$ on s, i.e. $\cdot s(D) = D(s)$. Obviously $\mathcal{D}^0(E) = \operatorname{End} E$. The double complexes given by taking check cochains read,

$$\begin{array}{ccc} \mathcal{C}^0(C, \mathcal{D}^i(E)) & \xrightarrow{\delta} & \mathcal{C}^1(C, \mathcal{D}^i(E)) \longrightarrow \cdots \\ \downarrow \cdot s & & \downarrow \cdot s \\ \mathcal{C}^0(C, E) & \xrightarrow{\delta} & \mathcal{C}^1(C, E) \longrightarrow \cdots \end{array}$$

where δ is the Check-coboundary operator.

Set

$$\mathcal{A}^p_{(i)} = \mathcal{C}^p(C, \mathcal{D}^i(E)) \oplus \mathcal{C}^{p-1}(C, E)$$

and define the operator

$$\delta_s^{(i)} : \mathcal{A}^p_{(i)} \longrightarrow \mathcal{A}^{p+1}_{(i)}$$
$$(D,u) \rightsquigarrow \delta_s^{(i)}(D,u) = (\delta D, \delta u + (-)^p Ds)$$

clearly

$$\delta_s^{(i)2} = (\delta^2 D, \delta(\delta u + (-)^p Ds) + (-)^{p+1}\delta D(s)) =$$
$$= (0, (-)^p \delta Ds + (-)^{p+1}\delta D(s)) = 0.$$

So we get hypercohomology groups denoted by $\mathbb{H}^p(d_s^i)$.

Now the coples

$$(X_{\alpha\beta}, h_\alpha) \in \mathcal{C}^1(\mathcal{C}, \mathcal{D}^1(E)) \oplus \mathcal{C}^0(\mathcal{C}, E)$$

and

$$(A_{\alpha\beta}, m_\alpha) \in \mathcal{C}^1(\mathcal{C}, \mathcal{D}^0(E)) \oplus \mathcal{C}^0(\mathcal{C}, E)$$

are actually 1-cocycles, indeed

$$\delta_s^{(1)}(X_{\alpha\beta}, h_\alpha) = ((\delta X)_{\alpha\beta\gamma}, h_\alpha - g_{\alpha\beta} h_\beta - X_{\alpha\beta} s_\alpha) = 0$$

$$\delta_s^{(0)}(A_{\alpha\beta}, m_\alpha) = ((\delta A)_{\alpha\beta\gamma}, m_\alpha - g_{\alpha\beta} m_\beta - A_{\alpha\beta} s_\alpha) = 0$$

the first components vanishing because $X_{\alpha\beta}$ and $A_{\alpha\beta}$ are our cocycles. The vanishing of the second components being equivalent to the relations (t) and (v) above.

Although the notation used here explicitely refers to the case of (C, E), being a curve and a vector bundle, it easily generalizes to higher dimensional varieties, or reduces to rank $E = 1$.

REFERENCES

[ADKP] E. Arbarello, C. DeConcini, V. Kac and C. Procesi "MOduli spaces of curves and representation theory" Commun. Math. Phys. **117** (1988) 1-36

[AGR1] L. Alvarez-Gaume, C. Gomez and C. Reina "Loop groups, Grasmannians and string theory" Phys. Lett. **190B** (1987) 55-62

[AGR2] L. Alvarez-Gaume, C. Gomez and C. Reina "New mwthods in string theory" in Superstrings '87 World Scientific (1987)

[DN] J. M. Drezet and M. S. Narasimhan "Groupe de Picard des varietes de modules de fibres semi-stable sur les courbes algebriques" Invent. math. **97** (1989) 53-94

[FR1] G. Falqui and C. Reina "Conformal field theory and moduli space of vector bundles over variable Riemann surfaces" Proc. XIX Differential Geometrical Methods in Physics (Rapallo 1990) (in press)

[FR2] G. Falqui and C. Reina "Atiyah algebras an moduli of vector bundles over curves" SISSA Preprint (1990)

[GO] P. Goddard and D. Olive "Kac-Moody and Virasoro algebras in relation to quantum physics" Int. J. Mod. Phys. **A1** (1986) 303-314

[H] N. Hitchin "Flat connections and geometric quantization" Commun. Math. Phys. **131** (1990) 347-356

[I] J. Igusa "Theta functions" Springer (1982)

[M] D. Mumford "Tata Lectures on Theta I" Birkhauser (1983)

[NR] M. S. Narasimhan and S. Ramanan "Deformations of the moduli space of vector bundles over an algebraic curve" Ann. Math. **101** (1975) 391-417

[PS] A. Pressley and G. Segal "Loop Groups' Claredon Press (1986)

[SW] G. Segal and G. Wilson "Loop groups and equationd of KdV type" Pub. Math. IHES **61** (1985) 5-65