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"Classical Chaos"

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LECTURES ON CHAOS

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LECTURE I

CLASSICAL CHAOS

1. INTRODUCTION

It is well known that the weather cannot be predicted accurately. Why is that? Henri Poincaré stated the problem already in 1910:

> Why have meteorologists such difficulty in predicting the weather with any certainty? . . . We see that great disturbances are generally produced in regions where the atmosphere is in unstable equilibrium. The meteorologists see very well that the equilibrium is unstable, that a cyclone will be formed somewhere, but exactly where they are not in a position to say; a tenth of a degree more or less at any given point and the cyclone will burst here and not there, and extend its ravages over districts it would otherwise have spared. Here again, we find the same contrast between a very trifling cause that is inappreciable to the observer, and considerable effects, that are sometimes terrible disasters.

Much of what I want to say in these lectures is there in that quotation, explicitly or implicitly; in particular apparently erratic behaviour when in some ways a physical situation is well understood, the possibly damaging effects of that behaviour, and implicitly in the reference to cyclones, the fact that the erratic phenomena, although unpredictable, can nonetheless be seen to be an essential element in the physical system.

Fluid dynamics, the science of meterorology, is based on

Newton's equations of motion in the form of the equation of conservation of momentum known as the Navier-Stokes equation, and on well-established theories of thermodynamics. Why are the solutions not as definite as the equations?

Solutions of differential equations also depend on initial conditions and boundary conditions.

Initial conditions and boundary conditions in meteorology and oceanography are not well known. There are not enough surface or upper air observations to define them - remember that three quarters of the surface of the Earth is sea.

If that were all then bigger computers, more spacecraft, more weather ships, might enable us to do better.

But that is not the whole matter, as was again recognised by Henri Poincaré:

> A very small cause that escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of the same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation *with the same approximation*, that is all we require, and we should say the phenomena had been predicted, that is, governed by law. But is is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomena.

Fortuitous behaviour we now call chaotic, strictly, classical chaos.

Aim of lectures:

To indicate general nature of chaos and how it comes about;
To show some examples in oceanography;
To mention some other consequences of non-linear dynamics.

2. DYNAMICS AS MAPS

Dynamics is study of evolution of a sysytem in time, that is of correspondences between states at different times, or maps of one state upon another.

The equations of motion of dynamics, whether Newton's laws, or the Lagrangian equations of motion, or Hamilton's equations, give general rules for the mapping; the maps in particular cases require empirical forms of the force in Newton's equations, or of the Lagrangian or Hamiltonian, which have to be found from the particular physics.

A state of a physical system can be represented by a vector of positions and momenta - (**q**,**p**) (6 components in all for N point masses). The rate of change with time is given by

$$\partial \mathbf{q}/\partial t = \partial H/\partial \mathbf{p}; \quad \partial \mathbf{p}/\partial t = -\partial H/\partial \mathbf{q}.$$

Diagrammatically, represent all vectors by point P_0 in 2-D, and mapping by change of that point to P_1. What sort of maps may there be? Poincaré guides us.

a) As we start from different P_0, the range of P_1 that we reach is of the same order as that of P_0;

b) The range of P_1 is much less than that of P_0 and may shrink to a single point, an attractor;

c) The range of P_1 greatly exceeds that of P_0 - chaos.

In the first two cases, useful predictions can be made about the mapping, but not in the third case.

Important variant of the third case, in which the state vector traces out orbits close to some asymptotic orbit but with random variations between them: the asymptotic orbit is called a *strange attractor*.

3. CONDITIONS FOR CHAOS

Consider a linear differential equation with constant coefficients

$$a_0 x(t) + a_1 \dot{x}(t) + a_2 \ddot{x}(t) + \ldots = f(t),$$

where $\dot{x}(t)$ is $\partial x/\partial t$ . .

The Laplace transform is

$$(1 + a_2 s)a_1 x_0 + a_2 \dot{x}_0 + \ldots + (a_0 + a_1 s + a_2 s^2 + \ldots)\xi(s) = \phi(s),$$

where x_0, $\dot{x}_0$, . . . , are the values at $t = 0$, $\xi(s)$ is the Laplace transform of $x(t)$ and $\phi(s)$ is that of $f(t)$

The transform of the solution is

$$\xi(s) = \frac{\phi(s)}{a_2 s^2 + a_1 s + a_0} - \frac{(1 + a_2 s)a_1 x_0 + a_2 \dot{x}_0}{a_2 s^2 + a_1 s + a_0},$$

whence

$$\frac{\partial \xi}{\partial \dot{x}_0} = -\frac{a_2}{a_2 s^2 + a_1 s + a_0},$$

$$\frac{\partial \xi}{\partial x} = -\frac{(1 + a_2 s)a_1}{a_2 s^2 + a_1 s + a_0}.$$

These results show:

variations of $x(t)$ are proportional to the initial conditions;

the variations are proportional to functions of time that are independent of the initial conditions.

Non-linear equations:

Even very simple equations show quite different behaviour.

Consider the simple equation

$$dy/dx = ay^2$$

$$\rightarrow \quad y = y_0/(1 - ay_0 x),$$

where $y_0 = y(x=0)$.

Then $\partial y/\partial y_0 = 1/(1 - ay_0 x)^2$,

which is infinite at $x = 1/ay_0$.

A more complex example - the logistic equation (originally a model of a breeding poulation subject to predators).

In difference form: $x_{n+1} = rx_n(1 - x_n)$,

equivalent to $dx/dt = (r-1)x - rx^2$,

solution: $x = x_0\, e^{st}[1 - (1+s^{-1})x_0(1 - e^{st})]$.

Some of the main features of chaos are shown by the solution.

1) $r<1$, $s<0$: $x \rightarrow 0$ as $t \rightarrow \infty$, whatever x_0.
   $x = 0$ is an *attractor*.

2) $r>1$, $s>0$: $x \rightarrow s/(1 + s) = (1 - r^{-1})$ as $t \rightarrow \infty$.
   $x = (1 - r^{-1})$ is an *attractor*.

3) Stability:

Suppose x is perturbed by e, and that consequent perturbation in $(x + dx)$ after time dt is e'. Then from the difference equation,

$$x + dx + e' = r(x + e)(1 - x - e).$$

But $x = dx = rx(1 - x)$,

and so $e' = r(1 - 2x)e$,

whence $|e'/e| > 1$ if $|r(1 - 2x)| > 1$.

At the limit, $x = (1 - r^{-1})$, $|r(1 - 2x)| = |2 - r|$,

and is > 1 if r > 3.

Bifurcation: If r > 3, there are two attractors and the solution oscillates between them, depending on the value of the random variable e; (analytical solutions are possible, but numerical ones are simpler).

With increase of r further solutions occur in pairs, with oscillations between them and eventually a random distribution over the whole range of possible states. The behaviour has some similarity to turbulence in fluids.

4 SUMMARY

Some non-linear differential equations have solutions with

attractors - values to which the solutions tend. Attractors are not necessarily points - in more than one dimension they may be curves round which orbits circulate in a random manner.

random switching between quasi-stable states, stimulated by random fluctuations - reversals of the magnetic field of the Earth are probably an example, for the dynamo equations that govern the field are highly non-linear.

development of a whole range of random states - in fluids, *turbulent* motion.

5) SOLITONS

Solitons are not chaotic, indeed highly coherent, but are another consequence of non-linear dynamics, first noticed in fluid mechanics but now widely used as models for many phenomena in physics and otherwise.

Recall: gravity waves on the surface of a liquid satisfy a linear boundary condition at the surface for small displacements from the undisturbed surface, while in the body of the liquid there is potential flow.

In consequence there are harmonic waves proportional to

$$\exp\{i(\omega t - kx)\},$$

with $$\omega^2 = gk \tanh kH.$$

(g is the value of gravity and H the depth of the water).

In deep water ω^2 tends to gk.

In shallow water, tanh kH becomes kH and the *velocity* tends to $\sqrt{(gk)}$.

The assumption that the displacement of the surface is small is often not realistic. Furthermore, viscosity has been neglected. With a displacement that is not small, the convective acceleration and viscosity have both to be included, and it can be shown that the surface elevation in a narrow shallow channel with waves travelling in one dimension satisfies the equation

$$\frac{\partial\xi}{\partial t} + \xi\frac{\partial\xi}{\partial x} + \frac{\partial^3\xi}{\partial x^3} = 0.$$

Note: non-linear acceleration through convective term $\xi\frac{\partial\xi}{\partial x}$,

balanced by diffusion $\frac{\partial^3\xi}{\partial x^3}$.

(the Kortveg-de Vries equation)

All constants, such as density, gravity and viscosity have been taken to be 1.

We try for a solution representing a wave travelling with a velocity v, that is with an argument (x - vt), the displacement proportional to f(x - vt), with a peak displacement X, say.

It is found that the equation is satisfied by

$$f(x - vt) = \operatorname{sech}^2[v^{1/2}(x - vt)],$$

with $v = X/3$.

The solution represents a single hump travelling at a speed proportional to the amplitude of the disturbance.

The width of the disturbance also decreases with greater amplitude. Consider the points at which the second derivative vanishes:

$$\frac{\partial^2\xi}{\partial t^2} = 0 \quad \text{at} \quad \phi'^2 \sinh^2\phi = \frac{1}{2},$$

where $\phi = v^{1/2}(x - vt)$ and $\phi' = \frac{d\phi}{dx} = v^{1/2}$.

Hence $\sinh^2\phi = \frac{1}{2v} = \frac{3}{2X}$.

Thus the hump is narrower the greater the amplitude.

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LECTURE 2

CHAOS IN HYDRONAMICS

1) INTRODUCTION

Previous lecture shows *two* aspects of chaos:

a) General random motion - an example is thermal motion in gases.

b) Erratic shifts between quasi-stable states - bifurcations.

Both arise from non-linear interactions and extreme sensitivity to initial conditions, which may sometimes seem like *insensitivity* to initial conditions.

Generally there is a competition between a non-linear interaction generating an effect and dissipation reducing it - the Kortveg-de Vries equation is a characteristic example, which applies to many physical phenomena.

The aim of this lecture is to show how chaotic effects arise in hydrodynamics.

2. NON-LINEAR TERMS IN FLUID MOTION

Fluid motion satisfies conservation of mass and of momentum.

Conservation of mass density:

$$\frac{\partial\rho}{\partial t} + \rho \operatorname{div}\mathbf{v} = 0.$$

Shall generally take water to be incompressible, so that

$$\frac{\partial\rho}{\partial t} = 0 \quad \text{and thus} \quad \operatorname{div}\mathbf{v} = 0.$$

Newton's Second Law, conservation of density of momentum:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}.\mathbf{grad})\mathbf{v} = -\frac{\partial p}{\partial x} + \eta\nabla^2\mathbf{v}$$

(the Navier-Stokes equation)

Origin of the terms in the Navier-Stokes equation:

a) The rate of change of momentum density:

The total change with time at any point (d/dt) is equal to the variation in time at that point (∂/∂t) together with the change due to convection in the flow of the fluid, (**v.grad**).

b) The force density:

equal to the divergence of the stress tensor; the components of the stress tensor are

potential energy: $p\delta_{ik}$,

viscous stress: $\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$.

The divergence gives

$$\frac{\partial}{\partial x_k}(p\delta_{ik}) = \frac{\partial p}{\partial x_i} = \mathbf{grad} p$$

and $$\eta\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) = \eta\frac{\partial^2 v_i}{\partial x_k \partial x_k} + \eta\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)$$

The second term vanishes because **divv** = 0 for an incompressible liquid.

If the convective term can be ignored, either because the velocity is small or because of geometrical constraints, the Navier-Stokes equation reduces to a linear equation that is ordinarily soluble. The forcing term is **-gradp** and the dissipative term is the viscosity times the Laplacian of the velocity.

The convective acceleration is inherently non-linear and is often not small.

3. SOME USEFUL FORMULAE

There is a number of useful transformations of terms in the Navier-Stokes equation.

(v.grad)v:

$$\begin{aligned}
\frac{1}{2}\mathbf{grad}(v^2) &= \frac{1}{2}\frac{\partial}{\partial x_i}(v_i^2 + v_j^2 + v_k^2) \\
&= v_i\frac{\partial}{\partial x_i}v_i + v_j\frac{\partial}{\partial x_i}v_j + v_k\frac{\partial}{\partial x_i}v_k \\
&= \left(v_i\frac{\partial}{\partial x_i}\right)v_i + v_j\frac{\partial}{\partial x_i}v_j - v_j\frac{\partial}{\partial x_j}v_i \\
&\qquad + v_k\frac{\partial}{\partial x_i}v_k - v_k\frac{\partial}{\partial x_k}v_i. \\
&= (\mathbf{v}.\mathbf{grad})\mathbf{v} + v_j(\mathbf{curl}\mathbf{v})_k - v_k(\mathbf{curl}\mathbf{v})_j \\
&= (\mathbf{v}.\mathbf{grad})\mathbf{v} + \mathbf{v}\times(\mathbf{curl}\ \mathbf{v}).
\end{aligned}$$

The result enables **(v.grad)v** to be written in terms of the density of kinetic energy, that is, the density times $v^2/2$, and the **curl** of the velocity.

The **curl** of the velocity is also known as the *vorticity* and is a measure of the local rotation of the fluid, for

$$\int_C \mathbf{v}.\mathbf{dl} = \int_S \mathbf{curlv}.\mathbf{dS}.$$

The integral on the left is known as the *circulation*. **curlv**, the vorticity, is denoted by $\boldsymbol{\omega}$.

An equation for the vorticity is obtained by taking the **curl** of the Navier-Stokes equation:

$$\rho\frac{\partial}{\partial t}(\mathbf{curlv}) + \mathbf{curl}[(\mathbf{v}.\mathbf{grad})\mathbf{v}] = -\frac{1}{\rho}\mathbf{curl\ grad}p + \eta\nabla^2\mathbf{curlv}.$$

Then using the relation for **v.gradv** obtained above, and the identity

$$\mathbf{curl\ grad}\ p = 0,$$

$$\rho\frac{\partial\boldsymbol{\omega}}{\partial t} - \rho\,\mathbf{curl}(\mathbf{v}\times\boldsymbol{\omega}) = \eta\nabla^2\boldsymbol{\omega}.$$

But $\mathbf{curl}(\mathbf{v}\times\boldsymbol{\omega}) = \mathbf{v}\,\mathrm{div}.\boldsymbol{\omega} - \boldsymbol{\omega}\,\mathrm{div}.\mathbf{v}$
$+ (\boldsymbol{\omega}.\mathbf{grad})\mathbf{v} - (\mathbf{v}.\mathbf{grad})\boldsymbol{\omega}$,

while $\mathrm{div}.\mathbf{v}$ and $\mathrm{div}.\boldsymbol{\omega}$ are both zero in incompresible fluids, so that

$$\rho\frac{\partial\boldsymbol{\omega}}{\partial t} + \rho(\mathbf{v}.\mathbf{grad})\boldsymbol{\omega} - \rho(\boldsymbol{\omega}.\mathbf{grad})\mathbf{v} = \eta\nabla^2\boldsymbol{\omega}.$$

Another form of $(\mathbf{v}.\mathbf{grad})\mathbf{v}$:

$$(\mathbf{v}.\mathbf{grad})\mathbf{v} = v_k\frac{\partial}{\partial x_k}v_i = \frac{\partial}{\partial x_k}(v_i v_k) - v_i\left(\frac{\partial v_k}{\partial x_k}\right).$$

But $(\partial v_k/\partial x_k)$ is the divergence of the velocity and is zero, so that

$$(\mathbf{v}.\mathbf{grad})\mathbf{v} = \partial(v_i v_k)/\partial x_k.$$

The total stress in a liquid is therefore made up of the stress corresponding t the potential energy and proportional to the pressure, the viscous stress, and the extra term equal to the density multiplied by the symmetrical tensor $(v_i v_k)$.

4. SEPARATION OF MEAN FLOW AND VARIATIONS

The behaviour of a liquid can often be described as a smooth flow on which are imposed fluctuations; but what are fluctuations and what is a smooth flow depend on the times and distances over which averages are taken.

Let $\mathbf{v} = \mathbf{v}_m + \mathbf{v}_d$;

the suffix m denotes the mean flow and d the fluctuating part, such that averaged over some time and distance $\mathbf{v}_d$ is zero.

Then
$$v_i v_k = (v_{mi} + v_{di})(v_{mk} + v_{dk})$$
$$= v_{mi}v_{mk} + v_{di}v_{mk} + v_{mi}v_{dk} + v_{di}v_{dk}.$$

On taking mean values, the second and third terms vanish. Thus

$$\langle v_i v_k\rangle = v_{mi}v_{mk} + \langle v_{di}v_{dk}\rangle.$$

The part $\langle v_{di}v_{dk}\rangle$ is called the Reynold's stress and will be denoted by R_{ik}. It vanishes if the velocity components are uncorrelated and consequently is proportional to the correlation coefficient of the components.

Now form the mean of the Navier-Stokes equation to give

$$\frac{\partial}{\partial t}\mathbf{v}_m + (\mathbf{v}_m.\mathbf{grad})\mathbf{v}_m + \mathrm{div}\,R_{ik} = -\frac{1}{\rho}\mathbf{grad}\;p + \eta\nabla^2\mathbf{v}_m.$$

The equation for the fluctuating velocity is

$$\frac{\partial}{\partial t}v_{di} + \frac{\partial}{\partial x_k}(v_{di}v_{mk} + v_{dk}v_{mi}) = \eta\nabla^2 v_{di},$$

or
$$\frac{\partial}{\partial t}v_{di} + \left(v_{mk}\frac{\partial}{\partial x_k}\right)v_{di} + \left(v_{dk}\frac{\partial}{\partial x_k}\right)v_{mi} = \eta\nabla^2 v_{di}.$$

These equations show, first that the Reynold's stress contributes to the development of the mean flow, and secondly, that the fluctuations are amplified by convection in the mean flow.

There is a related equation for the mean vorticity, for taking the mean of the vorticity equation, we have

$$\rho\frac{\partial\boldsymbol{\omega}}{\partial t} + \rho(\mathbf{v}.\mathbf{grad})\boldsymbol{\omega} = \eta\nabla^2\boldsymbol{\omega},$$

where in this equation both $\mathbf{v}$ and $\boldsymbol{\omega}$ stand for mean values and the mean value of $\boldsymbol{\omega}.\mathbf{grad}$ has been taken to be zero.

A further decomposition follows from the result that any vector field can be written as the sum of the gradient of a scalar potential and the curl of a vector potential. We write

$$\mathbf{v} = \mathbf{grad}\phi + \mathbf{curl}\mathbf{A}.$$

Hence

The Navier-Stokes equation will then become

$$\rho(\mathbf{grad}\dot\phi + \mathbf{curl}\dot{\mathbf{A}}) + \frac{1}{2}\rho\,\mathbf{grad}v^2 - \rho(\mathbf{v}\times \mathbf{curl v}) = -\mathbf{grad}p + \eta\nabla^2(\mathbf{grad}\phi + \mathbf{curl A})$$

Now take the divergence of that equation, using the result that

$$\mathrm{div}(\mathbf{v}\times \mathbf{curl v}) = \mathrm{div}(\mathbf{v}\times \boldsymbol{\omega}) = \omega^2 + \mathbf{v}.\mathbf{curl}\boldsymbol{\omega}$$

to give

$$\rho\frac{\partial}{\partial t}\nabla^2\phi + \frac{1}{2}\nabla^2 v^2 - \omega^2 = -\nabla^2 p + \eta\nabla^2\nabla^2\phi$$

and

$$\rho\frac{\partial}{\partial t}\mathbf{curl A} - \mathbf{v}.\mathbf{curlcurl A} = \eta\nabla^2\mathbf{curl A}.$$

Again we see that the fluctuating flow reacts back on the mean (potential) flow through the square of the vorticity. If the vorticity and viscosity are zero, the equation for the potential is the Laplacian of Bernouilli's equation.

The equation for the vector potential also shows in the second term how convection in the potential flow develops the rotational flow.

5. INSTABILITIES

The Navier-Stokes equation leads quite generally to unstable random motion, and the developed form of that, turbulent motion, is the subject of the next lecture. In the remainder of this lecture we shall see three particular ways in which instabilities develop.

A. Density inversion - Rayleigh-Taylor Instability.

A state of affairs in which a dense layer overlays a less dense one is unstable because of gravitational overturn, but it may be stabilised by viscous resistance.

The general way in which this and other types of instability is discussed is to assume a small harmonic displacement from equilibrium and to see if it increases or decreases in time. That is a linear procedure and while it yields the conditions for initial instability, it will not be applicable as soon as the motion becomes finite, so that it cannot be used to follow the development of the unstable motion.

The problem of the layer of dense fluid upon a less dense one can be considered in terms of the gravity waves that occur at the boundary between any two layers, internal gravity waves. Suppose that the upper and lower layers are both very deep.

Let the density of the upper layer be ρ_1 and of the lower layer, ρ_2.

Assume an harmonic motion proportional to $\exp[i(\omega t - \mathbf{k}.\mathbf{r})]$.

Then

$$\omega^2 = kg\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right).$$

Evidently the frequency is real and the motion is an harmonic wave if the upper layer is less dense than the lower, while if the density of the upper layer is the greater, the frequency is imaginary and one of the solutions corresponds to an harmonic variation in space (not a progressive wave) growing exponentially in time.

Viscosity has been neglected in the calculations that give those results and when it is included, a critical condition for the onset of the disturbance will be obtained.

As the disturbance grows, fingers of high density penetrate the lower layer until the layers are inverted commonly with turbulent mixing.

B. Thermal Convection

The onset of thermal convection is a classical problem first discussed by Lord Rayleigh, subsequently by Sir Harold Jeffreys.

Suppose that a fluid is heated below. The density decreases with temperature and the temperature falls with height at a rate that depends on the thermal conductivity. The difference of density drives the motion of the fluid which is opposed by viscosity. We again use exchange of stabilities to determine when the motion begins to increase exponentially with time.

Let the density in the steady state ($\mathbf{v} = 0$) be ρ_0 and let the temperature be T_0. Let the deviations be ρ' and T' when the velocity is $\mathbf{v}$.

Let the coefficient of thermal expansion be β so that

$$\rho' = -\rho_0 \beta T'.$$

Put $\eta/\rho = \nu$.

Then when the velocity and deviations are small, the Navier-Stokes equation reduces to

$$\nu\nabla^2\mathbf{v} = \frac{1}{\rho_0}\mathbf{grad}p' - \beta T' g$$

Take the z-coordinate to be in the vertical direction. In the steady state, the temperature is proportional to z:

$$T_0 = Az + \text{const}.$$

The equation of heat conduction in a moving fluid, including the convective term, is

$$\rho c_p\left(\frac{\partial T}{\partial t} + (\mathbf{v}.\mathbf{grad})T\right) = K\nabla^2 T.$$

c_p is the specific heat at constant pressure and K is the thermal conductivity.

With $\partial T/\partial t$ equal to zero, and $\partial T/dz$ equal to A,

$$Av_z = \frac{K}{c_p\rho}\nabla^2 T'$$

We assume, as usual, that the fluid is incompressible, $\text{div}\mathbf{v}$

The deviation of temperature then satisfies the following equation:

$$(\nabla^2)^3 T' = \frac{\gamma}{l^4}\nabla_2^2 T',$$

where ∇_2^2 denotes the two-dimensional Laplacian in the horizontal plane. γ and l are constants.

Suppose that T' has an harmonic variation with position $\mathbf{r}$ in in the horizontal plane

$$T' = \exp[i\mathbf{k}.\mathbf{r}\ f(z)].$$

The equation for $f(z)$ is then

$$\left(\frac{d^2}{dz^2} - k^2\right)^3 f(z) + \frac{\gamma k^2}{l^4} f(z) = 0.$$

The behaviour of $f(z)$ is determined by the Grashof number, $G = \beta g l^3 T'/\nu^2$.

When G is small, there is no motion and no deviation from the steady temperature T_0.

When G is about 100, steady motions begin in simple cellular patterns in the horizontal plane (Bénard cells).

When G is greater than about 1000 the motion becomes completely chaotic.

C. **Kelvin-Helmholtz instability**

Kelvin-Helmholtz instability occurs when two streams of fluid in contact, of different density, one above the other, are moving at different velocities, for example, air over the surface of water, or water from a river entering the sea.

An approximate treatment depends on Bernouilli's equation, by which the pressure is less in the faster stream. But the pressure must be the same on the two sides of the boundary and the difference is balanced by the displacement of the boundary in the gravitational field. The displacement of the fluid must also be the same on the two sides of the boundary. The equality of pressure and displacement are the boundary conditions for the solution of the eqautions of motion.

Let the motion be supposed harmonic with a wavevector $\mathbf{k}$ parallel to the boundary.

Let the velocities of the streams (parallel to the boundary) be $\mathbf{u}_1$ and $\mathbf{u}_2$ and let the densities of the streams be ρ_1 and ρ_2.

Instability sets in when

$$\rho_1\rho_2[\mathbf{k}.(\mathbf{u}_1 - \mathbf{u}_2)]^2 \;>\; kg(\rho_2^2 - \rho_1^2)$$

Examples of Kelvin-Helmholtz instability are waves driven by wind on the surfce of water and the herring-bone cloud patterns that form at the boundary of two streams of air at great heights in the atmosphere. Like other forms of instability, it may lead to completely random turbulence.

In each of the types of instability considered above the first motion is some coherent harmonic disturbance, but that is only sustained if the conditions are close to the boundary between stability and instability. More generally they develop into superpositions of many harmonic motions of random phase which finally become quite random.

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LECTURE 3

TURBULENCE

1. INTRODUCTION

Turbulent motion is a pre-eminent case of chaos, in which there is no organised behaviour but essentially random variations. At any one point in space the velocity varies in a random manner, although the mean square variation may be stable. There is usually some correlation between components in different directions and between velocities at different points. All such correlations are however essentially statistical.

We saw in the last lecture some ways in which instabilities can arise in fluid motion and how they can pass into fully developed turbulence.

We also saw how in general rotational motion is sustained by the steady flow.

We now look at developed turbulence more generally, and also at some of the consequences of turbulence. It is important to realise that there is still much that is not well understood about turbulent motion.

2. THEORIES OF DEVELOPMENT OF TURBULENCE

A. Cascades of harmonic quasi-modes

The fluctuating part of the Navier-Stokes equation is

$$\frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_m.\mathrm{grad}\,\mathbf{v}_d - (\mathrm{div}\,\mathbf{v}_m)\mathbf{v}_m = \nu\nabla^2\mathbf{v}_d.$$

$-\mathrm{div}\,\mathbf{v}_m$ has been written for $\mathrm{div}\,\mathbf{v}_d$ in the third term because $\mathrm{div}(\mathbf{v}_m+\mathbf{v}_d)$ is zero.

If v_m is supposed to be given that equation is a linear differential equation (actually three equations) for v_d. It will therefore have an harmonic solution, either growing or decaying according to the relative magnitudes of the forcing terms and the dissipation.

However the separation into mean flow and deviation is not complete - it only applies approximately when the non-linear terms in the full equation are small. In particular, if the fluctuating flow has a term with some speed, the Reynolds stress will have a part with twice that speed, and that will generate a cooresponding term in the mean flow and so in the fluctuations through the convective terms in the equation for the fluctuations.

The same point can be seen directly from the convective acceleration, (**v.grad**)**v**, for if **v** has an harmonic component with some speed, the convective aceleration will have a component with twice that speed. More generally, components of two speeds in the velocity will generate components with the sum and difference speeds.

It follows that the Navier-Stokes equation cannot be satisfied by a single harmonic term but only by an infinite set of terms. Thus we think of turbulence as described by an infinite set of quasi-modes which will in general have random phases. That is the Landau model of turbulence in terms of a cascade of quasi-modes.

The modes are called quasi-modes because there is no geometrical constraint that determines either the frequencies or the phases.

The Landau representation is in terms of harmonic waves but in view of the importance of vorticity it might be better to attempt a description in terms of cascades of vortices.

B. Bifurcations

We saw that the logistic equation $dy/dx = ry(1 - y)$ has

single stable solutions up to a certain value of r, then two solutions with random changes from one to the other driven by small external fluctuations, and then an infinite sequence of pairs of solutions. Initially the behaviour of the system is quasi-periodic in switching between the states, but as r increases with more and more bifurcations, the number of states becomes effectively unlimited and the changes between them random.

The Feigenbaum model of turbulence is based on the behaviour of the solutions of the logistic equation.

The model makes a specific prediction about the amplitudes of successive modes, namely, if A_n is the amplitude of the n^{th} mode, then

$$\mathrm{Lt}\ \frac{A_{n+1} - A_n}{A_n - A_{n-1}} = \frac{1}{2.5029}.$$

C. Three-mode model

A famous set of three coupled equations was derived by Lorenz from the Navier-Stokes equation on the asumption that the flow of a liquid could be represented by just the three slowest quasi-modes. The solutions have some very strange properties and have often been used as examples of chaotic behaviour.

The equations are

$$\frac{dx}{dt} = \sigma y - \sigma x$$

$$\frac{dy}{dt} = -xz + ry - y$$

$$\frac{dz}{dt} = -xy - bz.$$

r is a control parameter, its magnitude determines the character of the solutions.

The solutions have for the most part been obtained numerically and again have been used as models of turbulent behaviour.

Each of the three models makes a prediction about the spectrum of turbulence which can therefore be checked against observation. The results are not very clear. The initial spectra, when there are very few quasi-modes or bifurcations are distinctive and it seems that the Landau model does not correspond with any observations. Neither of the other two models seems to agree with all experiments but each does seem to represent some observations. Thus with fluid flowing between concentric cylinders rotating relative to each other shows distinct peaks in the velocity spectrum corresponding to quasi-modes and followed by broad band noise as the turbulence develops; the model with three coupled modes (C) seems to give reasonable agreement.

On the other hand, the spectra of heat transport in liquid helium seems to be better represented by the bifurcation model (B).

There seems at present no theory of general applicability for the spectra and cospectra of fully developed turbulence.

Empirically, laminar flow breaks down and gives way to turbulant flow at a velocity u determined by the balance between the convective acceleration and the viscous dissipation, as measured by the Reynolds number, R:

$$R = \frac{\rho u l}{\eta},$$

where ρ is the density, η the viscosity and l a characteristic length

3. NATURE OF TURBULENCE

Purely random spectra do not describe motions completely; there are structures, eddies or vortices on a wide range of scales, from something comparable with the scale of a mean flow down to very small dimensions.

The velocity at any point is the resultant of many superposed eddies.

Three points can be made here. First, eddies are self-contained structures with some persistence in time. Secondly, energy flows from the mean flow into the large eddies and from them to successively smaller ones until it is dissipated as heat in molecular friction. Thirdly, turbulence shows aspects of fractal geometry.

Fractal geometry is a consequence of non-linear dynamics. It is the repetition of the same structure at different scales of a phenomenon so that things look the same whatever the magnification. Structures in turbulent fluid flow often show fractal forms.

There are analogues in statistical mechanics to the description of turbulence either in terms of harmonic modes or in terms of eddies. The thermal motions in solids are most naturally considered as normal modes of elastic vibration that interact weakly, yet sufficiently to give the equilibrium Boltzmann distribution of energy.

Gases are best described as assemblies of almost isolated masses that interact only in colliding, and thus acquire a Maxwellian distribution of velocity.

If we knew how to describe the interaction of eddies, it might be possible to set up a model of a turbulent fluid as a collection of eddies rather than as a set of quasi-harmonic modes. A difficulty with all approaches is probably that, in contrast to the solid state or the classical gas, the interactions between modes or eddies in a turbulent fluid are not weak. It is also not clear what would be the equivalent of thermal equilibrium.

A further question, which will be taken up again in the final lecture, is what determines the energy in the turbulent part of a flow.

4. TURBULENT TRANSPORT

Recall that the motions of molecules in a gas are random with a distribution determined by the temperature but sometimes superposed on a mean motion. Thus if there is a gradient of momentum in a fluid, molecules passing from a region of high to low momentum have a momentum that on the average exceeds the mean momentum where they make their next collisions. Thus they carry momentum from the region of high to low momentum - the random motion of the molecules transports momentum down the momentum gradient. It is that transport that appears as viscosity in a fluid, specifically, it is *molecular* viscosity because the momentum is carried by the molecules.

In a similar way we think of the eddies in turbulent motion carrying fluid from a region of high to low mean flow momentum and so transporting mean momentum down a momentum gradient. Thus we can define a *turbulent viscosity* which will depend essentially on the spectrum of turbulence.

The turbulent velocity gives us an empirical way of writing the Reynolds stress which has the same form as the molecular viscous stress, in fact

$$R_{ik} = \eta_t\left(\frac{\partial v_{mi}}{\partial x_k} + \frac{\partial v_{mk}}{\partial x_i}\right) + \frac{2}{3}E\delta_{ik},$$

where η_t is the turbulent viscosity and E is the turbuklent kinetic energy equal to $R_{ii}/2$ and R_{ii} equals v_d^2.

This form for the Reynolds stress is known as the Boussinesq approximation. The turbulent viscosity is an empirical quantity of order $l\ \partial u/\partial n$, where l is a characteristic length over which fluid is mixed) the mixing length, of the dimension of the largest eddy, and $\partial u/\partial n$ is evaluted in the direction of the nearest boundary.

Molecules in a fluid also transport energy down a gradient, leading to molecular thermal conductivity; it is likewise possible to define a turbulent thermal conductivity and indeed other transport coefficients in turbulent flow.

The important fact is that turbulent viscosity and turbulent heat conduction are often much more effective than the molecular processes.

5. SOME EFFECTS OF TURBULENCE

A. Velocity profile above the sea bed.

At the sea bed, v is zero, while the stress on the solid surface is

$$\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

which determines the rate at which momentum flows into the sea bed. Let that rate be μ. Now the rate of change of v, horizontal velocity, with depth, z, has the dimensions of $(\text{time})^{-1}$ from which it follows that

$$\frac{dv}{dz} = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}\frac{1}{bz}$$

b is a numerical constant.

Put $\mu = \rho v_*^2$,

so that $\dfrac{dv}{dz} = \dfrac{v_*}{bz}$

and then $\dfrac{v}{v_*} = \dfrac{1}{b}\ln(z/z_0)$

where z_0, a constant of integration, is a scale height.

This is the logarithmic velocity profile, the scale of which is given by v_* which in turn depends on the viscosity, molecular or turbulenbt according to circumstances. If the flow is turbulent, then turbulent viscosity will determine the scale of the profile.

B. The Ekman Spiral

The preceding analysis says nothing about the direction of the flow in the horizontal plane, but in general the rotation of the Earth (which gives rise to Coriolis forces) must be taken into account.

Suppose that the motion is driven by a pressure gradient, gradp. The direction of flow then swings round from the direction of the pressure gradient far from the sea bed to one determined by the Coriolis force close to the sea bed. Again, the direction of the flow in the sea driven by a wind blowing over the surface changes in the upper laer of the sea. The form and scale of those spirals, the Ekman spirals, are determined by the viscosity, commonly the turbulent viscosity.

c. Friction at the sea bed

Here we are concerned with the work done on the sea bed by the frictional stresses exerted by the moving water. It is simpler to consider the flow over a flat solid plate that is oscillating in a viscous fluid.

Let the motion of the plate be harmonic with a velocity v in the x-direction equal to

$$v_0 \cos \omega t$$

The velocity in the fluid at distance z from the plate satisfies

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2},$$

the diffusion equation, which has the solution

$$v = v_0 \exp\left[i\left\{ (i-1)\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}} z - \omega t \right\} \right].$$

The stress on the plate is $\eta \frac{\partial v}{\partial z}|_{z=0}$.

that is $$v_0(\omega\eta\rho)^{\frac{1}{2}} \cos\left(\omega t + \frac{\pi}{4}\right).$$

In consequence of the phase shift of π/4 between the stress and the velocity of the plate, work is done on the fluid by the plate, or, if the fluid is oscillating and the plate stationary, work is done by the by the fluid on the plate.

The case of particular interest is the oscillating flow driven by the tide-raising forces. Here the plate, the sea bed, is attached to the whole Earth and the work done on it produces a torque about the polar axis that slows down the spin of the Earth, the by now well known effect of tidal friction. Here again the viscosity that is significant is the turbulent viscosity.

Since the power of the frictional forces is proportional to the square of the velocity, and also increases with the turbulent viscosity which itself increases with velocity, the greatest effect by far is in shallow seas where the tidal flows are strongest. G I Taylor, as long ago as 1915, first realised that friction in shallow seas might be sufficient to account for the observed slowing down of the Earth, and so it now seems to be.

d. Transport of solid material

In stationary liquid or fluid flowing steadily without turbulence, solid particles settle at rates given by Stokes's law. Turbulence, with substantial upwards velocities, keeps particles in suspension. It also increases the local velocities at the bed of a river or sea that bring particles into suspension or move them by saltation. Turbulence is therefore important in moving banks of solid material, especially in shallow water where velocities are greatest

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LECTURE 4

NON-LINEAR BEHAVIOUR OF SEAS

1. INTRODUCTION

My aim in this lecture is to consider how much is known of non-linear and chaotic effects in seas and how far and in what senses they may be predictable.

We have some direct knowledge of non-linear and chaotic behaviour of the atmosphere. The atmosphere is reasonably transparent, so that we can see turbulent motion, by plumes of smoke, clouds, leaves blown about and so on. We can see instabilities, such as Kelvin-Helmholtz instability, revealed by cloud formation. We can see the effects of thermal convection.

We can also see fractal phenomena, for example in the structures in clouds which repeat over a very wide range of scales in the same cloud.

On larger scales, there are sufficient observations of the atmosphere (pressure, temperature, humidity, wind) for eddy motion to be described and, in particular, for cyclones to be characterised as semi-permanent structures moving with little change of form, rather like solitons, though of course much more complex (Sir Harold Jeffreys, many years ago, called attention to that property of cyclones, but dealt with it in a linearised manner)

A feature of cyclones that is surely related to their non-linear structure is that they can behave unpredictably. When a cyclone develops in the Gulf of Mexico the United States Weather Service makes many observations of it and issues predictions of its course. Quite often those predictions are fulfilled, but sometimes they are not. [Example of Hurricane Hugo, September 1989]. Erratic changes of course are characterisitic of non-linear instabilities.

On a continental or global scale we see weather patterns established over a few years suddenly and unexpectedly change. In the last two or three years in Western Europe there have been very dry springs and summers which have persisted beyond forecasts. A similar sequence occurred about ten years earlier, but such episodes are not really exceptional, for there was a similar exceedingly hot dry period from England to Switzerland in 1900. What distinguishes these occurrences is that they succeed immediately to periods of quite different weather. A further instance is the present weather in California where a period of five years near drought has just given place to heavy rain storms as a high pressure system over the Pacific breaks down.

Yet again, the weather in northern latitudes is the result of waves circulating around the polar regions, but subject to apparently erratic changes of amplitude and mean position.

It may be argued that all these instances of apparent sudden change could be predicted if only we had enough data and big enough computers, but it may also equally be the case that they are cases of non-linear instability as first identified by Poincaré.

Leaving aside the question of the dynamical nature of some atmospheric phenomena, the fact I emphasise here is that they can be observed directly in various ways and on various scales.

Observation of similar dynamics in the oceans is much more difficult because sea water is not transparent over significant distances and because there are far far fewer contionuous observations of the seas and oceans than there are of the atmosphere.

In consequence there is bound to be a very great deal of guess work and speculation when we try to identify non-linear dynamics in the seas and oceans, and much appeal to arguments that because the mathematics says so, such effects must be there to be found.

2. NON-LINEAR MARINE PHENOMENA

With the foregoing reservations in mind, we try to identify some non-linear phenomena and chaotic behaviour apart from turbulence, which comes up again in the next section.

A. Solitons and shocks

Solitons were first identified by J Scott Russell on a canal in Scotland; they have subsequently been seen in more open water [Examples]

Shock waves on the surface of water are sudden changes of height - it is possible that solitons could be considered as two shocks back to back. Known also as hydraulic jumps or bores (in estuaries). The detailed dynamics of shocks is quite complex but important results follow from simple arguments of conservation.

Consider water in a channel of constant width. Since water is incompressible, ignore the density in the equations of conservation.

Let the depth of water in front of the shock be h and behind it let it be H. Suppose the shock front, the sudden change of depth moves into stationary water at velocity v while behind it the water is moving with velocity $(v-u)$. Imagine the shock brought to rest by giving the whole system a velocity $-v$.

Then water enters the shock with velocity v and leaves it with velocity u. Conservation of mass then gives

$$vh = uH.$$

The momentum of the water entering the shock is v per unit volume, so that the rate at which momentum enters is v^2h while the rate at which momentum leaves is similarly u^2H. The difference is equal to the difference of the integrals of pressure over the depths of water, namely $g(H^2 - h^2)/2$; g is the value of gravity.

Thus

$$v^2h - u^2H = g(H^2 - h^2)/2.$$

The two equations determine two of v, u, h, H if the other two are known.

The velocity v is $\sqrt{[gH(H+h)/2h]}$ and so is greater than the speed of gravity waves on water of depth h ($\sqrt{gh}$), but approaches that speed as H approaches h. u is equal to $\sqrt{[gh(H+h)/2H]}$.

The rate at which kinetic energy enters the shock is $v^3h/2$ and it leaves at $u^3H/2$. The difference is fixed by the previous equations and is greater than the work done by the flow of water against the difference of pressure. the excess energy goes in waves on the surface of the water, and turbulence in the flow.

It is likely that shocks and solitons occur in channels crossing shallow lagoons and may be involved in, for example, the acqua alta in Venice.

B. Fronts

We know that much of the weather is associated with movements of fronts, the boundaries between masses of air of different humidity and temperature. Because pressure changes across a front, Coriolis forces cause winds to have components parallel to a front, giving conditions for Kelvin-Helmholtz instability.

Fronts also occur in the oceans. As with other oceanic phenomena it is not easy to see them, but there is one place where they have been studied, at the western entry to the English Channel where the depth of the water becomes rapidly less at the continental shelf. The boundary of a front between cold Atlantic water and warmer inshore water can be mapped by observing plankton, which differs in thw two water masses, and by the infra-red observation of the temperature. The front moves in and out with the seasons, but seems to do so in a somewhat erratic way. The positions of fronts offshore are strongly controlled by geography in a way that weather fronts are not, but they still fluctuate

Yet another example is the El Nino phenomenon, an upwelling of deep water that occurs off the coasts of S Africa and S America. Every now and again it fails, which has a bad effect on the offshore fisheries. It is believed that El Nino is driven by, or at least closely related to the surface winds, but nonetheless it seems that whether or not it occurs is not well predicted.

3. AMPLITUDE OF TURBULENCE

Amplitude of a turbulent motion means the root mean square variation of velocity (or other parameter). The mean square variation gives the kinetic energy of the turbulence; in that it is analogous to the kinetic energy of thermal motion in a gas.

The equations of motion we have had so far do not determine the amplitude of turbulent motion. The equation for the fluctuating velocity was seen to be

$$\frac{\partial}{\partial t}v_{di} + \frac{\partial}{\partial x_k}(v_{di}v_{mk} + v_{dk}v_{mi}) = \nu\nabla^2 v_{di}$$

or, since $\mathrm{div}(\mathbf{v}_m + \mathbf{v}_d)$ is zero for an incompressible fluid,

$$\frac{\partial}{\partial t}v_{di} + (-\mathrm{div}\,\mathbf{v}_d)v_{di} + v_{mk}\frac{\partial}{\partial x_k}(v_{di}) + (\mathrm{div}\,\mathbf{v}_d)v_{mi} + v_{dk}\frac{\partial v_{mi}}{\partial x_k} = \nu\nabla^2 v_{di}$$

On multilying by v_{di} and taking mean values, we find

$$\frac{\partial}{\partial t}v_d^2 - (\mathrm{div}\,\mathbf{v}_d)v_d^2 = \nu\, v_{di}\nabla^2 v_{di}$$

This is a linear for v_d^2 without an independent forcing function, and can be satisfied by a mean square velocity of arbitrary magnitude because $-\mathrm{div}\mathbf{v}_d$ is equal to $\mathrm{div}\mathbf{v}_m$ and so independent of $\mathbf{v}_d$.

Thus the Navier-Stokes equation, together with the incompressible assumption, do not determine the magnitude of turbulence and some other conditions must be brought in if a definite value is to be established.

There is one very important case in which the turbulent amplitude is determined. About sixty years ago, Sir Harold Jeffreys showed that cyclones are an essential part of the general circulation of the atmosphere and not just parasitic upon it. The argument turns on two points, that without surface friction the general circulation, which is driven by zonal differences of temperature, would be indeterminate, and that as a consequence of surface friction, angular momentum would not be conserved in the genmeral circulation. There has to be a way of transporting angular momentum across zones and that is what cyclones do. Consequently there must be a relation between the intensity of cyclones and the strength of the general circulation of the atmosphere such that momentum is conserved.

Cyclones are however an example of chaotic motion, for where and when they start seems to be to some extent a matter of chance, as Poincaré observed. There must be something in the dynamics that ensures that cyclones do start up and develop so that on the average they maintain the balance of angular momentum.

Cyclones are well known. It is only quite recently that similar systems, gyres, have been identified in the deep oceans, in particular on the Gulf Stream, which is a major feature of the general circulation of the North Atlantic. Like cyclones, they seem to originate in the region of the Gulf of Mexico. The general circulation of the deep oceans is much more complex than that of the atmosphere on account of geographical constraints but it seems clear that the general thrust of Jeffreys's argument must apply here also and that the magnitude of the gyres in the deep oceans is determined by conservation of angular momentum.

In smaller seas it is likely that Jeffreys's argument would not apply, at least not in the form in which he set it out, for it depends upon the existence of the Coriolis force, which is not so important in small basins. I suppose therefore, that some other mechanism must operate there to determine the turbulent amplitudes.

4. PREDICTIONS

Much of the behaviour of seas and oceans is predictable. Tides are very regular, the general circulation is quite stable, the temperature of the sea often varies in a regular way. One of the reasons for that state of affairs is that the dynamical and thermal inertias of the seas are very great, so that the amplitudes of seasonal variations for example, are modest. Yet there are some events that are far less predictable and can do great damage, such as tidal waves, storm surges, hydraulic jumps, while the transport of sediment depends upon the turbulence in a stream.

One parameter that it would be very desirable to predict is the greatest amplitude of some phenomenon such as a wave in a great storm or an hydraulic jump. The mean square amplitude, averaged over some period of time, is not sufficient for planning defences against exceptional events. The mean square amplitude is however what we do want to know when thinking about the turbulent regime of some flow.

We would also like to know something of quasi-periods. For how long may we expect a certain weather regime, whether in the atmosphere or oceans, to last before changing to some other, seven fat years and seven lean, or some other numbers?

To make succesful predictions you need a good theory of the dynamics but you also need good observations. The seas are a particular physical system, particular forces act on them. Theory can organise empirical observations, it cannot produce then out of nothing. As we learn more of the types of behaviour that non-linear dynamical systems undergo, we can see how to understand some of the data, but we still need the data and as I have emphasised, it is much more difficult to obtain it at sea than for the atmosphere.

We are far from being able to make useful predictions of chaotic and non-linear behaviour in oceanography.

5. CONCLUSION

I have set out in these four lectures some account of what we mean by classical chaos and why chaotic behaviour may occur in the dynamics of the oceans. I have also described some other examples of non-linear dynamics, such as solitons.

I gave a cursory account of the main features of turbulence in fluid dynamics and discussed briefly how the level of turbulence might be determined dynamically.

In this last lecture I have called attention to difficulties in observing the behaviour of the oceans and consequently in obtaining data from which it might be possible to make useful predictions of chaotic or other non-linear phenomena.

The inertia of the oceans is great, changes in behaviour probably take very much longer than changes of the atmospheric weather and if we are to obtain good ideas of the inherent variability of the oceans we need observations extending over considerable periods. That will be the subject of my lecture on historical oceanography.