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Course on Oceanography of Semi-Enclosed Seas
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"Notes On Geophysical Fluid Dynamics"

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Notes on Geophysical Fluid Dynamics

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For the Course on Oceanography of Semi-enclosed Seas
International Centre for Theoretical Physics
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Fundamentals

Quasigeostrophy

Notes and References

1. Conservation Equation of Geophysical Fluids

The *basis* is classical mechanics (Newton's law) and thermodynamic (the first law and the Equation of State) cast in a form appropriate to the *fluid continuum*. This requires also an explicit statement of the conservation of *mass* (which is not required in particle mechanics).

The CONSERVATION EQUATIONS:

1. Conservation of Momentum based on Newton's second law

2. $\mathbf{F} = m\mathbf{a}$ $\quad \mathbf{a} = \frac{d\mathbf{v}}{dt}$

3. $\mathbf{v} = (u$ eastward, v northward, w upward)

4. Conservation of Mass ρ, density $\frac{m}{L^3}$

5. * Conservation of Energy (Heat Equation): Since Internal Energy

$$e \sim \rho c_v T$$

6. * Conservation of Salt: S usually expressed as mass of (dissolved) sal per mass of seawater. Salt density $\sim \rho S$.

7. Equation of State: $\rho \equiv \rho(T, S, p)$.

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* Under many circumstances of interest, these equations can be combined, using 7, to a single equation for the conservation of "*Density*" nd the same as 4 or *Apparent Temperature* or *Buoyancy*, especially for large scale motions.

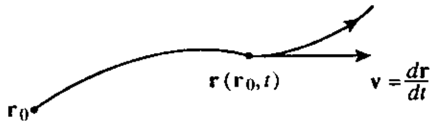
Acceleration, etc.

We require for the fluid either an *Eulerian* (*field*) description

$$\varphi(\mathbf{r},t)=\varphi(x,y,z,t)$$

or

Lagrangian description



particle trajectory

"Mark" the fluid particles at an initial moment and follow trajectories,

etc.

$$\mathbf{r}_0=(x_0,y_0,z_0)\quad \text{at}\quad t=0$$

Find $x(x_0,y_0,z_0,t)$, $y(x_0,y_0,z_0,t)$, etc.

... Floats, tracers ...

Total derivative following the motion (Substantial Derivative)

$$\frac{d}{dt}\varphi(\mathbf{r}(\mathbf{r}_0,t),t)=\frac{\partial\varphi}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{u}+\frac{\partial\varphi}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{v}+\frac{\partial\varphi}{\partial z}\underbrace{\frac{\partial z}{\partial t}}_{w}+\frac{\partial\varphi}{\partial t}\underbrace{\frac{\partial t}{\partial t}}_{1}$$

$$\frac{D\varphi}{Dt}=\underbrace{u\varphi_x+v\varphi_y+w\varphi_z}_{\text{advective rate of change}}+\underbrace{\varphi_t}_{\text{local rate of change}}$$

Thus a component of the *acceleration* is

$$\frac{Du}{Dt}=u_t+uu_x+vu_y+wu_z$$

and vectorially the acceleration is:

$$\frac{D\mathbf{v}}{Dt}=\frac{\partial\mathbf{v}}{\partial t}+\underbrace{(\nabla\times\mathbf{v})\times\mathbf{v}+\frac{1}{2}\nabla\mathbf{v}\cdot\mathbf{v})}_{''\mathbf{v}\cdot\nabla\mathbf{v}''}$$

Rotating reference system

The earth is rotating relative to an inertial reference system with approximately constant angular frequency $\mathbf{\Omega}$. Newton's law $\mathbf{F}=m\mathbf{a}$ holds in an inertial frame of reference and the time derivative of vectors in an inertial reference system $\left(\frac{d}{dt}\right)_I$ and a rotating reference system (fixed to earth) $\left(\frac{d}{dt}\right)_\Omega$ are related by

$$\left(\frac{d}{dt}\right)_I=\left(\frac{d}{dt}\right)_\Omega+\mathbf{\Omega}\times$$

In particular

$$\mathbf{v}_I=\left(\left(\frac{d}{dt}\right)_\Omega+\mathbf{\Omega}\times\right)r=\mathbf{v}_\Omega+\mathbf{\Omega}\times\mathbf{r}$$

and

$$\mathbf{a}_I=\left(\left(\frac{d}{dt}\right)_\Omega+\mathbf{\Omega}\times\right)\left(\left(\frac{d}{dt}\right)_\Omega+\mathbf{\Omega}\times\right)\mathbf{r}$$

$$=\mathbf{a}_\Omega+\underbrace{2\mathbf{\Omega}\times\mathbf{v}}_{\text{Coriolis}}+\underbrace{\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}}_{\text{Centrifugal}}$$

Mass Conservation:

Mass in a volume $V=\int_V\rho dv$

Rate of change of mass in volume V

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho\, dv = \int_V \frac{\partial \rho}{\partial t}\, dv$$

No internal sources of mass only surface flux $\rho\mathbf{v}$

$$\frac{dM}{dt} = -\int_{\partial V} \rho\mathbf{v}\cdot\mathbf{n}\, ds \qquad (\mathbf{n}\ \text{ outer normal})$$

divergence theorem

$$= -\int_V \nabla\cdot(\rho\mathbf{v})\, dv$$

i.e.,

$$0 = \int\left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) dv \quad \text{all volumes } V$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Internal Energy (Temperature) and Salt Conservation

If scalar φ is now internal energy e or salt S

Rate of change of scalar φ in volume V

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_V \varphi\, dv = \int_V \frac{\partial \varphi}{\partial t} dv$$

Internal sources q, surface fluxes

$\varphi\mathbf{v}$	advective
$-K\nabla\varphi$	diffusive

we get

$$\frac{d\Phi}{dt} = -\int_{\partial V}(\varphi\mathbf{v} + K\nabla\varphi)\cdot\mathbf{n}\, ds + \int_V q\, dv$$
$$= -\int_V \nabla\cdot(\varphi\mathbf{v} - K\nabla\varphi)dv + \int_V q\, dv$$

all volumes V

$$\frac{\partial \varphi}{\partial t} + \nabla\cdot(\varphi\mathbf{v}) = \nabla\cdot(K\nabla\varphi) + q$$

(in water $\partial\rho/\partial t \simeq 0,\ \nabla\cdot\mathbf{v} = 0$

for e: $\dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = D = q + \nabla\cdot(K_T\nabla T)$

for S: $\dfrac{\partial S}{\partial t} + \mathbf{v}\cdot\nabla T = D_S = \nabla\cdot(K_S\nabla S)$

$K_T \neq K_S$)

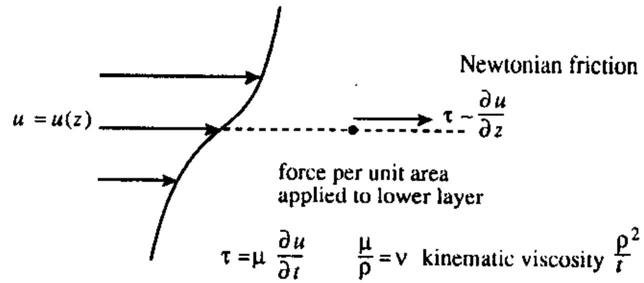
Force Description:

Internal Fluid-Fluid Forces are represented by the pressure and by stresses (viscosity).

hydrostatically $-p$ is force per unit area

$$\equiv \left[\frac{m\ell}{t^2}\,\frac{1}{\ell^2}\right] = \frac{m}{\ell t^2}$$

n.b. $\quad \dfrac{p}{\rho} = \dfrac{m}{\ell t^2}\,\dfrac{1}{\frac{m}{\ell^3}} = \dfrac{\ell^2}{t^2}$



External Forces commonly derivable from a potential function (earth gravity, sun and planets gravity)

$$\mathbf{F_{ext}} = -\nabla\Phi$$

The geopotential Φ in the rotational reference system associated with earth gravity (whose constant surfaces are called level surfaces) is dominated by gravity with a small distortion due to centrifugal effects. With z direction pointing upwards

$$d\Phi = g dz$$

and

$$\mathbf{F_{ext}} = -g\hat{\mathbf{z}}$$

Model Equations

Momentum : $$\rho\frac{D\mathbf{v}}{Dt} + \rho 2\Omega \times \mathbf{v} + \rho g\hat{z} + \nabla P = \mathbf{F} \qquad (1.1, 2, 3)$$

Mass : $$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad (1.4)$$

Energy : $$T_t + \mathbf{v}\cdot\nabla T = \frac{DT}{Dt} = \mathcal{D} \qquad (1.5)$$

Salt : $$\frac{DS}{Dt} = \mathcal{D}_S \qquad (1.6)$$

State : $$\rho = \rho(T, S, p) \qquad (1.7)$$

Remarks:

- Seawater is almost incompressible, density changes are only $\sim 1/10^3$. Buoyancy effects etc. can nonetheless be of *primary* importance **BUT** other aspects of density variation are trivial and can be totally *ignored*.

(Mass) Equation (4) becomes

$$\nabla\cdot\mathbf{v} = u_x + v_y + w_z = 0 \qquad (1.4a)$$

- Equation of State (1.7)

Two versions:

(i) $\rho = \frac{1}{\alpha} = \rho_0(1 - \sigma \times 10^{-3})$

$$\alpha(T, S, p) = \alpha(0°c, 35.00‰, p) + \delta$$
$$\equiv \alpha_{35,0,p} + \delta$$

in terms of α the specific volume and δ the specific volume anomaly.

Real Seawater complex nonlinear subtle behavior.

(ii) The Theoreticians "GFD" Ocean

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_0))$$

$$\alpha = \left(\frac{\partial\rho}{\partial T}\right)_{S,p} \quad \text{etc.}$$

Boussinesq *Approximation*

$\rho \equiv \rho_0$ everywhere EXCEPT in term $\rho g\hat{z}$ (the buoyancy *acceleration*)

2. The Boussinesq Approximation

The density of seawater is almost constant, with fractional variations $O(10^{-3})$. For the scales of motion of interest the kinematics effects of compressibility are negligibly important both qualitatively and quantitatively. However, the small density differences which occur are of zero-order consequence in that they give rise to one of the largest and most important driving forces for ocean currents and circulation, the buoyancy forces associated with differential gravitational acceleration. Boussinesq first recognized this in the late 19th century and introduced an approximation which he stated in *ad hoc* physical terms as the limit of $\alpha \to 0$ but the product αg remaining finite, where α is the thermal expansion coefficient and density differences were assumed to arise only from temperature differences with a linear equation of state. We will derive the Boussinesq approximation from the full Navier-Stokes equations via a formal perturbation expansion in the framework of modern scale analysis. The key idea is that two independent nondimensional parameters involving $\Delta\rho$, the maximum density difference in the fluid will occur, one characterizing buoyancy accelerations and the other characterizing kinematic effects. For buoyancy driven flows, the former parameter is set equal to unity while the latter is the small parameter of the expansion. To illustrate the physics a two-dimensional (vertical and one horizontal) nonrotating version of the momentum and mass conservation equations (1.1–1.3) are sufficient. We will not explicitly consider the heat and salt equations (1.7) and (1.8) which do not involve ρ directly but simply assume constant kinematic diffusivities.

Assume a linear equation of state of the form

$$\rho = \rho_0[1 - \alpha(T - T_0) + \beta(S - S_0)] \equiv \rho_0[1 + \epsilon\eta] \qquad (2.1)$$

where

$$\alpha = -\frac{\partial\rho}{\partial T}\bigg|_{S,P}, \qquad \beta = -\frac{\partial\rho}{\partial T}\bigg|_{T,P}, \qquad \epsilon = \frac{\Delta\rho}{\rho_0}$$

and η is an order unity nondimensional density anomaly. Note that the omission of pressure dependence in the equation of state is equivalent to assuming infinite sound speed and serves to filter sound waves from the resulting approximate model equations. For simplicity we scale both coordinates similarly with scale L, the time advectively with LU_0^{-1}, the velocity components by U_0 and express the pressure as

$$P_D = -\rho_0 gz + \rho_0 P_0 P_{\mathrm{ND}}(x, y, z, t) \quad . \qquad (2.2)$$

We will use the same symbols for dimensional (D) and nondimensional (ND) versions of the same fields and drop the subscripts except for occasional clarification.

Nondimensionally then the x, z momentum equations and the continuity equation appear as

$$(1 + \epsilon\eta)\left[\frac{u_0^2}{L}(u_t + uu_x + wu_z) - \frac{\nu u_0}{L^2}\nabla^2 u\right] + \frac{P_0}{L}P_x = 0 \qquad (2.3a)$$

$$(1 + \epsilon\eta)\left[\frac{u_0^2}{L}(w_t + uw_x + ww_z) - \frac{\nu u_0}{L^2}\nabla^2 w\right] + g\epsilon\eta + \frac{P_0}{L}P_z = 0 \qquad (2.3b)$$

$$\epsilon\eta_t + \epsilon[u\eta_x + w\eta_z] + (1 + \epsilon\eta)(u_x + w_z) = 0 \qquad (2.3c)$$

The term $g\epsilon\eta$ in (2.3b) is the basic driving force. Now multiply (2.3a,b) by LP_0^{-1} and choose

$$P_0 = Lg\epsilon = gL\frac{\Delta\rho}{\rho_0} = u_0^2 \tag{2.4}$$

and define the Reynolds number by $R = Lu_0\nu^{-1}$. Then

$$(1+\epsilon\eta)[u_t + uu_x + wu_z - R^{-1}\nabla^2 u] + P_x = 0 \tag{2.5a}$$

$$(1+\epsilon\eta)[w_t + uw_x + ww_z - R^{-1}\nabla^2 w] + \eta + P_z = 0 \tag{2.5b}$$

$$\epsilon\eta_t + \epsilon[u\eta_x + w\eta_z] + (1+\epsilon\eta)(u_x + w_z) = 0 \tag{2.5c}$$

Now as is necessary and usual in perturbation analysis we assume that all fields are smooth functions of their arguments, i.e., an order unity upper bound exists for the function and all derivatives so that the size of the individual terms in the equations is indicated by their coefficients. The coefficient of the buoyancy driving term, η, in (2.5b) is 1. The perturbation expansion for all fields is

$$\varphi = \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2 + \cdots \quad .$$

and to zeroth order in ϵ (2.5a,b,c) become, dropping the 0-subscript on all fields,

$$u_t + uu_x + wu_z - R^{-1}\nabla^2 u + P_x = 0 \tag{2.6a}$$

$$w_t + uw_x + ww_z - R^{-1}\nabla^2 w + \eta + P_z = 0 \tag{2.6b}$$

$$u_x + w_z = 0 \tag{2.6c}$$

The fluid is driven by buoyancy but otherwise incompressible.

Note that (2.4c) to first order in ϵ appears as

$$\eta_{0t} + u_0\eta_{0x} + w_0\eta_{0z} + u_{1x} + w_{1z} = 0 \tag{2.7}$$

so that density is not conserved following the motion as a consequence of (2.3c). This is sometimes misstated for the Boussinesq approximation. Density conservation statements to zero order must come from a consideration of the temperature (and salinity) equations. The extension to three dimensions and inclusion of rotation are straightforward. The simple statement of the result is that it is correct to treat the density as constant everywhere except in the gravitational force term.

3. The Thinness Approximation

The fact that the oceans and the atmosphere are thin shells of fluid whose vertical extent is much less than their horizontal extent leads to an important approximation in their dynamics. Many large-scale, synoptic scale and mesoscale motions of interest are in approximate hydrostatic balance. In order to illustrate this fact, prior to a full treatment of the ocean basins geometry on the earth, we consider a simple two-dimensional (vertical scale H, horizontal scale L) Cartesian coordinate system that has an extreme aspect ratio

$$\frac{H}{L} = \lambda << 1 \quad . \tag{3.1}$$

We retain u_0 for the horizontal velocity scaling but introduce an independent scale w_0 for the component. We retain the expression (2.2) for the pressure and the scaling $P_0 = u_0^2$. Then the nondimensional form equations (2.6) become

$$u_t + uu_x + \left(\frac{w_0 L}{u_0 H}\right) wu_z - R^{-1}\left[\frac{\partial^2}{\partial x^2} + \lambda^{-2}\frac{\partial^2}{\partial z^2}\right] u + P_x = 0 \tag{3.1a}$$

$$\left(\frac{w_0 H}{u_0 L}\right)(w_t + uw_x) + \left(\frac{w_0^2}{u_0^2}\right) ww_z - R^{-1}\left(\frac{w_0 H}{u_0 L}\right)\left[\frac{\partial^2}{\partial x^2} + \lambda^{-2}\frac{\partial^2}{\partial z^2}\right] w$$

$$+ \left(\frac{g\Delta\rho H}{\rho_0 u_0^2}\right)\eta + P_z = 0 \tag{3.1b}$$

$$u_x + \left(\frac{w_0 L}{u_0 H}\right) w_z = 0 \tag{3.1c}$$

Consider first the continuity equation (3.1c). The upper bound physically sensible scaling for the vertical velocity is that for which the vertical

divergence and the horizontal divergence are of the same order of magnitude. Thus we choose

$$w_0 = \frac{u_0 H}{L} \quad . \tag{3.2}$$

If w were larger than this, the larger derivative in the thin direction (z) could not be balanced, there could only be a trivial thru-flow. Note that this choice of scaling ensures that the vertical and horizontal advections, i.e., terms in the substantial derivatives, are of the same order of magnitude. This will be true for *all* conservation equations. Equations (3.1a,b,c) now appear as

$$u_t + uu_x + wu_z - R^{-1}\left(\frac{\partial^2}{\partial x^2} + \lambda^{-2}\frac{\partial^2}{\partial z^2}\right) u + P_x = 0 \tag{3.3a}$$

$$\lambda^2\left[w_t + uw_x + ww_z - R^{-1}\left(\frac{\partial^2}{\partial x^2} + \lambda^{-2}\frac{\partial^2}{\partial z^2}\right) w\right] + \left(\frac{g\Delta\rho H}{\rho_0 u_0 z}\right)\eta + P_z = 0 \tag{3.3b}$$

$$u_x + w_z = 0 \tag{3.3c}$$

For $\lambda^2 << 1$ the viscous and inertial terms in the vertical momentum equation (3.3b) are much less important relative to the vertical pressure gradient than are the corresponding terms relative to the pressure gradient in (3.3a). The thin system is in approximate hydrostatic balance

$$\left(\frac{g\Delta\rho H}{\rho_0 u_0^2}\right)\eta + P_z = 0 \tag{3.4}$$

In essentially all cases of interest the Reynolds number and is large $R^{-1}\lambda^2$ will be small. The almost hydrostatic approximate model equations are known in meteorology and oceanography as the *primitive equations*. The generalization to three dimensions is trivial. If the motions are primarily buoyancy driven then

$$u_0^2 = g\Delta\rho\rho_0^{-1}H$$

is appropriate.

A special case of considerable interest are the homogeneous density ($\eta = 0$) ideal fluid ($R \to \infty$) in three dimensions: the shallow water equations:

$$u_t + uu_x + vu_y + wu_z + P_x = 0 \qquad (3.5a)$$

$$v_t + uv_x + vv_y + wv_z + P_y = 0 \qquad (3.5b)$$

$$P_z = 0 \qquad (3.5.c)$$

$$u_x + v_y + w_z = 0 \qquad (3.5d)$$

Since $\partial^2 P/\partial x\partial z = \partial^2 P/\partial y\partial z = 0$; the vertical structure of the flow is constant horizontal velocities with a vertical component that varies linearly viz.

$$w = w_0(x,y,t) + w_1(x,y,t)z \qquad (3.6)$$

whence (3.5a,b,d) become

$$u_t + uu_x + vu_y + P_x = 0 \qquad (3.7a)$$

$$v_t + uv_x + vv_y + P_y = 0 \qquad (3.7b)$$

$$u_x + v_y + w_1 = 0 \qquad (3.7c)$$

4. β-plane Approximation

$$\mathbf{\Omega} = \Omega\cos\theta\hat{\mathbf{t}} + \Omega\sin\theta\hat{\mathbf{z}}$$

$\hat{t}$ = tangential to Earth surface (horizontal)

$\hat{z}$ = vertical

For latitudinal range $L < R_E$ radius of Earth $\theta - \theta_0$ small and

$$\sin\theta \simeq \sin\theta_0 + \cos\theta_0(\theta - \theta_0)$$

$$f = 2\Omega\sin\theta \cong 2\Omega\sin\theta_0 + 2\Omega\cos\theta_0 y/R_E$$

$$y = R_E(\theta - \theta_0)$$

On sphere

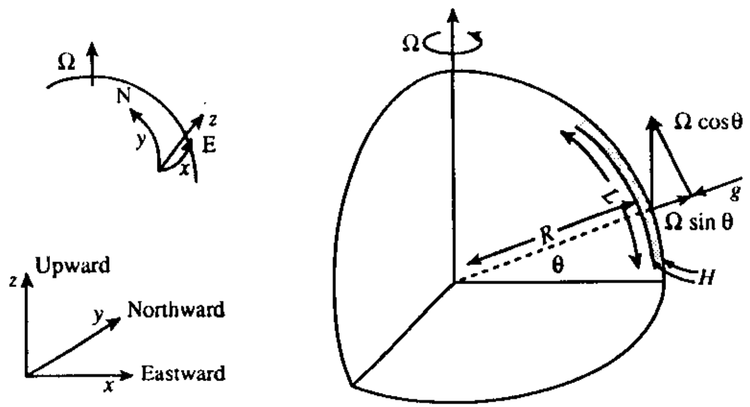
$$ds^2 = R_E^2\,d\theta^2 + (R_E\cos\theta)^2\,d\lambda^2 + dr^2$$

$$R_\theta\,d\theta = dy$$

$$(\text{small }\theta - \theta_0) \Rightarrow R_E\cos\theta\,d\lambda \cong R_E\cos\theta_0\,d\lambda = R_E\,dx$$

$$dr = dz$$

These will be the final coordinates, x → eastward on the surface, y → northward on the surface and z vertically upwards. The result will be Cartesian but varying vertical rotation, viz



$$u_t + uu_x + vu_y + \omega u_z - fv + \frac{1}{\rho}p_x = \mathcal{F}_x \qquad 4.1a$$

$$v_t + uv_x + vv_y + \omega v_z - fu + \frac{1}{\rho}p_y = \mathcal{F}_y \qquad 4.2a$$

$$g + \frac{1}{\rho}p_z = 0 \qquad 4.3a$$

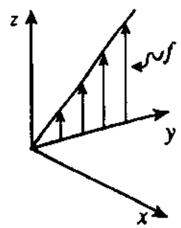
$$u_x + v_y + w_z = 0 \qquad 4.4a$$

where

$$f = f_0 + \beta y \qquad \text{The Coriolis Parameter}$$

$$= 2\Omega \sin\theta$$

$$f_0 \equiv 2\Omega\sin\theta_0\,, \quad \beta = \frac{2\Omega\cos\theta_0}{R}\,.$$



ONLY Remaining effect of earth's curvature

But starting point is spherical equations

r momentum

$$u_t + \frac{u}{R\cos\theta}u_\lambda + \frac{v}{R\cos\theta}(u\cos\theta)_\theta + \frac{w}{R}(ru)_r - 2\Omega v\sin\theta + 2\Omega w\cos\theta \;\frac{1}{\rho_0 R\cos\theta}P_\lambda = \mathcal{F}_1 \qquad 4.2a$$

$$v_t + \frac{u}{R\cos\theta}v_\lambda + \frac{v}{R}v_\theta + \frac{w}{R}(rv)_r + \frac{u^2}{R}\tan\theta + 2\Omega\sin\theta + \frac{1}{\rho R}P_\theta = \mathcal{F}_2 \qquad 4.2b$$

$$w_t + \frac{u}{R\cos\theta}w_\lambda + \frac{v}{R}w_\theta + ww_r - \frac{u^2+v^2}{R} - 2\Omega\cos\theta + \frac{1}{\rho}P_r = \mathcal{F}_3 \qquad 4.2c$$

or mass

$$\frac{1}{R\cos\theta}u_\lambda + \frac{1}{R\cos\theta}(v\cos\theta)_\theta + \frac{1}{R^2}(r^2 w)_r = 0 \qquad 4.2d$$

Scale $x,y \to L \quad z \to H \quad u,v \to U_0 \quad w \to U_0\frac{H}{L}$

The scaled variables are used to set up the ND equations and the assumptions

$$\frac{H}{R} \ll 1 \quad \left(\frac{L}{R}\right)^2 \ll 1 \quad (\tan\theta_0)\frac{L}{R} \ll 1 \quad \frac{HR}{L^2} \ll 1 \quad \frac{H}{L}\cot\theta_0 \ll 1$$

Reduce Equations 4.2 to 4.1

The approximations are <u>geometrical</u> - We separate dynamical approximations.

5. Geostrophic Hydrostatic Motion of an Ideal Fluid

Ideal. In (1.1 3.5,6) $F = D = D_S \approx 0$ i.e., no momentum dissipation, heat or salt sources. Coriolis Acceleration dominates relative accelerations: e.g.,

$$u_t + uu_x + vu_y + \omega u_z - 2\Omega V \ ,$$

Compare typical term uu_x to $2\Omega V$. Ratio is

$$\frac{uu_x}{2\Omega V} \approx \frac{u_0^2}{L}\,\frac{\quad}{2\Omega u_0} = \frac{u_0}{2\Omega L} \ .$$

Rossby number $\frac{u_0}{2\Omega L} << 1$.

Pressure gradient balances approximately gravity acceleration. A key point is $\frac{H}{L} << 1$ where H is the vertical extent of motion and L is the horizontal extent of motion. Then momentum equation

$$\rho 2\mathbf{\Omega} \times v + \rho g \hat{z} + \nabla p = 0 \ ,$$

$$\nabla \cdot \mathbf{v} = 0 \ .$$

Also,

$$\frac{DS}{Dt} = 0 \qquad \frac{DT}{Dt} = 0 \ , \qquad \text{implying}$$

$$\frac{D\rho}{Dt} = 0 \ .$$

Geostrophic Hydrostatic Motion:

i) The "Real Ocean"

$$-fv + \alpha p_x = 0$$

$$fu + \alpha p_y = 0$$

$$g + \alpha p_z = 0$$

Go from u, v, p dependent variables
x, y, z independent variables

to u, v, z dependent variables
x, y, p independent variables

If: $F(x, y, z)$ or $F(x, y, p(x, y, z))$

$$\begin{aligned} dF &= F_x dx + F_y dy + F_z dz \\ &= (F_x)_p dx + (F_y)_p dy + F_p dp \\ &= (F_x)_p dx + (F_y)_p dy + F_p[p_x dx + p_y dy + p_z dz] \\ &= \{(F_x)_p + F_p p\}dx + \{(F_y)_p + p_y\}dy + F_p p_z dz \ . \end{aligned}$$

Thus,

$$\left(\frac{\partial F}{\partial x}\right)_{y,z} = \left(\frac{\partial F}{\partial x}\right)_p + \left(\frac{\partial F}{\partial p}\right)_{x,y} p_x$$

$$\left(\frac{\partial F}{\partial y}\right)_{x,z} = (F_x)_p + (F_p)_x P y$$

$$\left(\frac{\partial F}{\partial z}\right)_{x,y} = F_p p_z \ .$$

If $F = p$,

$$p_x = 0 + 1 \cdot p_x \ .$$

If $F = z$,

$$(z_x)_{y,z} = 0 = z_x + z_p p_x$$
$$0 = z_y + z_p p_y$$
$$1 = z_p p_z \, .$$

We get $p_x = -\frac{z_x}{z_p}$.

"Real ocean" equations

$$-fv + \alpha\left(-\frac{z_x}{z_p}\right) = 0$$
$$fu + \alpha\left(-\frac{z_y}{z_p}\right) = 0$$
$$g + \alpha\,\frac{1}{z_p} = 0 \qquad \frac{1}{z_p} = -\frac{g}{\alpha}\,.$$

Thus (recalling $d\Phi = g\,dz$):

$$\begin{aligned} -fv + gz_x &= 0 \qquad & -fv + \Phi_x &= 0 \\ fu + gz_y &= 0 \qquad & fu + \Phi_y &= 0 \\ \alpha + gz_p &= 0 \qquad & \alpha + \Phi_p &= 0 \end{aligned}$$

where $\Phi_x \equiv \left(\frac{\partial \Phi}{\partial x}\right)_{y,p}$ etc....

and

$$v_p = -\frac{1}{f}\,\alpha_x = -\frac{1}{f}\,\delta_x\,,$$
$$u_p = +\frac{1}{f}\,\delta_y\,.$$

Dynamic height:

Concept $g dz = -\frac{1}{\rho}dp = d\Phi$.

Units $\quad$ 1 dyn m = 10 m^2S^{-2}

1 dyn decim = 10^{-1} dyn m

1 dyn cm = 10^{-2} dyn m

1 cm^2S^{-2} = 10^{-4} m^2S^{-2}

1 dyn cm = 10^{-1} m^2S^{-2}.

$$dD \quad = \quad \alpha_{S\theta p} \quad dp\,,$$

dyn m $\quad \left(\frac{cm^3}{gm} = \frac{m^3}{ton}\right) \quad$ decibars.

Also

$$dD \quad = \quad \tfrac{1}{10}g \quad dz\,,$$

(m S^2) $\quad$ (m)

$$V \quad = \quad \frac{10}{f} \quad \frac{\Delta D}{\Delta x}\,.$$

(m/sec) $\quad$ (s) $\quad \left(\frac{\text{dyn m}}{\text{m}}\right)$

Remember: 1 ton = 10^6 gm

1 dyne = 1 gm $\frac{cm}{sec^2}$

1 bar = 10^6 dyne cm^{-2}

1 decibar = 10^5 dyne cm^{-2}

ii) The GFD ocean

$$\rho = \rho_0(1 - \alpha T) ,$$
$$-fv + \frac{1}{\rho_0}\, p_x = 0$$
$$fu + \frac{1}{\rho_0}\, p_y = 0$$
$$-\alpha g T + \frac{1}{\rho_0}\, p_z = 0$$

P.g., due to ρ_0 is subtracted out, henceforth also "$\frac{p}{\rho_0}$" is called "p" $\left[\frac{\ell^2}{t^2}\right]$, and $T_0 = 0$. Also, $u_x + v_y + w_z = 0$.

Case a

$$T = \beta = 0 \qquad u_z = v_z = 0$$
$$f(v_x + v_y) = 0 \quad \Longrightarrow \quad w_z = 0$$

(Taylor Proudman Theorem).

Case b

$$T = 0 \qquad f(u_x + v_y) + \beta v = 0$$
$$\beta v = f w_z$$

Planetary Divergence

Sverdrup Dynamics

Case c

$$\beta = 0 \qquad v_z = \frac{\alpha g}{f}\, T_x \qquad u_z = -\frac{\alpha g}{f}\, T_y$$

"Thermal Winds'

Case d Gerneal c, b hold with $f = f_0 + \beta y$ and

$$\frac{\partial^2 w}{\partial z^2} = \frac{\beta \alpha g}{f^2}\, \frac{\partial T}{\partial x}$$

"Thermocline Dynamics"

6. Hydrodynamic Instability

The nonlinearities inherent in the advective terms of the substantial derivative form of the acceleration etc. which characterize our description of the continuum nature of fluid physics result in profound effects including spontaneous internal instabilities and turbulence. Unlike more familiar branches of linear physics, solutions of our model conservation equations are not necessarily unique, and a given exact solution may or may not be observed in the laboratory or in nature under the requisite parametric conditions. Again, for simplicity of exposition only, we will illustrate some concepts for the two dimensional flow of a Boussinesq fluid. Two very simple but very important exact solutions exist for a fluid contained between two infinite parallel plates separated by distance H in the vertical. The first is for a homogeneous fluid driven by a constant pressure gradient $p_x = G\rho_0$. The flow is a steady parabolic sheet symmetric about the midpoint.

$$U = \frac{U_0}{H^2}\, Z(Z - H) , \quad w = 0 \quad \text{and} \quad U_0 = \frac{H^2 G}{2\nu} , \tag{6.1}$$

with nonlinearities identically equal to zero. The second example is for the two plates held at constant temperatures, T_L at the lower plate and $T_L + \Delta T$ at the upper plate. The exact solution here is hydrostatic with

$$T = T_L + \Delta T\, \frac{Z}{H} , \qquad u = w = 0 \tag{6.2}$$

which is an exact solution of (.1, .6a,b,c) and

$$T_t - K\nabla^2 T + uT_x + wT_z = 0 , \tag{6.3}$$

for all ΔT positive or negative. The parabolic flow is sometimes observed in the laboratory and sometimes not; for large U_0 and H, or small ν, the flow is turbulent with a flatenned profile. The hydrostatic state (6.2) is observed for positive ΔT, but for negative values, convective motion ensues.

The nonlinear physical processes are very difficult to deal with conceptually and theoretically. Lack of uniqueness means that the physical model is inadequate as constructed, and requires additional physics. The first consideration is to address the question of when a given exact solution of the model equations will or will not be realized or observed. The classical approach is to test the stability of the solution to infinitesimal disturbances. If the circumstances are stable, a slight perturbation of the state of the system, from the noise which is always present in nature, will decay away in time. If such a disturbance were to grow, due to its triggering the release of energy via an internal dynamical process, then the instability would indicate that under these circumstances the solution would not be observed, without necessarily indicating what the new state of the system would be at the near finite amplitude equilibrium state. The initial value problem for the small disturbance is linearized in the perturbation amplitude, thereby reducing the mathematical difficulties, although most stability problems remain quite difficult.

Let us now assume that there is, in general, a basic steady state whose stability is to be investigated given by $U(x,z)$, W, P, $\mathbf{T}$ which satisfies the steady model equations. Now let the total or composite (c) fields be written as

$$u^c = U(x,z)+u(x,z,t) \quad w^c = W+w\ , \quad p^c = P+p\ , \quad T^c = \mathbf{T}+T\ . \quad (6.4)$$

Now equations for the (lower case) fluctuation contributions are obtained by substituting the composite fields into the momentum, mass and heat conservation equations and subtracting the mean field conservation equations. viz

$$u_t - \nu\nabla^2 u + Uu_x + uU_x + wU_z + uu_x + wu_z + \frac{1}{\rho_0}P_x = 0\ , \qquad (6.5a)$$

$$w_t - \nu\nabla^2 w + Uw_x + uW_x + wW_z + uw_x + ww_z + g\eta + \frac{1}{\rho_0}P_z = 0\ , \quad (6.5b)$$

$$u_x + w_z = 0\ , \qquad (6.5d)$$

$$T_t - K\nabla^2 T + UT_x + u\mathbf{T}_x + w\mathbf{T}_z + WT_z + uT_x + wT_z = 0\ . \qquad (6.5e)$$

So far there has been no approximation. The stability equations are obtained from (6.5a–d) by neglecting the quadratic self-interactions of the fluctuations but retaining the fluctuation-mean field interactions which are generally non-contact coefficient but linear in the fluctuation fields.

The basic state fields satisfy all forcing functions including any inhomogeneous boundary conditions. Thus, the disturbance problem is an initial value problem with homogeneous boundary conditions. We may now assume simply that all fields vary exponentially in time i.e.,

$$u(x,z,t) = e^{\sigma t}u'(x,z)\ , \quad w = e^{\sigma t}w'\ , \quad p = e^{\sigma t}p'\ , \quad T = e^{\sigma t}T'\ . \quad (6.6)$$

Inserting these expressions into the linearized form of (6.5) we obtain

$$\sigma u - \nu\nabla^2 u + Uu_x + uU_x + wU_z + Wu_z + \frac{1}{\rho_0}P_x = 0\ , \qquad (6.7a)$$

$$\sigma w - \nu\nabla^2 w + U w_x + u W_x + w W_z + W w_z + g\eta + \frac{1}{\rho_0} P_z = 0 \,, \qquad (6.7b)$$

$$u_x + w_z = 0 \,, \qquad (6.7d)$$

$$\sigma T - K\nabla^2 T + U T_x + u \mathbf{T}_x + w \mathbf{T}_z + W T_z = 0 \,. \qquad (6.7e)$$

The problem is in the nature of an eigenvalue problem with eigenvalue, σ, the growth rate. Assume, as is often the case and in any event provides an interesting example, that σ is real. Then, the only eigenvalue of interest is the value $\sigma = 0$, which indicates the transition from $\sigma < 0$ (a stable flow situation) to $\sigma > 0$, an instability. The trivial solution of zero fluctuations always satisfies equations (6.7a–d). For a nontrivial solution to exist for $\sigma = 0$, there must be an eigenparameter identified from the physical parameters of the fluid.

Consider the simplification of a homogeneous density fluid and a one dimensional basic state $U(z)$ only. The stability equations are

$$u_t - \nu\nabla^2 u + U u_x + w U_z + \frac{1}{\rho} P_x = 0 \,, \qquad (6.8a)$$

$$w_t - \nu\nabla^2 w + U w_x + \frac{1}{\rho} P_x = 0 \,, \qquad (6.8b)$$

$$u_x + w_z = 0 \,. \qquad (6.8d)$$

Form the energy equation by multiplying (6.8a) by u, (6.8b) by w and adding. Thus,

$$K_t - \nu[u\nabla^2 u + w\nabla^2 w] + U K_x + uwU_z + \frac{1}{\rho_0}(uP)_x + \frac{1}{\rho_0}(wP)_z = 0 \,, \quad (6.9)$$

with $K = \frac{u^2+w^2}{2}$. Integrate over the closed volume of the fluid and impose the homogeneous condition of no flow through the boundaries. Then, the 3rd term (advection of K) and the last terms (rate of pressure work) will vanish, and the 2nd frictional term can be integrated by parts using the identity

$$\nabla \cdot (\phi\nabla\phi) = (\nabla\phi)\cdot(\nabla\phi) + \phi\nabla^2\phi \,. \qquad (6.10)$$

Then,

$$\langle K \rangle_t = -\Phi - \langle uwU_z \rangle \,, \qquad (6.11)$$

where $\langle\cdot\rangle = \int \cdot\, d\mathrm{Vol}$, and $\Phi = \nu\langle \nabla u \cdot \nabla u + \nabla w \cdot \nabla w\rangle$ is the positive definite dissipation of fluctuation kinetic energy.

Equation (6.11) shows that a net growth of perturbation kinetic energy, or instability, can occur only if the fluctuation components are so correlated as to $\langle uwU_z\rangle$ negative, and that the source of the energy of the perturbations in this case can only be the shear of the mean flow. It can be shown easily that a *necessary* condition for a shear flow to release energy to infinitesimal perturbations is the existence of a point of inflection in the profile of U. Introducing a scale speed u_0 and length L, (6.11) becomes non-dimensionally and with $\frac{\partial}{\partial t} = \sigma$,

$$\sigma\langle K\rangle = -R^{-1}\langle \nabla u \cdot \nabla u + \nabla w \cdot \nabla w\rangle - \langle uwU_z\rangle \,, \qquad (6.12)$$

with $R = u_0 L \nu^{-1}$. Thus, at $\sigma = 0$, the Reynolds number is the critical eigenvalue for which the rate of release of energy by the shear can overcome the frictional loss to dissipation of the motion.

7. Convection and Salt Fingers

As an example of an hydrodynamic instability problem, we will consider the simplest case of thermal convection between two parallel plates, starting with the fluid at rest (6.2) with uniform temperature gradient $(\Delta T)H^{-1}$. The plates are assumed kinematically rigid ($w = 0$), but incapable of supporting tangential stress ($\nu \frac{\partial u}{\partial z} = 0$). Then, (6.7a–d) with $\sigma = 0$ become

$$-\nu\nabla^2 u + \frac{1}{\rho_0} P_x = 0\,, \qquad (7.1a)$$

$$-\nu\nabla^2 w - \alpha g T + \frac{1}{\rho_0} P_z = 0\,, \qquad (7.1b)$$

$$u_x + w_z = 0\,, \qquad (7.1d)$$

$$-K\nabla^2 T + \frac{\Delta T}{H}\, w = 0\,. \qquad (7.1c)$$

Equation (7.1d) implies a streamfunction ψ such that

$$u = \psi_z\,, \qquad w = -\psi_x\,. \qquad (7.5)$$

In terms of the streamfunction and upon eliminating the pressure between (7.1a,b)

$$\nu\nabla^4\psi - \alpha g T_x = 0\,, \qquad (7.2a)$$

$$K\nabla^2 T + \frac{\Delta T}{H}\psi_x = 0\,, \qquad (7.2b)$$

with boundary conditions

$$T = \psi = \psi_{zz} = 0 \qquad \text{at} \qquad z = 0\,, H\,. \qquad (7.2d)$$

Problem (7.2) is solved by

$$\psi = A \sin\frac{a\pi x}{H}\,\sin\frac{n\pi}{H}\,z \qquad (7.3)$$
$$T = B \cos\frac{a\pi x}{H}\,\sin\frac{n\pi}{H}\,z$$

where n is integer and a is unspecified. Then, (7.2a,b) become

$$\nu\,\frac{\pi^4}{H^4}\,(n^2+a^2)^2 A + \alpha g\,\frac{a\pi}{H}\,B = 0\,, \qquad (7.4a)$$

$$-K\,\frac{\pi^4}{H^4}\,(n^2+a^2)B + \frac{\Delta T}{H}\,\frac{a\pi}{H}\,A = 0\,, \qquad (7.4b)$$

which yields in terms of the Rayleigh number

$$\text{Ra} = \frac{\alpha g \Delta T H^2}{K\nu} = -\pi^4\,\frac{(n^2+a^2)^3}{a^2}\,. \qquad (7.4d)$$

For instability, ΔT must be negative and large enough to overcome frictional dissipation and the tendency of the perturbation to be diffused away. For $n = 1$ the critical value of the Rayleigh number $\text{Ra}(a^2)$ is determined by the minimum condition

$$\frac{\partial(\text{Ra})}{\partial a^2} = 0 = 3\,\frac{(1+a^2)^2}{a^2} - \frac{(1+a^2)^3}{a^4}\,. \qquad (7.5)$$

The result is

$$a^2 = \frac{1}{2}\,, \qquad \text{Ra} = \frac{-27\pi^4}{4}\,. \qquad (7.6)$$

These are two-dimensional rolls. Rectangles, triangles and hexagons are close packed plan forms in 3-D and hexagons are observed in laboratory conditions.

Double diffusive effects can cause convection to occur in the ocean under conditions of a *stable* density gradient when warm, salty water overlies cooler, fresher water. Still under the Boussinesq approximation, but with

$$S = S_L + \Delta S \, \frac{Z}{H} \, , \tag{7.7a}$$

$$\eta_D = -\alpha g T + \beta g S \, , \tag{7.7b}$$

and

$$-K_S \nabla^2 S + (\Delta S) w = 0 \, , \tag{7.7d}$$

with

$$S = 0 \qquad \text{at} \qquad Z = 0 \, , H \, , \tag{7.7e}$$

put now

$$S = C \cos \frac{a \pi x}{H} \, \sin \frac{n \pi z}{H} \, , \tag{7.7e}$$

into (7.7c),

$$-K_S \, \frac{\pi^2}{H^2} \, (n^2 + a^2) C + \frac{\Delta S}{H} \, \frac{a\pi}{H} \, A = 0 \, . \tag{7.7e}$$

Because of (7.7b), the twisting term in (7.2a) now has two terms

$$-\alpha g T_x + \beta g S_x \, , \tag{7.8a}$$

and thus the last term of (7.4a) is modified to

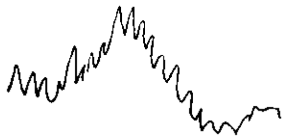
$$\frac{a\pi}{H} \, g(\alpha B - \beta C) \, . \tag{7.8b}$$

The net effect is simply to replace the Rayleigh number by the double diffusive version,

$$\mathrm{Ra}_{DD} = \left[\frac{\alpha g \Delta T}{K} - \frac{\beta g \Delta S}{K_S} \right] H^3 \, .$$

Now with positive ΔT and positive ΔS, Ra_{DD} can be negative. This possibility is favored by the fact that $K_S < K$ for sea water by almost two orders of magnitude. The diffusivities do not, of course, enter the basic state density difference $\rho_0(-\alpha\Delta T + \beta\Delta S)$. What happens is that the heat is essentially diffused away, and only the density difference due to salt remains which is then unstable. In actuality thin columns (salt fingers) occur. The base of the Mediterranean outflow into the Atlantic creates such favorable conditions.

8 Turbulence and Reynolds Stresses



All fields $\varphi = \varphi' + \overline{\varphi}$

If $\varphi_t + \frac{\partial}{\partial x}[u\varphi - K\varphi_x] = 0$

Then $\overline{\varphi}_t + \frac{\partial}{\partial x}[\overline{u}\,\overline{\varphi} - K\overline{\varphi}_x + \overline{u'\varphi'}]$

↑ A new mean flux from correlated fluctuations.

All directions, x, y, z, all fields u, v, w, T, S,

The Navier Stokes Equations can be written

$$\frac{\partial v_i \rho}{\partial t} + \frac{\partial}{\partial x_j}[\rho v_i v_j - \tau_{ij} + p] + \rho g + 2\rho\epsilon_{ijk}\Omega_j v_k = 0$$

For $\tau_{ij} = \rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \nu_* \delta_{ij}\frac{\partial v_k}{\partial x_k}$ $\quad \nu \geq 0$, $\nu_* \geq -\frac{2}{3}\nu$

(incompressibility of course $\frac{\partial v_k}{\partial x_k} = 0$)

Now $\tau_{ij\,\text{total}} = \rho\nu\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right) - \rho\overline{v_i' v_j'} \equiv \tau_{ij} + r_{ij}$

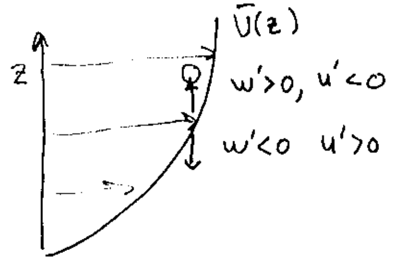
Reynolds Stress,

Closure Problem

Eddy Viscosities and Diffusivities

by "Analogy" $r_{ij}(x\,y\,z\,t) = A(x,y,z,t)\frac{\partial \overline{v}_i(x\,y\,z\,t)}{\partial x_j}$

New field functions



$\overline{u'w'}$ correlated and < 0

$-\overline{u'w'} > 0$, same sign as $\rho\nu\frac{\partial \overline{u}}{\partial z}$

A down gradient process.

Mixing lengths.

$$Ri = \frac{g}{\rho}\frac{\partial \rho}{\partial z} \Big/ \left(\frac{\partial u}{\partial z}\right)^2$$

Common Model

Vertical & Horizontal Eddy Viscosities & Diffusivities

Thus, for Eqns 4.1 & 4.2a

$$F_x = +\frac{\partial}{\partial x}\left(\nu_{EH}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_{EH}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_{EV}\frac{\partial u}{\partial z}\right)$$

$$F_y = \frac{\partial}{\partial x}\left(\nu_{EH}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_{EH}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_{EV}\frac{\partial v}{\partial z}\right)$$

and similarly $\frac{\partial}{\partial x}\left(K_{EH}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{EH}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{EV}\frac{\partial T}{\partial z}\right)$

Estimates for vertical coefficients range from 10^{-1}–10^{4} $\frac{cm^2}{sec}$ and horizontal 10^{5}–10^{8} $\frac{cm^2}{sec}$

Thus overwhelming the molecular processes.

EPS 200B

9 Ekman Sverdrup Flow

A SUBTROPICAL AND SUBPOLAR GYRE MODEL EXAMINING THE EKMAN, SVERDRUP AND GEOSTROPHIC TRANSPORT IN A HOMOGENEOUS OCEAN MODEL

Our mathematical model consists of the three fundamental (time-averaged) balance equations for large scale flow in the ocean:

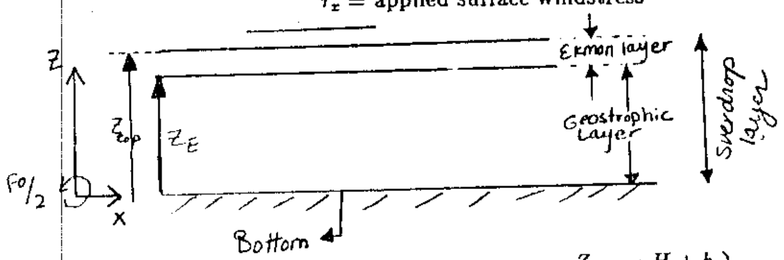
$$-(\nu U_z)_z - fV + P_x = 0 \qquad (1a)$$

$$-(\nu V_z)_z + fU + P_y = 0 \qquad (1b)$$

$$U_x + V_y + W_z = 0 \ , \qquad (1c)$$

The Physical Picture of the situation we are attempting to study is

τ_x = applied surface windstress

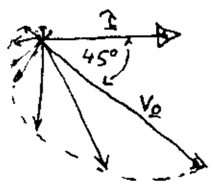


$$\left.\begin{aligned} Z_{top} &= H + h \\ Z_E &= H \end{aligned}\right\} \text{ and } h/H << 1$$

With $D_E = \left(\frac{2\nu}{f}\right)^{\frac{1}{2}}$

$$u = \frac{\tau^{(x)}H}{\sqrt{2}\,\nu} e^{z/} \sin\left(\frac{z}{D_E} + \frac{\pi}{4}\right)$$

$$v = \frac{\tau^{(x)}H}{\sqrt{2}\,\nu} e^{z/} \cos\left(\frac{z}{D_E} + \frac{\pi}{4}\right)$$



See below

ASSUMPTIONS

One of the primary quantities of interests is the vertical transport at the boundary of the Ekman region in terms of the applied surface stress τ_x and τ_y at the ocean surface. The vertical transport is obtained by integrating 1) with respect to z *within* the Ekman Layer and invoking the conditions

(i) $\tau_x = \tau_y = 0$ at $z = z_E$

(ii) $\frac{\partial P}{\partial x} \simeq 0, \frac{\partial P}{\partial y} \simeq 0$ *within* the Ekman Layer

(iii) The eddy stresses $\tau_x \equiv \nu U_z$ and $\tau_y \equiv \nu V_z$ and satisfy the conditions $\tau_x = \tau_y \cong 0$ at $z = z_E$

(iv) $W_{top} = 0$

A very important feature of this problem is that the latitudinal variation of the Coriolis parameter is necessary due to the large horizontal extent of the general circulation in the gyre regions. The variation in the Coriolis parameter can be adequately described by a linearly varying Coriolis parameter. This approximation combined with approximating the spherical geometry by a planar geometry (centered at a particular mid latitude) constitutes the so called β plane approximation for the general circulation in the ocean. Strictly speaking, this is not a good approximation since $(L/R)^2 \sim 0(1)$ and not a small quantity but for our illustrative purposes the planar approximation is sufficient.

EKMAN SOLUTION

Integrating 1) in the Ekman Layer and invoking the assumptions (i)–(iii) gives:

$$-\tau_x(x,y)\,|_{z=z_{top}} - fV_E = 0 \Rightarrow V_E = \frac{-1}{f(y)}\tau_x(x,y) \qquad (2a)$$

$$-\tau_y(x,y)\,|_{z=z_{top}} + fU_E = 0 \Rightarrow U_E = \frac{1}{f(y)}\tau_y(x,y) \qquad (2b)$$

$$\frac{\partial U_E}{\partial x} + \frac{\partial V_E}{\partial y} + \underbrace{W_{top}}_{=0} - W_E = 0$$

where we have defined

$$(U_E, V_E) \equiv \int_{z_E}^{z_{top}} (U, V)\, dz \quad \text{and } W_E \equiv W \big|_{z=z_E},\ f(y) \doteq f_0 + \beta_0 (y - y_0)$$

Equation 2) can be simplified to obtain a statement conerning the vertical Ekman velocity:

$$W_E = \frac{\partial U_E}{\partial x} + \frac{\partial V_E}{\partial y} = \frac{\partial}{\partial x}\left(+\frac{1}{f}\tau_y\right) - \frac{\partial}{\partial y}\left(\frac{1}{f}\tau_x\right)$$

$$= \hat{k} \cdot \left(\vec{\nabla} \times \vec{\tau}/f\right) \tag{3}$$

In other words, the vertical Ekman velocity is equal to the vertical component of the vector curl of applied surface stress divided by the Coriolis parameter.

Application to a "double gyre":

$$\tau_y \cong 0, \tau_x = \tau_0 \cos\left(\frac{\pi y}{L}\right)$$

$$\Rightarrow V_E = -\frac{\tau_0}{f(y)} \cos\left(\frac{\pi y}{L}\right) \tag{2a}$$

$$\Rightarrow U_e = 0 \tag{2b}$$

$$\Rightarrow W_E = -\frac{\partial}{\partial y}\left[\tau_0/f \ \cos\left(\frac{\pi y}{L}\right)\right] \tag{2c}$$

$$= +\frac{T_0}{f} \sin\left(\frac{\pi y}{L}\right) \cdot \frac{\pi}{L} + \frac{T_0}{f^2} \beta \cos\left(\frac{\pi y}{L}\right)$$

$$= \frac{T_0 \pi}{f \cdot L}\left[\sin\left(\frac{\pi y}{L}\right) + \frac{\beta \cdot L}{f \cdot \pi} \cos\left(\frac{\pi y}{L}\right)\right]$$

Characteristic Values are $L = 5 \times 10^8$ cm, $\beta = 10^{-13} \frac{1}{\text{cm sec}}$ $f_0 = 10^{-4}$ sec^{-1} and $\tau_0 = 1 \frac{\text{cm}^2}{\text{sec}^2}$. Plugging in Values gives:

$$\frac{T_0 \pi}{f L} = \frac{1 \cdot 3}{10^{-4} \cdot 5 \cdot 10^8} = .6 \times 10^{-4} = 6 \times 10^{-5}\ \frac{\text{cm}}{\text{sec}}$$

$$\frac{\beta L}{f \pi} = \frac{10^{-13} \cdot 5 \cdot 10^8}{10^{-4} \cdot 3} = 1.6 \times 10^{-1} \sim .16 \sim .20 \quad \text{for practical purposes}$$

The above expressions become

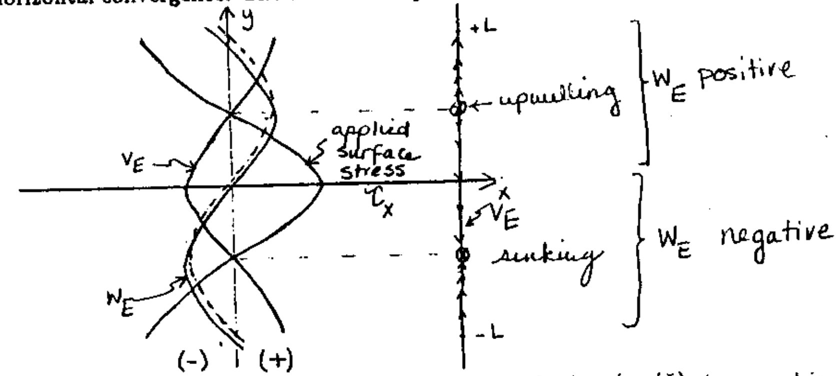
$$V_E = \frac{-10^4}{(1 + 10^{-7}(y - y_0)} \cos\left(\frac{\pi y}{L}\right) \frac{\text{cm}^2}{\text{sec}} \quad \text{(mass flux modulo the density } \rho_0)$$

$$U_E = 0$$

$$W_E = 6 \times 10^{-5}\left[\sin\left(\frac{\pi y}{L}\right) + .2\phi \cos\left(\frac{\pi y}{L}\right)\right] \text{ cm/sec}$$

Note:

Regions where $W_E > 0$ represent horizontal divergence and regions where $W_E < 0$ represent horizontal convergence. The Results are plotted below:



Note: W_E has a slight phase shift due to the additional $2\phi \cos(\pi y/L)$ term and is obtained from the dotted line by shifting the dotted curve downward a little bit.

II) GEOSTROPHIC SOLUTION

Up to this point we have discussed only the Ekman flow and not the interior geostrophic flow. To determine the Geostrophic flow we demand that the fundamental balance of forces be between the pressure gradient and Coriolis force discarding the small correction due to eddy stresses in the *interior.* Thus to first order in the Rossby number we obtain

$$-fV_g = -\frac{\partial P}{\partial x} \tag{4a}$$

$$fU_g = -\frac{\partial P}{\partial y} \quad \text{where } g \text{ is shorthand for geostrophic} \tag{4b}$$

$$\frac{\partial U_g}{\partial x}+\frac{\partial V_g}{\partial y}+\frac{\partial W_g}{\partial z}=0$$

$$\frac{\partial P}{\partial z}=0 \tag{4d}$$

The Geostrophic relation (4a,b) implies

$$\frac{\partial}{\partial x}(fU_g)+\frac{\partial}{\partial y}(fV_g)=0$$

$$\Rightarrow f\left(\frac{\partial U_g}{\partial x}+\frac{\partial V_g y}{\partial z}\right)+\beta V_g=0 \quad , \tag{4e}$$

since for the problem at hand f is not a constant. Mass conservation 4c) implies that $\frac{\partial U_g}{\partial x}+\frac{\partial V_g}{\partial y}=-\frac{\partial W_g}{\partial z}$ so (4e) reduces to

$$\beta V_g=f\frac{\partial W_g}{\partial z} \tag{4f}$$

Since $\frac{\partial P}{\partial z}=0 \Rightarrow P$ is independent of z and differentiating 4a) and 4b) with respect to z gives $\frac{\partial V_g}{\partial z}=\frac{\partial U_g}{\partial z}=0$, hence V_g and U_g are independent of z as well (Taylor Proudman Revisited again!). Differentiating 4f) with respect to z gives $fW_gzz=0 \Rightarrow W_g$ is linear in z! Neglecting the bottom friction layer and assuming $W=0$ at $z=0$ gives $W_g=(z/H)\ R(x,y)$, where R is an arbitrary function. *Matching* the solution with the Ekman solution at $z=H=z_E$ requires $R(x,y)=W_E$ so that

$$W_g=(z/H)\,W_E \tag{4g}$$

$$V_g=f/\beta\frac{W_E}{H}=\frac{\tau_0\cdot\pi}{\beta LH}\left[\sin\left(\frac{\pi y}{L}\right)+\frac{\beta L}{\pi f}\cos\left(\frac{\pi y}{L}\right)\right] \tag{4h}$$

To determine U_g we use the continuity equation and solve for U_g by quadrature:

$$\frac{\partial U_g}{\partial x}=-\frac{\partial V_g}{\partial y}-\frac{\partial W_g}{\partial z}$$

$$=-\frac{\tau_0\pi}{\beta LH}\frac{\pi}{L}\cos\left(\frac{\pi y}{L}\right)+\frac{\tau_0\pi}{\beta LH}\cdot\frac{\beta L}{\pi f}\frac{\pi}{L}\cdot\sin\left(\frac{\pi y}{L}\right)$$

$$-\frac{\tau_0\pi}{HfL}\sin\left(\frac{\pi y}{L}\right)-\frac{\tau_0\beta}{Hf^2}\cos\left(\frac{\pi y}{L}\right)$$

$$=-\cos\left(\frac{\pi y}{L}\right)\left[\frac{\tau_0}{\beta H}(\pi/L)^2+\frac{\tau_0\beta}{Hf^2}\right]$$

$$\Rightarrow\frac{\partial U_g}{\partial x}+\left(\frac{\tau_0}{\beta H}(\pi/L)^2+\frac{\tau_0\beta}{f^2H}\right)\cos\left(\frac{\pi y}{L}\right)=0$$

Integrating in x and imposing the condition that $U=0$ at $x=L$ gives:

$$U_g=(L-x)\left\{\frac{\tau_0}{\beta H}(\pi/L)^2+\frac{\tau_0\beta}{f^2H}\right\}\cos\left(\frac{\pi y}{L}\right) \tag{5}$$

III SVERDRUP SOLUTION

The final aspect of this exercise is to relate the Ekman and geostrophic solutions to the Sverdrup solutions. The Sverdrup Solution is commonly referred to as the "total transport solution" since it is obtained by vertically integrating the fundamental balance equations 1a)–1c) from $z=0$ to $z=z_{\text{top}}$. In this integration we cannot neglect the horizontal pressure gradients since they are a primary field quantitity in the geostrophic region: The Geostrophic region accounts for the major portion of the ocean in the vertical and neglecting the pressure gradients in the vertical averaging procedure would lead to serious errors.

Integrating Equations 1) in the vertical gives the Sverdrup Relations

$$-\tau_x-fV_s+P_x=0 \tag{6a}$$

$$0+fU_s+P_y=0 \quad (\text{assuming } \tau_y<<\tau_x) \tag{6b}$$

$$\frac{\partial U_s}{\partial x}+\frac{\partial V_s}{\partial y}=0 \quad \text{since } W=0 \text{ at both } z=0 \text{ and } z=z_{\text{top}} \tag{6c}$$

Cross differentiating 6a) and 6b) to ***eliminate*** the vertically averaged Pressure yields:

$$+\frac{\partial}{\partial y}\tau_x+\beta V_s+\underbrace{f\frac{\partial V_s}{\partial y}+f\frac{\partial U_s}{\partial x}}_{\text{zero by 6c)}}=0$$

$$\Rightarrow V_s=\frac{-1}{\beta}\frac{\partial}{\partial y}\tau_x \quad . \tag{7}$$

This result implies that the zeros of $\frac{\partial}{\partial y}\tau_x$ in the meridional coordinate correspond to the gyre boundaries since the net vertically averaged northward transports are zero on these meridians.

Substituting our expression for τ_x into (7) gives:

$$V_s = -\frac{1}{\beta}\frac{\partial}{\partial y}\left(\tau_0 \cos\frac{\pi y}{L}\right) = +\frac{\tau_0 \pi}{\beta L}\sin\left(\frac{\pi y}{L}\right) \quad ,$$

so that for $y > 0 \Rightarrow V_s > 0$ and $y < 0 \Rightarrow V_s < 0$

We anticipate that the Sverdrup meridional mass transport is equivalent to the sum of the geostrophic and Ekman meridional mass transports. To show this explicitly we note that V_g is independent of z so that $V_g = \int_0^{z_s} v_g \, dz = HV_g$ therefore

$$V_g + V_E = H \cdot V_g + V_E = \frac{H\tau_0\pi}{\beta L H}\left[\sin\frac{\pi y}{L} + \frac{\beta L}{\pi f}\cos\frac{\pi y}{L}\right] - \frac{\tau_0}{f}\cos\left(\frac{\pi y}{L}\right)$$

$$= \frac{\tau_0 \pi}{\beta L}\sin(\pi y/L) + \underbrace{\frac{\tau_0}{f}W(\pi y/L) - \frac{\tau_0}{f}\cos(\pi y/L)}_{\text{zero!}}$$

$$= V_S$$

10. Potential Vorticity

A generalization of *vorticity conservation*: Simplest example is 2-d homogeneous fluid

$$u_t + uu_x + vu_y + P_x = 0$$

$$v_t + uv_x + vv_y + P_y = 0$$

$$u_x + v_y = 0$$

Cross-differentiation of momentum equations:

$$\frac{\partial}{\partial t}(v_x - u_y) + u(v_x - u_y)_x + v(v_x - u_y)_y + \underbrace{(u_x + v_y)}_{0}(v_x - u_y) = 0 \quad \text{or}$$

$$\frac{D}{Dt}\zeta = 0 \qquad \zeta \equiv v_x - u_y = \nabla^2\psi \quad \text{vorticity}$$

where $v = \psi_x$, $u = -\psi_y$. If $\frac{\partial}{\partial t} = 0$, $u \cdot \nabla\zeta = J(\psi, \nabla^2\psi) = 0$ and $\nabla^2\psi = F(\psi)$ is a first integral.

There are *many* extensions, generalizations and special cases in GFD useful for rotating, stratified, thin spherical shell, etc. fluid situations (not so obviously related to each other and not all in same units).

Generally are variants of *ERTELS Theorem*: *if*

$$\frac{D\mathbf{v}}{Dt} + 2\mathbf{\Omega} \times \mathbf{v} + \nabla\Phi + \frac{1}{\rho}\nabla P = 0 \qquad \text{(3-d Euler equations)} \quad \text{and}$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0$$

AND "s" is a *conserved scalar*, i.e. $\frac{Ds}{Dt} = 0$.

Then

$$\boxed{\frac{D}{Dt}\left[\frac{(2\mathbf{\Omega} + \nabla \times \mathbf{v})}{\rho} \cdot \nabla \mathbf{s}\right] = 0}$$

11. Shallow Water Approximation: Homogeneous, Hydrostatic Fluid Layer on Variable Depth β-Plane

$$\rho = \rho_0$$

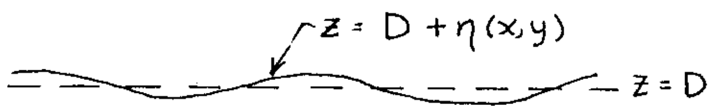
Equations are

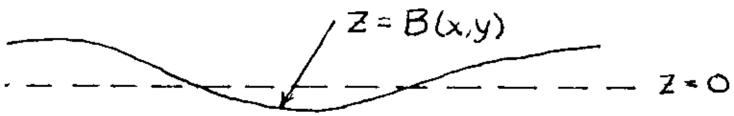
$$u_t + uu_x + vu_y + wu_z - (f_0 + \beta y)v + P_x = 0 \qquad (11.1)$$

$$v_t + uv_x + vv_y + wv_z + fu + P_y = 0 \qquad (11.2)$$

"shallowness' is basis of almost *hydrostatic* $\quad g\left(\frac{\rho_0}{\rho_0}\right) + P_z = 0 \qquad (11.3)$

$$u_x + v_y + w_z = 0 \qquad (11.4)$$





B.C. is $\quad P = 0 \quad$ at $\quad z = h(x, y, t) = D + \eta$

$$\text{and} \quad \eta_t + u\eta_x + v\eta_y - w = 0 \quad \text{at} \quad z = h \qquad (11.5)$$

$$\text{and} \quad uB_x + vB_y - w = 0 \quad \text{at} \quad z = B$$

Then

$$P = g[\eta + D - z]\,, \qquad P_x = g\eta_x\,, \qquad \text{etc.}$$

w is a *linear* fit of z and $u_z = v_z \equiv 0$. Then cross-differentiate and eliminate

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1) = \frac{\partial}{\partial t}(v_x - u_y) + u\frac{\partial}{\partial x}(v_x - u_y)$$
$$+ v\frac{\partial}{\partial y}(v_x - y_y) + [(v_x - u_y) + f]u_x + v_y + \beta v = 0$$

$$\frac{D}{Dt}(\zeta + f) + \underbrace{(\zeta + f)(-w_z)}_{} = 0 \qquad (11.6)$$

the so-called STRETCHING TERM

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \qquad \zeta = v_x - u_y \ \ (= \hat{k} \cdot \nabla \times \mathbf{v})$$

Then integrate (11.6) and invoke (11.5)

$$H\frac{D}{Dt}(\zeta + f) - [w(h) - w(B)][\zeta + f] = \sigma$$

$$H\frac{Dq}{Dt} - \left[\frac{D\eta}{Dt} - \frac{DB}{Dt}\right] q = 0 \qquad (11.7)$$

or

$$H\frac{Dq}{Dt} - q\frac{DH}{Dt} = 0 \quad \Rightarrow \quad \underbrace{\frac{D}{Dt}\left(\frac{q}{H}\right) = 0}_{\text{Conservation of Potential Vorticity}}$$

when

$$q = \zeta \underbrace{+f}$$

(but "f_0"

is irrelevant

Explicitly

$$\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]\left[\frac{v_x - u_y + \beta y}{\eta + D - B}\right] = 0$$

where

$v_x - u_y =$ relative vorticity

$\beta y =$ planetary vorticity

$\eta =$ sea surface distribution (negligible)

$D =$ mean depth

$B =$ bottom topography

$y =$ planetary vorticity

i) There is a "quasigeostrophic version"

Introduce the approximation

$$V = \frac{P_x}{f_0} = \frac{g\eta_x}{f_0} \qquad u = -\frac{P_y}{f_0} \qquad (11.8)$$

and evaluate the full nonlinear expression

$$\left[\frac{\partial}{\partial t} + \frac{P_x}{f}\frac{\partial}{\partial y} - \frac{P_y}{f_0}\frac{\partial}{\partial x}\right]\left[\frac{\frac{1}{f_0}\nabla^2 P + f}{H}\right] = 0 \qquad (11.9)$$

A nonlinear prognostic equation for the geostrophic pressure

ii) Relative Vorticity is negligible

$\left(\frac{f(y)}{H(x,y)}\right)$ is conserved following the motion

. (geostrophic contour flow)

iii) Linearized Version. From (11.7)

$$H(\zeta_t + \beta v) - f[\eta_t - uB_x - vB_y] = 0 \qquad (11.10)$$

Or with (11.8): AND ASSUMING $B_x = 0$

$$\frac{\partial}{\partial t}\left[\nabla^2 - \frac{f_0^2}{gH}\right] P + \left[\beta + \frac{f_0}{H}B_y\right] P_x = 0 \qquad (11.11)$$

NOTE equivalency here of β and B_y --

B_y is positive if slope is upward to the North.

Let $\beta + \frac{f_0}{H}B_y \approx \tilde{\beta} \sim$ constant.

Free Waves:

If

$$P \sim e^{i\left[\frac{kx}{L} + \frac{\ell y}{L} - \sigma t\right]}$$

$$(11) \Rightarrow \sigma \left[\frac{k^2 + \ell^2}{L^2} + \frac{f_0^2}{gH}\right] + \beta \frac{k}{L} = 0 \qquad (11.12)$$

or

$$\sigma = -(\beta L) \frac{k}{k^2 + \ell^2 + F}$$

$$F \equiv \frac{f_0^2 L^2}{gH} \equiv \frac{L^2}{R_e^2}$$ R_e is *external* (Rossby) deformation radius

A *horizontal* transverse wave with phase propagation to the *west* and with *short waves* slow and *long waves fast*!

Stratified Analogy

$$\sigma_n = \frac{(L\beta)k}{k^2+\ell^2+\lambda_n}$$ "nth mode

...

$$\lambda_n = \frac{f_0 L^2}{gH_n} = \frac{L^2}{R_n^2}$$

↑ equivalent depth ↖ internal deformation radius

12. Quasigeostrophic Approximation

We first sketch a derivation that emphasizes the physical basis of the quasigeostrophic approximation for continuously stratified fluids in the β-plane with the hydrostatic, thinness and Boussinesq approximation implied.

$$u_t + uu_x + vu_y + wu_z - (f_0 + \beta y)v + P_x = 0 \qquad (12.1)$$

$$v_t + uv_x + vv_y + wv_z + (f_0 + \beta y)u + P_y = 0 \qquad (12.2)$$

$$g\frac{\rho}{\rho_0} + P_z = 0 \qquad (12.3)'$$

$$\rho = \rho_0[1 - \underbrace{\alpha(T - T_0) + \beta(S - S_0)}_{\Theta(z) + \sigma(x,y,z,t)}]$$

$$(12.3)' \Rightarrow g(1 - \Theta(z)] + \mathcal{P}(z) = 0\ ;$$

and, with

$$P = \mathcal{P} + p \qquad P_x = p_x \qquad P_y = p_y$$

$$-g\sigma + p_z = 0 \qquad (12.3)$$

$$u_x + v_y + w_z = 0 \qquad (12.4)$$

$$\sigma_t + u\sigma_x + v\sigma_y + wS(z) + w\sigma_z = 0 \qquad (12.5)$$

$$S(z) \equiv \left(\frac{\partial \Theta}{\partial z}\right)_{\textbf{adriabatic}}$$

(12.1–5) are the starting point version of the ideal fluid, β-plane, primitive equations.

Synoptic/Mesoscale Motions are almost f_0 geostrophic, almost hydrostatic over almost horizontally nondivergent; i.e.,

$$-f_0 v_g + P_{gx} = 0 \qquad (12.1a)$$

$$f_0 u_g + P_{gy} = 0 \qquad (12.2a)$$

$$-g\sigma_g + P_{gz} = 0 \qquad (12.3a)$$

$$u_{gx} + v_{gy} = 0 \qquad (12.4a)$$

Cross-differentiate and combine (12.1a, 12.2a) to get

$$f_0(u_{gx} + v_{gy}) = 0 \qquad (12.6a)$$

which agree with (12.4a) but (geostrophic degeneracy) has *no* information content. It is a *trivial* vorticity statement.

What dynamics governs the evolution and distribution of the geostrophic pressure field, P_g?

Answer: It is the (small) geostrophic momentum balance, and (small) horizontal divergence. That gives a *non*trivial vorticity equation.

Now consider perturbations from geostrophy:

$$u = u_g + u_1 + \cdots \qquad P = P_g + P_1 + \cdots$$

$$w = 0 + w_1 + \cdots$$

first in the momentum equations

$$\begin{aligned} u_{gt} + u_{1t} + \cdots + (u_g + u_1 + \cdots)(u_{gx} + u_{1x} + \cdots) + \\ + v_g u_{gy} + \cdots + w_1(u_{gz} + u_{1z} + \cdots) - \\ - f_0(v_1 + \cdots) - \beta v_g + p_{1x} = 0 \end{aligned} \qquad (12.1b)'$$

where we have subtracted the geostrophic balance, (12.1a).

(12.1b)' is similar to (12.1a)'.

$$u_{1x} + v_{1y} + w_{1z} = 0 \qquad (12.4b)$$

The *largest* ageostrophic terms are the acceleration and advections by the *geostrophic* motion: Thus approximately by (12.1a)′ and (12.2b)′ become

$$u_{gt} + u_g u_{gx} + v_g u_{gy} - f_0 v_1 - \beta v_g + p_{1x} = 0 \qquad (12.1b)$$

$$v_{gt} + u_g u_{gx} + v_g v_{gy} + f_0 u_1 + \beta u_g + p_{1y} = 0 \qquad (12.2b)$$

Forming $\frac{\partial}{\partial x}$(12.2b) − $\frac{\partial}{\partial y}$(12.1b) *and* invoking (12.4a) we obtain

$$\underbrace{(v_{xy} - u_{yy})_t}_{\zeta} + u_g \zeta_x = v_g \zeta_y + \beta v_g + f_0(u_{1x} + v_{1y}) = 0$$

or

$$\frac{D_H}{Dt}\zeta + \beta v_g - f_0 w_{1z} = 0 \qquad (12.6b)$$

where

$$\zeta = v_{gx} - u_{gy} = \frac{1}{f_0}(P_{gxx} + P_{gyy})$$

and

$$\frac{D_H}{Dt} \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

Since

$$\frac{D_H}{Dt}(f) \equiv \beta v_g$$

we can also write (12.6b) in the form

$$\frac{D_H}{Dt}(\zeta + f) - f_0 w_{1z} = 0$$

Now, to find w_{1z} in terms of the g(eostrophic)-fields:

Consider the density equation

$$\sigma_{gt} + \sigma_{1t} + \ldots + (u_g + u_1 + \ldots)(\sigma_{gx} + \sigma_{1x} + \cdots) + v_g g \sigma_{gy} + \cdots$$
$$+ w_1 S + w_1(\sigma_{gz} + \cdots) = 0 \qquad (12.5b)'$$

The consistent approach is to neglect the vertical advection of $\sigma_g J$ by w_1 relative to the horizontal advection of σ_g by the geostrophic velocity —
BUT

the vertical advection of the basic stratification by w_1 acting on $S(z) \equiv (\frac{\partial\Theta}{\partial z})_{\mathbf{adiabatic}}$ *cann*[illegible] neglected

Thus, approximately:

$$\sigma_{gt} + u_g \sigma_{gx} + v_g \sigma_{gy} + w_1 S = 0 \qquad (12.5b)$$

or

$$w_1 = -\frac{1}{S(z)} \frac{D_H}{Dt} \sigma_g = -\frac{D_H}{Dt}\left(\frac{1}{S}\sigma_g\right)$$

[$N^2 \equiv gS$: N Brunt-Vaisala frequency]

Thus

$$w_{1z} = -\frac{D_H}{Dt}\left[\frac{\partial}{\partial z}\left(\frac{1}{S}\sigma_g\right)\right]$$

This has the alternative expression via (12.3a)

$$w_{1z} = -\frac{D_H}{Dt}\left[\frac{\partial}{\partial z}\frac{1}{gS}P_{gz}\right] = -\frac{D_H}{Dt}\left[\frac{\partial}{\partial z}\frac{1}{N^2}P_{gz}\right]$$

So (12.6b) becomes

$$\frac{D_H}{Dt}(\zeta + f) + f_0 \frac{D_H}{Dt}\left[\frac{\partial}{\partial z}\left[\frac{1}{N^2}P_{gz}\right]\right] = 0$$

$$\frac{D_H}{Dt}\left[\zeta + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}P_{gz}\right) + f\right] = 0$$

which is the conservation of Quasigeostrophic Potential Vorticity or (*Pseudo* Potential Vorticity).

Dropping the "g" subscript we state our final results:

$$\frac{D_H}{Dt}\left[\nabla_H^2 P + \frac{\partial}{\partial z}\left[\frac{f_0^2}{N^2(z)}\frac{\partial P}{\partial z}\right) + \beta y\right] = 0 \qquad (12.7a)$$

where

$$\frac{D_H}{Dt} \equiv \left[\frac{\partial}{\partial t} + \frac{P_x}{f_0}\frac{\partial}{\partial y} - \frac{P_y}{f_0}\frac{\partial}{\partial x}\right] \qquad (12.7b)$$

and

$$u \equiv -\frac{P_y}{f_0} \qquad v \equiv \frac{P_x}{f_0} \qquad \sigma \equiv \frac{P_z}{g} \qquad (12.7c)$$

$$w = -\frac{D_H}{Dt}\left(\frac{1}{N^2}P_z\right) \qquad (12.7d)$$

Thus we have a prognosis equation (12.7a) for the geostrophic pressure P, which acts as a single scalar from which *all fields* (12.7c), (12.7d) are derivable.

Formalize this derivation as follows:

L horizontal length ~ eddy size ~60 km

H vertical length ~ thermocline depth ~500 m

t_0 time scale

v_0 velocity ~ eddy velocity ~0.5 m/s

$$(x, y) = L(x', y'), \quad z = H'z, \quad t = t_0 t' \qquad (12.8a)$$

Density is split

$$\rho = \rho_0(1 - \Theta(z) - \sigma(x, y, z, t)) \qquad (12.9)$$

$\Theta(x)$: horizontal and time average of $(1 - \rho/\rho_0)$

σ: fluctuation

and scale:

$$\sigma = \frac{v_0 f_0 L}{gH}\sigma' \qquad (12.8b)$$

$$N^2(z) = N_0^2 N'^2(z) = N_0^2\left(-g\frac{d\Theta}{dz}\right)_{\mathbf{ad}} \qquad (12.8c)$$

N_0: characteristic Brunt-Vaisala frequency.

The total pressure is similarly split

$$p = g\rho_0\int_0^z(\Theta(z) - 1)dz + \rho_0\rho_0 v_0 L\psi' \qquad (12.8d)$$

with

$$\psi = v_0 L\psi' \qquad (12.8e)$$

is the streamfunction

$$(u, v, w) = v_0\left(u', v', \frac{L}{H}w'\right) \qquad (12.8f)$$

$$f = f_0\left(1 + \frac{1}{t_0 f_0}\beta y'\right); \quad \beta = Lt_0\frac{\partial f}{\partial y} \qquad (12.8g)$$

Boundary conditions:

$$\text{At surface} \quad z' = 0 \quad w' = 0 \quad \text{rigid line} \qquad (12.10a)$$

$$\text{At bottom} \quad z' = -H_0/H \quad w' = u'\nabla B' \quad \text{impermeable} \qquad (12.10b)$$

$$(z = -H_0 + B(x, y), \text{ Taylor expansion})$$

(12.10b) determines the strength of topography appropriate to QG expansion.

The QG expansion is usually made in terms of a basic Rossby number,

$$R_0 = V_0/(f_0 L) << 1$$

but we will adopt a somewhat more general approach which will reduce to the usual approximation when the time scale is advective. Here the QG expansion is made in terms of the ratio of the inertial time scale $1/f_0$ to a scaling time t_0:

$$\varepsilon' = 1/(t_0 f_0) \qquad (12.11a)$$

This ratio should be small for problems for which this model is formally valid. The scaling time will be defined in terms of

$$\alpha = t_0/t_a \qquad (12.11b)$$

the ratio of scaling time to the advective time scale $L/V_0 = t_a$, and

$$\beta = t_0/t_\beta \qquad (12.11c)$$

the ratio of the scaling time to the planetary wave time scale

$$t_\beta = 1/(\beta_0 L) \quad . \qquad (12.11d)$$

When α is unity the traditional expansion is regained. The strength of stratification is characterized by

$$\Gamma^2 = \frac{f_0^2}{N_0^2}\frac{H^2}{L^2} \qquad (12.11e)$$

Insert (12.8), using (12.9) and (12.11) into (12.1-5) dropping primes to set

$$\epsilon[u_t + \alpha(uu_x + vu_y + wu_z)] - (1+\epsilon\beta y)v + \psi_x = F_1 \ , \qquad (12.1c)$$

$$\epsilon[v_t + \alpha(uv_x + wv_z)] + (1+\epsilon\beta y)u + \psi_y = F_2 \ , \qquad (12.2c)$$

$$\sigma - \psi_z = 0 \ , \qquad (12.3c)$$

$$u_x + v_y + w_z = 0 \ , \qquad (12.4c)$$

$$\epsilon[\sigma_t + \alpha(u\sigma_x + v\sigma_y + w\sigma_z)] + \frac{1}{\Gamma^2\sigma}\, w = Q \quad . \qquad (12.5c)$$

The flow variables are represented by a perturbation expansion $\varphi = \varphi_0 + \epsilon\varphi_1 + \cdots$.

The zeroth-order equations are geostrophic

$$v_0 = \psi_{0x} \qquad (12.1(0))\ (12.1c_0)$$

$$-u_0 = \psi_{0y} \qquad (\text{~~12.2(0)~~}\ 12.2c_0)$$

$$\sigma_0 = \psi_{0z} \qquad (\text{~~12.3(0)~~}\ 12.3c_0)$$

$$u_{0x} + v_{0y} + w_{0z} = 0 \qquad (\text{~~12.4(0)~~}\ 12.4c_0)$$

$$\frac{1}{\sigma}\, w_0 = 0 \quad . \qquad (\text{~~12.5(0)~~}\ 12.5c_0)$$

The first-order equations are

$$u_{0t} + \alpha(u_0u_{0x} + v_0u_{0y}) - v_1 - \beta y v_0 = -\psi_{1y} \qquad (\text{~~12.1(1)~~}\ 12.1c_1)$$

$$u_{0t} + \alpha(u_0v_{0x}) + v_0v_{0y} + u_1 + \beta y u_0 = -\psi_{1y} \qquad (\text{~~12.2(1)~~}\ 12.2c_1)$$

$$\sigma_1 = \psi_{1z} \qquad (\text{~~12.3(1)~~}\ 12.3c_1)$$

$$u_{1x} + v_{1y} + w_{1z} = 0 \qquad (\text{~~12.4(1)~~}\ 12.4c_1)$$

$$\sigma_{0t} + \alpha[u_0\sigma_{0x} + v_0\sigma_{0y}] + \frac{N^2}{\Gamma^2}\, w_1 = 0 \quad . \qquad (\text{~~12.5(1)~~}\ 12.5c_1)$$

From the vertical momentum equation (12.~~3(0)~~ 3c_0), and the equation for conservation of density, (12.~~5(1)~~ 5c_1), we obtain an equation relating the vertical velocity to the change in the vertical gradient of the stream function, (dropping now the order subscript),

$$\Gamma^2\,\frac{D_H}{Dt}(\psi_z\sigma) = -w \qquad (12.12)$$

where as before

$$\frac{D_H}{Dt} = \frac{\partial(\cdot)}{\partial t} + \alpha J(\psi,\cdot) \ , \left(J(a,b) \equiv \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial b}{\partial y}\right)$$

referred to here as the thermal equation. Define the relative vorticity as

$$\xi = \psi_{xx} + \psi_{yy} \quad \text{and} \quad \xi = \psi_{xx} + \psi_{yy} + \Gamma^2(\sigma\psi_z)_z \ . \qquad (12.13)$$

Cross-differentiating (12.1(1)) and (12.2(1)) and using (12.4(1)) gives an equation for the evolution of the relative vorticity:

$$\frac{D}{Dt}\,\xi - w_z + \beta\psi_x = 0$$

or using (12.13)

$$\frac{D}{Dt}\,\xi + \beta\psi_x = 0 \ . \qquad (12.14)$$

Equation (12.14) may also be expressed as

$$\frac{D}{Dt}\,q = 0 \ ;$$

where q, the quasigeostrophic potential vorticity, equals the sum of the relative, thermal and planetary vorticities:

$$q = \nabla_h^2\psi + \Gamma^2(\sigma\psi_z)_z + (1 + \beta y) \ . \qquad (12.15)$$

When computing model diagnostics (12.14) will be expressed as

$$\underset{(T)}{\xi_t} + \underset{(A)}{\alpha J(\psi,\xi)} + \underset{(C)}{\Gamma^2(\sigma\psi_z)_{zt}} + \underset{(D)}{\Gamma^2\alpha J(\psi,(\sigma\psi_z)_z)} + \underset{(B)}{\beta\psi_x} = 0$$

where

(T) is the local rate of change of relative vorticity

(A) is the advection of relative vorticity

(C) is the local rate of change of thermal vorticity

(D) is the advection of thermal vorticity

(B) is the advection of planetary vorticity.

Together equations (12.13) and (12.14), along with (12.10) appropriate and lateral boundary conditions, form a complete set of equations for determining the evolution of the stream function and vorticity.

In open ocean models the horizontal boundary conditions are of the CFvN (Charney For von Neumann) type. The stream function is specified along all four boundaries and the vorticity is specified along inflow points. The boundary conditions at the top and bottom (12.10) are used to determine w in (12.12). The integration of (12.5c) given heat sources Q if any, provide ψ_z (12.3c). Vertical velocities imposed by Ekman pumping in the surface layer and bottom layer can be imposed replacing (12.10) with

$$\text{At surface} \qquad z' = 0 \quad w' = -w_E = -\text{curl}\left(\frac{\tau_{\text{WIND}}}{\rho_0 f}\right) \qquad (12.10\text{a})'$$

$$\text{At bottom} \qquad z' = -H_0/H \qquad w' = u'\nabla B' + E\xi \qquad (12.10\text{b})'$$

13. Energy Analyses, Baroclinic & Barotropic Instabilities

Consider Equations QG (1)-(5)

write
$$u = V + u', \quad v = v', \quad w = w'$$

$$\frac{\rho}{\rho_0} - 1 = \Theta(z) + \vartheta(y) + \sigma'$$

geostrophically
$$f\,\frac{\partial V}{\partial z} = -g\vartheta_y \,.$$

So
$$V = -\frac{g}{f}\int^z \vartheta(y)\,dz + V_0(y)\,.$$

Dropping the primes (and in the "spirit of the QG approximation")

$$u_t + Vu_x + vV_y - fv + p_x = 0\,, \qquad (13.1)$$

$$v_t + Vv_x + fu + p_y = 0\,, \qquad (13.2)$$

$$\sigma_t + V\sigma_x + v\varphi_y + wS = 0\,. \qquad (13.3)$$

Also

$$u_x + v_y + w_z = 0\,, \qquad (13.4)$$

$$-g\sigma + p_z = 0\,. \qquad (13.5)$$

Multiply (1) & (2) by u, v & add.

$$\underbrace{\frac{\partial}{\partial t}\frac{u^2+v^2}{2}}_{\equiv K} + V\frac{\partial}{\partial x}\frac{u^2+v^2}{2} + uvV_y + up_x + vp_y = 0\,.$$

And in the same vein, $-gw\sigma + wp_z = 0$. Adding we get the "pressure work" $(up)_x + (vp)_y + (wp)_z$. So, integrating over a closed domain

$$\langle\dot K\rangle + \langle uvV_y\rangle - \langle gw\sigma\rangle = 0\,. \qquad (13.6)$$

Multiply the bouyancy equation (13.3) by σ:

$$\sigma\sigma_t + V\sigma\sigma_x + v\sigma\vartheta_y + \sigma wS = 0\,.$$

For simplicty, assume $S \equiv$ constant, multiply by g and divide then by S

$$\underbrace{\frac{\partial}{\partial t}\frac{g\sigma^2}{2S}}_{\equiv \text{"}A\text{" available (gravitational) Potential Energy}} + \frac{V\partial}{\partial x}\frac{g\sigma^2}{2S} + \frac{g}{S}\,v\sigma\vartheta_y + g\sigma w = 0\,.$$

$\equiv$ "A" available (gravitational) Potential Energy

Now integrate over the closed domain

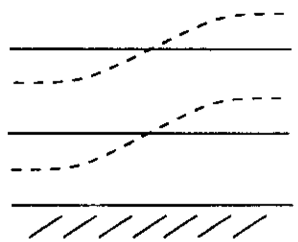
$$\langle\dot A\rangle + \langle\frac{g}{s}v\sigma\vartheta_y\rangle + \langle g\sigma w\rangle = 0\,. \qquad (13.7)$$

Interpretation $\langle -uvV_y\rangle$ Barotropic Instability makes "K" (better, K_E).

$\langle -\frac{g}{s}v\sigma\vartheta_y\rangle$ Baroclinic Instability makes "A" (better, A_E).

$\langle g\sigma w\rangle$ Buoyancy works, *converts* between K & KA.

NB: $V_y = \vartheta_y = 0$, 13.6 and 13.7.



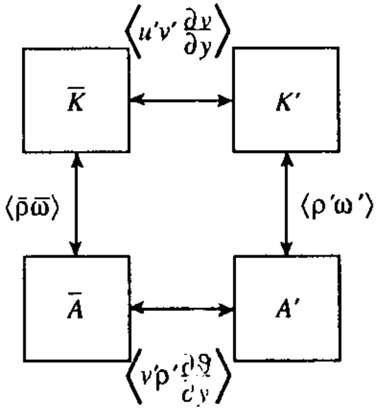
$$\langle\dot K\rangle + \langle\dot A\rangle = 0\,.$$

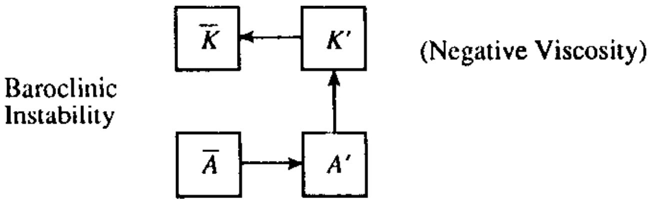
"level" the distorted density surfaces.

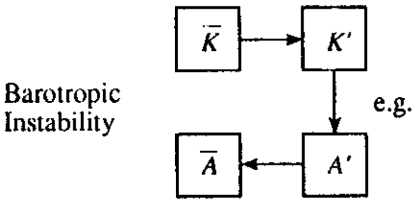
More generally "Mixed Instablility," finite amplitude version of the process, and

$$K = K_n + K_E$$

$$A = A_n + A_E$$







Most common is ***Mixed Instabilities*** (*Open* Domain Problems).