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Course on Oceanography of Semi-Enclosed Seas
15 April - 3 May 1991

"Long-Waves Along Coasts & Over Continental Shelves"

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LONG WAVES ALONG COASTS AND OVER CONTINENTAL SHELVES
M. HENDERSHOTT TRIESTE, 1991

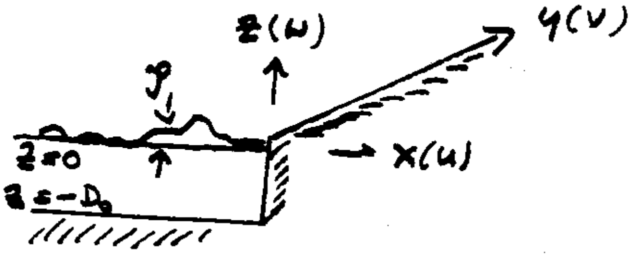
A variety of wave motions are trapped against the coast in a rotating, stratified ocean with a continental shelf. These two lectures summarize, by examples, the most important features of the dispersion relation for such waves travelling in the along-shore (y) direction in which flow variables have the travelling wave form $\exp(-i\sigma t + iky)$function (x,z) and the dispersion relation is $\sigma=$ function(k).

In the shallow water homogeneous fluid (unstratified) approximation, the solutions may usually be divided into (i) a **Poincaré continuum** of long gravity waves incident from the deep sea and reflected at the coast-continental shelf, (ii) an infinite family of **edge waves** long gravity waves refractively trapped in a waveguide whose sidewalls are the coast and the seaward edge of the continental shelf, (iii) a **Kelvin mode** trapped at the coast by rotation, (iv) an infinite family of **topographic Rossby waves** usually called continental shelf waves trapped over the continental shelf and slope. The gravity waves are all superinertial; their frequencies σ exceed the local Coriolis parameter f. In shallow water approximation, the topographic Rossby waves are all subinertial; $\sigma < f$. The Kelvin mode may be either sub or super inertial.

At subtidal frequencies, the topographic Rossby modes and the Kelvin mode are strongly forced by coastal winds and may travel great distances along coasts. The shallow water approximation represents them adequately when the stratification is gentle (the Brunt-Vaisala frequency N is small) and the shelf slope a is small; i.e. when aN/f is small. Values of aN/f less than one characterise the broad and gentle continental shelf of the Atlantic coast of the United States.

But when aN/f is the order of one or greater (e.g. the Pacific coast of the United States or of Mexico), the shallow water approximation fails to give a realistic representation of the topographic Rossby modes and the Kelvin mode. As aN/f is increased, the topographic rossby modes increaingly take on the character of internal Kelvin modes trapped at the coast; their frequencies may become superinertial. Remarkably, analysis of the strongly stratified case (when aN/f is of order one or much greater) is facilitated by reviewing the non rotating homogeneous fluid problem *without* the shallow water approximation; there is a simple correspondence between the solutions of this problem (a continuum of deep water waves incident from the deep sea and a finite family of refractively trapped edge waves) and the solutions of the low frequency rotating and stratified problem that must be solved when aN/f is order one or greater.

Coordinates



Get the Linearised Shallow Water Eqs (LSW)

Start with 3-D linearised eqs in homogeneous fluid (density ρ_0).

$$u_t - f_0 v = -P_x/\rho_0$$
$$v_t + f_0 u = -P_y/\rho_0$$
$$w_t = -P_z/\rho_0 - g$$
$$u_x + v_y + w_z = 0$$

(a) $u_x + v_y + w_z = 0 \Rightarrow$ (Size of w) = (Size of u)(Depth / Horizontal Scale)

i.e. size of $w \to 0$ if Depth << Horiz. Sc.

(b) Size of $w \to 0 \Rightarrow$ hydrostatic $0 = -P_z - g\rho_0$

i.e. $P(x,y,z,t) = g\rho_0[\eta(x,y,t) - z]$

(c) Now $u_t - f_0 v = -P_x/\rho_0 = -g\eta_x$

$\Rightarrow u = u(x,y,t) \neq u(x,y,z,t)$ etc

(d) Hence $\int_{-D_0}^{0} (u_x + v_y + w_z)\,dz = D(u_x + v_y) + w\big|_{-D}^{0}$,

$w(x,y,0,t) \simeq \eta_t(x,y,t)$.

(e) Finally, the LSW eqs are

$$\boxed{\begin{aligned} u_t - fv &= -g\eta_x \\ v_t + fu &= -g\eta_y \\ \eta_t + D_0(u_x + v_y) &= 0 \end{aligned}}$$

Fluid moves in columns

(f) By cross differentiation to eliminate u, v obtain

$$\boxed{\partial_t\left[\nabla_H^2\eta - (\eta_{tt} + f_0^2\eta)/gD_0\right] = 0}$$

First Plane Wave Solutions of the LSW eqs.

They have the form

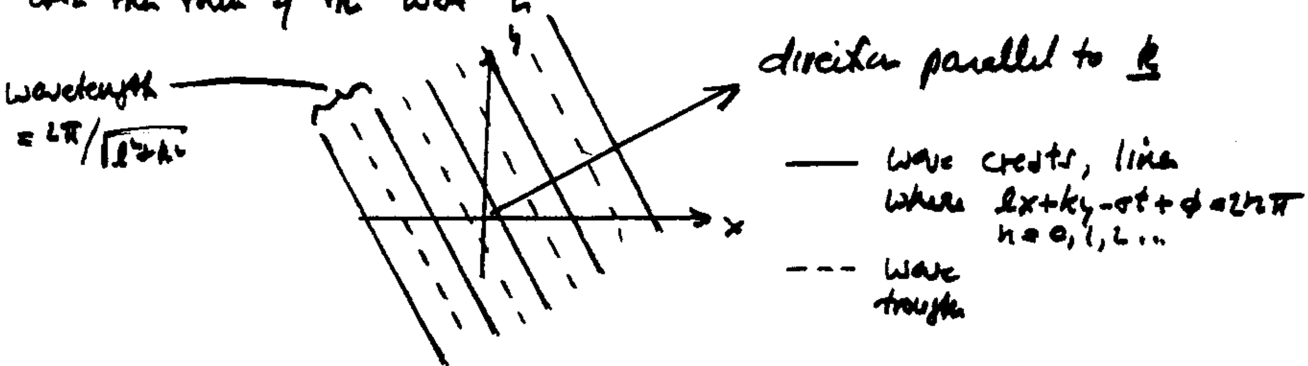
$$\eta(x,y,t) = |A| \cos(lx+ky-\sigma t+\phi)$$

where l, k = component of wave number vector $\underline{k}$

σ = wave frequency

ϕ = phase $|A|$ = amplitude

Viewed from above, the pattern of sea-surface displacement associated with this form of the wave is



The crests + troughs are $\perp$ to the direction in which they move. That direction is parallel to l, k

To find when the plane wave is a solution of the LSW eqs it helps to say

$$\eta = \mathrm{Re}\left[|A|e^{i\phi}\, e^{-i\sigma t+ilx+iky}\right]$$

In fact, because our equations are linear, we just look now for solutions of the form

$$\boxed{\eta = e^{-i\sigma t+ilx+iky}}$$

(took $|A|=1$, $\phi=0$) + later determine A, ϕ from forcing ... .

So put $\eta = e^{-i\sigma t+ilx+iky}$ into

$$\nabla^2\eta - (\eta_{tt} + f_0^2\eta)/gD_0 = 0$$

The last eq. $\neq 0$ unless

$$\boxed{\sigma^2 = f_0^2 + gD_0\,(l^2+k^2)}$$

NB: $\sigma > f_0$ ie superinertial

This is the dispersion relation. Only plane waves when σ, l, k obey it solve the LSW equations. If $f_0=0$, the phase speed $c = \sigma/\sqrt{l^2+k^2} = \sqrt{gD_0}$.

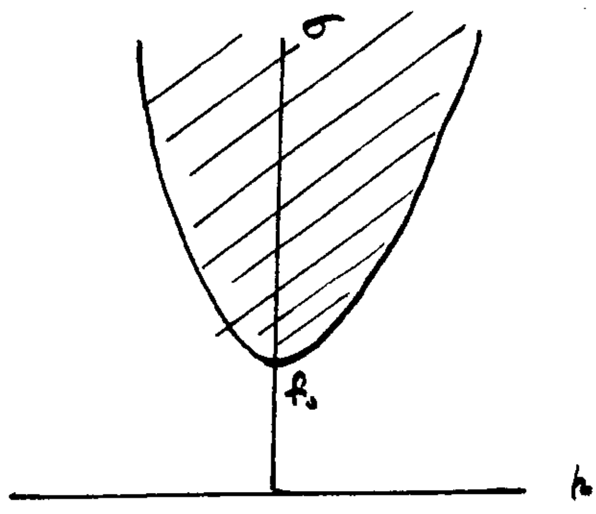
You should work out how the water moves as the wave goes by, i.e. for this plane wave ψ find u, v, w & draw the flow.

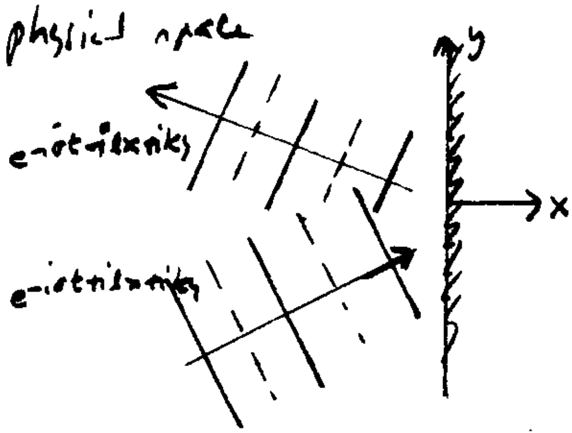
Now we've found that $\psi = e^{-i\sigma t + ilx + iky}$ solves the SW eqs if $\sigma^2 = f_0^2 + gD_0(l^2+k^2)$. But for most such waves, $u \neq 0$ at a coast. We have to add two plane waves together to satisfy $u=0$ at our coast, at $x=0$. Thus take

$$\boxed{\psi = e^{-i\sigma t + ilx + iky} + \text{some constant } e^{-i\sigma t - ilx + iky}}$$

If $f_0 = 0$, you should be able to see that the choice constant $= 1$ makes $\psi = e^{-i\sigma t + iky} \cos lx$ & satisfies $u=0$ at $x=0$; if $f_0 \neq 0$ the constant must be chosen differently but we can still make $u=0$ at $x=0$.

It is customary to display the dispersion relation $\sigma^2 = f_0^2 + gD_0(l^2+k^2)$ which this solution (still!) obeys by plotting σ vs k. Since l can be any value all we can say is that $\sigma^2 > f_0^2 + gDk^2$ ie our plot is as sketched; the dispersion relation fills the Poincaré Continuum ////. In physical space





ie an incident plane wave has been reflected from the coast. All /// there is made up of incident + reflected waves.

Kelvin Wave It is a special solution with $u \equiv 0$ in our geometry. Thus it satisfies

$$-f_0 v = -g\eta_x \quad \text{ie geostrophic in cross coast direction}$$
$$v_t = -g\eta_y$$
$$\eta_t + D_0 v_y = 0$$

Seek solution $\eta = e^{-i\sigma t + iky}\phi(x)$ ie plane wave along coast (suggested by last two eqs) with on-offshore variation yet to be determined. Last two eqs. say $\eta_{tt} - gD_0\eta_{yy} = 0$. A solution is $e^{-i\sigma t + iky}$ provided that $\sigma^2 = gD_0k^2$ (dispersion relation). So

$$\eta = e^{-i\sigma t + iky}\phi(x) \qquad \sigma = \pm k\sqrt{gD_0}$$

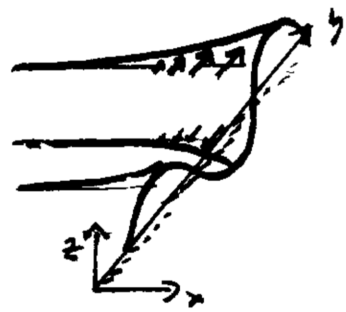
satisfies all but first equation. First & second eqs say

$\eta_{xt} = f_0\eta_y$ i.e. $-i\sigma\phi_x = f_0 ik\phi$

i.e. $\phi_x - f_0 k/\sigma\,\phi = 0$. Solution of this is

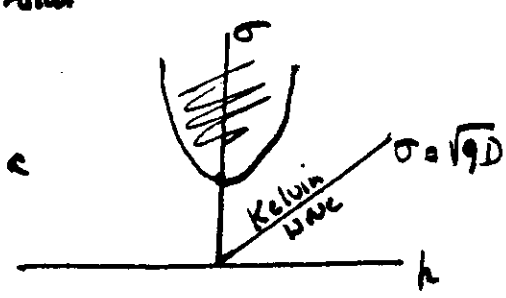
$\phi = e^{(f_0k/\sigma)x}$. We need solution that don't explode when $x \ll 0$, so we need $k/\sigma > 0$. Thus, finally

$$\eta = A\,e^{-i\sigma t + iky + f_0x/\sqrt{gD_0}}, \quad \sigma = k\sqrt{gD_0}$$
$$v = (gk/\sigma)\eta \qquad u \equiv 0$$



Crests/troughs move poleward at $c = \sqrt{gD_0}$
Fluid surges forward under crests, backward under troughs.
Crests/troughs decay like $e^{-x/R}$ seaward where $R = c/f_0 = \sqrt{gD_0}/f_0$ = Rossby radius
Seaward decay of crests/troughs is because η_x has to balance $f_0 v$.

On dispersion relation, $\sigma = k\sqrt{gD}$ is a straight line.



The fact that $\sigma/k = \sqrt{gD_0}$ doesn't depend on k means that all Kelvin waves move at same speed regardless of wavelength. This means that an initial distribution $\eta_0(y, 0)$ of coastal sea level composed only of Kelvin waves will move without changing form; $\eta(y,t) = \eta_0(y - \sqrt{gD_0}t, 0)$.

More technically:

$$-f_0 V = -g\eta_x \qquad V_t = -g\eta_y \qquad \eta_t + D_0 V_y = 0$$

Take $\eta = \zeta(y,t)\,\phi(x) \qquad V = v(y,t)\,\phi(x)$

Second + third eqs say $\zeta_{tt} - gD_0\zeta_{yy} = 0$

ie $\zeta(y,t) = \left[a(y-ct) + b(y+ct)\right], \quad c = \sqrt{gD_0}$

$v(y,t) = \frac{g}{c}\left[a(y-ct) - b(y+ct)\right]$

At this stage solution do appear to move without change of shape, but can move both north + south. As earlier, the first equation makes the choice

$$f_0 V = g\eta_x \Rightarrow f_0\phi v = g\zeta\phi_x \Rightarrow \phi_x - \frac{f_0}{g}\frac{v}{\zeta}\phi = 0$$

This gives us solution $\phi(x)$ only if v/ζ is constant (i.e. doesn't depend on y or t); this means either $a=0$ or $b=0$. To get decaying solutions we need $v/\zeta > 0$ ie $b = 0$.

Thus
$$\boxed{\begin{aligned} &\eta(x,y,t) = a(y-ct)\,e^{+f_0 x/c} \\ &V(x,y,t) = g\eta/c \\ &u = 0 \end{aligned}}$$

+ indeed, a satisfies the simple equation

$$\boxed{a_t + c\,a_y = 0}$$

These eqs describe the motion of a whole bunch of Kelvin waves all added together to give any initial coastal disturbance

$$\eta(0, y, 0) = a(y)$$

Waves in a Stratified Constant Depth Ocean

When the ocean is at rest, $u = v = w = 0$ and density $= \rho_0(z)$. It is convenient to use not $\rho_0(z)$ but the Väisälä Frequency $N^2(z) \equiv g\rho_{0z}/\rho_0$. Typically in the open ocean $N(z)$ is a few cph. At rest the pressure P_0 obeys $0 = -P_{0z} - g\rho_0$.

Let $\rho(x,y,z,t)$ + $p(x,y,z,t)$ be perturbation away from rest: total density $= \rho_0(z) + \rho$
total pressure $= p_0(z) + p$.

Then (check this)

$$\begin{array}{ll} 1. & u_t - fv = -p_x/\rho_0 \\ 2. & v_t + fu = -p_y/\rho_0 \\ & 0 = -p_z - g\rho \\ & \rho_t + w\rho_{0z} = 0 \\ 4. & u_x + v_y + w_z = 0 \end{array} \quad \Big\} \quad \boxed{N^2 w = -p_{zt}/\rho_0} \quad 3.$$

govern u, v, w, p, ρ. We've assumed motion small enough to be linear, motion also slow enough to be hydrostatic. ρ_0 is just the average density; you could take $\rho_0 = 1$ in cgs units. The density equation is got this way:

$$\frac{D\rho_{total}}{Dt} = 0 = \underset{\rho_{0z_t}=0}{(\rho_{0z}+\rho)_t} + \underbrace{u(\ )_x + \cdots + w(\rho + \rho_{0z})_z}_{\text{small}} = 0$$

$$\Rightarrow \quad \rho_t + w\rho_{0z} = 0$$

Now do some algebra, but with a useful result.

Take $\quad u = \mathcal{U}(x,y,t)\,F(z) \qquad v = \mathcal{V}(x,y,t)\,F(z) \qquad$ 5.

$p = \rho_0 g\, Z(x,y,t)\, F(z) \qquad$ 6.

$w = \mathcal{W}(x,y,t)\, G(z) \qquad$ 7.

$$1+5 \Rightarrow \quad \boxed{\begin{array}{l} U_t + fV = -gZ_x \\ V_t - fU = -gZ_y \end{array}}$$
$$2+5 \Rightarrow$$

ie we are getting the LSW eqs again!!! Keep going.

$$4+5+7 \Rightarrow \quad (U_x + V_y)F + WG_z = 0$$

ie if we take

$$F = G_z D, \quad W = Z_t$$

we get

$$\boxed{Z_t + D(U_x + V_y) = 0}$$

Here we've introduced a constant D that we don't yet know.

We haven't used 3 yet, & it forces us to make all the foregoing choices for F, G, W, Z self consistent. So

$$3+6+7 + W = Z_t + F = G_z D \Rightarrow \quad \boxed{G_{zz} + [N^2/gD]G = 0}$$

To make $w=0$ at $z=0$ (a good approximation for internal waves) as well as at $z=-D_0$, $\boxed{G=0, \; z = 0, -D_0}$

So here is how we get the internal modes:

(1) solve $G_{zz} + [N^2(z)/gD]G = 0$, $G=0$ $z=0, -D_0$
to get a sequence of solutions $G_n(z)$, D_n, n = mode #.

(2) For each member of that sequence solve the LSW eqs
$$U_t - f_0 V = -gZ_x \quad \text{etc}$$
$$Z_t + D_n(U_x + V_y) = 0$$

& then, i.e. $u(x,y,z,t) = U_n(x,y,t)\, G_{nz}(z)$.

NB: if $N(z) = N_0$, then $G_n(z) = \sin(n\pi z/D)$, $\sqrt{gD_n} = \dfrac{N_0 D}{n\pi}$

Notice: the shallow water equations are relevant both for homogeneous shallow water over possibly variable depth, & for stratified fluid over constant depth (provided only that $\sigma \ll N$ so that eq. 3 of previous page hold).

As an example, get the internal Kelvin modes if $N = N_0$. I'll get $v(x,y,z,t)$, you should get $\rho(x,y,z,t)$, $p(x,y,z,t)$ + verify (?) my sketch.

$$Z(x,y,t) = e^{-i\sigma t + iky + f_0 x/c_n} \qquad c_n = \sqrt{gD_n}$$

$$V = (g/c_n)Z \quad , \quad U = 0$$

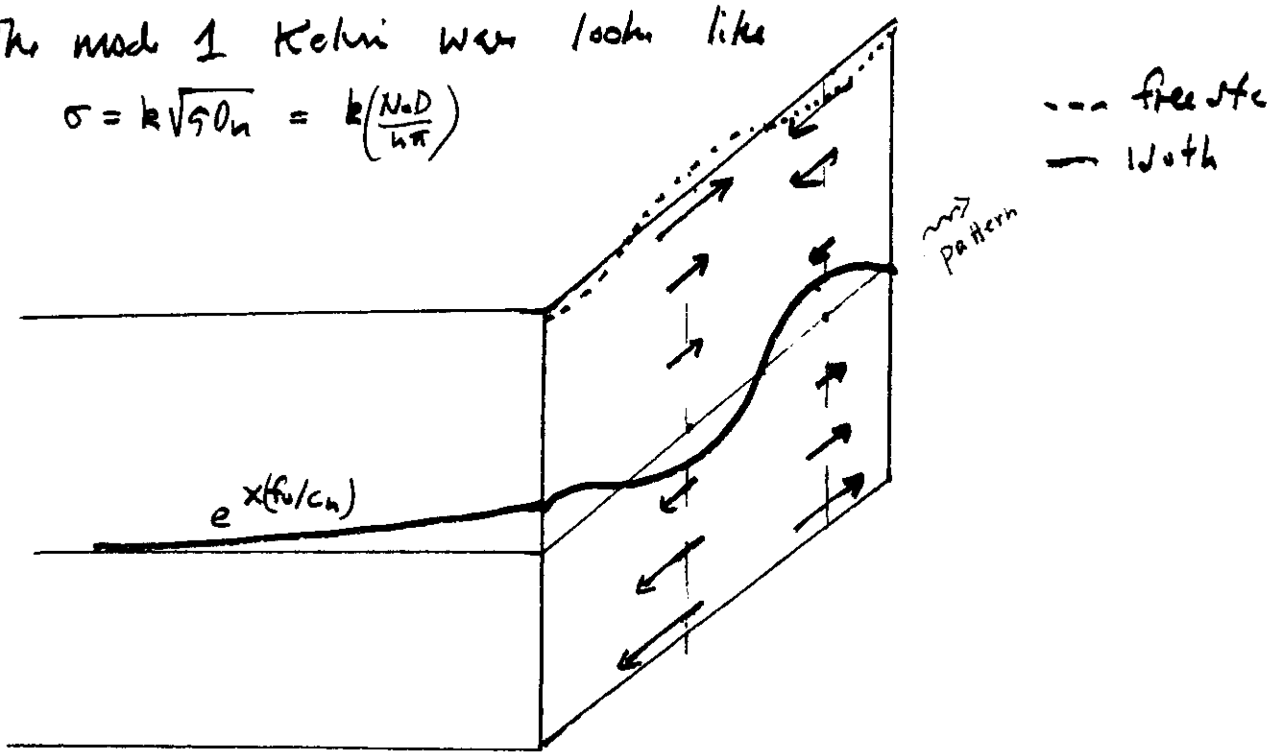
$$v(x,y,z,t) = \text{constant}\; e^{-i\sigma t + iky + f_0 x/c_n} \cos\left(\frac{n\pi z}{D_0}\right)$$

The one thing you can't easily get is $\eta(x,y,t)$ because I took $w=0$ at $z=0$. Yet the free surface does move slightly (cm) as isotherms move vertically (10's of meters). A useful rule is, if $N = N_0$

$$w(\text{sfc})/w(\text{max interior}) = N_0^2 D_0/n\pi g \simeq \frac{\Delta\rho}{\rho}\frac{1}{n\pi}$$

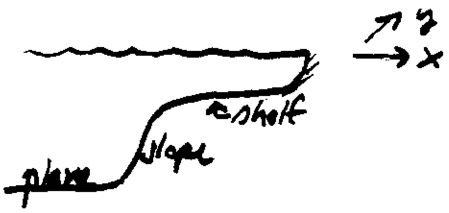
The mode 1 Kelvin wave looks like

$$\sigma = k\sqrt{gD_n} = k\left(\frac{N_0 D}{n\pi}\right)$$



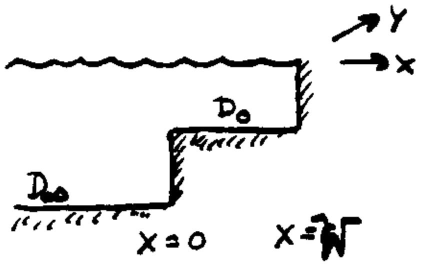
LSW Waves over the continental shelf.

The ocean deepens away from the coast. The shallow & sometimes broad region near the coast is the continental shelf, the region of rapid depth change is the continental slope, the adjoining deep sea will be idealised as the abyssal plain. We look for long waves that travel up/down coast, $e^{-i\sigma t+iky}$. Refractive trapping of LSW gravity waves at the slope gives rise to edge waves.



Topographic stretching of vortex lines associated with the earth's rotation gives rise to topographic Rossby waves trapped over the sloping relief; they are usually called continental shelf waves.

Edge Waves To see them in their simplest form, consider a step shelf with no rotation ($f_0 = 0$). Seek waves of the form



$\eta(x,y,t) = e^{-i\sigma t+iky} Z(x)$. Then

$$Z_{xx} + (\sigma^2/gD - k^2)Z = 0 \qquad D = D_0 \text{ or } D_\infty$$

At $x = W$ require $u = 0$ e.g. $Z_x = 0$

As $x \to -\infty$ require Z finite.

At the step $x = 0$ match Z and uD e.g. require

$$Z_0(0) = Z_\infty(0) \quad , \quad D_0 Z_{0x}(0) = D_\infty Z_{\infty x}(0).$$

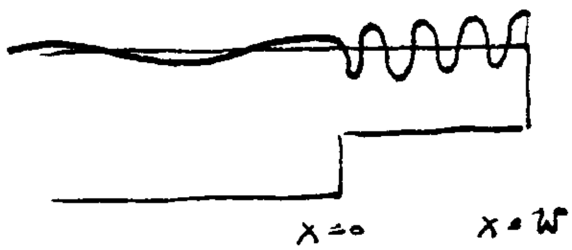
There are two families of solutions finite as $x \to -\infty$; they are waves incident from $x = -\infty$ that are reflected at $x = W$, and waves refractively trapped within $(0, W)$.

Wave incident from $x = -\infty$ has the form

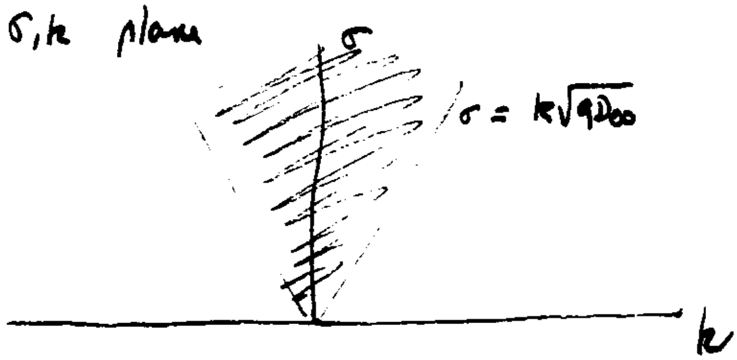
$$Z_0(x) = A\cos l_0 x + B \sin l_0 x, \qquad l_0^2 = \sigma^2/gD_0 - k^2 > 0$$
$$Z_\infty(x) = C\cos l_\infty x + D \sin l_\infty x, \qquad l_\infty^2 = \sigma^2/gD_\infty - k^2 > 0$$

$$Z_0(0) = Z_\infty(0) \Rightarrow A = C$$
$$D_0 Z_{0x}(0) = D_\infty Z_{\infty x}(0) \Rightarrow D_0 l_0 B = D_\infty l_\infty D$$
$$Z_{0x}(W) = 0 \Rightarrow B = A \tan l_0 W$$

Thus
$$Z_0 = A'\left[\cos l_0 x + \tan l_0 W \sin l_0 x\right]$$
$$Z_\infty = A'\left[\cos l_\infty x + (D_0 l_0 / D_\infty l_\infty)\tan l_0 W \sin l_\infty x\right]$$



These solutions were constructed supposing $l_0^2 > 0$, $l_\infty^2 > 0$ i.e. they require $\sigma^2/gD_\infty > k^2$. Just like the plane wave reflected from a straight coast in a flat ocean, these solutions fill the continuum $\sigma^2/gD_\infty > k^2$ in the σ, k plane



For refractively trapped mode

$$Z_0 = A\cos l_0 x + B\sin l_0 x \qquad Z_\infty = C e^{l_\infty x}$$

where $l_0^2 = \sigma^2/gD_0 - k^2 > 0 \qquad l_\infty^2 = k^2 - \sigma^2/gD_\infty > 0$

Now $Z_0(0) = Z_\infty(0) \Rightarrow A = C$

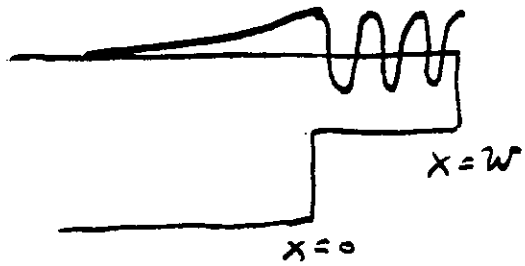
$$D_0 Z_{0x}(0) = D_\infty Z_{\infty x}(0) \Rightarrow D_0 l_0 B = D_\infty l_\infty C$$

so that

$$Z_0 = A'\left[l_0 D_0 \cos l_0 x + l_\infty D_\infty \sin l_0 x\right]$$

$$Z_\infty = A' \, l_0 D_0 \, e^{l_\infty x}$$

where A' is an arbitrary constant.



$Z_{0x}(W) = 0 \Rightarrow$

$$-l_0^2 D_0 \sin l_0 W + l_0 l_\infty D_\infty \cos l_0 a = 0$$

$$\frac{l_\infty D_\infty}{l_0 D_0} = \tan l_0 W$$

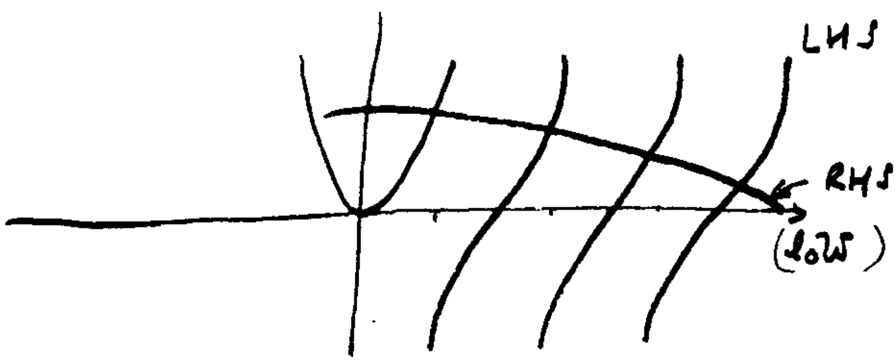
The two dispersion relations may be combined to yield

$$l_\infty^2 = (1 - D_0/D_\infty)k^2 - (D_0/D_\infty)l_0^2$$

This plus the last yield

$$(l_0 W)\tan(l_0 W) = \sqrt{\frac{D_\infty}{D_0}\left(\frac{D_\infty}{D_0} - 1\right)k^2 W^2 - \frac{D_\infty}{D_0}(l_0 W)^2}\,.$$

This gives $l_0 W$ as a function of k, the dispersion relation is then $\sigma^2/gD_0 = k^2 + (l_0 W)^2/W^2$. Obtain the solution for $l_0 W$ graphically:



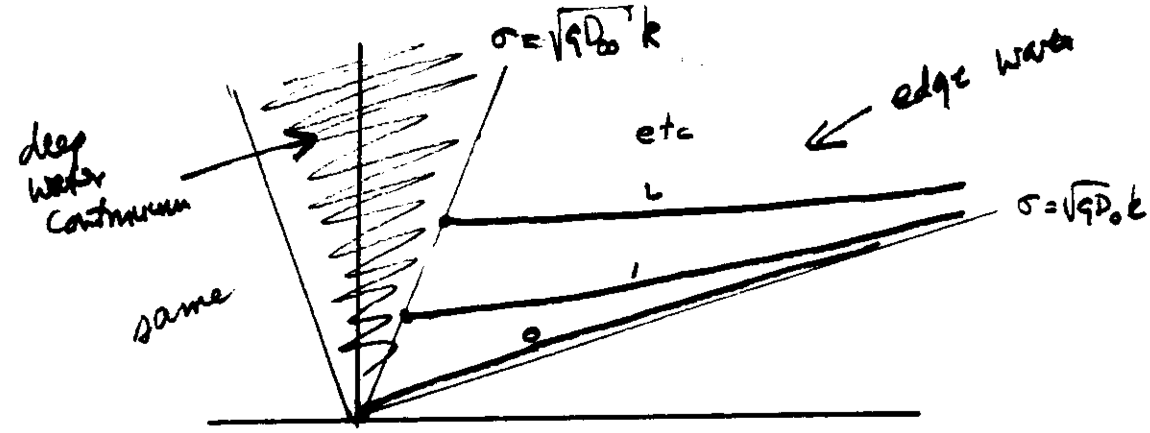
Notice (1) as $kW \to \infty$ RHS plot rises, roots are
$\ell_0 W \to (n\pi + \pi/2)$ as $kW \to \infty$

(2) as $kW \to 0$ RHS plot drops, root that was $n\pi + \pi/2$ drops to $n\pi$ + then ceases to exist.

(3) so $\sigma^2/gD_0 = k^2 + (n\pi + \pi/2)^2/W^2$ $k \to \infty$

(4) if k decreases to k_n' where root is $n\pi$ then
$(D_\infty/D_0 - 1)(k_n'^2 W^2) = (n\pi)^2$ +
$\sigma^2 = gD_0 \left(k_n'^2 + (D_\infty/D_0 - 1) k_n'^2 \right) = gD_\infty k_n'^2$

(5) so plot σ vs. k i

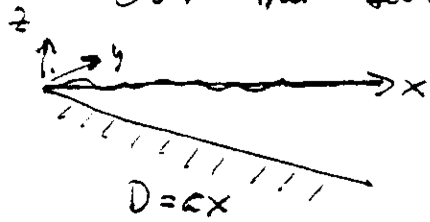


The step shelf w/o rotation then allows a continuum of waves incident from $x = -\infty$ + reflected at the coast (the shelf only changes the phase of their reflection), and the step shelf refractively traps an infinite family of edge waves.

This problem can be solved with rotation, but we learn more by doing other problems.

<u>The uniformly sloping beach</u>

Over this beach $D(x) = \alpha x \quad (x>0)$. The LSW eqs are



$$u_t - f_0 v = -g\eta_x$$
$$v_t + f_0 u = -g\eta_y$$
$$\eta_t + (uD)_x + (vD)_y = 0$$

They have solution of the form $\eta(x,y,t) = e^{-i\sigma t + iky} Z(x)$ where

$$Z_{xx} + (D_x/D) Z_x + \left(\frac{\sigma^2 - f_0^2}{gD} - \frac{f k D_x}{\sigma D} - k^2 \right) Z = 0$$

Here $D = \alpha x$ so that

$$Z_{xx} + Z_x/x + \left[\frac{\sigma^2 - f_0^2}{g\alpha x} - \frac{f_0 k}{\sigma x} - k^2 \right] Z = 0 \qquad x > 0.$$

Take $x = \eta/2k \qquad \lambda = (\sigma^2 - f_0^2)/(2g\alpha k) - f_0/2\sigma$ to find

$$Z_{\eta\eta} + Z_\eta/\eta + [\lambda/\eta - 1/4] Z = 0 \qquad Z(0), Z(\infty) \text{ finite.}$$

This eigenvalue ~~pr~~ appears in the QM treatment of the hydrogen atom (Schiff, Quantum Mechanics, p 88). The solution are

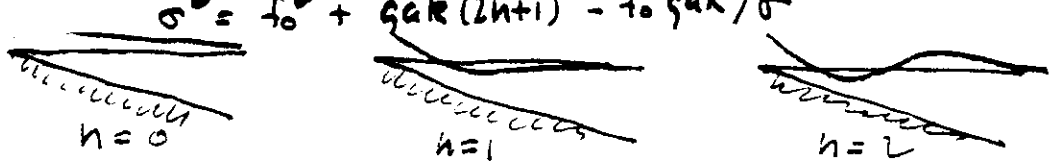
$$Z = e^{-\eta/2} L_n(\eta) \qquad \lambda = n + 1/2 \qquad n = 0, 1, \ldots$$

where $L_n(\eta)$ are Laguerre Polynomials

$$L_0 = 1 \qquad L_1 = 1 - 2\eta \qquad L_2 = 1 - 2\eta + \eta^2 \ldots$$

Thus the solution are

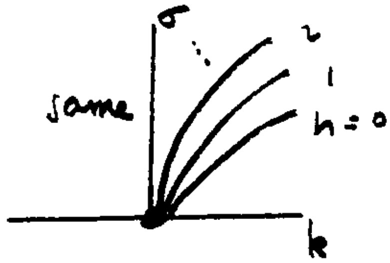
$$\eta(x,y,t) = e^{-i\sigma t + iky} e^{-kx} L_n(2kx) \qquad x<0$$
$$\sigma^2 = f_0^2 + g\alpha k(2n+1) - f_0 g\alpha k/\sigma$$



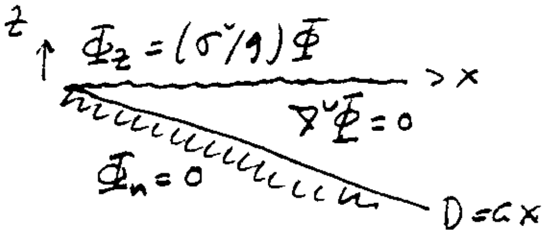
The dispersion relation is noteworthy. It is <u>cubic</u> in σ so that there are more than just edge waves.

→ When there is no rotation ($f_0 = 0$), $\sigma^2 = gak(2n+1)$.

Now there are only edge waves, there is no deep water continuum because there is no flat bottom deep ocean. Another way to say this is that if wave energy leaves the beach at $x = 0$ it is always ultimately refracted back to the beach because the wave speed $\sqrt{gD}$ grows without limit offshore.



But it is inconsistent to let $D \to \infty$ in the shallow water equations. Ursell (1952) relaxed the shallow water approximation & solved



$$\phi_z = (\sigma^2/g)\,\phi \qquad z = 0$$

$$\phi_{xx} - k^2\phi + \phi_{zz} = 0$$

$$\phi_n = 0 \qquad z = ax \qquad x < 0$$

where $\underline{u} = \nabla\Phi$, $\Phi = e^{-i\sigma t + iky}\,\phi(x,z)$. Solutions were

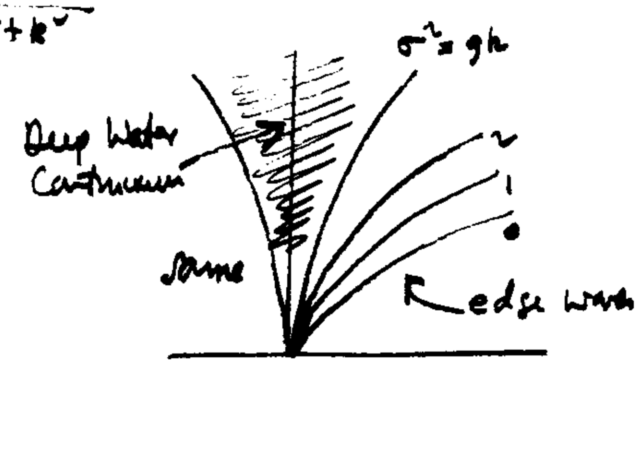
Edge waves: $\sigma^2 = gk \sin\left[(2n+1)\tan^{-1} a\right]$ $\quad n = 0, 1 \cdots \leq \left[\frac{\pi}{4\tan^{-1} a} - \frac{1}{2}\right]$

These are the shallow water solutions for small a, but the S.W. approximation erroneously gives an infinite family of edge waves.

Deep water continuum: far from the coast, the waves don't feel the bottom

$$\Phi = e^{-i\sigma t + iky} \cos(lx + \text{phase})\, e^{z\sqrt{l^2+k^2}}$$

$$\sigma^2 = g\sqrt{l^2+k^2}$$



→ If we restore rotation but make the sea surface rigid, we must solve $u_t - f_0 v = -p_x/\rho_0$ $\quad v_t + f_0 u = -p_y/\rho_0$ $\quad (uD)_x + v_y D = 0$.

Seek solution $p(x,y,t) = e^{-i\sigma t + iky}\, Z(x)$

where $$Z_{xx} + Z_x/x - \left[-\frac{f_0 k}{\sigma x} + k^2\right] Z = 0 \qquad x<0.$$

As above $Z = e^{kx} L_n(-2kx)$, $\sigma = f_0/(2n+1)$.

There are thus still waves even though the sea surface is rigid.

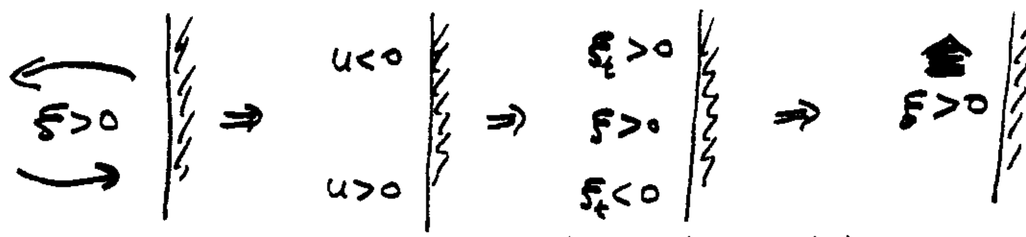
The physical basis for their existence is most easily seen from the vorticity equation:

$$u_t + uu_x + vu_y - f_0 v = -g\eta_x$$
$$v_t + uv_x + vv_y + f_0 u = -g\eta_y \quad \Rightarrow \quad \xi = v_x - u_y,\quad \xi_t + u\xi_x + v\xi_y + (\xi+f_0)(u_x+v_y) = \ldots$$
$$\eta_t + \big(u(\eta+D)\big)_x + \big(v(\eta+D)\big)_y = 0 \quad \Rightarrow \quad \frac{D}{Dt}\left(\frac{\xi+f_0}{\eta+D}\right) = 0$$

· Over the linear beach $D = -\alpha x$ with rigid sea surface

$$\frac{D}{Dt}\left(\frac{\xi+f_0}{-\alpha x}\right) = 0 \quad \xRightarrow[|\xi|\ll f_0]{} \quad \xi_t = uf_0/x$$

The sense of ξ is that $f_0 > 0 \Leftrightarrow$ ccw



i.e. an initial disturbance travels in the Kelvin wave sense

→ Now we can plot the full dispersion relation

$$\sigma^2 = f_0^2 + g\alpha k(2n+1) - f_0 g\alpha k/\sigma .$$

Take $S = \sigma/f_0$ $\quad K = g\alpha k/f_0^2$; $\quad \underline{S^3 - S[1+(2n+1)K] - K = 0}$

Take $K>0$, $S \gtrless 0$ & later redraw as $S>0$, $K \lessgtr 0$.

Points to notice

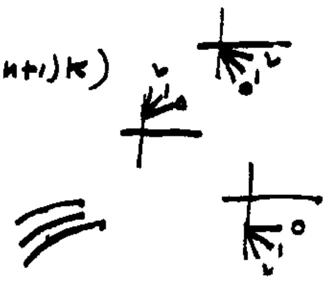
(1) $n=0$ $\quad \left[S-\left(\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+K}\right)\right]\left[S-\left(\tfrac{1}{2}-\sqrt{\tfrac{1}{4}+K}\right)\right]\left[S+1\right]=0$

(2) any n $\quad S = 0, \pm 1$ for $K = 0$

(3) near $S = \epsilon$ $\quad K \to 0$ $\quad \epsilon = -K/(1+(2n+1)K)$

$S = 1+\epsilon$ $\quad K \to 0$ $\quad \epsilon = (n+1)K$

$S = -1+\epsilon$ $\quad K \to 0$ $\quad \epsilon = -nK$

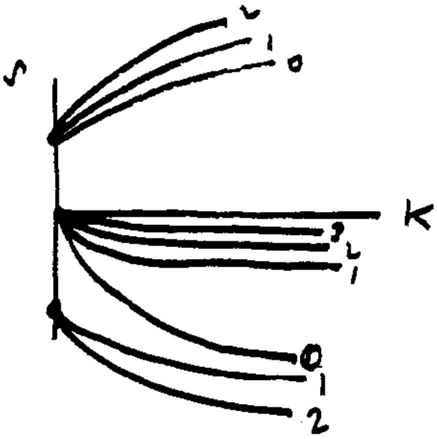


(4) $S^2 \to \infty$ $\quad K \to \infty$ $\quad S^2 = (2n+1)K$

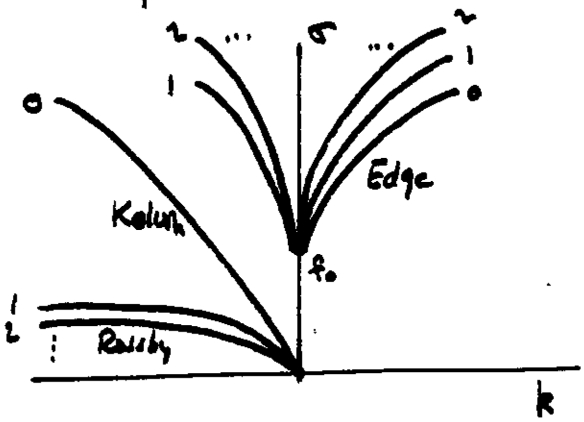
$S \to 0$ $\quad K \to \infty$ $\quad S = 1/(2n+1)$

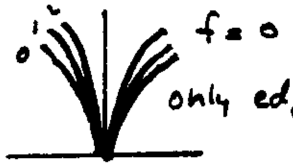
(5) The root $S=-1$ does not correspond to a physically realisable solution

(6) Full plot is



Finally then (for shallow water case)

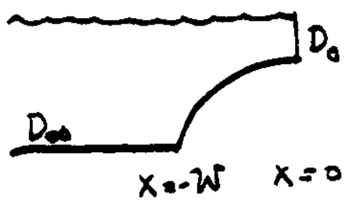




$\nabla \cdot uD = 0$

The exponential beach

One last situation that admits of simple solutions & illustrates a general property of the solutions in the case of divergence-free ($\nabla\cdot\underline{u}D = 0$) flow over the relief



$$D(x) = D_0 e^{-ax} \qquad -W<x<0$$
$$= D_\infty \qquad x<-W \qquad (D_\infty = D_0 e^{aW})$$

We want to solve

$$u_t - f_0 v = -P_x/\rho_0 \qquad v_t + f_0 u = -P_y/\rho_0 \qquad (uD)_x + v_y D = 0$$

It is quickest to take $vD = \Psi_x$, $uD = -\Psi_y$. Then

$$\nabla\cdot(\nabla\Psi/H)_t + (f_0 H_x/H^2)\Psi_y = 0$$

We solve subject to $\Psi(0,y) = 0$, $\Psi(-\infty,y) = 0$

$$\Psi(-W^-,y) = \Psi(-W^+,y), \quad \Psi_x(-W^-,y) = \Psi_x(-W^+,y)$$

Take $\Psi = e^{-i\sigma t + iky}\psi(x)$

$$\psi_{0xx} - a\psi_{0x} + \left(-\frac{f_0 ak}{\sigma} - k^2\right)\psi_0 = 0 \qquad \psi_0(0) = 0$$
$$\psi_{\infty xx} - k^2\psi_\infty = 0 \qquad \psi_\infty(-\infty) = 0$$
$$\psi_0(-W) = \psi_\infty(-W), \quad \psi_{0x}(-W) = \psi_{\infty x}(-W)$$

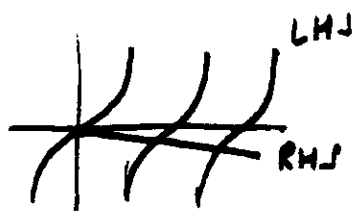
Solution

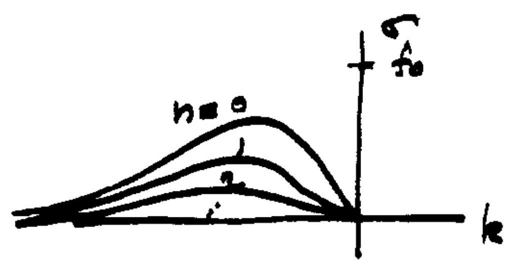
$$\psi_0 = e^{a(x+W)/2}\sin lx \qquad l^2 = \frac{-f_0 ak}{\sigma} - k^2 - \frac{a^2}{4}$$
$$\psi_\infty = A e^{|k|(x+W)}$$

$$\sin(-lW) = A \qquad a/2\,\sin(-lW) + l\cos(-lW) = |k|A$$

$$\Rightarrow \tan lW = \frac{-lW}{(a/2+|k|)W} \qquad \sigma = -f_0 ak\big/(k^2 + a^2/4 + l^2)$$

Solve for l graphically



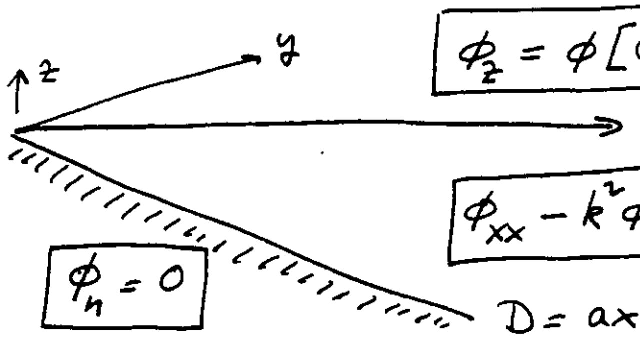


As $a, W \to \infty$, $lW \to n\pi$

NOW the low frequency modes have a zero in group velocity. Generally true if D_x/D bounded.

URSELL PROBLEM

(A) Infinite beach, no rotation, drop shallow-water approximation



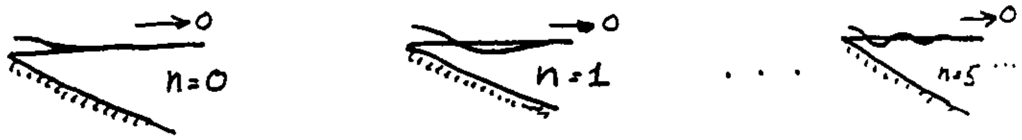
$$\boxed{\phi_z = \phi[\sigma^2/g]}$$

$$\boxed{\phi_{xx} - k^2\phi + \phi_{zz} = 0}$$

$$\boxed{\phi_n = 0}$$

(B) EDGE WAVES $\quad \sigma^2 = gk\sin\left[(2n+1)\tan^{-1}a\right]$

$$n = 0, 1, \dots \leq \left[\frac{\pi}{4\tan^{-1}a} - \frac{1}{2}\right]$$



For each mode n, ϕ is a complicated function of x, z and is not separable. It decays in an oscillatory manner away from the corner. <u>Both</u> along-coast and offshore length scales go as k^{-1}.

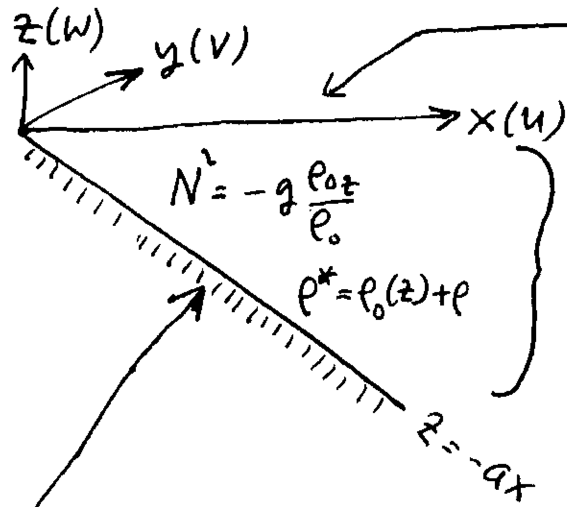
(C) DEEP WATER CONTINUUM $\quad \sigma^2 > gk$

Far from corner, ϕ is just the standing deep water wave

$$\phi = e^{-i\sigma t + iky}\cos(lx + \text{phase})\, e^{+z\sqrt{l^2+k^2}}$$

$$\sigma^2 = g(k^2+l^2)^{1/2}$$

$\boxed{\text{LOW FREQUENCY PROBLEM}}$ $\quad N^2 > f^2 > \sigma^2$



$$W = 0 \Rightarrow \rho = 0 \Rightarrow \boxed{P_z = 0}$$

$$\left\{\begin{aligned} -i\sigma u - fv &= -P_x/\rho_0 \\ -i\sigma v + fu &= -P_y/\rho_0 \\ -i\sigma w + 0 &= -P_z - g\rho \\ u_x + v_y + w_z &= 0 \\ -i\sigma\rho + w\rho_{0z} &= 0 \end{aligned}\right\}$$

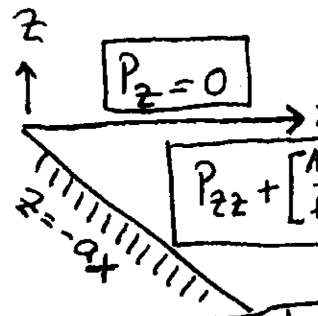
$$\boxed{P_{zz} + \frac{N^2-\sigma^2}{f^2-\sigma^2}\left(P_{xx} + P_{yy}\right) = 0}$$

$$\underline{u}\cdot\hat{n} = 0 \Rightarrow aU + W = 0 \Rightarrow \boxed{a\left\{\frac{\sigma P_x - fP_y}{f^2-\sigma^2}\right\} + \frac{\sigma}{N^2-\sigma^2}P_z = 0}$$

Seek solutions of the form $\boxed{P = e^{-i\sigma t + iky}\ \text{FUNCTION}(x,z)}$
but unfortunately the FUNCTION is not separable.

However, if N is constant, this problem may be transformed into the Ursell problem whose spectrum is known.

Ou's Transformation

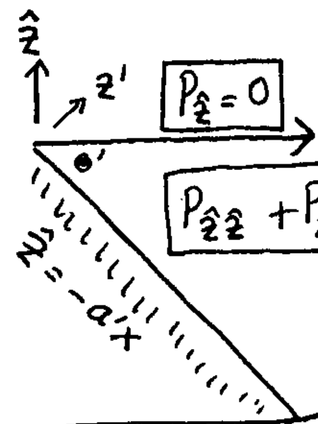


$$P_z = 0$$

$$P_{zz} + \left[\frac{N^2-\sigma^2}{f^2-\sigma^2}\right]\left[P_{xx} - k^2 P\right] = 0$$

$$a\left[\sigma P_x - fkP\right] + \sigma\left[\frac{f^2-\sigma^2}{N^2-\sigma^2}\right] P_z = 0$$

① Low Frequency Stratified Problem



$$P_{\hat z} = 0$$

$$P_{\hat z \hat z} + P_{xx} - k^2 P = 0$$

$$z = \hat z (f^2-\sigma^2)^{1/2}/(N^2-\sigma^2)$$
$$a = a' \;\; " \;\; ""$$

$$a\left[\sigma P_x - fkP\right] + \sigma\left[\frac{f^2-\sigma^2}{N^2-\sigma^2}\right]^{1/2} P_{\hat z} = 0$$

x'

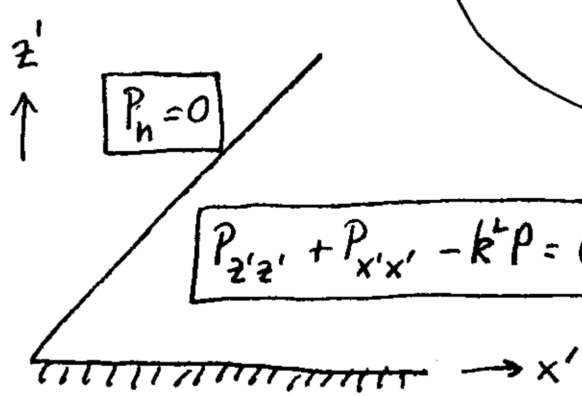
② Result of Stretching z

$$a' P_x + P_{\hat z} = a' \frac{kf}{\sigma} P$$

$$-\sin\theta' P_x + \cos\theta' P_{\hat z} = \frac{kf}{\sigma} P \sin\theta'$$

a'
1

$$P_{z'} = \frac{kf}{\sigma} \frac{a'}{(1+a'^2)^{1/2}} P$$



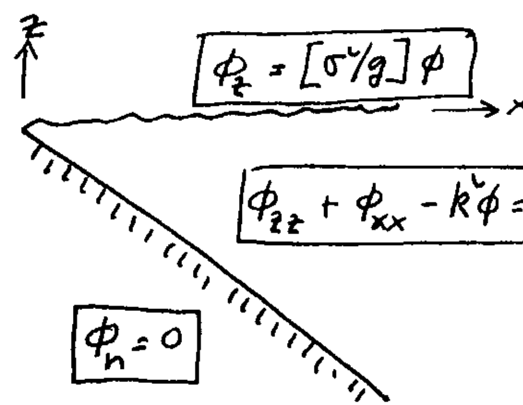
$$P_n = 0$$

$$P_{z'z'} + P_{x'x'} - k^2 P = 0$$

$$P_n = \left[kf/\sigma\right]\left[\frac{a'}{(1+a'^2)^{1/2}}\right] P$$

③ Result of Rotation

Note
$k/\sigma < 0$



$$\phi_z = \left[\sigma^2/g\right]\phi$$

$$\phi_{zz} + \phi_{xx} - k^2\phi = 0$$

$$\phi_n = 0$$

④ Ursells Problem

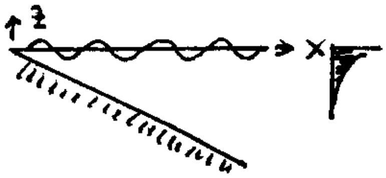
Note: $\sigma^2/g > 0$

Results of Ou's Transformation – Infinite Uniformly Sloping Beach

Ursell – for surface waves

Ou – for low frequency shelf waves

$$\frac{\sigma^2}{g} \leftrightarrow -\frac{kf}{\sigma}\frac{a'}{\sqrt{1+a'^2}}$$



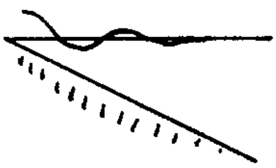
Deep water continuum

$e^{-i\sigma t + iky} \cos(lx + \text{phase})\, e^{z\sqrt{l^2+k^2}}$

$\sigma^2 = g\sqrt{l^2+k^2} > gk$ $\quad \boxed{\sigma^2 > gk}$ $\quad \Rightarrow$

Bottom trapped continuum

$\frac{f}{\sigma}\frac{a'}{\sqrt{1+a'^2}} > 1$ i.e. $\boxed{\sigma < N\sin(\tan^{-1}a)}$



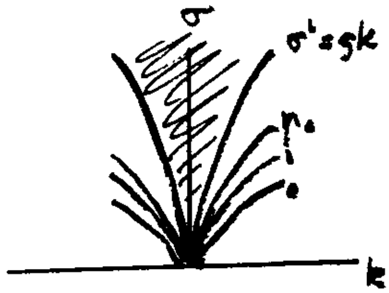
Edge waves

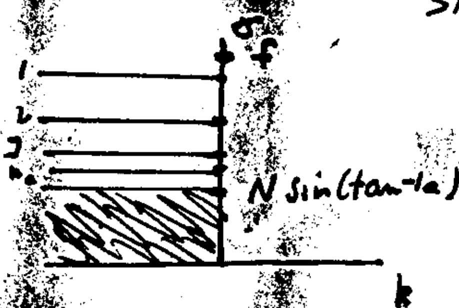
$\sigma^2 = gk\sin[(2n+1)\tan^{-1}a] < gk$

Refractively Trapped Bottom waves

$\sigma = fa'/[(1+a'^2)^{1/2}\sin(2n+1\tan^{-1}a')] < f$

$> N\sin(\tan^{-1}a)$





NB (a) $\sigma < fa'/\sqrt{1+a'^2} \Rightarrow \sigma^2 < \frac{f^2a^2(N^2-\sigma^2)}{(f^2-\sigma^2)+a^2(N^2-\sigma^2)}$

i.e. $\sigma^2(f^2-\sigma^2+a^2(N^2-\sigma^2)) < f^2a^2(N^2-\sigma^2) \Rightarrow \sigma^2(f^2-\sigma^2) < a^2(f^2-\sigma^2)(N^2-\sigma^2)$

$\Rightarrow \sigma^2 < N^2a^2/(1+a^2)$

NB (b) $\left.\begin{array}{l}(2n_c+1)\tan^{-1}a' \le \pi/2 \\ \sigma_c = N\tan^{-1}a\end{array}\right\} \Rightarrow (2n_c+1) \le \pi/2\Big/\tan^{-1}a\sqrt{\frac{1-\sin^2\tan^{-1}a}{f^2/N^2-\sin^2\tan^{-1}a}}$

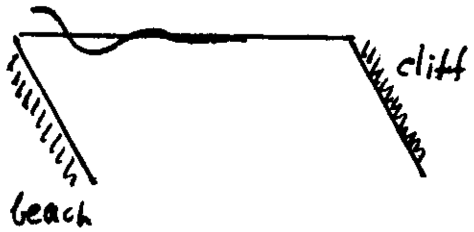
$n_c \to \infty$ $f/N \to \sin^{-1}\tan^{-1}a$ but $n_c = 0$ if $f/N < \sin\tan^{-1}a$

i.e. need $f > N\sin\tan^{-1}a$ to get any refractively trapped modes

NB (c) if $\sigma \ll N, f$ $\quad a' = aN/f \quad \sigma = f/(2n+1)$ for small n

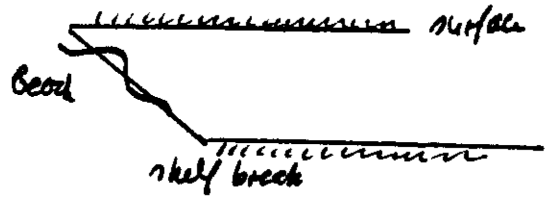
i.e. SW dispersion relation OK for small n, small a altho not for large n, a.

Finite Uniformly Sloping Beach



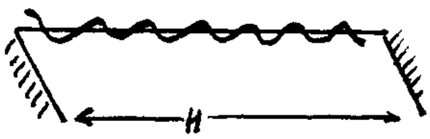
Edge waves not affected by cliff:

$\sigma^2 = gk \sin[(2n+1)\tan^{-1}\alpha]$



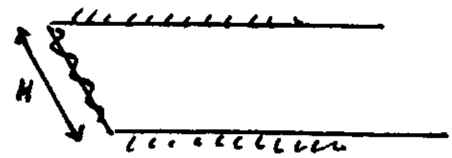
Refractively trapped bottom wave not affected by shelf break:

$\sigma = f\alpha'/(1+\alpha'^2) \sin[(2n+1)\tan^{-1}\alpha']$



Deep water continuum 'quantised' by repeated reflection beach ↔ cliff.
Even $k \to 0$ edge wave feel cliff:

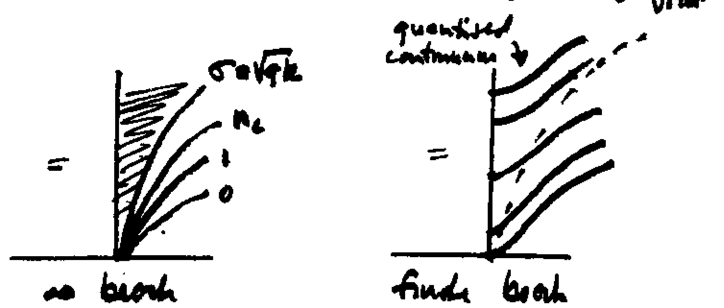
$$\sigma^2 = g\sqrt{\left(\frac{n\pi}{H}\right)^2 + k^2}$$

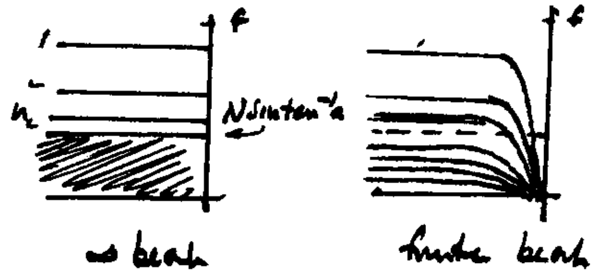


Bottom trapped continuum quantised by repeated reflection beach ↔ break
Even refractively trapped mode feel break as $k \to 0$

$$\frac{\sigma^2}{g} \leftrightarrow \frac{-kf}{\sigma}\frac{\alpha'}{\sqrt{1+\alpha'^2}}$$

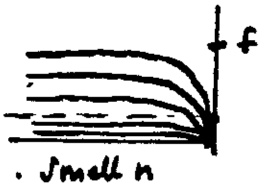
$$\sigma \simeq \frac{f\alpha'}{(1+\alpha'^2)} \frac{-k}{\sqrt{\left(\frac{n\pi}{H}\right)^2 + k^2}}$$





NB (a) In real world

$\alpha N/f \ll 1$ (US east coast)



$n_c > 0$. Small n edge waves well approximated by SW equations

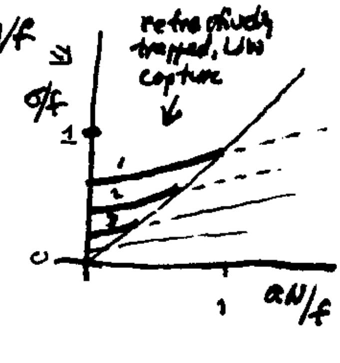
$\alpha N/f \lesssim 1$

$n_c = 0$
no modes well approx by SW eqs.

$\alpha N/f > 1$ (California, Peru)

$\sigma \gtrless f$ dispersion curves pass smoothly through f

NB(b) Since $\sigma \neq \sigma(k)$ except as $k \to 0$ can plot σ vs $\alpha N/f$ ⇒



NB(c) If $\alpha \ll 1$, $\sigma \ll f$, $\alpha' = \alpha N/f$

$\sigma \simeq -\alpha N k/\sqrt{(n\pi/H)^2 + k^2} \to -\frac{ND}{n\pi} k$

i.e. as $k \to 0$ modes over uniformly sloping beach propagate like internal Kelvin wave trapped at vertical coastline of depth $D = \alpha H$.

Observations of coastally trapped waves along the coast of Peru, from Smith (1987), JGR 83, C12, p. 6083 and Huyer(1980), JPO, 10, p. References are to figure numbers from these papers.

A. Along this coast, where bottom slope $a = 400m/5\ km \approx .05$, $f=2\Omega\sin(14^o)\approx 3.5x10^{-5}$rps, $N_o=(g\Delta\rho/H\rho_o)^{.5}=(10ms^{-2}\ x\ .001/\ 400m)^{.5} \approx .005$ rps, the parameter aN/f is about 7. We consequently expect stratification to be very important in determining the properties of long waves over the shelf.

B. Evidence for wave propagation, Smith Fig. 1 and Fig. 2 plus Table 1.

> Events in wind stress (τ) series at widely separated locations along coast are far closer to being simultaneous than events in sea level (ζ) or in along shore current (v).

> Events in ζ and in v propagate poleward.

> Events in ζ propagate with but little change in shape.

C. Estimating the wave speed and dispersion, Smith Figs. 4, 6, 7, 9.

> Simple estimation of time of travel of identifiable events in ζ and v leads to wave speed of about 200 km/day (Smith Fig. 4).

> Let $\zeta = \cos[ky-\sigma t] = \cos[\sigma(t-c^{-1}y)]$. Then at location a, $\zeta_a= \cos[\sigma(t-c^{-1}y_a)]$ so that the phase difference between the two time series $\zeta_a(t)$ and $\zeta_b(t)$ at locations a and b is $-\sigma c^{-1}(y_a-y_b)$. It is linear in σ if c is constant, e.g. if waves are dispersionless. See that linearity in estimates of coherence and phase at frequencies from very low to about 0.25cpd (Smith, Figs 6, 7, 9).

D. Vertical structure of v and T fields is first internal mode.

> When sealevel at the coast rises, isotherms near the coast go down and viceversa Huyer, Fig. 5, Fig. 3).

> Over depth sampled, isotherms all move up/down together with some tendency for deepest sampled (about 600 m) isotherms to display greatest vertical excursions.

> Currents tend to be along-shore, greatest at surface and nearly zero at deepest current meter (412 m) (Smith, Fig 11)

E. Temperature-correlated-with v and v are in thermal wind relationship.

> Vertical excursions of isotherms decay offshore with a decay scale of tens of km.

> The part of T-correlated-with-v and v itself ~~and~~ are coherent in the geostrophic sense (Smith Table 2):

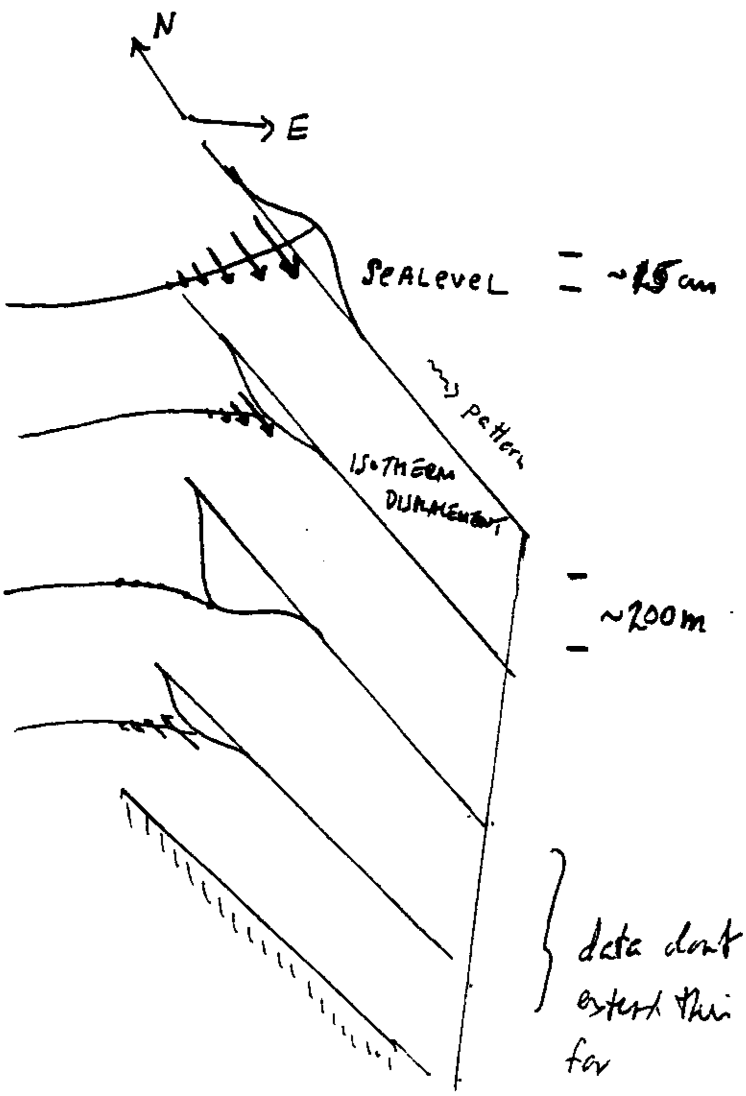
> The thermal wind equation $fv = g\alpha(dT/dx)$ is satisfied if the offshore decay scale of T is 53 km, in ok agreement with the observations above (Huyer, Fig. 5).

F. Kinematics of V, ζ and T are thus in qualitative accord with coastally trapped first mode internal Kelvin wave: $\zeta \approx \exp(-i\sigma t+ky+\lambda x)$, $v \approx \exp(-i\sigma t+ky+\lambda x)\cos(\pi z/D)$, $T \approx -\exp(-i\sigma t+ky+a\lambda x)\sin(\pi z/D)$.

The ratio of surface displacement ($\approx$15 cm) to displacement (200m) of isotherms at roughly the depth where v=0 (about 500 m) is about .00075, of order $\Delta\rho/(\rho\pi) \approx .0003$ over this depth.

The offshore decay scale is given by $\lambda = f/c = 3.5\text{x}10^{-5}\text{rps} / 200\text{kpd} \approx (60\ \text{km})^{-1}$, in reasonable accord with the estimate from observations.

The observed wave speed c (200 km/day) is only about 40% of the speed $c=N_0D/\pi$ along a vertical coast where the depth is the offshore depth (about 4000 m) if the Vaisala frequency is taken at a constant value of 2.8 cph appropriate to the stratification in the upper 500 m (Smith, Fig 1). But this may well be an overestimate for the deeper water.



Smith JGR 83 1978

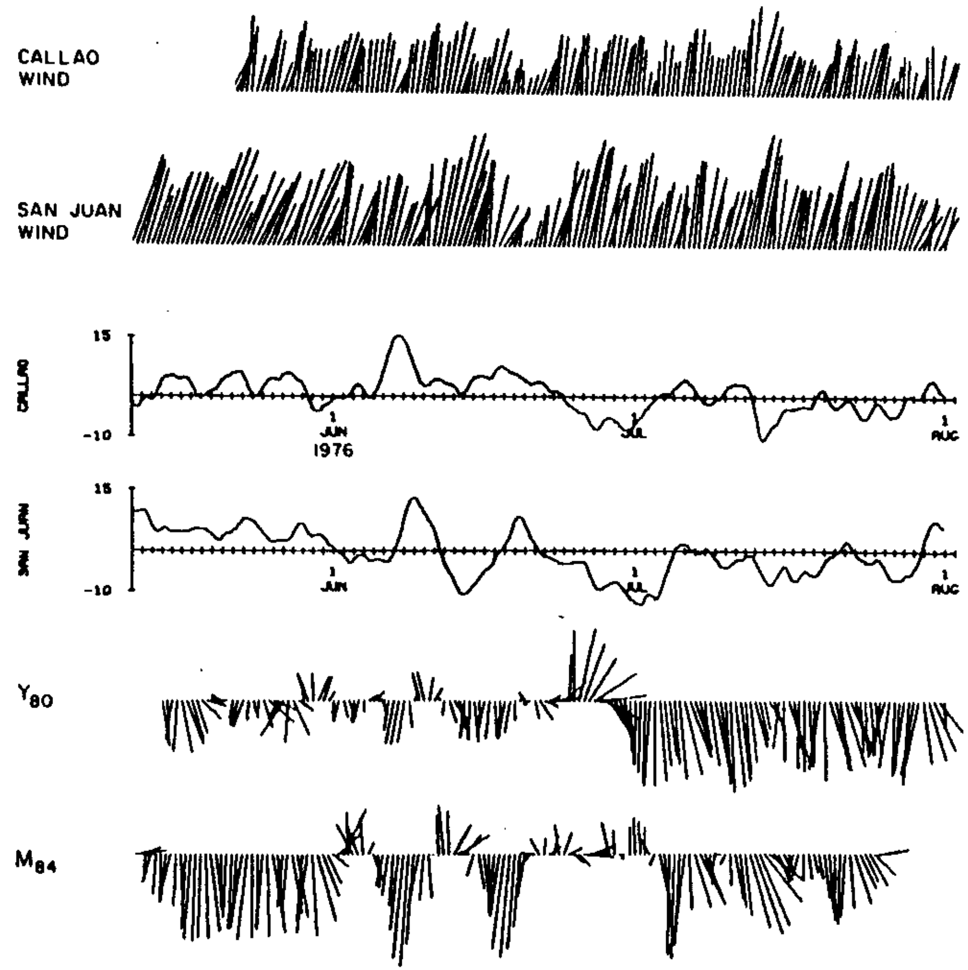


Fig. 2. Low-pass-filtered wind vectors and adjusted sea level records from Callao (12°04'S) and San Juan (15°20'S) and current vectors from Y (80 m below surface at 12°05'S, 77°22'W; bottom depth of 120 m) and M (84 m below surface at 15°06'S, 75°30'W; bottom depth of 128 m). Scale: 10 days on the time axis corresponds to 25 cm on the vertical sea level scale, 10 m s^{-1} in winds and 50 cm s^{-1} in currents. Vectors are rotated into individual principal axes, and vertical axis is toward 315°T for winds and M_{84} current and toward 335°T for Y_{80}.

	CC_0 unlagged	CC_m lagged	Lag, hours
Wind at Callao (12°S) versus			
Wind at San Juan (15°20'S)	0.57*	$\boxed{0.73*}$	$\boxed{12}$
Sea level at Callao	−0.16†	−0.43	36
Current at Y_{80} (12°S)	0.09†	0.18†	18
Sea level at Callao versus			
Sea level at San Juan	0.50	$\boxed{0.67*}$	$\boxed{42}$
Current at Y_{80}	−0.31†	−0.37	−18
Current at Y_{80} versus current at M_{84} (15°S)	0.24†	$\boxed{0.66*}$	$\boxed{54}$
Wind at San Juan versus			
Sea level at San Juan	[illegible]	−0.41	30
Current at M_{84}	[illegible]	0.21†	18
Sea level at San Juan versus current at M_{84}	−0.59*	−0.72*	−24

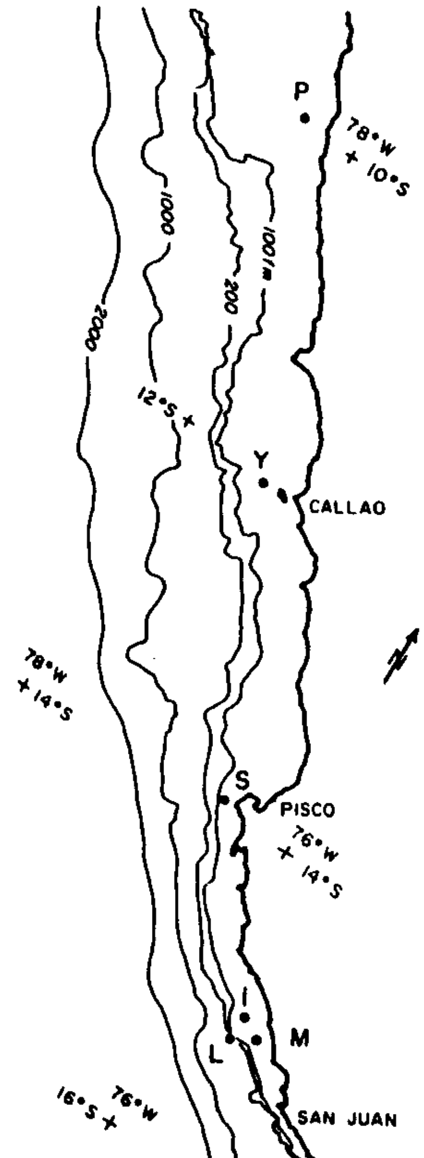


Fig. 1. Location of current measurements used in this study. The 100-, 200-, 1000-, and 2000-fm (183-, 366-, 1830, and 3660-m) isobaths are shown. Tide gages and anemometers were located at Callao and San Juan.

Smith JGR 83 1978

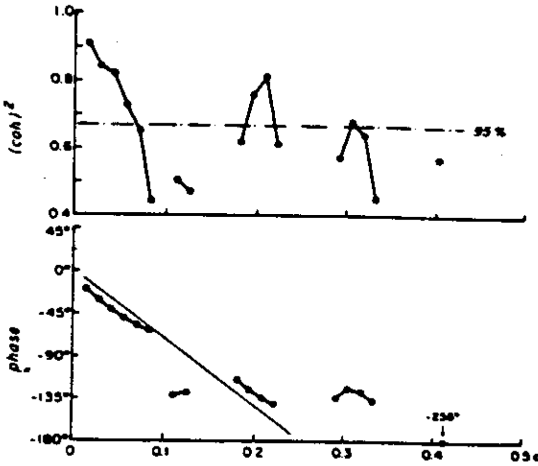


Fig. 6. Coherence squared and phase between alongshore currents Y_{m} and M_{m} for the same records as in Figure 5. The line on the phase versus frequency diagram represents 200 km d^{-1} phase speed for poleward propagating (i.e., from Y_{m} toward M_{m}) nondispersive waves.

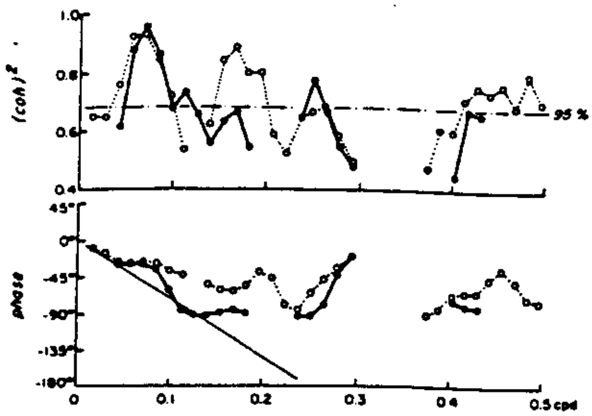


Fig. 7. Coherence squared and phase between the Callao and San Juan alongshore winds (open circles, dotted lines) and between the Callao and San Juan adjusted sea levels (solid circles and lines) during May–July 1976.

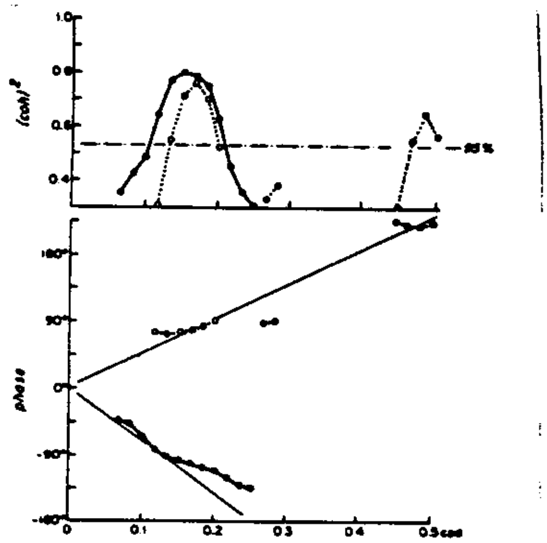


Fig. 9. Coherence squared and phase between alongshore currents at Y_{w} and P_{w} (open circles, dotted lines) and between Y_{w} and M_{m} (solid circles and lines) for the same records as in Figure 8.

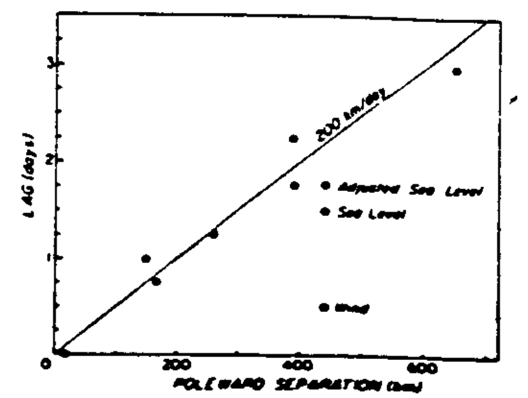


Fig. 4. Lag (with equatorward series leading) at maximum correlation between alongshore current measurements at different separations along the Peru coast from 10°S to 15°S and between the Callao and San Juan series of wind and sea level.

Huyer JPO 10 1980

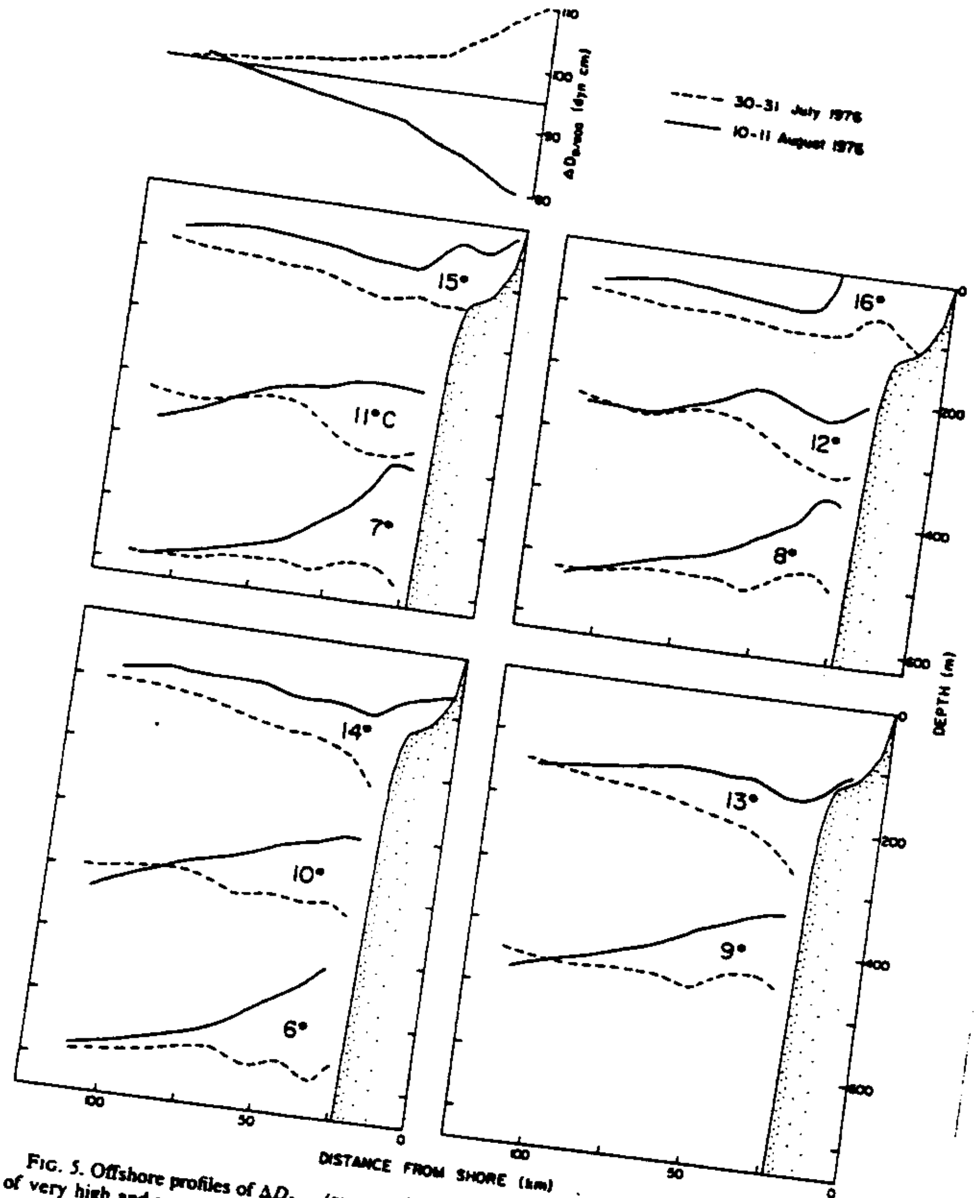


FIG. 5. Offshore profiles of $\Delta D_{0/500}$ (representing the sea surface) and of selected isotherms at times of very high and very low sea level (on 31 July and 10 August, respectively). Isotherms have been separated for clarity.

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Huyer JPO 10 1980

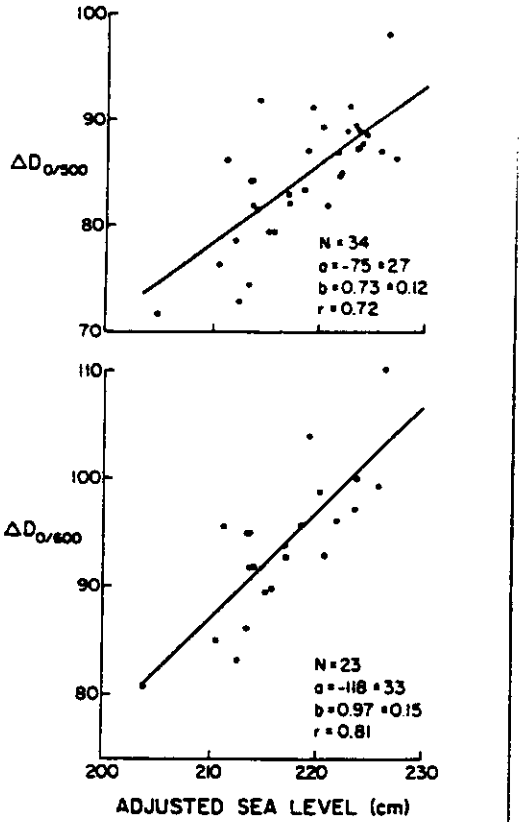


FIG. 3. The dynamic height of the sea surface at the inshore station of each section, relative to the 500 and 600 db pressure surfaces, plotted as a function of the simultaneous value of adjusted sea level at San Juan. The number of points (N), the correlation coefficient (r) and the intercept (a) and slope (b) of the regression line are shown for each comparison.

3. Dynamic height as a measure of sea level

The contribution of density variations to changes in the height of the sea surface can be determined from the density profile if the ocean is assumed to be in hydrostatic balance, i.e.,

$$gdz = \alpha dp,$$

where g is the acceleration due to gravity, α the specific volume, and dz and dp are small increments in depth and pressure. Integrating over depth, and converting to common oceanographic units (depth in m, pressure in db, specific volume in cm^3 g^{-1} and acceleration in m s^{-2}), we obtain the vertical separation between any two isobars:

$$z = 10 \int_{p_1}^{p_2} \frac{\alpha}{g} dp = \frac{10}{g} \int_{p_1}^{p_2} \alpha dp$$

$$= \frac{10}{g} \int_{p_1}^{p_2} \alpha_{35,0,p} dp + \frac{10}{g} \int_{p_1}^{p_2} \delta dp,$$

where $\alpha_{35,0,p}$ is the specific volume of sea water at 0°C and 35‰ and δ is the specific volume anomaly. The first of the two terms on the right is constant, so variations in the separation z' depend only on the second term. The integral in the second term generally is called the geopotential anomaly or dynamic height anomaly ($\Delta D_{p_1/p_2}$); it is routinely computed from hydrographic data and has units of dynamic meters (dyn m). Thus,

$$z' = \frac{10}{g} \Delta D_{p_1/p_2},$$

i.e., changes in the separation between isobars are directly proportional to changes in dynamic height, and the constant of proportionality is very close to 1. If the upper isobar is the sea surface ($p_1 = 0$ db) and the lower isobar remains at a constant depth, variations in $\Delta D_{0/p_2}$ represent changes in the sea level due to density variations.

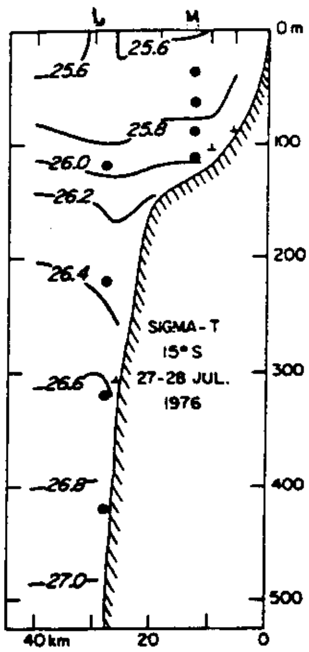


Fig. 10. Vertical section of sigma-t near 15°S from CTD measurements. The section was taken along a line perpendicular to the coast and passing through the M and L mooring positions. The current meters on M and L are indicated by circles.

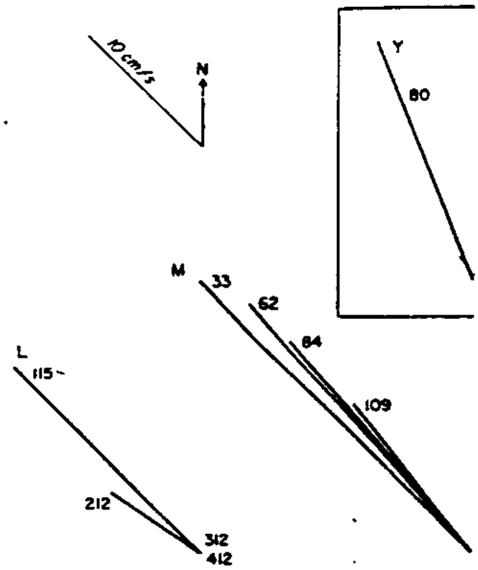


Fig. 11. Components, shown as vectors at each depth and location, of the first 'overall' empirical orthogonal mode computed from all current meters on the M and L moorings (May–July 1976). The normalized eigenvector components have been multiplied by the standard deviation of the mode. The L_{312} and L_{412} vectors are indistinguishable from zero in this plot. For comparison the Y_{80} record is treated equivalently but separately; this gives a vector in the direction of the major principal axis with a length equal to the standard deviation of the Y_{80} velocity in that direction.

$$v(x, t) = \frac{g}{f} \frac{\partial s(x, t)}{\partial x}$$

where $v(x, t)$ is the alongshore near-surface current and $\partial s(x, t)/\partial x$ is the offshore component of the sea level slope.

If the fluctuations represent a baroclinic Kelvin wave, we would expect v and s to fall off exponentially with distance from the coast on a scale given by the baroclinic radius of deformation. If the waves were barotropic continental shelf waves, v would fall off as $H^{-1/2}$ as the bottom depth H increases with distance offshore [*Kundu and Allen*, 1976]; an exponential depth profile is a reasonable approximation for the shelf and slope region off Peru, and thus both v and s would fall off exponentially with distance from the coast. We therefore assume that the sea level at distance offshore $(-x)$ is related to the coastal sea level S_0 by

$$s(-x, t) = S_0(t)e^{-x/l}$$

where l is the offshore scale to be determined. Then the alongshore velocity at some distance x offshore and coastal sea level should be related as

$$\frac{v(t)}{S_0(t)} = \frac{g}{f} \frac{e^{-x/l}}{l}$$

TABLE 2. M-L Empirical Orthogonal Modes for Temperature

	First Mode	Second Mode
Eigenvalue (variance in mode), (°C)²	1.62	0.54
Percent of total variance	59	20
M_{33}	0.61	−0.39
M_{62}	0.50	−0.15
M_{84}	0.43	0.13
M_{109}	0.33	0.29
L_{115}	0.26	0.23
L_{212}	0.08	0.42
L_{312}	0.03	0.41
L_{412}	0.09	0.58

The second M-L temperature mode is, however, significantly correlated with the first M-L velocity mode at a significance level of 95% but not with the wind. This mode is dominated by the temperature over the slope. Again, this is consistent with the results of *Brink et al.* [1978], who found strong evidence for baroclinic flow over the slope.

If we assume that the first M-L velocity mode (V_1) represents the wave velocity field and the second M-L temperature mode (T_2) represents the temperature variations associated with the wave, we can use these modal time series to obtain another estimate of the offshore scale. The hypothesis is made that the temperature fluctuations at the moorings represent fluctuations in the slope of the isotherms (relative to some mean temperature field) as the wave passes. We find the regression coefficient between $V_1(t)$ and $T_2(t)$ and interpret it in terms of

$$\frac{\partial v(t)}{\partial z} = \frac{g\alpha}{f} \frac{\Delta T(t)}{L}$$

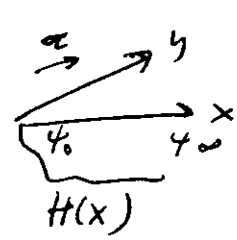
Smith JGR 83 1978

A prototype # for wind forcing of coastally trapped waves is the barotropic (LSW) rigid lid problem with wind, friction

$$u_t - fv = -p_x/\rho_0 + (\tau^x/H) - ru/H$$

$$v_t + fu = -p_y/\rho_0 + (\tau^y/H) - rv/H$$

$$(uH)_x + (vH)_y = 0$$



It is easiest to work in terms of $vH = \psi_x$, $uH = -\psi_{0y}$:

$$\left(\nabla\cdot\frac{1}{H}\nabla\psi_0\right)_t + f\left(\frac{\psi_{0y}H_x - \psi_{0x}H_y}{H^2}\right) = \left(\frac{\tau^y}{H}\right)_x - \left(\frac{\tau^x}{H}\right)_y - r\left[\left(\frac{\psi_{0x}}{H^2}\right)_x + \left(\frac{\psi_{0y}}{H^2}\right)_y\right]$$

Solve this subject to $\psi_0 = 0$ at $x = 0$ coast.

In the (flat bottom) deep sea where $H = H_0$, $\tau = r = 0$,

$$\nabla^2\psi_\infty = 0$$

At the seaward edge of the shelf must match u, p (since H is continuous) i.e. u, v, i.e. ψ_x, ψ_y i.e. ψ, ψ_x

The deep sea solution for free waves is of the form

$$\psi_\infty \sim e^{-i\sigma t + iky - |k|x}$$

so that matching $\psi_{0x} = \psi_{\infty x}$, $\psi_0 = \psi_\infty$ at shelf edge produces $\psi_{0x} = -|k|\psi_\infty = -|k|\psi_0$ i.e

$$\psi_{0x} = -|k|\psi_0 \quad \text{at shelf edge}$$

If we consider sufficiently long waves, $|k| \to 0$ and

$$\psi_{0x} = 0 \quad \text{at shelf edge}$$

As $k \to 0$ ~~and~~ $H = H(x)$, $(\psi_x/H)_t + f\psi_{0y}H_x/H^2 = -r(\psi_x/H^2)_x$ ...

We'd get this from the start if we'd taken $-fv = -p_x/\rho_0$ i.e. if we'd assumed cross-shelf geostrophy.

The unforced # then reduces to

$$(\psi_{0x}/H)_{xt} + f(H_x/H^2)\psi_{0y} = 0 \qquad \psi_0 = 0 \ \ x=0, \quad \psi_{0x} = 0 \ \ x = L.$$

Solutions are separable + of the form $\psi_0(x,y,t) = \sum \Phi_n(x)\, \phi_n(y,t)$

since $\sum_n \left[(\Phi_{nx}/H)_x \phi_{nt} + f(H_x/H^2)\Phi_n \phi_{ny} \right] = 0 \quad \ldots$

is satisfied if

$$\phi_{nt} + c_n \phi_{ny} = 0 \qquad (\Phi_{nx}/H)_x + f(H_x/H^2)(1/c_n)\Phi_n = 0$$
$$\Phi_n = 0 \ \ x = 0, \quad \Phi_{nx} = 0 \ \ x = L$$

Solutions are dispersionless because we took $|k| \to 0$. The crosshelf eigensolutions obey $\int_0^L \Phi_n (H_x/H^2)\Phi_m \, dx = \delta_{nm}$,

the eigenvalue is the along-shore wave speed.

Add forcing $(\tau^y \neq 0)$ + friction $(r>0)$:

$$(\psi_{0x}/H)_{xt} + f(H_x/H^2)\psi_{0y} = (\tau^y/H)_x - r(\psi_x/H^2)_x.$$

$$\psi_0 = \sum \Phi_n(x)\, \phi_n(y,t)$$

$$\sum_n \left\{ \phi_{nt} (\Phi_{nx}/H)_x + f(H_x/H^2)\Phi_n \phi_{ny} + r\phi_n (\Phi_{nx}/H^2)_x \right\} = (\tau^y/H)_x$$

Multiplying by $\int_0^L \Phi_m \Rightarrow$

$$f\phi_{mt}/c_m + f\phi_{my} + r\sum_n \phi_n \int \Phi_m (\Phi_{nx}/H^2)_x = \int \Phi_m \left(\frac{\tau^y}{H}\right)_x dx$$

i.e. $$\phi_{mt} + c_m \phi_{my} + (rc_m/f)\sum_n \phi_n \int_0^L \Phi_m (\Phi_{nx}/H^2)_x dx = \frac{c_m}{f}\int \Phi_m \left(\frac{\tau^y}{H}\right)_x dx$$

The projection $\int \Phi_m (\tau^y/H)_x dx$ drives ϕ_m + friction couples the modes. If we'd taken $u_t - fv = -P_x/\rho_0 + (\tau^x/H) - ru$ then friction would just damp each mode but wouldn't couple them. But constant r is overly schematic anyway

Put wind forcing & friction into stratified # by ekman layers

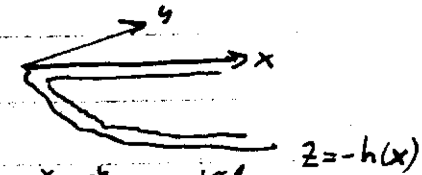
$u_t - fv = -P_x + \tau^x_z \rightarrow$ cross ~~shelf~~ shelf geostrophy $\rightarrow \boxed{-fv = -P_x}$

$v_t + fu = -P_y + \tau^y_z$

$0 = -P_z - g\rho$

$\rho_t + w\rho_{0z} = 0$ $\Big\}$ $\boxed{w = -P_{zt}/N^2}$

$u_x + v_y + w_z = 0$



$\tau^x = \tau^y = 0$ outside ekman layers

$\boxed{u = -P_{xt}/f^2 - P_y/f}$

$\nabla \cdot u = 0 \Rightarrow \boxed{\boxed{\left[P_{xx} + f^2 (P_z/N^2)_z\right]_t = 0}}$ Governing eq.

at $z = 0$ $\quad w = w_{ekman} = -(\bar{U}_{ex} + \bar{V}_{ey}) \qquad f\bar{U}_e = \tau^y \quad f\bar{V}_e = -\tau^x$

$= -\frac{1}{f}(\tau^y_x - \tau^x_y)$

$P_{zt} = (N^2/f)(\tau^y_x - \tau^x_y)$

velocities just above bottom ekman layer

at $z = -h(x)$ $\quad w = -uh_x + w_{ekman} = -uh_x + D_E(V_x - U_y) \qquad D_E = \sqrt{A/f}$

$-P_{zt}/N^2 + h_x(-P_{xt}/f^2 - P_y/f) - D_E P_{xx}/f = 0$

$(P_{xt} + fP_y)h_x + (f^2/N^2)P_{zt} + rP_{xx} = 0, \quad r = fD_E$

at $x = b$, no on-offshore flow in $h(b)\,u_{geostrophic}$ + top ekman + bottom ekman = 0

$h(-P_{xt}/f^2 - P_y/f) + \tau^y/f - \tau^{y\,bottom}/f = 0 \qquad \tau^{y\,bottom} = -rV$

$P_{xt} + fP_y + rP_x/h = f\tau^y/h$

$x \to \infty \quad P \to 0$

The stratified problem with wind / bottom friction is

$$P_{xx} + f^2 (P_z/N^2)_z = 0$$

$$P_{zt} = (N^2/f)\, \tau^y_x \qquad z = 0$$

$$(P_{xt} + fP_y) h_x + (f^2/N^2) P_{zt} + rP_{xx} = 0 \qquad z = -h(x)$$

$$(P_{xt} + fP_y) + rP_x/h = f\tau^y/h \qquad x = 0 \quad \text{near apex}$$

$$P \to \infty \qquad x \to \infty$$

Without τ, r seek solution $\boxed{P(x,y,z,t) = F(x,z)\,\phi(y,t)}$

$$P_{xx} + f^2(P_z/N^2)_z = 0 \qquad \boxed{F_{xx} + f^2 (F_z/N^2)_z = 0}$$

$$P_{zt} = 0 \qquad z = 0 \qquad \boxed{F_z = 0 \qquad z = 0}$$

$$P_{xt} + fP_y = 0 \qquad x = 0 \qquad \boxed{\phi_t + c\phi_y = 0} \qquad \boxed{F_x - (f/c)F = 0 \qquad x = 0}$$

$$(P_{xt} + fP_y) h_x + (f^2/N^2) P_{zt} = 0 \qquad z = -h \qquad \boxed{(f^2/N^2)F_z + h_x F_x = \frac{f h_x}{c} F \qquad z = -h}$$

$$P \to 0 \qquad x \to \infty \qquad \boxed{F \to 0 \qquad x \to \infty}$$

As in homogeneous problem $\phi_t + c\phi_y = 0$ and c is found by solving the eigenvalue problem in F.

With forcing + damping $\qquad p = \sum F_n(x,z)\, \phi_n(y,t)$

$$F_{nxx} + f^2 (F_{nz}/N^2)_z = 0$$

$$F_{nz} = 0, \quad z = 0 \qquad F_{nx} + (f/c_n) F_n = 0, \quad x = 0 \qquad F_n \to 0, \quad x \to \infty$$

$$F_{nz} + (N^2/f^2) h_x (F_x + f/c_n\, F) = 0 \qquad z = -h(x)$$

$$\phi_{ny} - \frac{1}{c_n}\phi_{nt} + \sum a_{nm}\phi_m = b_n \tau^y(y,t)$$

$$b_n = \frac{1}{h(0)} \int_{-h(0)}^{0} F_n(0,z)\, dz$$

The orthogonality relation is

$$\int_{-h(0)}^{0} F_n F_m \Big|_{x=0} dz + \int_0^\infty h_x (F_n F_m)\Big|_{z=-h} dx = \delta_{nm}$$

Coastally Trapped Waves - Bibliography

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