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DISCRETE AND CONTINUOUS-TIME MODELS OF CHAOTIC DYNAMICS IN ECONOMICS

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Continuous-time models of chaos in economics

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In this paper, we shall discuss some applications of chaos theory to the study of continuous-time economic dynamic models, i.e., models represented by systems of ordinary differential equations. Two such applications will be considered. The first, discussed in Part I of the paper, is a continuous-time generalization of a class of non-linear, one-dimensional maps which encompasses the majority of existing economic models of chaotic dynamics. The second, discussed in Part II, is a model of inventory cycles of Keynesian inspiration, represented by a system of three differential equations, including a single 'one-humped' non-linearity. Two points will be given particular emphasis, namely: in Part I, the relationship between discrete- and continuous-time representation of economic phenomena; in both Parts I and II, the combined role of lags and non-linearity in generating chaotic output.

PART I

1. Discrete and continuous representations of economic processes

The majority of applications of chaos theory to economics produced so far,[1] consist of variations of the one-dimensional difference equation

$$x_{t+T}=f(x_t), \tag{1}$$

where $x\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ is a smooth, unimodal (or 'one-humped') function. The potential complexity of the dynamical system represented by eq. (1) has

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[1]The list includes, among others, macroeconomic models [e.g., Stutzer (1980) and Day (1982)]; models of rational consumption [e.g., Benhabib and Day (1981)]; models of overlapping generations [e.g., Benhabib and Day (1982); Grandmont (1985)]; models of optimal growth [e.g., Deneckere and Pelikan (1986)]. Recent overviews of the matter, with further instances of one hump functions derived from economic problems can be found, in Baumol and Benhabib (1988) and Hans Walter Lorenz (1989).

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been brilliantly illustrated by the biologist R. May (1976) and its mathematical properties have been investigated in innumerable papers and books.[2] On the contrary, continuous-time models of chaos in economics are relatively rare. This fact and the attempt to find a rational explanation for it, have renewed the old controversy on the superiority, in general, of continuous versus discrete representations of reality. This has often led the participants in this debate to discuss the 'true' nature of time, or the continuity (or lack of it) of reality, whatever the latter concept may mean.

Undoubtedly, these are very profound and unsettled philosophical questions. We believe, however, that, as far as economics is concerned, the argument over them has often been ill-posed and, in most cases of practical interest, unnecessary. The essential question here is what are the mechanisms responsible for dynamic complexity, whether these mechanisms are present in certain specific economic circumstances, and how they can be more effectively represented. The answer to these questions, and especially to the last one, cannot be given a priori, once and for all. As many examples from other scientific disciplines show, the working of the same mechanism may often be investigated by both a continuous and a discrete dynamical system, and a number of illuminating relations can often be established between the two representations.

As this author has observed elsewhere,[3] the occurrence of chaotic dynamics in economic models derived from the May equation is the combined result of the presence of non-linearities of the unimodal, 'one-hump' type and of the lag structure implicit in those models. Such non-linearities have in the works in question been given more or less solid economic justifications, based on explicitly-stated hypotheses concerning, for example, technology or utility functions. On the contrary, the rôle of the lag structure in producing complex behaviour of the system has been systematically neglected.

A more careful investigation of this question indicates that aggregate, discrete-time models like those quoted above suffer from a fundamental weakness as they imply a crude form of dynamic aggregation with no economic underpinning. In fact, such an aggregate representation of an economic dynamic process could only be justified if some 'natural period' could be postulated, based on the technological or psychological characterizations of the economy. A moment's consideration will indicate that this extreme simplification would only hold under rather unrealistic or uninteresting conditions, e.g., in a single-crop agricultural economy in which there is a constant, uniform delay between input and output.

[2]Thorough treatments of this subject together with a rich bibliography can be found, for example, in Collet and Eckmann (1980) and Devaney (1986).

[3]See Invernizzi and Medio (1991) and Medio (1991), to which we make constant reference for the rest of this section.

If this unwarranted assumption is dropped, we have to recognize that different economic agents or units behave differently, and, in particular, they react with different speeds to economic stimuli. If so, we must wonder whether the interesting results obtained in the analysis of the aggregate models in question and in particular the occurrence of chaotic dynamics for certain values of the controlling parameter, still hold under more general and realistic assumptions.

2. Probability distribution of lags and continuous-time lag operators

To answer the question raised in the previous section, instead of a single 'representative' economic agent (or unit), we postulate an economy consisting of an indefinitely large number of agents, who respond to a certain signal with given discrete lags. The lengths of the lags are different for different agents, and are distributed in a random manner over all the population. In this situation, the economy's aggregate time of reaction to the signal can be modelled as a real, non-negative random variable T.

On a purely a priori ground, the probability distribution of T can take any of many different forms. However, when the only priori constraints, besides the essential positivity of the random variable, are given by the mean and a second indicator such as the geometric mean (or, equivalently, the mean of the logarithm of the random variable), there exist good reasons for choosing a two-parameter gamma distribution.[4] As is well-known, if we denote by α the shape parameter and by β the scale parameter, the density function of a gamma distributed random variable can be written as

$$g(\alpha,\beta;t)=\begin{cases}0, & \text{if } t\leqq 0,\\ \{\beta^{\alpha}\Gamma(\alpha)\}^{-1}t^{\alpha-1}\,\mathrm{e}^{-t/\beta}, & \text{if } t>0,\end{cases} \tag{2}$$

where $\Gamma(\alpha)$ indicate the gamma function, i.e.,

$$\Gamma(\alpha)=\int_0^{\infty} t^{\alpha-1}\,\mathrm{e}^{-t}\,\mathrm{d}t,$$

and $\alpha,\beta>0$. The mean of the distribution is equal to $\alpha\beta$, its geometric mean to $\beta\exp\{\Gamma'(\alpha)/\Gamma(\alpha)\}$, and its variance to $\beta^2\alpha$. Notice that, for a two-parameter gamma distribution, fixing the mean and the geometric mean is equivalent to fixing the mean and the variance.

[4]Cf. Invernizzi and Medio (op. cit).

Keeping in mind these considerations, we shall now look at the question of lags from a different angle and shall recall certain basic properties of lag operators and their weighting functions. Broadly speaking, a lag is a mechanism relating the value of a certain (possibly vector-valued) variable at a given time to the values of the same or another variable at different (past) times. This relationship is sometimes referred to as the 'memory' of the system. Under certain conditions, it is possible to represent the lag structure in the form of a polynomial in D, $D=\mathrm{d}/\mathrm{d}t$ being the derivative with respect to time, or equivalently in the form of an ordinary differential equation whose order depends on the degree of the polynomial. A *multiple exponential lag* (*MEL*) of order n can be written as

$$G(D)=(D/\tau n+1)^n.$$

The time-constant τ can be interpreted as the overall speed of adjustment associated with the lag, its reciprocal $1/\tau$ being therefore the overall length of the lag, whereas n denotes its order.

A MEL of order n results from n successive applications of a simple exponential lag. If a variable $Y(t)$ lags behind another variable $Z(t)$ (which may well be the same variable at a different time) according to a MEL, we shall denote this dynamic relationship by the expression

$$G(D)\,Y(t)=Z(t), \tag{3}$$

which, mathematically, is an nth order differential equation. In the economic literature the case most often encountered is the MEL of order one.[5] If we put n equal to one, eq. (3) can be rewritten as

$$(\mathrm{d}/\mathrm{d}t)\,Y(t)=\tau[Z(t)-Y(t)]. \tag{4}$$

In economics, the variable Z sometimes represents the desired, or equilibrium value of Y, defined in another part of the model, so that eq. (4) depicts an adaptive mechanism of an 'Achilles and the Tortoise' kind, through which the actual magnitude *chases* the desired one, approaching it at an exponentially slowing speed, and catching up with it only in the limit for $t\to+\infty$.

The 'weighting function' of the MEL (3) represents the different impact that different values of Z in the more or less distant past have on the value of Y now. It is known[6] that, for a lag represented by a polynomial in $D=\mathrm{d}/\mathrm{d}t$ such as $G(D)$, the weighting function can be simply found by calculating its inverse Laplace Transform. In particular, if we denote the weighting function of MEL by $w(t)$, we shall have

$$w(t)=L^{-1}G(D)=\left(\frac{1}{\tau n}\right)^n\frac{t^{n-1}}{(n-1)!}\,\mathrm{e}^{-t\tau n}, \tag{5}$$

where L^{-1} indicates the inverse Laplace Transform.

If we now put $\alpha=n$ and $\beta=1/\tau n$ and compare eqs. (2) and (5), we can verify that the weighting function of a continuous-time, multiple exponential lag is the same as the density function of a two-parameter gamma distribution. The overall length of the lag $1/\tau$ (i.e., the reciprocal of the overall speed of adjustment) corresponds to the mean of the distribution, while the variance is proportional to the inverse of the order of the lag n (the proportionality factor being equal to $1/\tau^2$).

This result shows that a fundamental equivalence can be established between a system with an indefinitely large number of agents reacting to inputs with different *discrete* lags whose lengths are randomly distributed among agents according to a gamma distribution, and a system with one single 'representative' agent reacting to inputs with a continuous, multiple exponential lag. In particular, the order of the lag n can be interpreted as the 'degree of homogeneity' of the system in question, insofar as agents' speed of response is concerned.

Fig. 1 shows the weighting functions of different MELs as functions of n. It will be observed that, as the order of the lag – or, equivalently, the degree of homogeneity of the system – increases, the weighting function becomes progressively steeper. It can be shown that in the limit for $n\to\infty$, the function tends to a Dirac-δ function situated at $t=1/\tau$. This means that the current value of output only depends on the value of input $1/\tau$ time units ago. This interesting limit case can also be dealt with more formally, by considering that

$$\lim_{n\to\infty}\left(\frac{D}{\tau n}+1\right)^{-n}=\mathrm{e}^{-D/\tau}. \tag{6}$$

The expression on the R.H.S. of eq. (6) is called *shift operator* and, when applied to a continuous function of time, it has the effect of translating the entire function forward in time by an interval equal to $1/\tau$.[7] Thus, the fixed delay lag employed in models of the type (1) can be seen as a special, limit case of a multiple exponential lag when the order of the lag tends to infinity. Equivalently, the aggregate fixed delay of those models can be viewed as a

[5]MELs of orders two and three have been discussed by Allen (1967), but MELs of higher orders are rarely encountered.

[6]Cf., for example, Doetsch (1971, Ch. 3).

[7]Cf. Yosida (1984, Part III, Ch. VIII).

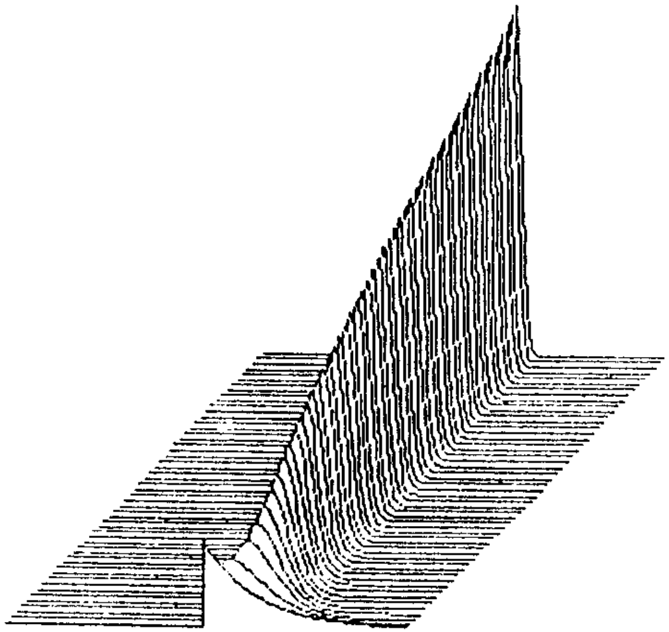


Fig. 1. The density function of a gamma distribution [from Invernizzi and Medio (1991)].

limit case of a system characterized by gamma distributed individual reaction times, which is obtained when the dispersion around the mean (the variance) tends to zero.

3. A continuous-time generalization of fixed delays

Since the hypothesis of an economy which is absolutely homogeneous with respect to agents' speed of reaction is usually unacceptable, the question arises of how general is the main result obtained in the economic applications of eq. (1), namely, the occurrence of chaotic dynamics for certain values of the controlling parameter. In the present context, it is immaterial which of the particular applications quoted above is selected, since the aspect we wish to criticize and amend is common to them all.

In order to answer that question, we shall replace the fixed delay lag implicit in (1) with a MEL. That is to say: We shall replace the discrete-time dynamical system (1) with its continuous-time generalization

$$(D/n+1)^n x = f(x), \tag{7}$$

where, for simplicity's sake, we have put $\tau = 1$, and shall investigate its

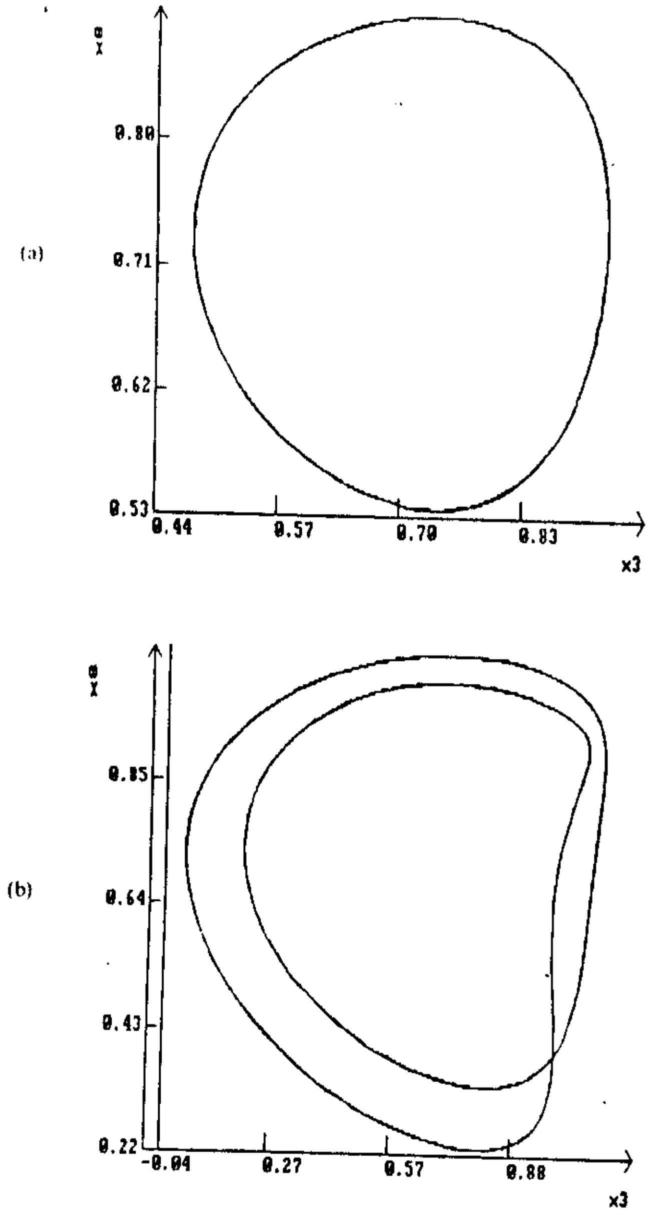
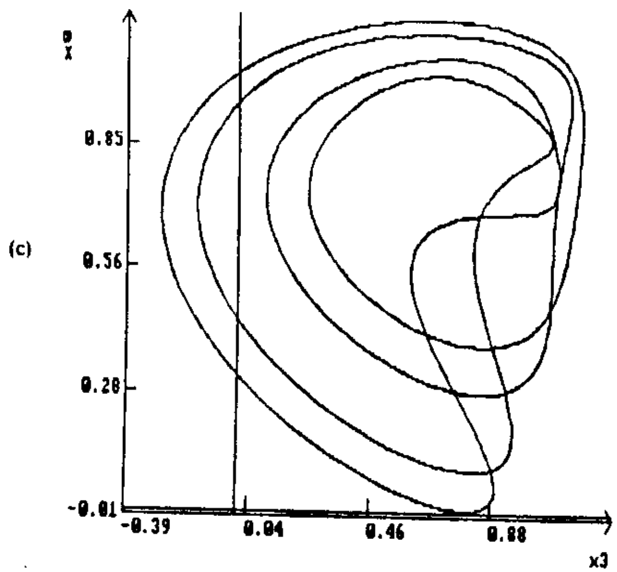


Fig. 2(a)–(b). Final orbits of system (8-9) for increasing values of r.



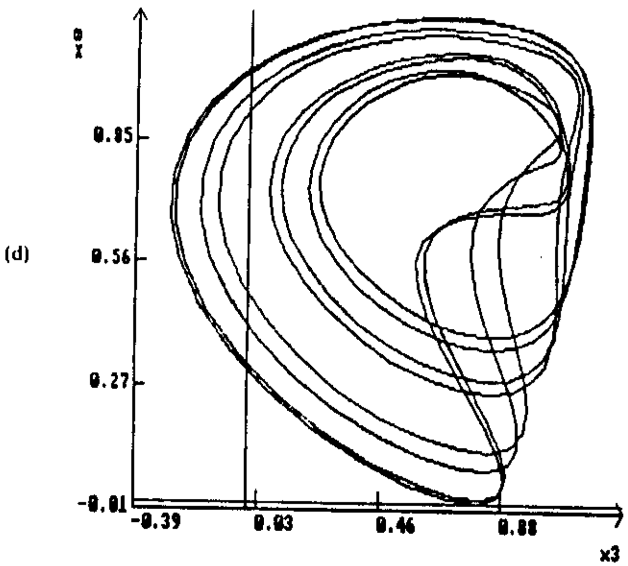


Fig. 2(c)–(d). Final orbits of system (8-9) for increasing values of r.

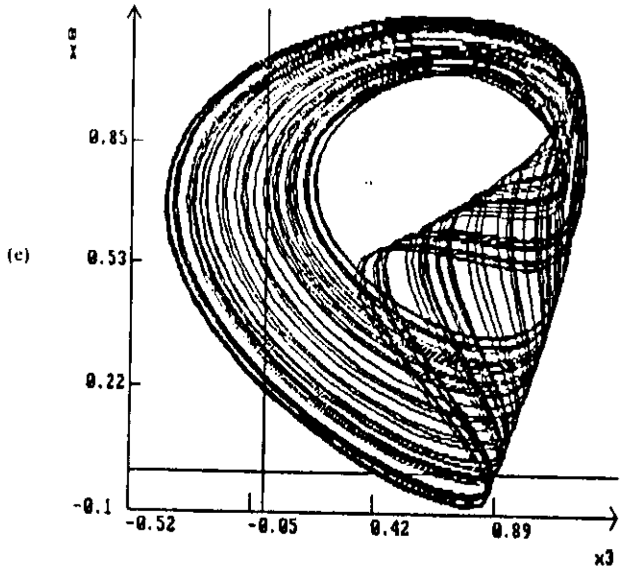


Fig. 2(e). Final orbits of system (8-9) for increasing values of r.

behaviour in the general case $1 \leqq n < \infty$. To make things more specific, we shall henceforth put

$$f(x) = rx(1-x).$$

Consider, first of all, that the nth order differential equation (7) can be equivalently written in the form of a system of n first-order differential equations, thus:

$$(D/n+1)x_j = x_{j-1}, \quad j=2,\ldots,n, \tag{8}$$

$$(D/n+1)x_1 = f(x_n). \tag{9}$$

We shall recall here certain results of this investigation already reported elsewhere,[8] and shall add some new ones.

The equilibrium conditions of system (8)–(9) are

$$x_1 = x_2 = \cdots = x_n = \bar{x},$$

and

$$\bar{x} = r\bar{x}(1-\bar{x}).$$

[8] Cf. Sparrow (1980); Medio (1991).

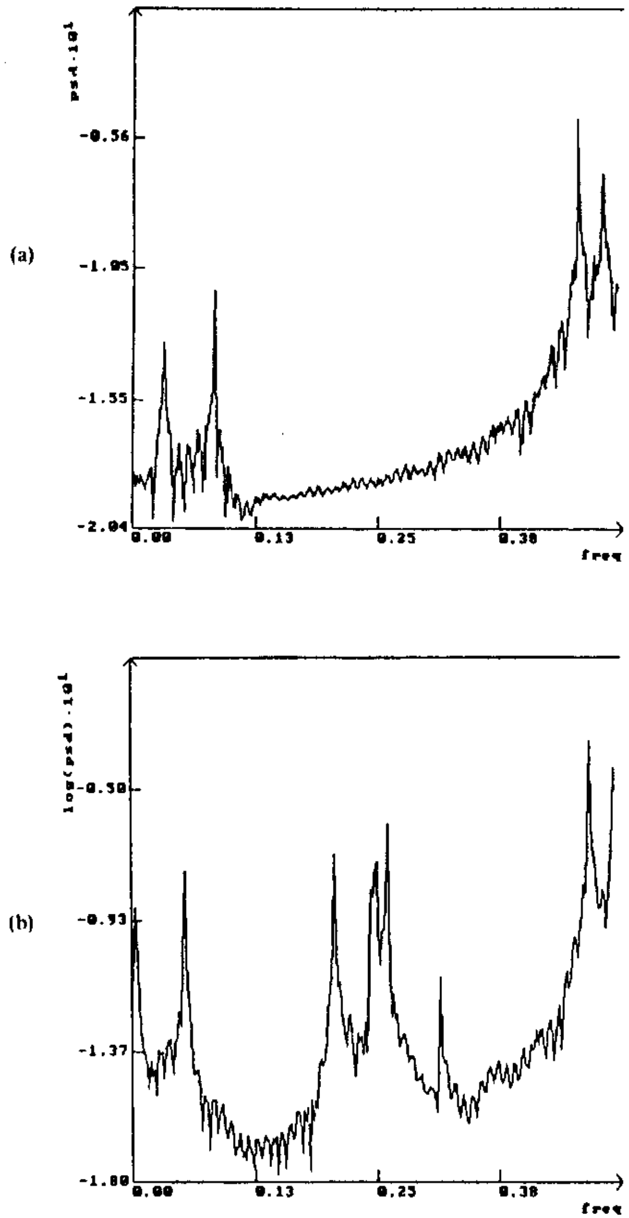


Fig. 3(a)–(b). Power spectral density of trajectories of system (8-9) for increasing values of r.

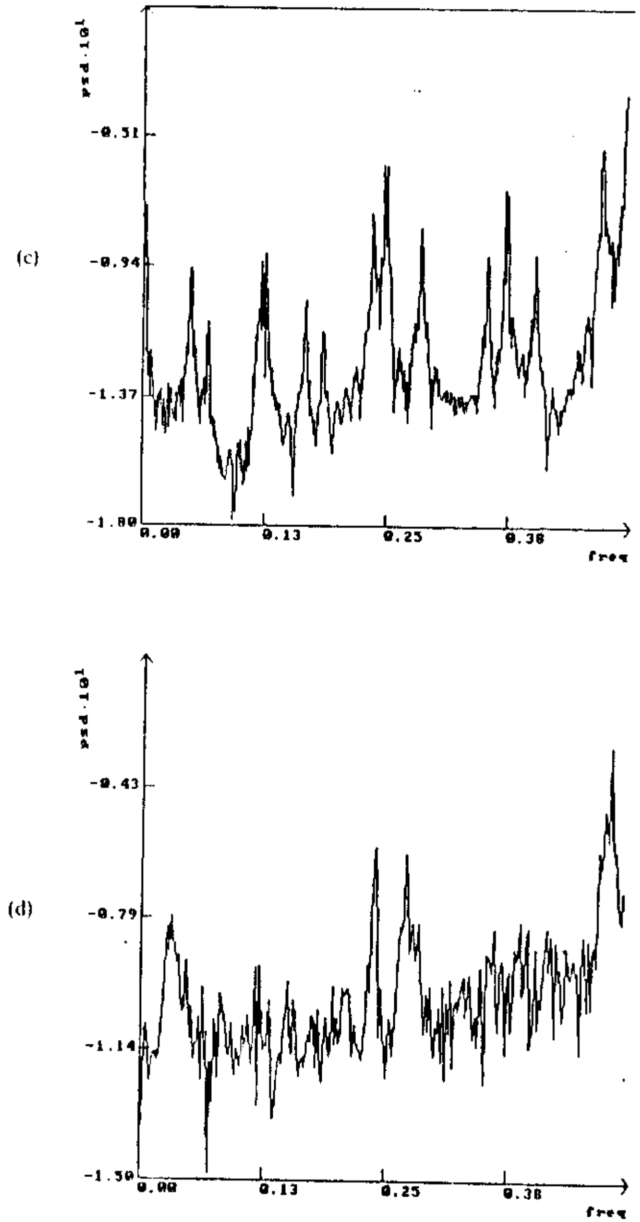


Fig. 3(c)–(d). Power spectral density of trajectories of system (8-9) for increasing values of r.

(e)

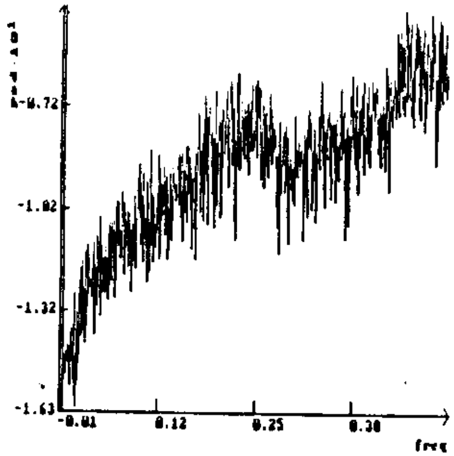


Fig. 3(e). Power spectral density of trajectories of system (8-9) for increasing values of r.

whence we obtain the two equilibrium solutions:

$$\bar{x}_1 = 0; \qquad \bar{x}_2 = 1 - (1/r). \tag{10}$$

As concerns the stability of equilibria, consider that the auxiliary equation of the system can be written in the simple form:

$$(1 + \lambda/n)^n = f'(\bar{x}), \tag{11}$$

where λ indicates an eigenvalue of the Jacobian matrix calculated at equilibrium and $f'(\bar{x}) = \mathrm{d}f(\bar{x})/\mathrm{d}x = r(1 - 2\bar{x})$.

Hence we have:

$$f'(\bar{x}_1) = r; \qquad f'(\bar{x}_2) = 2 - r. \tag{12}$$

From (10)–(12) we gather that, for $r < 1$, the origin is the only non-negative equilibrium point, and it is stable. At $r = 1$, we have a transcritical bifurcation: the equilibrium point at the origin loses its stability and a second, initially stable, equilibrium point bifurcates from it in the positive orthant of the phase space.

For $n < 3$, little happens when we increase the parameter r: the positive equilibrium point remains stable for all values of $r > 1$ (for $n = 2$, damped oscillations occur for $r > 2$). For $n \geq 3$, however, as r increases past a certain value which depends on n, a Hopf bifurcation takes place and a periodic orbit bifurcates from the stable equilibrium point that becomes unstable. Successive bifurcations can be detected for greater values of r, although their

(4)

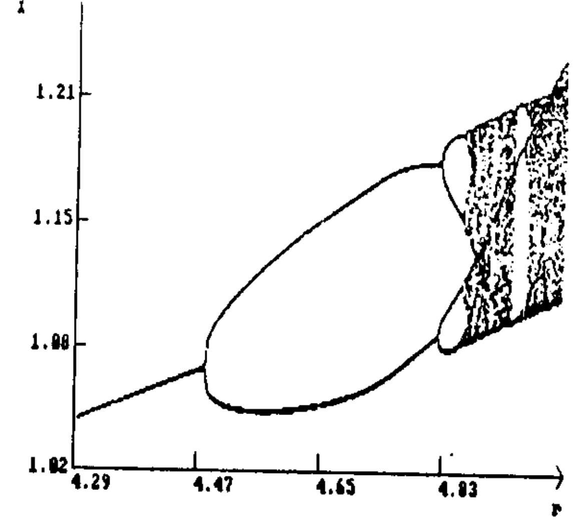


Fig. 4. Bifurcations diagram of system (8-9).

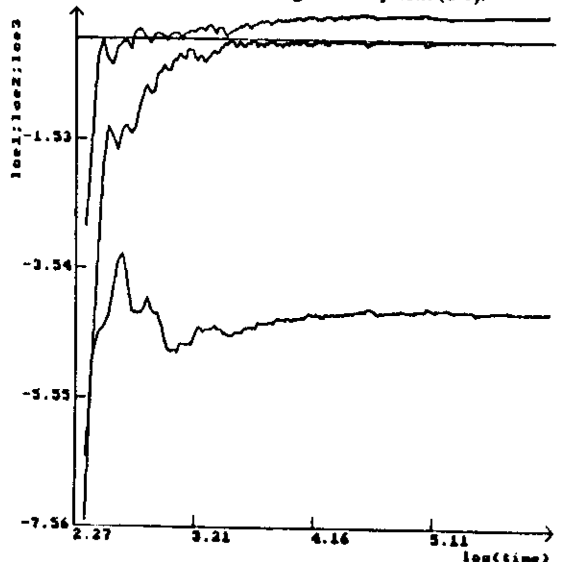
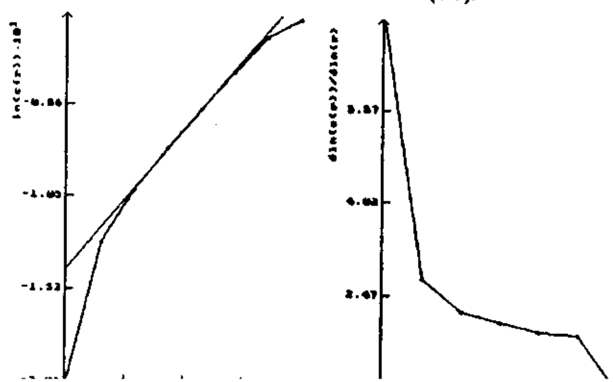


Fig. 5. The first three LCEs of (8-9).



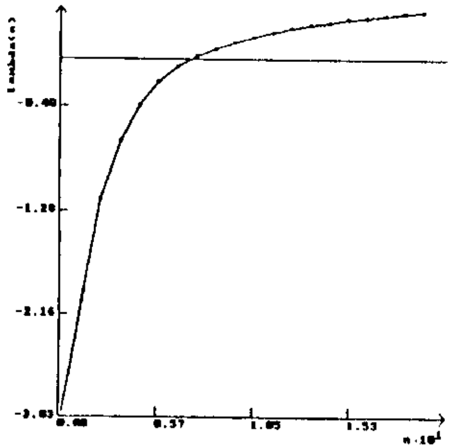


Fig. 7. The function $\Lambda(n)$.

(a)

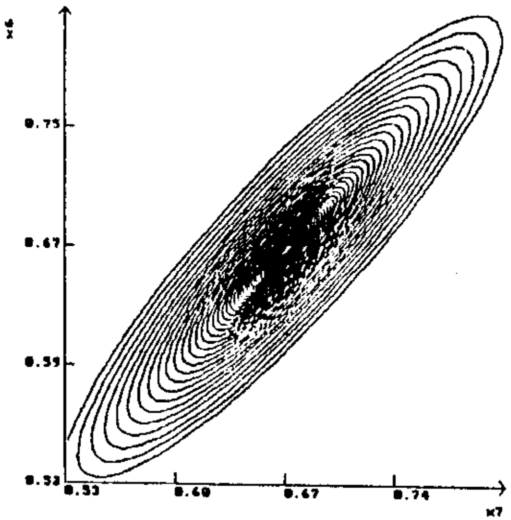


Fig. 8(a). Orbits of system (16) for $n=7$.

(b)

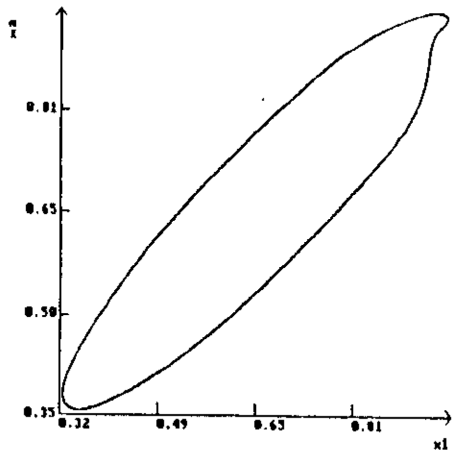


Fig. 8(b). Orbits of system (16) for $n=8$.

(c)

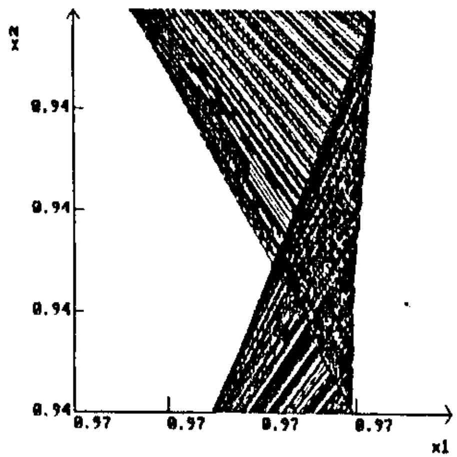


Fig. 8(c). Orbits of system (16) for $n=8$ (enlargement).

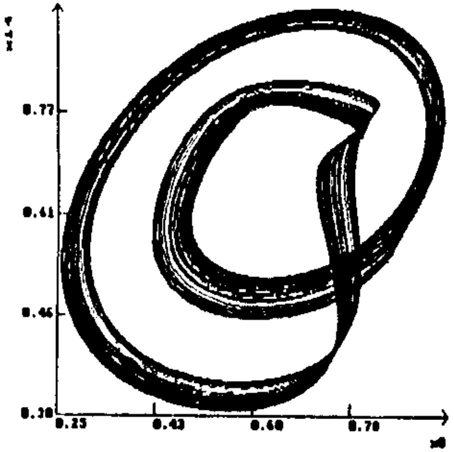


Fig. 9. Two-band chaotic attractor for $n = 15$.

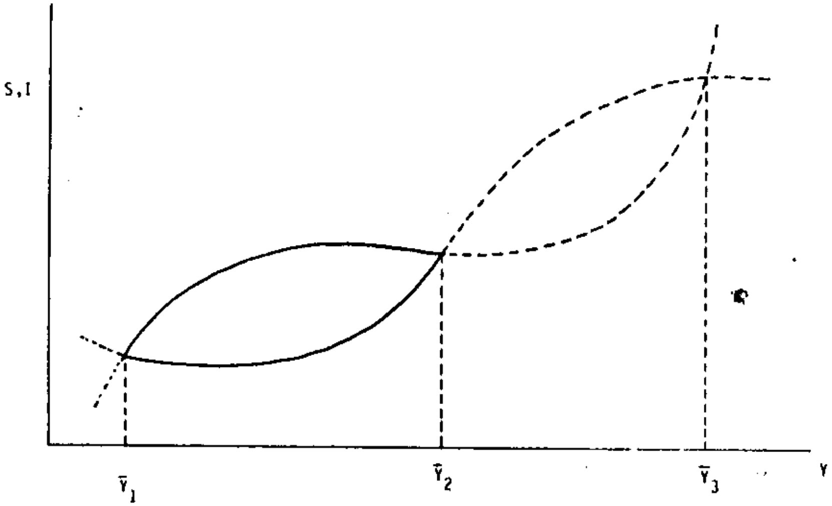


Fig. 10. Kaldorian saving and investment functions.

exact structure still escapes us. Whatever value the parameter r may take, however, nothing more complicated than periodic orbits seem to occur for low values of n. However, when the order of the exponential lag becomes sufficiently large, there does appear to exist a value of r beyond which the system gives chaotic output.

4. Numerical investigation of chaotic behaviour

At present, a complete analytical understanding of non-linear dynamical

systems is restricted to a very small number of special cases. To further pursue our investigation we shall therefore have to make recourse to numerical simulation.

In order to analyze this case in detail, we select $n = 10$.

In this case, the first Hopf bifurcation occurs for $r \approx 3.6$ and our numerical study indicates that, when the control parameter r is increased further, the system undergoes a period-doubling sequence of bifurcations, eventually exhibiting chaotic behaviour. This can be appreciated by looking at fig. 2, which shows plots of post-transient trajectories of the system for different values of r. The reader will observe the progression of complexity from a limit cycle to period 2, 4, 8 cycles, and finally, for $r \approx 5$, to a chaotic attractor. This sequence can also be detected by means of spectral analysis of the same trajectories, as illustrated by fig. 3. Starting from fig. 3(a) we observe a basic frequency, call it f_0, (plus a number of harmonics). Then, increasing the parameter, we have the appearance in the spectrum of frequencies $f_0/2, f_0/4, f_0/8$. For even higher values of r, a 'noise floor' associated with chaotic behaviour appears and successively destroys the subharmonics $f_0/2^n$ in the opposite order of their appearance. In the last diagram of fig. 3(e), corresponding to the chaotic attractor of fig. 2(e), no distinct 'peak' is left in the spectrum, which looks like a broad band.

Another interesting representation of the period-doubling route to chaos can be observed in the bifurcation diagram of fig. 4.[9] The familiar period-doubling scenario is quite evident as is the presence of periodic windows within the chaotic zone, which makes the diagram very similar to that obtained for the one-dimensional map (1), and reproduced in innumerable works.

Finally, numerical evidence of (and quantitative information about) chaotic behaviour of the system under investigation has been provided by the estimate of the LCEs and the fractal dimension, calculated for $r = 5$, i.e., within the chaotic zone. As is known, the existence of an attractor with one or more positive LCE indicates that the motion of the system on the attractor has sensitive dependence on initial conditions, i.e., it is chaotic. Since in our case the attractor looks approximately two-dimensional, we would expect the sign pattern of the ten LCEs to be $(+, 0, -, \ldots, -)$.

And this is precisely what we get from our computations. The essential results are shown by fig. 5, which shows the first three LCEs (the other seven are all strongly negative). Notice that the convergence is pretty strong and the exponent is distinctly positive (≈ 0.35). The interpretation is that the motion of the system is strongly convergent toward the attractor from all directions but two. The zero Lyapunov exponent is associated with the

[9]To construct this diagram (and the similar diagrams of Part II) we have plotted, for each value of the parameter r (from 4.3 to 5), the maxima of the relevant variable, after discarding some transients.

direction of the motion along the flow. The presence of one positive exponent indicates that, on the attractor, there exists a direction along which nearby trajectories, on the average, diverge exponentially.

A non-integer fractal dimension is often (though not always) an indication of the chaotic nature of an attractor. The fractal dimension of the attractor of system (8)–(9) has been computed (always for $r=5$) by means of the Grassberger and Procaccia method. The results are shown in fig. 6. The estimated dimension is ≈ 2.15, which is consistent with the geometrical aspect of the attractor. It is also close to the so-called Lyapunov dimension, whose simple computation is based on the knowledge of LCEs. Suppose we have computed the n LCEs $(\chi_1, \chi_2, \ldots, \chi_n)$ of a dynamical system and listed them in a decreasing order, thus:

$$\chi_1 > \chi_2 > \cdots > \chi_n.$$

Suppose also that s is the largest number for which

$$\sum_{i=1}^{s} \chi_i > 0,$$

then the Lyapunov dimension D_L is equal to

$$s + \frac{\sum_{i=1}^{s} \lambda_i}{|\lambda_{s+1}|}.$$

In the present case, we have $\chi_1 \approx 0.35$; $\chi_2 \approx 0.00$; $\chi_3 \approx -4.2$. Consequently, we obtain:

$$D_L = 2 + \frac{\chi_1 + \chi_2}{|\chi_3|} \approx 2.08.$$

5. Analytical evidence of chaos: A semi-linear case

It would be interesting at this point to investigate the dependence of the behaviour of the system on the parameter n (the 'degree of homogeneity'). However, it is one thing to fix n and then study the effect of variations in r, and quite a different thing to fix r and then treat n as a varying parameter. In the latter case, we do not simply change the value of a parameter of a system with a given structure, but we change the dimension of the problem by adding one equation to system (8)–(9).

In order to make this question more tractable, we shall therefore introduce

a useful simplification, taking a semi-linear approximation of the function $f(x)$ and putting

$$f(x) = \begin{cases} 2x, & \text{for } 0 \leq x < 1/2, \\ 2-2x, & \text{for } 1/2 \leq x \leq 1. \end{cases} \tag{13}$$

When used in one-dimensional, discrete-time systems like (1), function (13) is referred to as 'symmetrical tent map' and it is known that there exists a dynamical equivalence between such a map and the 'logistic' map $rx(1-x)$ for $r=4$, which is also known to have a chaotic behaviour.

As we shall promptly see, once this simplification is adopted, it becomes possible to evaluate the Lyapunov Characteristic Exponents of the system analytically, as functions of n taken as a parameter. In particular it is possible to establish the conditions for the dominant LCE to be positive, which we take here as an indication that the dynamics of the system is chaotic.[10]

In order to proceed in our discussion, let us concisely[11] recall the definition of Lyapunov Characteristic Exponent. Let $\phi(t,x)\colon \mathbb{R} \times M \to \mathbb{R}^n$ denote the solution of a system of ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{14}$$

through a point $x \in M$, where $M \in \mathbb{R}^n$ is the phase space of system (14).

Then, if we indicate by w a vector of $T_x M$ (the tangent space to M at x) the LCE of the vector w (or first-order LCE) can be defined as

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{\|D_x \phi(t,x) w\|}{\|w\|} = \chi(x,w), \tag{15}$$

where $D_x\phi(t,x)$ is the (time-varying) matrix of partial derivatives with respect to x.[12]

Consider now that, when a 'tent' function is adopted, the phase space of our system is divided into two zones, i.e., Zone 1::$(0 \leq x < 1/2)$ and Zone 2::$(1/2 \leq x \leq 1)$, within each of which the system is in fact linear. Notice also

[10] We have not been able to prove analytically the presence of a bounded attracting set for the system in question, but numerical simulations strongly suggest that this is indeed the case.

[11] For a more rigorous and complete characterization of LCEs, cf. Benettin et al. (1980).

[12] Notice these facts: (i) the existence of limit (15) is guaranteed under rather mild conditions by a theorem of Oseledec (1968) (see Benettin et al., op. cit.); (ii) the LCEs depend on w, but if we compute (15) for a vector w chosen at random (i.e., generally), we shall obtain the *dominant* LCE; (iii) in general, the LCEs depend on x. However, if they are computed for points belonging to an attractor of a system which is ergodic, then the LCEs will be the same irrespective of the choice of x.

that the unique non-trivial equilibrium point of the system is located in Zone 2.

For each zone, therefore, it is possible to find the exact solution of system (8)–(9). For this purpose, we can re-write it as

$$\dot{x} = A_i x, \qquad i = 1, 2, \tag{16}$$

where A_1, A_2 are constant $(n \times n)$ matrices governing the motion of the system in Zone 1 and 2, respectively.

The general solution of (16) through a point x can now be written as

$$\phi(t, x) = e^{A_i t} x,$$

and consequently we have

$$D_x \phi(t, x) = e^{A_i t}.$$

Suppose now for a moment that the motion of the system in question is entirely controlled by a single constant matrix A, i.e., the system is actually linear. In this case, a simple relation could be promptly established between the LCEs and the eigenvalues of the matrix A. In particular, for a randomly-chosen vector w, we shall have

$$\chi_1 = \mathrm{Re}\,\sigma_1,$$

where χ_1 is the largest LCE of order one and σ_1 is the eigenvalue of A with the largest real part.

From this fact we can conclude that, when both the dominant eigenvalues of the matrices A_1 and A_2 are positive, then, if the system has a bounded indecomposable attractor, this must possess sensitive dependence to initial conditions, i.e., it must be chaotic.

Prompted by these considerations, we now turn to the evaluation of the eigenvalues of the matrices A_i as functions of the parameter n. From eqs. (11)–(13), we obtain

$$\lambda(A_i) = \begin{cases} n(2^{1/n} - 1), & \text{for } i = 1, \\ n((-2)^{1/n} - 1), & \text{for } i = 2, \end{cases} \tag{17}$$

where by $\lambda(A)$ we of course denote an eigenvalue of A. It can immediately be seen that, since there exists a real solution of $2^{1/n}$ which is greater than 1 for $n < \infty$, the dominant eigenvalue of A_1 is positive for all finite (positive) n. We are then left when the case $i = 2$. In this case, there will be one real (negative)

eigenvalue for n odd and no real eigenvalues for n even. By making use of de Moivre's theorem, and considering that, for any θ, $\cos(\theta) = \cos(-\theta)$, it can be established that the real part of the m complex eigenvalues will be equal to

$$\lambda(A_2) = n\left[2^{1/n}\cos\left(\frac{\pi}{n} + \frac{2k\pi}{n}\right)\right], \tag{18}$$

where $k = 0, 1, \ldots, (m-1)$, and $m = n$ if n is even and $m = n - 1$ if n is odd. Moreover, in expression (18), $2^{1/n}$ denotes the (unique) real solution of the nth root of 2.

Simple calculations show that the pair of complex eigenvalues with the largest real part corresponds to $k = 0$ and $k = (m-1)$. Henceforth, therefore, we put $k = 0$, and we study the function

$$\mathrm{dom}\,\lambda(A_2) = \Lambda(n) = n\left[2^{1/n}\cos\left(\frac{\pi}{n}\right) - 1\right]. \tag{19}$$

The values of $\Lambda(n)$ for integer, positive values of $n = 1, 2, \ldots$ can easily be calculated, and it can be shown that $\Lambda(n)$ increases monotonically with n. The shape of the function is illustrated in fig. 7.

It can also be shown that, for values of $n < 8$, $\Lambda(n) < 0$ which means that the non-trivial equilibrium value situated in Zone 2 is stable. For values of $n \geq 8$, $\Lambda > 0$. This indicates that the numerically-detected attractor, for those values of n has sensitive dependence on initial conditions and it is chaotic. We have looked for numerical confirmation of our analytical findings and the results are shown in figs. 8, which show bidimensional plots of trajectories of system (16) for $n = 7$ and $n = 8$.

In the first case [fig. 8(a)] the system spirals towards the non-trivial equilibrium point, as expected. In the second case ($n = 8$), the (presumably) post-transient trajectories of the system would at first sight [fig. 8(b)] seem to indicate that the asymptotic motion is periodic, thus apparently contradicting our analytical conclusions. However, by performing and enlargement of a section of the attractor [fig. 8(c)] the complex nature of its dynamics and its fractal structure are revealed. The fact that the chaotic motion is confined within a narrow 'tube' around a closed curve is not surprising since, for $n = 8$, the dominant LCE although positive is very small (of the order 10^{-3}). The fact that the 'degree of chaoticity' grows with n is clearly supported by the simulation shown in fig. 9, where calculations have been performed for $n = 15$.

A final valuable result can be obtained by evaluating the limit

$$\lim_{n\to\infty} A(n) = \lim_{n\to\infty} n\left[2^{1/n}\cos\left(\frac{\pi}{n}\right) - 1\right] = \ln 2 > 0. \tag{20}$$

This is very interesting since $\ln 2$ is precisely the well-known value of the (single) LCE corresponding to the discrete-time, one-dimensional map (1) for $r=4$, or equivalently to the 'symmetrical tent' map. And this further corroborates our argument that system (1) can be viewed as a special, limit case of a continuous, multidimensional system.

PART II

1. Introduction: A model of inventory cycles

In the previous section we have investigated a particularly important class of models for which chaotic output appears to be the combined results of a continuous-time MEL and a single non-linearity of a 'one-hump' type. We have shown that chaotic dynamics may only occur in this case if the order of the lag n is rather large.[13] When the lag structure and/or the (non-linear) input to it are of a different kind, however, complicated dynamics may occur at a much lower order of the lag.[14]

To see this, we shall investigate a continuous-time model whose economic motivation can be found in the early analysis of inventory business cycles [Metzler (1941)], and, more recently, in the works of Gandolfo (1983) and Lorenz (1989). The latter author was the first to point out the relation between the economic model in question and certain mathematical results of Arneodo et al. [see, for example, (1981), (1982)], concerning the so-called 'spiral chaos'.

Our own model closely follows Lorenz's, but we shall manipulate it somewhat with a view to establishing a relation with the previous discussion of lags.

Let the notation be the following:

Y = actual net national income (output),
B = inventory stock,
S = global (net) saving,

[13]We have seen that, with a non-linearity of the tent type, chaos may occur when $n \geq 8$. If the non-linearity is of an exponential type [i.e., $f(x)=rxe^{-x}$], a much greater order ($n\approx 50$) is required [cf. Sparrow (1980), Invernizzi and Medio (1991)].

[14]Of course 3 is the minimum order of a system of o.d.e. for which complex behaviour may occur anyway.

I = global (net) investment,
b = desired stock/output ratio, constant.

Let also an index 'e' denote the expected value of the relevant variable.

Suppose now that output adjusts in response to discrepancies between desired and actual inventory stock, through a first order exponential lag, thus:

$$\dot{Y} = \tau(bY^e - B), \quad \tau > 0, \tag{21}$$

τ being the speed of adjustment, and $1/\tau$ the 'length' of the lag.

Suppose further that expected output is a linear function of the level, velocity and acceleration of actual output,[15] thus:

$$Y^e = G_1(D)Y, \tag{22}$$

where $G_1(D) = a_2D^2 + a_1D + 1$, $D = \mathrm{d}/\mathrm{d}t$, and a_1 and a_2 are positive constants.

The increase in inventory is equal to the excess of (net) saving over (net) investment, i.e.,

$$\dot{B} = S - I. \tag{23}$$

Finally, we shall adopt Kaldorian saving and investment functions [cf. Kaldor (1940, pp. 78–92)] as depicted in fig. 10.

The original (verbal) formulation of Kaldor, seems to assume a simple adaptive mechanism such as

$$\dot{Y} = \theta[I(Y) - S(Y)], \quad \theta > 0,$$

which, in turn, implies that $\bar{Y}_1$ and $\bar{Y}_3$ are stable equilibria and $\bar{Y}_2$ is an unstable one. In Lorenz's model, owing to the different specification of the adjustment mechanism, both the equilibria $\bar{Y}_1$ and $\bar{Y}_2$ may be simultaneously unstable. His model may in fact be interpreted as an investigation of the dynamics of the economy when output is 'trapped' at, or in the vicinity of the low level interval $(\bar{Y}_1, \bar{Y}_2)$. It can easily be seen that the function

$$F(Y) \equiv S(Y) - I(Y), \quad Y \in [\bar{Y}_1, \bar{Y}_2],$$

belongs to the unimodal class and can be formulated, for example, as:

[15]Notice that, eq. (22) may be interpreted as a simplified version of the hypothesis that agents' expectations on income are positively affected by the level, the rate of growth and the changes in the rate of growth of income. If the rate of growth of income (and its rate of change) are small, the two hypotheses are roughly equivalent.

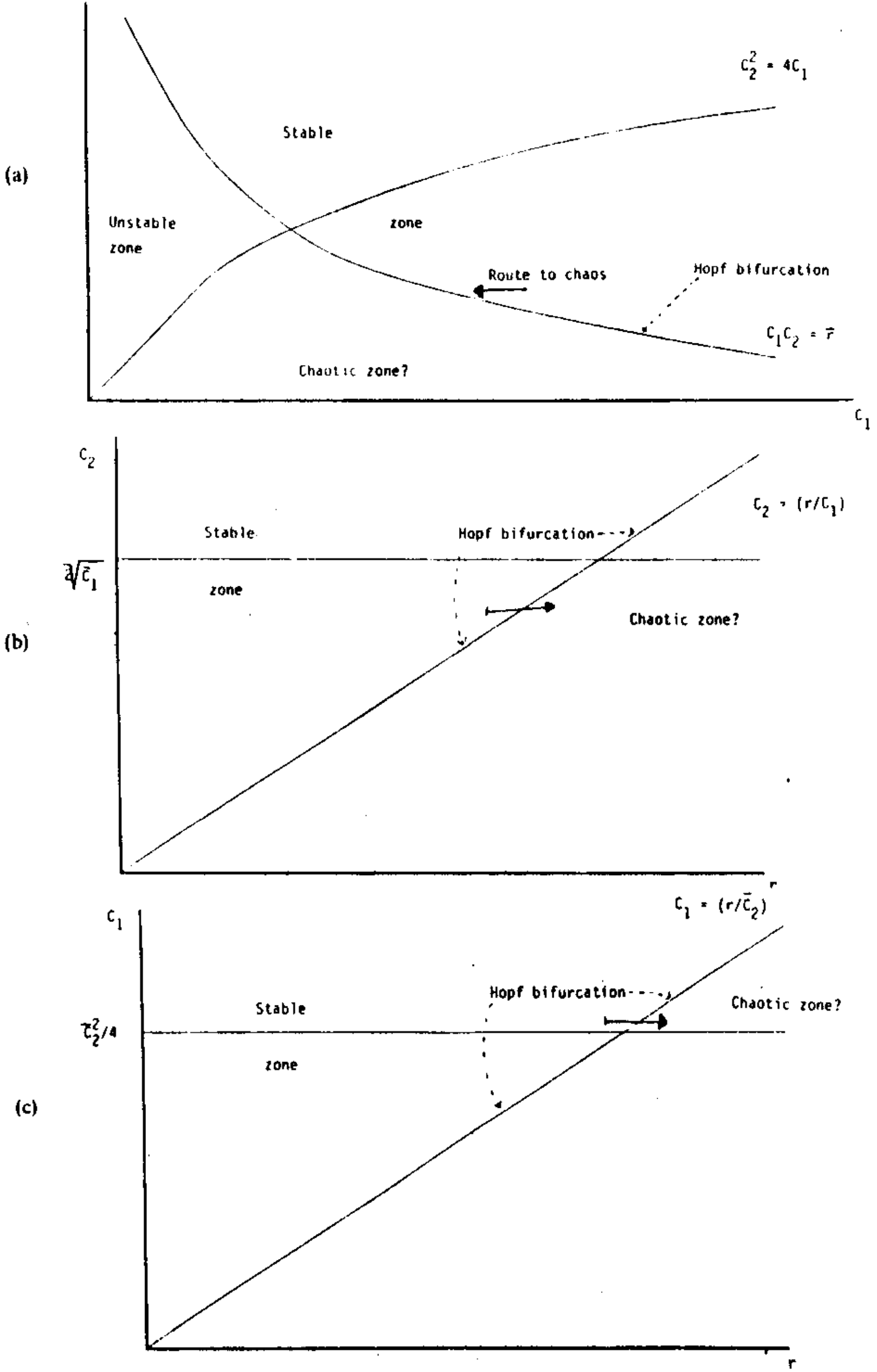


Fig. 11(a)–(c). Routes to chaos for system (25).

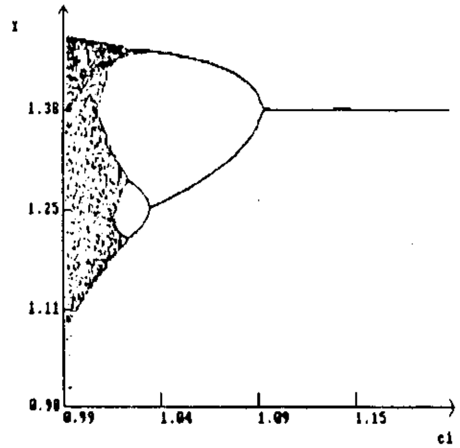


Fig. 12. Bifurcations diagram I for system (25).

$$F(Y) = mY(1-Y), \quad m>0. \tag{24}$$

Combining eqs. (21)–(23) and putting $Y_1 = 0$ and $Y_2 = 1$, we can write[16]

$$G_2(D)Y = \hat{F}(Y), \tag{25}$$

where

$$G_2(D) = D^3 + c_2D^2 + c_1D, \quad c_1 = 1/a_2 > 0,$$

$$c_2 = [ba_1 - (1/\tau)]ba_2 \gtreqless 0, \quad \hat{F}(Y) = rY(1-Y),$$

$$r = m/ba_2 > 0.$$

Under the postulated assumptions, the Lie derivative (the divergence) of (25) is constant and equal to $-c_2$. Since we are interested here in the study of dissipative systems, we shall henceforth only consider the case $c_2 > 0$. System (25) has two equilibrium points (i.e., points at which $D^3Y = D^2Y = DY = 0$), namely: E_1, located at $Y=0$, and E_2, located at $Y=1$. It can easily be seen that E_1 is always unstable[17] whereas E_2 is stable iff $r < c_1c_2$. If $c_2 > 0$, the Jacobian matrix calculated at E_2 cannot have positive real roots and one of the following situations occurs:

(i) There are three negative real roots.

[16]Notice that in this case $Y<0$ does not necessarily mean that output is negative.

[17]The reader can verify that, at E_1, the third Hurwitzian determinant Δ_3 is equal to $-\Delta_2 r$, Δ_2 being the second Hurwitzian determinant. Consequently, since $r>0$, either Δ_2 or Δ_3 must be negative, which implies instability.

(a)

(b)

(c)

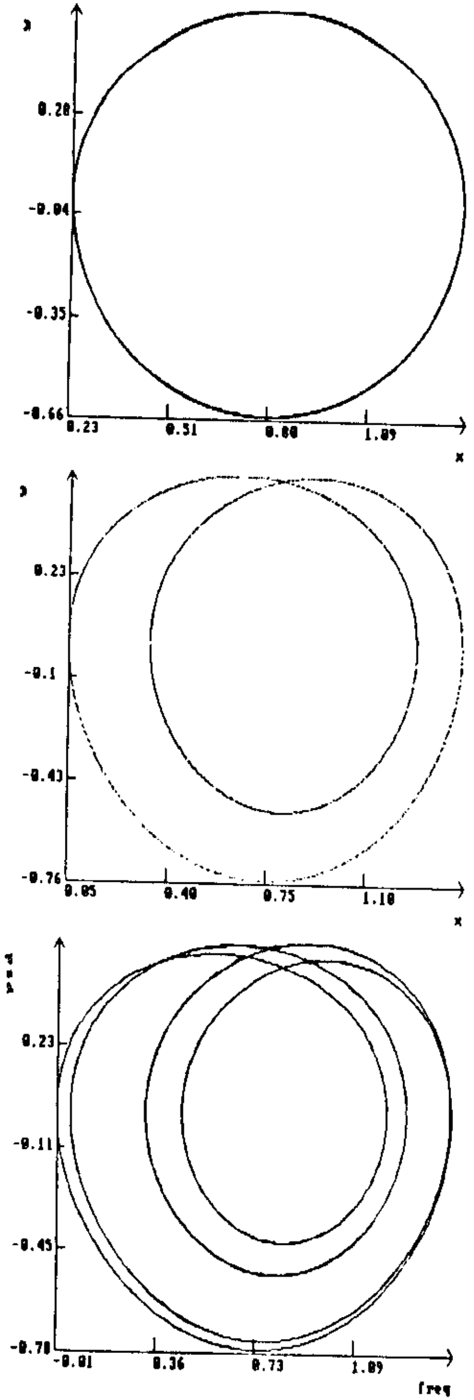


Fig. 13(a)–(c). Period doubling transition to chaos for system (25).

(d)

(e)

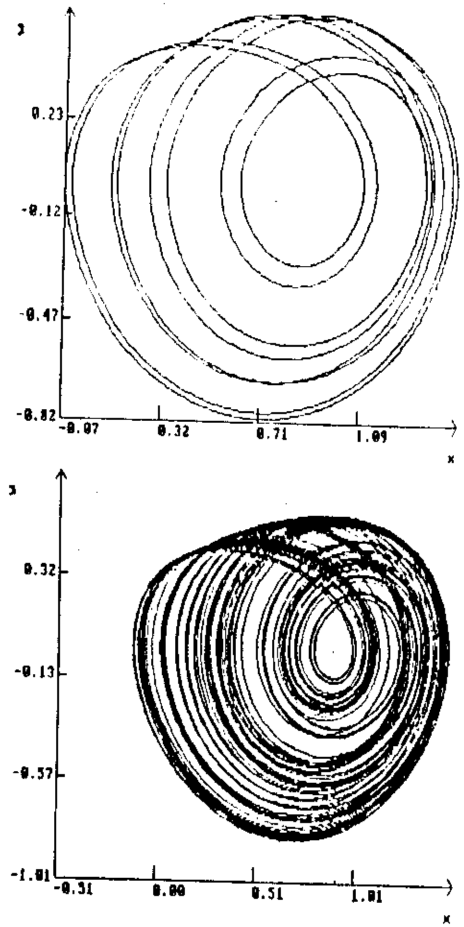


Fig. 13(d)–(e). Period doubling transition to chaos for system (25).

(ii) There is one negative real root and a pair of complex conjugate roots[18] whose real part is greater or smaller than zero according to whether c_1c_2 is greater or smaller than r.

It follows that, when c_1c_2 becomes greater than r, the system loses its

[18] Notice that complex roots will occur whenever c_2 is sufficiently small vis-à-vis c_1, independently of r. To see this consider the auxiliary equation of (25) at E_2, i.e.,

$$\lambda^2 + c_2\lambda_2 + c_1\lambda + r = 0. \qquad (*)$$

Any algebra textbook dealing with third degreee equations, will show that a sufficient condition for (*) to have two complex roots is that

$$q = \frac{1}{3}c_1 - \frac{1}{9}c_2^2 > 0,$$

which is clearly obtained if c_2 is small vis-à-vis c_1.

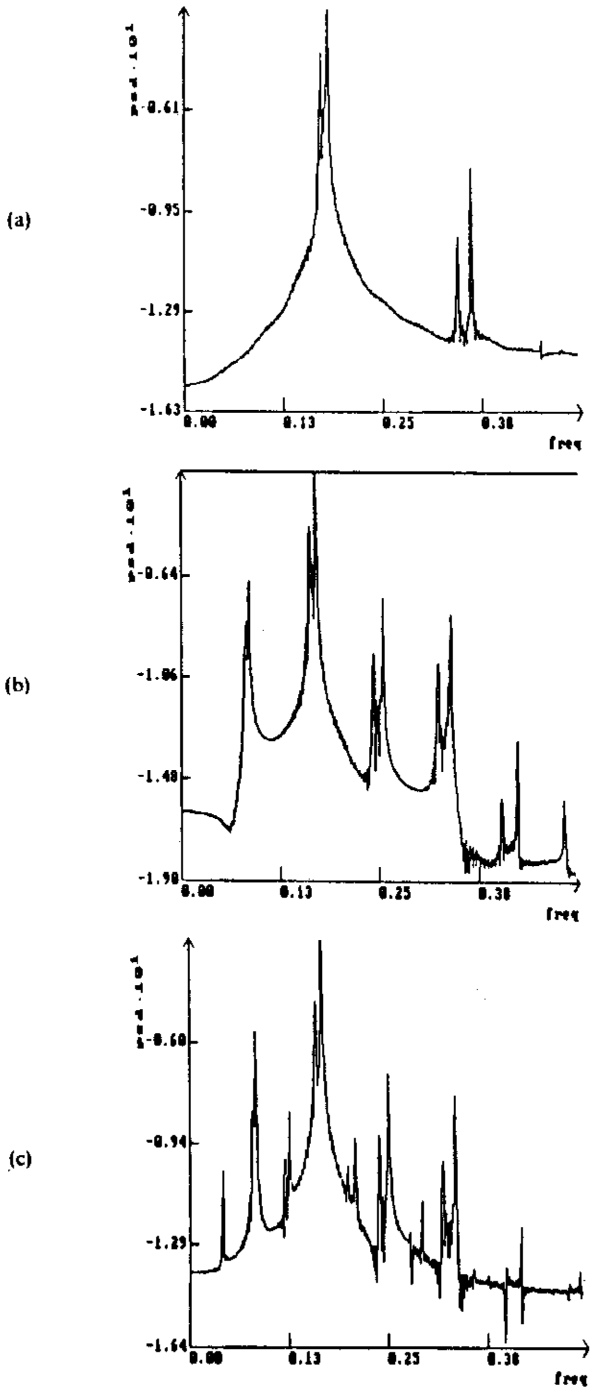


Fig. 14(a)–(c). Power spectral density of trajectories of system (25).

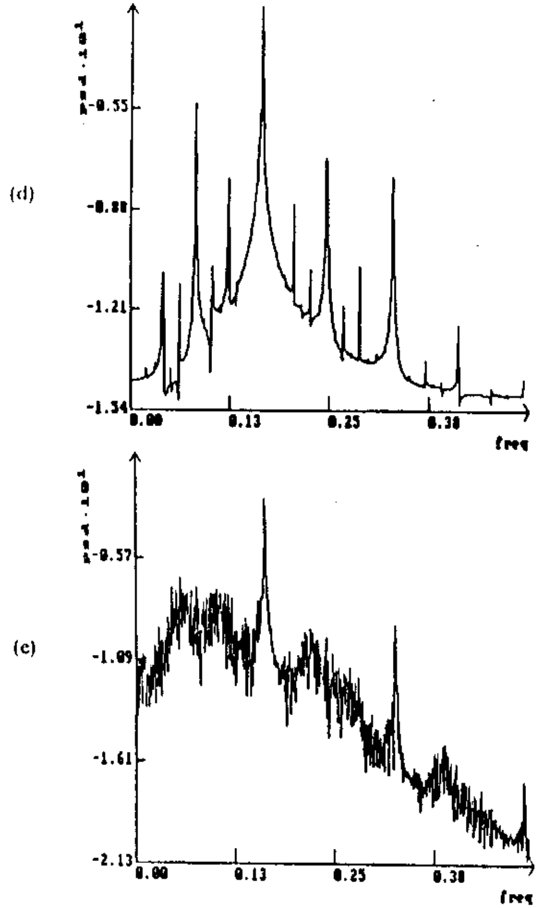


Fig. 14(d)–(e). Power spectral density of trajectories of system (25).

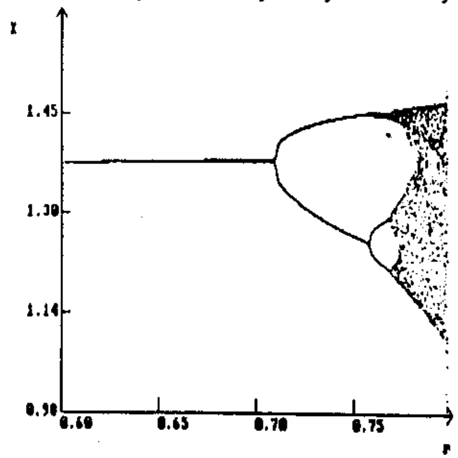


Fig. 15. Bifurcations diagram II for system (25).

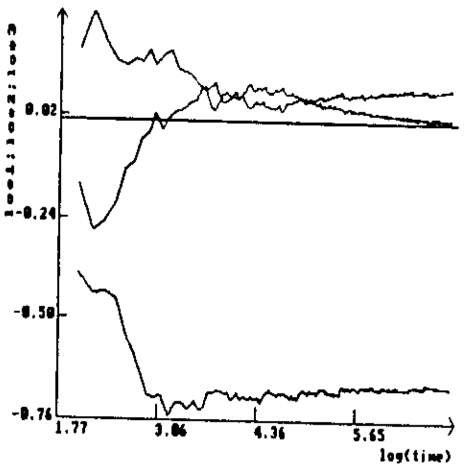


Fig. 16. The first three LCEs of (25).

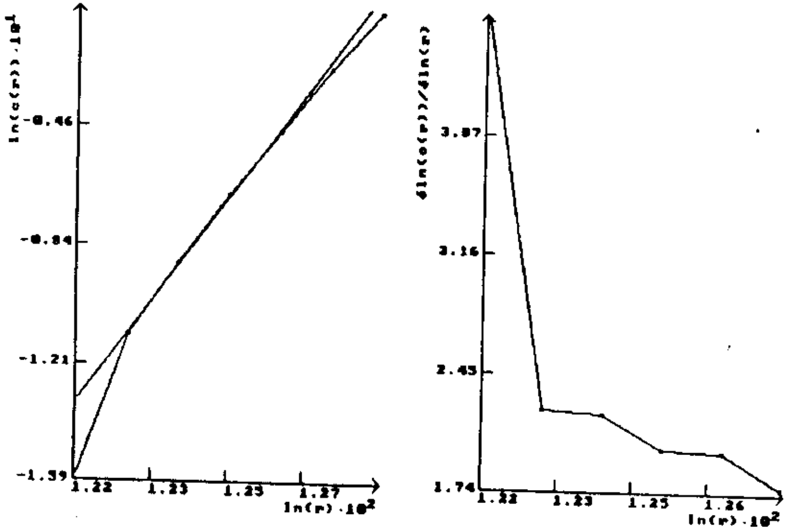


Fig. 17. Continuous-time models of chaos in economics.

stability through a Hopf bifurcation. Consequently, in this case, the loss of stability leads to the appearance of periodic solutions.

2. Lag structure and complex behaviour

Arneodo et al. (op. cit.) proved numerically[19] that, for certain values of the parameters, system (25) satisfies the conditions of the Silnikov theorem[20] and therefore possesses a horseshoe-like (chaotic) invariant set. Those authors' simulations also indicate that the chaotic set is attracting.

These results suggest the following interesting observation:

> There exist low-order lag structures [e.g., the polynomial $G_2(D)$], different from a MEL, such that, when coupled with a non-linearity of a 'one hump' type, their interaction results in chaotic dynamics. In this case, chaos may occur at values of the order of the lag and/or of the steepness of the 'hump' much lower than would be necessary for a MEL.

It would be a promising research project to classify different classes of lags which, in combination with given classes of non-linearities, may produce complex behaviour. Such an ambitious goal could not be pursued here, so we shall limit ourselves to addressing some specific questions concerning the present model.

First of all let us pose the following question: What makes the lag mechanism $G_2(D)$ capable of producing a chaotic output, when it is coupled to a non-linearity of a logistic type such as (24)? We do not have a fully-developed mathematical answer but we shall surmise a conjecture.

The model under discussion can be described in terms of a closed, single-loop feedback system, consisting of a lag mechanism [the polynomial $G_2(D)$], and a non-linear function $\hat{F}(Y)$.

In Part I we have seen that, under certain conditions, a correspondence can be established between a lag operator of a dynamical system and a 'weighting function' representing the impact on current output of inputs delivered at different times in the past. We have also seen that, when the operator takes the form of a MEL, this weighting function can be interpreted as a probabilistic version of a one-dimensional, difference equation.

Consider now the lag operator $G_2(D)$. As we know from Part I, the corresponding weighting function $w(t)$ can be found by means of the Laplace Transform (L.T.) machinery, by calculating

$$w(t) = L^{-1}[G_2(D)],$$

[19] In fact, Arneodo et al. (1982), adopting a semilinear approximation of $F(Y)$, proved *analytically* the presence of an homoclinic orbit and of the other conditions of the Silnikov theorem.

[20] On the Silnikov theorem, see, for example, Guckenheimer and Holmes (1983, pp. 318–325).

where L^{-1} is the inverse of the L.T. operator.

Now let λ be a root of the polynomial $G_2(D)$, and let us consider that Arneodo et al. found chaotic dynamics for the system (25) for values of the parameters $c_1=1$, $r\approx 0.35$ and $c_2\approx 0.4$. Since for those values of the parameters $c_1>c_2>0$ and $c_1^2<4c_2$, they correspond to one zero and two complex conjugate roots with negative real parts.[21]

The weighting function of the lag operator $G_2(D)$ in this case will have the general form:

$$w(t)=1/c_1+A\,\mathrm{e}^{-\sigma t}\cos(\omega t+\theta), \tag{26}$$

where we have put $\lambda=-\sigma\pm i\omega$, $\sigma>0$, and A and θ are real numbers which depend only on the parameters. The weighting function corresponding to the lag $G_2(D)$, therefore, is a damped sinusoid.[22]

If we compare (26) with the weighting function (5) associated with a MEL, we observe that, whereas the latter is unimodal for any order $n>2$, the weighting function of $G_2(D)$ is multimodal whenever its roots are complex. It is this structure of the time profile of the lag, we believe, that, when coupled to a non-linear input function of a 'one-hump' kind like $\hat{F}(Y)$, can produce a chaotic output. A hint for understanding this fact (although by no means a rigorous explanation) may be found in the following considerations.

Taking the same approach as discussed in Part I, the weighting function (26) may be viewed as a probabilistic representation of an infinite-dimensional, discrete-time lag. However, if the damping factor σ is sufficiently large, all but the first few dominant lags (corresponding to the dominant modes) can be neglected. The following, simple example involving only two lags will help clarify the issue.

Suppose we have a discrete-time dynamical system such that a linear relation exists between the value of a variable Z at time t and the values of another variable Y at times $t+T_1$, $t+T_2$ $(T_1, T_2>0)$. In general, Z may be a (linear or non-linear) function of Y. We can then write

$$b_2 Y_{t+T_2}+b_1 Y_{t+T_1}=Z_t, \tag{27}$$

where $b_i>0$ and constant for $i=1,2$. Recalling our discussion of Part I, a continuous-time generalization of (27) can be written as

$$G_*(D)Y=Z, \tag{28}$$

[21] In the following discussion, the reader should keep in mind the distinction between the roots of the lag polynomial and the roots of the auxiliary equation of the entire system.

[22] Cf. G. Doetsch, loc. cit. The reader should notice that complex roots of a lag polynomial always entail a multimodal weighting function.

where

$$G_*(D)=\left\{b_2\left(\frac{T_2 D}{n}+1\right)^n+b_1\left(\frac{T_1 D}{n}+1\right)^n\right\}.$$

A possible (but by no means the only) interpretation of (28) is that it refers to an economy characterized by two classes of agents, each comprising an indefinitely large number of members, and differing from one another as concerns their speeds of reaction to economic stimuli. Each class is characterized by a different expected value T_i $(i=1,2)$, of the overall reaction time and by a variance which represents the 'spread' of individual reaction-times of the members of each class around the mean. (For simplicity's sake, we have assumed here that the two classes have the same variance.) We know that, as $n\to\infty$, (28)→(27) and we are back to the case of two fixed delays. This corresponds to the situation in which the agents belonging to each class are perfectly homogeneous, whereas the two classes are different from each other. By putting $G_*(D)=0$ and solving for D, it is easy to see that, the polynomial $G_*(D)$ has at most one real root, all the others being complex conjugate pairs. Therefore, for $n\geqq 2$, the corresponding 'weighting function' is multimodal and as $n\to+\infty$, it tends to *two* Dirac delta functions situated at T_1 and T_2.

Therefore, whereas a MEL can be seen a continuous-time generalization of a discrete-time lag of order one, the lag operator $G_2(D)$ can be viewed as a similar generalization of discrete-time lags of order two or higher. Now the potential complexity of non-linear maps is known to exhibit a 'leap' whenever the dimension is increased by one unit. Then, if our reasoning so far is correct, it is little wonder that the lag operator $G_2(D)$ can generate a more complex dynamic behaviour than does a MEL of the same order, coupled to the same non-linearity.

3. Parameter analysis and numerical simulations

In order to explore this point and seek confirmation of our hypothesis, let us go back to system (25) and perform some parametric analyses. There are three parameters in the system: c_1, c_2 and r. Recalling that the condition for instability of (25) is $c_1c_2<r$ and the condition for complex roots of the lag polynomial is $c_2^2<4c_1$, we can combine them as indicated in the diagrams of fig. 11.

These diagrams can be used as a practical tool to locate zones of potential complex behaviour in the three-dimensional parameter space. Consider, for example, fig. 11(a). If the values of the three parameters are chosen so as to position the system in the lower part of the stable zone, and, keeping c_2 and

r fixed, c_1 is progressively decreased, we expect the system to undergo a Hopf bifurcation and, possibly, a period-doubling transition to chaos. This conjecture is supported by the bifurcation diagram shown in fig. 12, which has been calculated fixing $c_2=0.4$, $r=0.8$ and decreasing c_1 from 1.2 to 0.99.

The period-doubling sequence is quite evident. There is no sign of periodic windows, however, but this could be the consequence of the coarseness of the numerical simulation and its graphical representation. For low values of c_1 (≈ 1), there seems to be a sudden increase in the size of the chaotic attractor. This suggests the presence of the so-called 'interior crisis', or 'interior catastrophe', a phenomenon whose occurrence has also been detected for the logistic map (1). For even lower values of c_1, the system 'explodes' and the solutions become unbounded.

The period-doubling route to chaos can be further appreciated by considering the diagrams of fig. 13 which show the final trajectories of eq. (25) for different values of the controlling parameter c_1.

The changes in the behaviour of the system can also be detected by power spectrum analysis (performed for the same values of the parameters), as illustrated in fig. 14, where evidence of an evolution from periodic to aperiodic chaotic behaviour is quite strong and qualitatively similar to that discussed in Part I. As we decrease the control parameter c_1, we see peaks appear in the power spectrum in correspondence to sub-multiples of the fundamental frequency. For even lower values of c_1, a 'noise floor' associated with chaotic behaviour appears and successively destroys the sub-harmonics in the opposite order of their appearance. Notice, however, that, insofar as our numerical analysis is correct, the 'noise floor' does not seem to destroy the fundamental frequency (and some of its harmonics). This type of chaos characterized by the co-existence in the power spectrum of broad band components and sharp peaks is sometimes called 'non-mixing'. An example of non-mixing chaos is given by the Rössler attractor.[23]

A similar procedure can be followed with regard to the route to chaos illustrated in fig. 11(b), fixing $c_1=0.99$, $c_2=0.4$ and increasing r from 0.6 to 0.8. The resulting bifurcation diagram is shown in fig. 15.[24] The results are qualitatively the same as in the previous experiment: a period-doubling route to chaos terminating in an 'explosion' of the system; no clear sign of periodic windows in the chaotic zone; some indication of an 'interior crisis', immediately before the 'explosion'.

Finally, we have estimated the LCEs and the (Grassberger and Procaccia) fractal dimension for the chaotic attractor corresponding to $c_1=0.99$, $c_2=0.4$, $r=0.8$. The results are shown in figs. 16 and 17.

The dominant LCE is small but positive (≈ 0.08) and the convergence is

[23]See Rössler (1976); Oono and Osikawa (1980).

[24]In order not to overburden the presentation, we shall omit the discussion of the diagram of fig. 11(c), which gives entirely similar results.

good. The fractal dimension is equal to ≈ 2.1, which is in harmony with the other qualitative information we have on the attractor.

Altogether, our exercises strongly suggest that, for wide zones of the parameter space, a 'multimodal' structure the lag operator $G_2(D)$ associated to a 'one-hump' non-linearity in the excess saving function may generate cycles and chaos in the dynamics of the system.

4. A few complications and conclusions

The constraints put on the lag structure [the polynomial $G_2(D)$] in order to obtain chaotic behaviour of the system could be substantially relaxed if the function $\hat{F}(Y)$ depended on both the level and the rate of change of the representative variable Y. To see this, let us modify the investment function of the Arneodo–Lorenz model by postulating that investment depends not only on the level but also on the rate of increase of income: in other words, by introducing an 'acceleration' effect. The saving function will be left unchanged. For simplicity's sake, we shall assume that the investment function $I(Y,\dot{Y})$ is separable, convex in the first argument and linear in the second, namely:

$$I=I(Y,\dot{Y})=I_1(Y)+I_2(\dot{Y}),$$

$$I_1'(I)>0,\quad I_1''(Y)>0,\quad I_1(0)>0,$$

$$I_2(\dot{Y})=v\dot{Y},$$

where $v>0$ is the acceleration coefficient, assumed to be constant.

The rôle of the 'accelerator' in producing chaotic dynamics can be readily seen. Including the acceleration effect, our system will now be written as

$$G_2(D)=F_1(Y,DY), \tag{29}$$

where $F_1(Y,DY)=\hat{F}(Y)-\hat{v}DY$, and $\hat{v}=(v/ba_2)$, which, in turn, is equivalent to

$$G_3(D)=\hat{F}(Y), \tag{30}$$

where $G_3(D)=G_2(D)+\hat{v}DY$.

Recalling our discussion of the diagram of fig. 11(a) above, let us now fix $r=\bar{r}$ and choose $c_2<\bar{c}_2=(4\bar{r})^{1/3}$, $\hat{v}=0$ and c_1 large so that we are out (to the right) of the chaotic zone. Clearly, by increasing $\hat{v}$ sufficiently, we can move into the chaotic zone. In fact, for example, by putting $r=0.8$, $c_2=0.4$ and $c_1>1$ we can always choose $\hat{v}=1-c_1$ so that the exact numerical results of section 3 above are reproduced, and chaotic output is generated. If c_1

becomes too large, however, the system will 'explode' and its solutions will become unbounded.

We conclude that the addition of a *linear* acceleration component to the investment function of a macromodel of inventory cycles can bring about chaotic behaviour of the economy. More generally, when the input to a lag mechanism is velocity dependent, the time profile of the input–output interaction may be altered so as to lead to a more complex motion than would otherwise be the case. And this even when the additional input in question is linear.

One may also wonder what difference it would make if we used 'complete' Kaldorian saving and investment functions (cf. fig. 9).[25] The numerical simulation performed indicate that the 'complete' system has the same broad dynamic characteristics as the 'reduced' one, the main difference being that chaotic attractors now retain their stability for values of parameters which, with 'reduced' saving–investment functions, would have made the system 'explode'.

From an economic point of view, the exercises performed in the preceding pages have a mainly pedagogic significance and should not be taken too seriously. However, they give us an interesting clue as concerns the rôle played by lags and non-linearities in producing irregular fluctuations and chaos and confirm the possibility of complex behaviour in simple, low-dimensional continuous-time systems. Stronger results will perhaps be obtained when the broad analytical and numerical findings are combined with a deeper understanding of the mechanisms generating those lags and non-linearities in real economy. For this purpose, economists should perhaps rely more on empirical observation and classification and less on a priori deduction from very general first principles.

[25]Formally this could be achieved, for example, by putting $\bar{Y}_3=2$ and replacing the function $F(Y)$ with

$$F(Y)=\begin{cases} F(Y), & \text{for } 0\leq Y\leq 1,\\ r(Y-1)(2-Y), & \text{for } 1<Y\leq 2.\end{cases}$$

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J.E.B.O.: A. Medio, Continuous-Time Models of Chaos in Economics

ERRATUM

P=page; para.=paragraph; L.=line; eq.=equation

	Original Text	Corrected Text
P.117, para.3, L.2	priori	*a priori*
P.119, eq.(5)	$\left(\frac{1}{\tau n}\right)^{n}$	$\left(\frac{1}{\tau n}\right)^{-n}$
P.120, para.2, L.2	"arises of"	"arises:"
P.120, para.3, L.2	"We"	"we"
P.133, para.3, last L.	$M \in \mathbb{R}^n$	$M \subset \mathbb{R}^n$
P.133, last two L.	::	:
P.134, para.5, L.2	"bounded"	"compact"
P.134, last L.	"when"	"with"
P.135, eq.(18), R.H.S.	$= n\left[2^{1/n}\cos\left(\frac{\pi}{n}+\frac{2k\pi}{n}\right)\right]$	$= n\left[2^{1/n}\cos\left(\frac{\pi}{n}+\frac{2k\pi}{n}\right)-1\right]$
P.139, eq.(25), L.3	$c_2 = [ba_1 - (1/\tau)]ba_2$	$c_2 = [ba_1 - (1/\tau)]/ba_2$
P.141, L.3	"greater or smaller"	"smaller or greater"
P.141, L.4	"greater"	"smaller"
P.146, L.4	$r \approx 0.35$	$r \approx 0.8$
P.149, L.5	"to a 'one-humped' "	"with a 'one-humped' "
P.149, last para., L.2	"c_1 large"	"c_1 small"
P.149, last para., L.3	"right"	"left"
P.149, last para., L.5	$c_1 > 1$	$c_1 < 1$
P.149, last L./P.150 1st L.	"If c_1 becomes too large"	"If $c_1 + \hat{v}$ is too small"

References: Invernizzi, S. and A.Medio, 1991, Lags, etc. should read: Invernizzi, S. and A.Medio, 1991, On lags, etc.

Figure 10 on P.130: in the interval $[\bar{Y}_1, \bar{Y}_2]$, the two curves should be labelled thus: "S(Y)" the curve above and "I(Y)" the curve below.

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DISCRETE AND CONTINUOUS-TIME MODELS OF CHAOTIC DYNAMICS IN ECONOMICS

ALFREDO MEDIO

The paper discusses applications to economics of non-linear one-dimensional maps. The hypothesis implicit in a fixed delay lag is criticized and a continuous-time generalization of it is suggested. The resulting system of differential equations is investigated by means of analytical and numerical tools. A 'symptomatology' of chaos is developed and it is shown that the combination of non-linearity and exponential lags may indeed produce chaos in a continuous setting. It is also shown that the essential qualitative properties of the full flow can be captured by a 'reconstructed 1-D map'.

1. INTRODUCTION

In the present discussion the phrase 'chaotic dynamics' broadly refers to a system characterized by orbits more complex than periodic or quasi-periodic ones. The existence of complex behavior in low-dimension deterministic models has been known to mathematicians and physicists since the end of the 19th century, thanks to the pioneering work of Poincaré. The subject has experienced a tremendous revival in the last two decades and there now exists a large and rapidly growing literature. Economists soon realized that the new ideas and results in the theory of dynamical systems offered great potentialities for their own research. The study of economic dynamics can now be carried out with much more powerful and sophisticated tools of analysis.

This progress is particularly relevant as far as the theory of business cycles is concerned. In this area, mathematical economists have long been (and still are) divided into two schools. Some of them maintain that economic laws *per se* would lead to an equilibrium position, but their operation is continuously subjected to disturbances of more or less random nature, which keep the economy fluctuating. Consequently, according to this view, deviations from equilibrium are better studied by means of probabilistic instruments. This position was held in the past by such eminent economists as Slutzky, Frisch, the Adelmans and, more recently, by Lucas and the 'equilibrium business cycle' school.

The 'random shock' theory of the cycle was criticized, however, on the ground that it relegates the cause of fluctuations to factors that, by definition, cannot be the object of scientific explanation. As John Hicks put it a long time ago (1950, pp. 90–91), to state that business cycles are essentially due to erratic shocks is tantamount to saying that we do not know how they come about. This position

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was held, for example, by economists belonging to the Keynesian tradition, like Kalecki, Kaldor, Goodwin, Samuelson and Hicks himself.[1] The results of their efforts were various models of the multiplier–accelerator type that, under certain conditions on the parameters, can generate limit cycles and thus effectively describe persistent regular oscillations of the relevant variables, typically income, investment and employment.

The multiplier–accelerator (and, more generally, the disequilibrium) theories of the cycle fell out of favour with the economic profession during the 1960s for a number of reasons that need not detain us here. We would like to stress, however, a fundamental flaw of those theories, which equally concerns any economic representation of business cycles as regular, easily recognizable periodic orbits. Indeed if real economies behaved as those theories imply, economic agents would not fail to notice it, and even the most prudent among them would not miss the opportunity to speculate with total safety. However, in so doing, they would destroy those very mechanisms that produced the cycle. The latter would consequently disappear and its theory with it.

This particular criticism, however, would not apply to deterministic models describing fluctuations of the relevant variables which are persistent and bounded, but highly irregular, so as to make accurate economic forecasts impossible, except in the very short run. It is not surprising, therefore, that the recent developments in the field of dynamical system theory were greeted with enthusiasm by the supporters of endogenous theories of business cycles, who promptly realized that the new ideas and results now available offered great potentialities for their line of investigation.

2. AGGREGATE, DISCRETE-TIME DYNAMICAL MODELS IN ECONOMICS

If one looks at the applications of those analyses to economics, one will notice that they almost exclusively[2] consist of adaptations (with various amounts of value added) of the celebrated first-order, non-linear difference equation, so brilliantly studied by Robert May (1976), and subsequently investigated in a large number of papers and books.

As is well known, the May equation has the general form

$$x_{t+T} = G(x_t), \tag{1}$$

where $x \in \mathbf{R}$, $G : \mathbf{R} \to \mathbf{R}$ is a smooth 'one-hump' function, and T is the length of a fixed delay, which of course can always be made equal to one by appropriately choosing the unit of measure of time. As is well-known, the dynamics of discrete-time dynamical systems of type (1) essentially depend on a single parameter quite independently of the specific form of $G(\cdot)$ and, over a certain interval of values of this parameter, may be very complex or chaotic. The question which immediately leaps to mind is the following: how general are the

[1] However, some of Kalecki's models of business cycles include both deterministic and stochastic elements.

[2] Some of the relatively few exceptions are discussed in Hans Walter Lorenz (1989).

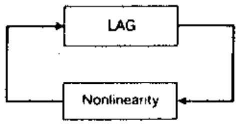

FIG. 1. A closed-loop, non-linear feedback system.

(essentially mathematical) results derived from the investigation of the discrete dynamical system (1) and how relevant are they to economic theory?

Notice, first of all, that a system described by equation (1) may be conceived as a highly aggregate mechanism, consisting of two parts, namely a non-linear functional relationship and a lag, the latter being, in this particular case, a fixed delay (see Fig. 1). Let us then consider these two elements in turn.

The non-linearities implied by single-hump functions are of a rather common kind, and their presence has been detected in situations pertaining to practically all branches of the natural and social sciences. Economists may (and indeed do) disagree on the likelihood of such non-linearities actually affecting the operation of real economic systems, and some of them are busy proving (or disproving) their existence in specific cases (see, e.g. Brock and Sayer, 1988). There should be consensus, we believe, that the general laws governing rational economic behaviour, as postulated by the prevailing theory, do not exclude *a priori*, and indeed in some cases logically imply, the presence of nonmonotonic functional relationships of the one-hump type.

This point can be appreciated by considering some of the applications to economics of equations of type (1), including, amongst others, macroeconomic models (e.g. Stutzer 1980; Day, 1982); models of rational consumption (e.g. Benhabib and Day, 1981); models of overlapping generations (e.g. Benhabib and Day, 1982; Grandmont, 1985); models of optimal growth (e.g. Deneckere and Pelikan, 1986). Recent overviews of the matter, with further instances of one-hump functions derived from economic problems, can be found in Baumol and Benhabib (1989) and Hans Walter Lorenz (1989).

On the contrary, the role of the second element of the May model (i.e. the lag) has been rather neglected in economists' discussions of chaotic dynamics. Comments on this point are often limited to some *passim* observations that most mathematical theorems utilized in this case only apply to discrete, one-dimensional systems and that the most interesting result—the occurrence of chaotic dynamics—disappears when (supposedly) equivalent continuous-time formulations of the same problems are considered.

This neglect is particularly surprising since there exists in the economic literature a lively and intellectually stimulating debate on the relative virtues and shortcomings of discrete and continuous models, which is very relevant to the point in question. Within the limits of this paper, we cannot explore the issue exhaustively. It will be sufficient here to briefly recall the main difficulties that arise in the study of an aggregate model (economic or otherwise) by means of

so-called 'period analysis', i.e. in discrete time.[3] First of all, even though economic transactions of a given type do not take place continuously and are therefore discrete, in general they will not be perfectly synchronized as period analysis implicitly assumes, but overlap in time in a random manner. Only in very rare circumstances (e.g. in an agricultural, single-crop economy), could one define a 'natural period' for the economic activity under investigation. Whenever this is not possible, there is a danger that the implicit assumption underlying the fixed delay hypothesis may yield misleading conclusions.

The possibility that some of the conclusions obtained by means of period analysis may be mere artifacts owing to the mispecification of the model is clearly present in virtually all the existing applications of equation (1) to economic problems, for which no aggregate 'natural period' could be defined. Those who think that chaotic dynamics do exist in real economies (this author is among them) are therefore under obligation to show how this particular mispecification can be avoided.

To satisfy this requirement, in the following sections we shall develop a procedure that, in our opinion, will contribute to clarify some not fully understood aspects of the connection between discrete and continuous-time representations of a process.

3. PROBABILITY DISTRIBUTION OF LAGS[4]

Let us consider any of the economic models discussed in the works quoted in the preceding section (perhaps Benhabib and Day's model of rational consumer, or Grandmont's overlapping generations model[5]) and suppose that, although we accept the authors' arguments in any other respects, for the reason discussed above we reject the hypothesis of a single fixed delay as an unduly crude way of aggregating the economy. Instead of a single 'representative' economic agent (or unit), as implicitly postulated by those models, we consider a hypothetic economy consisting of an indefinitely large number of agents, who respond to a certain signal with given discrete lags. The lengths of the lags are different for different agents and are distributed in a random manner over all the population. In this situation, the economy's aggregate time of reaction to the signal can be modelled by a real, nonnegative random variable T, the overall length of the delay.

In principle, T can be distributed in a number of different ways, depending on the specific problem at hand. There exist, however, certain general criteria for choosing an 'optimal' probability distribution, given certain constraints. In the attempt to estimate the true distribution of a random variable, a statistician, in order to avoid unjustified biases, should formulate his assumptions so as to

[3] This paragraph is closely related to the discussion developed in J. May (1970) (a different May!), Foley (1975) and Turnovsky (1977). In order to facilitate reference to economic literature, in what follows we use the concepts of 'period' and 'fixed delay lag' as synonyms.

[4] In this section we follow Invernizzi and Medio (1991) (henceforth IM), to which we refer the reader for a more detailed discussion and bibliographical reference.

[5] As concerns this particular model, however, we only refer here to so called 'backward dynamics', leaving aside any questions concerning its relation with 'true', forward dynamics.

maximize the uncertainty about the system, subject to the constraints deriving from his prior knowledge of the problem. A rigorously defined measure of uncertainty is provided by 'entropy'.

The concept of entropy was introduced after World War II in the context of Information Theory by Shannon [1948], but the idea can be traced back to Boltzmann and earlier. Broadly speaking, if we consider a random variable X taking a finite number of values with probability $p_1, \ldots, p_N$, we define the entropy of X as:

$$H = -\sum_{i=1}^{N} p_i \log(p_i).$$

The quantity H measures the uncertainty concerning X, or equivalently the amount of information we get on the average by making an observation. An equivalent, more complicated definition can be derived for continuous random variables.[6]

The 'principle of maximum entropy' for selecting probability distribution was put forward in the economic literature by Theil and Fiebig (1981) and can be looked at as a generalization of the famous Laplace's 'principle of insufficient reason'.

As this author has discussed elsewhere (see IM quoted), when the random variable in question is essentially positive, and the only *a priori* information on its distribution is that there exists a certain expected value (mean) and a certain geometric mean (or, equivalently, the mean of the logarithm of the random variable in question), then the principle of maximum entropy requires that a two-parameter gamma distribution be chosen.

If we indicate by α the shape parameter and β the scale parameter, the density function of a two-parameter gamma random variable can in general be written thus:

$$g(\alpha, \beta; t) = \begin{cases} 0, & \text{if } t \leq 0, \\ (\beta^\alpha \Gamma(\alpha))^{-1} t^{\alpha-1} e^{-t/\beta}, & \text{if } t > 0, \end{cases} \tag{2}$$

where Γ is the gamma function, i.e.

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt,$$

and $\alpha, \beta > 0$.

From the knowledge of the parameters α and β, we can derive the mean $(\alpha\beta)$, the variance $(\beta^2\alpha)$ and the geometric mean $(\beta \exp(\Gamma'(\alpha)/\Gamma(\alpha)))$ of the distribution.

Conversely, it can be shown that, for the gamma distribution, fixing mean and geometric mean is equivalent to fixing mean and variance. Since the latter two statistical indicators are the most commonly used in economics, we shall henceforth use them as the parameters of the distribution.

[6] *Cf.* e.g. Eckmann and Ruelle (1985, p. 638).

4. A FEED-BACK REPRESENTATION OF LAGS

We shall now temporarily abandon the probabilistic aspect of lags and shall consider the problem from the point of view of feedback theory.

Suppose a variable $Y(t)$ is related with a continuously distributed lag to another variable $Z(t)$, where of course $Z(t)$ may well indicate the same variable Y at some time different from t. The equation of the lag can be written in its time-form thus:

$$Y(t) = \int_0^\infty f(\tau) Z(t-\tau)\, d\tau, \tag{3}$$

where $f: \mathbf{R} \to \mathbf{R}$ is continuous and we have:

$$f(\sigma) = 0, \text{ for } \sigma \leq 0; \quad f(\sigma) \geq 0, \text{ for } \sigma > 0; \quad \text{and} \quad \int_0^\infty f(\tau)\, d\tau = 1.$$

Thus f can be thought of as a 'weighting function': it indicates the different impact that different values of Z in the more or less distant past have on the current value of Y. In principle any kind of time-profile of such an impact can be modelled by (3). Under rather general circumstances, the lag can equivalently be represented in the form of a differential operator. If we restrict the variable Z to be zero for $t < 0$, then, for each fixed t, $f(\tau)Z(t-\tau)$ is also zero for all $\tau > t$. Therefore, equation (3) can be re-written thus:

$$Y(t) = \int_0^t f(\tau) Z(t-\tau)\, d\tau. \tag{4}$$

The expression on the right-hand side of (4) arises in the theory of the Laplace Transform as the *convolution* of f and Z, and it is denoted by $f * Z$. We recall that $f * Z = Z * f$. Moreover, under general conditions (see, e.g. Kaplan, 1962, p. 40), we have:

$$\mathcal{L}[f * Z] = \mathcal{L}[f] \cdot \mathcal{L}[Z], \tag{5}$$

where $\mathcal{L}$ is the usual Laplace Transform.

If, moreover, $\mathcal{L}[f]$ is the reciprocal of a polynomial

$$V(p) = a_0 p^n + \ldots + a_n,$$

that is

$$f(\tau) = \mathcal{L}^{-1}[1/V(p)],$$

then $Y = f * Z$ is exactly the unique solution of the differential equation

$$V(D)Y(t) = Z(t), \tag{6}$$

where $D = d/dt$, with initial conditions $Y = Y' = \ldots = Y^{n-1} = 0$ at $t = 0$. (*cf.* Kaplan 1962, p. 358, Th. 20).

In particular, if the lag is of a multiple exponential type of n-th order, we shall have:

$$\frac{1}{V(p)} = \left(\frac{pT}{n} + 1\right)^{-n},$$

and consequently equation (6) will take the form:

$$\left(\frac{DT}{n}+1\right)^n Y(t) = Z(t)$$

or

$$Y(t) = \left(\frac{DT}{n}+1\right)^{-n} Z(t), \tag{7}$$

where n is a positive integer and T is the time-constant of the lag.

If we now calculate the 'weighting function' $f(t)$ corresponding to a multiple exponential lag by performing the inverse Laplace Transform operation, we shall obtain:[7]

$$f(t) = \left(\frac{n}{T}\right)^n \frac{t^{n-1}}{(n-1)!} e^{-nt/T}. \tag{8}$$

Considering now equations (2) and (8), and putting $\alpha = n$ and $\beta = T/n$, it can be shown that the weighting function of a multiple exponential lag of order n is the same as the density function of a two-parameter gamma distribution.[8] The time constant T of the lag operator corresponds to the mean and the order of the lag, n, is inversely related to the variance. It can also be promptly seen that, for $n \geq 2$, $f(t)$ will have a 'one-hump' form, with a maximum at $t = T[1-(1/n)]$. The greater the order of the lag, the smaller will be the variance around the mean T. For large values of n, therefore, the weighting function will have a peaked form, indicating that the value of the output at any given instant t mostly depends on the input delivered at a given instant of the past, $t - T$, or in a certain small neighborhood of it.

Two simple cases are most common in the economic literature.

4.1. *The Simple Exponential Lag*

This corresponds to $n = 1$. In this case, the reaction to the input starts immediately at $t = 0$, but the temporal evolution of the output, that is the growth of Y, is proportional to the excess of the input over the output. This leads to the following ordinary differential equation:

$$\dot{Y}(t) = \gamma[Z(t) - Y(t)], \tag{9}$$

where $\gamma > 0$ has dimension *time*$^{-1}$ and represents the *speed of adjustment*, whereas its inverse $1/\gamma = T$ can be considered the length of the simple exponential lag. In economic applications, the variable Z sometimes represents the desired, or equilibrium value of Y, so that equation (9) depicts an 'Achilles and the Tortoise' situation in which the actual magnitude 'chases' the desired one, approaching it at an exponentially slowing speed, and catching up with it only in the limit for $t \to +\infty$. The simple exponential lag, or exponential lag of order one,

[7] For the details of this operation see, e.g. Kaplan (1962).

[8] To the best of our knowledge, this result was first established in IM quoted above.

corresponds to a rather special (and crude) formalization of economic reaction-mechanisms.

4.2. *The Fixed Delay*

When n becomes indefinitely large, the weighting (and the distribution) function tends to a Dirac delta function on T, and the continuous exponential lag tends to a fixed delay of length T. In fact, we have:

$$\lim_{n\to\infty}\left(\frac{DT}{n}+1\right)^{-n} = e^{-DT}. \tag{10}$$

The expression on the right-hand side of equation (10) is called the *shift operator*, which, when applied to a continuous function of time, has the effect of translating the entire function forward in time by an interval equal to T (*cf.* Yosida, 1984, Part III, Ch. VIII). Thus the fixed delay lag employed in models of the type (1) can be seen as a special, limit case of a multiple exponential lag when the order of the lag tends to infinity. Equivalently, the aggregate fixed delay of those models can be viewed as a limit case of a system characterized by gamma distributed individual reaction times, which obtains when the dispersion around the mean (the variance) tends to zero.

The question now naturally arises whether some or all the interesting results obtained by the investigation of one-dimensional maps of type (1) can be reproduced by means of models adopting less drastic and more flexible assumptions concerning the weighting function, and consequently the time profile of the lag.

Mathematically, this can be done by replacing equation (1) with the more general, continuous-time n-th order differential equation:[9]

$$x_n = \left(\frac{D}{n}+1\right)^{-n} G(x_n) \tag{11}$$

or, equivalently, by the system of n first-order differential equations:

$$\left(\frac{D}{n}+1\right)x_j = x_{j-1}, \quad (j = 2, \ldots, n), \tag{12}$$

$$\left(\frac{D}{n}+1\right)x_1 = G(x_n), \tag{13}$$

where T has been put equal to 1, $G(\cdot)$ is a one-humped function like (1) and again $D = d/dt$.

From what we have said before, it appears that system (12)–(13) can be considered a continuous approximation of (1) and, for $n \to \infty$, the lag structures of the two systems become equivalent. The main purpose of what follows is to show that some of the most interesting features of equation (1) can be reproduced by

[9] A similar approach to this problem can be found in a paper by Colin Sparrow (1980) from the study of which we have greatly benefited. In fact, some of the results in this section and the following one can be considered as further developments of the same line of research. See also R. May's comments on this point (1983, pp. 548–549 and 555).

system (12)–(13) at a reasonably low level of dimensionality (i.e. for relatively low values of n). In particular, we wish to show that chaotic behaviour, which is known to exist for the former, also exists for (relatively) low-dimensional specifications of the latter. To make things more specific, we shall select, from now on a particular form of the function $G(x_n)$ which appears in (13), namely we put:

$$G(x_n) = rx_n(1 - x_n).$$

Qualitatively similar results can be obtained with different specifications of the one-hump function (*cf.* IM quoted).

5. SYMPTOMATOLOGY OF CHAOS: ANALYSIS AND NUMERICAL SIMULATIONS

At present, complete global information on the structure of trajectories of continuous dynamical systems of dimension greater than two can only be obtained by integrating numerically the differential equations of the systems, studying their geometry and computing the values of certain crucial quantitative properties. Although theoretical knowledge is not sufficient to provide a complete picture of the dynamics of the system, it is nevertheless indispensable to guide numerical computations and to interpret their results. In most applications, and specifically in economics, therefore, the only promising research strategy seems to be at the moment an association of analysis and numerical simulation, through which one can establish a series of 'symptoms' whose concurrence is the best available indicator of the presence of those patterns of behaviour one is looking for.

In the sequel, we shall search for 'symptoms' of chaotic behaviour by inspecting certain crucial qualitative and quantitative features of the system, such as the time profile of the variables, the morphology of the attracting set, the power spectrum as estimated by the Fast Fourier Transform, the fractal dimension, as measured by one or another of the various existing algorithms, and the so-called 'sensitive dependence on initial conditions' as indicated by the presence of positive Lyapunov Characteristic Exponents (LCEs).

Equipped with these ideas and tools of analysis, we first discuss some properties of the system (12)–(13) which are independent of the order of the lag n.

From (12)–(13) we gather that the equilibrium conditions are the following:

$$x_1 = x_2 = \ldots = x_n = \bar{x}$$

and

$$\bar{x} = r\bar{x}(1 - \bar{x}),$$

whence we obtain the two equilibrium solutions:

$$\bar{x}_1 = 0; \qquad \bar{x}_2 = 1 - (1/r). \tag{14}$$

As concerns the stability of equilibria, consider that the auxiliary equation can be

written in the simple form:

$$(1 + \lambda)^n = f'(\bar{x}), \tag{15}$$

where $n\lambda$ indicates an eigenvalue of the Jacobian matrix and $f'(\bar{x}) = dG(\bar{x})/dx = r(1 - 2\bar{x})$.

Hence we have:

$$f'(\bar{x}_1) = r; \qquad f'(\bar{x}_2) = 2 - r. \tag{16}$$

From equations (14)–(16), we gather that, for $r < 1$, the origin is the only non-negative equilibrium point and it is stable. At $r = 1$, we have a transcritical bifurcation: the equilibrium point at the origin loses its stability and a second, initially stable, equilibrium point bifurcates from it in the positive orthant of the phase space.

For $n < 3$, nothing much happens when we increase the parameter r: the positive equilibrium point remains stable for all values of $r > 1$ (for $n = 2$, damped oscillations occur for $r > 2$). For $n \geq 3$, however, as r increases past a certain value which depends on n, a Hopf bifurcation takes place and a periodic orbit bifurcates from the stable equilibrium point that becomes unstable. Successive bifurcations can be detected for greater values of r, although their exact structure still escapes us. Whatever value the parameter r may take, however, nothing more complicated than periodic orbits seem to occur for low values of n. However, when the order of the exponential lag becomes sufficiently large, there seems to exist a value of r beyond which the system gives chaotic output.

In order to analyze this case in detail, we select $n = 10$.

In this case, the first Hopf bifurcation occurs for $r \approx 3.6$ and, for $r \approx 5$, we have been able to detect some quife strong symptoms of chaos, which are illustrated in the following figures.[10]

To begin with some 'visual dynamic analysis', Fig. 2(a) illustrates a time series for the variable x_1, in which fluctuations *prima facie* appear to be irregular. Figure 2(b) shows the superposition of two trajectories with slightly different starting conditions (the difference is of the order 10^{-3}). The reader will observe that the trajectories at the beginning are indistinguishable, but, after a short time they diverge and, for a while, become apparently unrelated.

More impressive are the diagrams of Fig 3(a) and (b), which illustrate 2-D and 3-D projections of the motion of the system in the phase-space, after elimination of transients. The pictures show a sheet-like, approximately 2-D object which looks very much like Rössler's Band. The latter is a chaotic attractor yielded by a simple system of three differential equations (see Rössler, 1976). Our simulations indicate that all orbits starting in the positive quadrant rapidly move toward the attractor and, as time goes by, tend to fill it (but not quite), without settling on any apparently periodic trajectory.

This is a particularly interesting result, all the more so since we know that certain basic features of the dynamics of the Rössler model can be effectively

[10] Notice that, although strictly speaking there are 10 variables in the model, essentially this is a one-variable model. At any rate, the choice of the variable(s) in what follows is not essential.

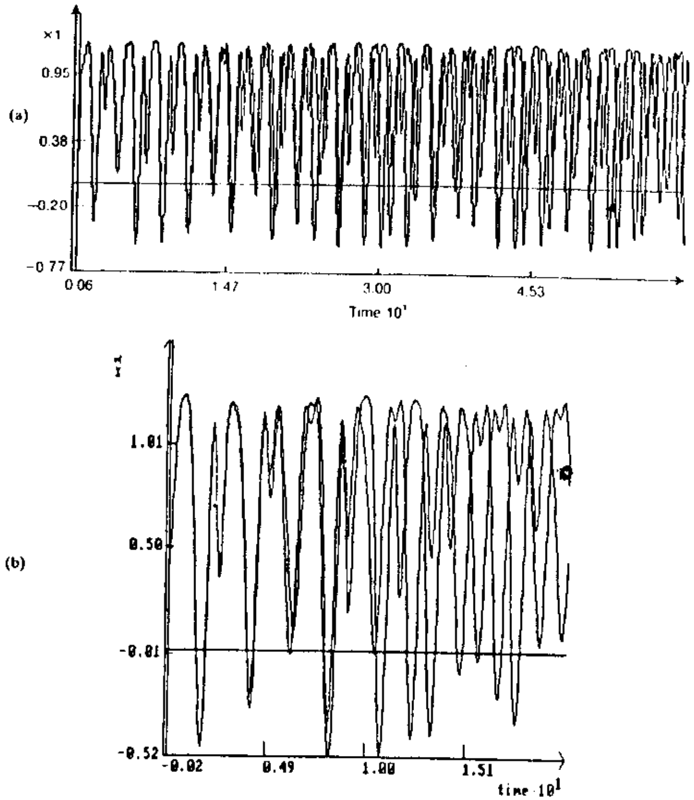

FIG. 2. (a) A time series $x_1(t)$. (b) Superposition of two trajectories with slightly different initial conditions.

approximated by means of a 1-D map, similar to the one we started with. We shall return to this point later.

A third symptom is illustrated in Fig. 4, which represents the power spectrum of a final trajectory of our model in the (presumably) chaotic zone, estimated by means of Fourier Fast Transform. (We measure log spectral density on the ordinate axis, and frequency on the abscissa.) We observe that the spectrum is characterized by a broad band with no isolated peaks. This provides a strong

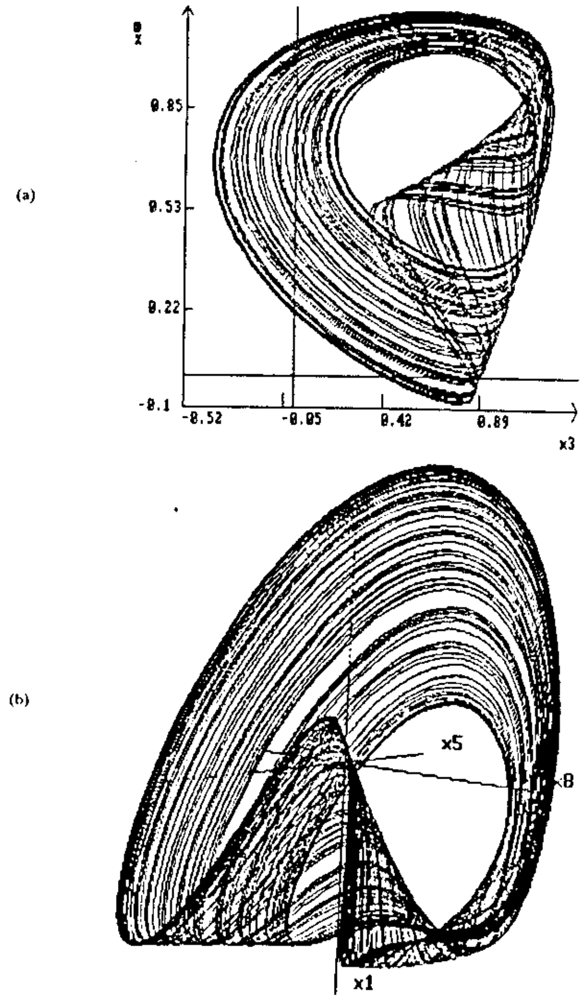

FIG. 3. (a) 2-D projection of the attractor. (b) 3-D projection of the attractor.

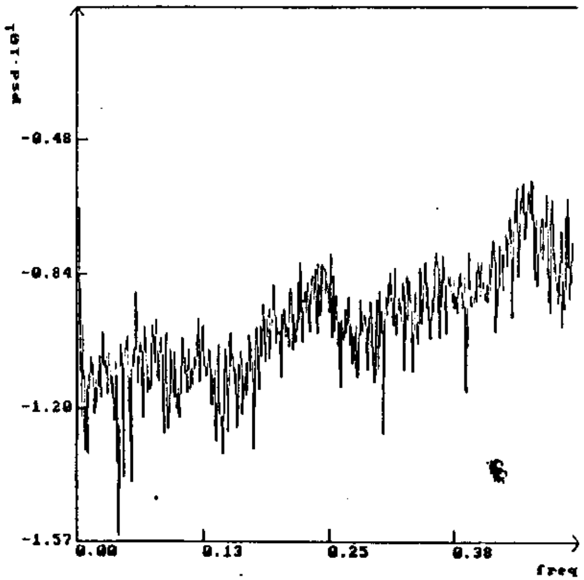

FIG. 4. The power spectrum of the final trajectory of the system.

presumption (although not conclusive evidence) of chaotic behaviour, especially so when the result is strengthened by the other indicators of chaos.

The fourth, and strongest evidence of chaotic behaviour of our model is provided by the calculation of LCEs[11] to evaluate. As is well known, the presence of an attractor with positive LCEs indicates that the motion of the system on the attractor has sensitive dependence on initial conditions, i.e. it is chaotic. Since in our case the attractor looks approximately 2-D, we would expect the sign pattern of the 10 LCEs to be $(+, 0, -, \ldots, -)$.

And this is precisely what we get from our computations. The essential results are shown in Fig. 5 which shows the first three LCEs (the other seven are all strongly negative). One will notice that the convergence is pretty strong and the dominant exponent is distinctly positive (≈0.35).

The interpretation is that the motion of the system is strongly convergent toward the attractor from all directions but two. The zero Lyapunov exponent is associated with the direction of the motion along the flow. The presence of one

[11] Basic reading on the concept of LCEs and the related computing techniques is Benettin *et al.* (1980).

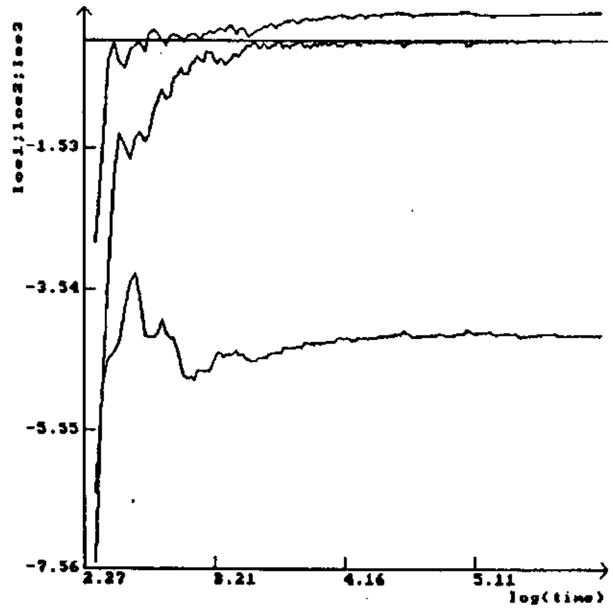

FIG. 5. The first three LCEs of the attractor.

positive exponent indicates that, on the attractor, there exists a direction along which nearby trajectories, on the average, diverge exponentially. This is tantamount to saying that the system has sensitive dependence of initial conditions; therefore it is chaotic.

From the knowledge of the LCE spectrum, we can also obtain an estimate of the fractal dimension of the attractor. By making use of a method first suggested by Kaplan and Yorke (1979), we can derive a measure of the Hausdorff dimension of the attractor from the calculated LCEs. Let us denote the 10 LCEs as χ_i $(i = 1, 2, \ldots, 10)$, and let us list them in descending order, namely:

$$\chi_1 > \chi_2 > \ldots > \chi_n.$$

Let j be the largest integer for which we have:

$$\chi_1 + \chi_2 + \ldots + \chi_j > 0.$$

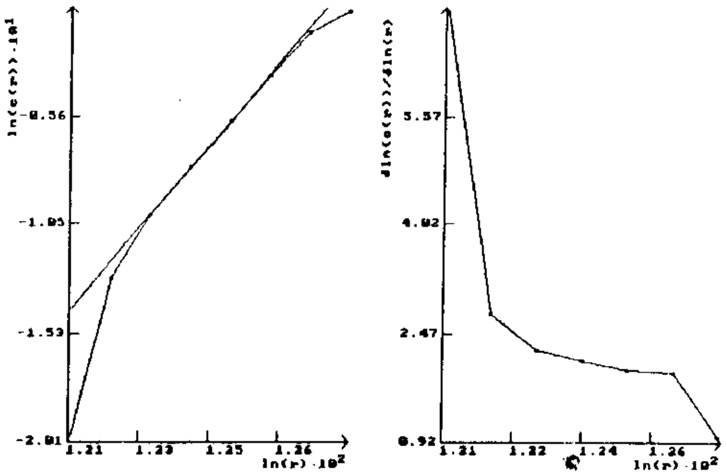

FIG. 6. Fractal dimension by the Grassberger and Procaccia method.

Then a measure of the fractal dimension can be written as:

$$d = j + \frac{\sum_{i=1}^{j} \chi_i}{|\chi_{j+1}|}.$$

In our case, we have $\chi_1 \approx 0.35$; $\chi_2 \approx 0.00$; $\chi_3 \approx -4.2$. Consequently, we obtain:

$$d \approx 2.08,$$

which is in harmony with the geometrical shape of the attractor, as it appears in the phase-space diagrams, and is of the same order of magnitude as the Hausdorff dimension estimated for the Rössler attractor.

The fractal dimension has also been computed (always for $r = 5$) by means of the Grassberger and Procaccia method. The results are shown in Fig. 6. The estimated dimension is about 2.15 which is close to the Lyapunov dimension.

6. A RECONSTRUCTED 1-D MAP

Recalling a comment made in the previous section concerning the Rössler attractor, one may wonder whether the chaotic nature of our (continuous multidimensional) system (12)–(13) could be captured by means of a 1-D map. After all, this was the starting point of our investigation.

To answer this question, we have adopted a procedure which is sometimes

called the '1-D Map Approach' and which, although not yet fully justified from a mathematical point of view, has become common in the literature on continuous chaos. This procedure was successfully employed by Edmund Lorenz in his celebrated paper (1963) and, more recently, by Rössler (1976), Roux *et al.* (1983) and others. In all these cases, the authors showed that an auxiliary 1-D map can be derived from the original continuous model, which preserves much of the behaviour of the full flow, in particular with regard to the chaotic nature of its attractor. This procedure is particularly appropriate when the rate of dissipation, as measured by the trace of the Jacobian matrix, averaged over the attractor of the system, is strong. This indeed applies to our case, where, as it can be promptly verified, the trace is constant and equal to $-n^2$.

In order to apply the '1-D Map Approach' to our system, let us consider the chaotic case $r = 5$. Then let us construct a (planar) Poincaré surface of section, transversal to a 3-D projection of the attractor, and study the successive intersections of a trajectory with this surface. This is illustrated in Fig. 7. We observe that if a suitable projection of the attractor is chosen and, if the surface is suitably positioned across the flow, the intersection looks like a (fractal) collection of points lying on an open curve. This is an interesting result and provides in itself a symptom of chaotic behaviour.[12]

Next, let us measure the values of either of the co-ordinates (in the section plane) of each intersecting point, and let z_n denote the sequence in time of these values. Clearly, there are no *a priori* reasons for there to be any definite relation between such consecutive values. In general, we would expect them to be scattered throughout the plane, within the limits of the intersecting curve. However, if we plot z_{n+1} as a function of its antecendent z_n, we obtain a fairly well defined curve, which has a familiar one-hump shape, with a rather steep slope (see Fig. 8).

To complete our experiment, we can try to estimate the (unique) LCE associated with the reconstructed 1-D map. The LCE in this case is defined by:[13]

$$\chi = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln |f'(z_i)|.$$

where $f(z)$ is the reconstructed map and $f'(z_i)$ is the derivative of the map at z_i. In practice, one takes n large enough to show convergence of the LCE to a limit value. By fitting the data (i.e. the points of the first return map) by cubic splines, it is possible to calculate the derivative at each point, and subsequently compute the LCE. The result we have obtained is $\chi \approx 0.68$, which indicates that the dynamics generated by the reconstructed 1-D map is chaotic.

[12] Notice that, in principle, the choice of a particular cut transversal to the flow should in no essential way affect the result we are pursuing. Indeed, taking a different cut corresponds to a co-ordinate transformation on the return map, and a theorem by Oseledec (1968) guarantees that the LCEs spectrum is invariant with respect to co-ordinate transformation. See, on this point, Lichtenberg and Lieberman (1983, p. 12).

[13] See Lichtenberg and Lieberman (1983, p. 11).

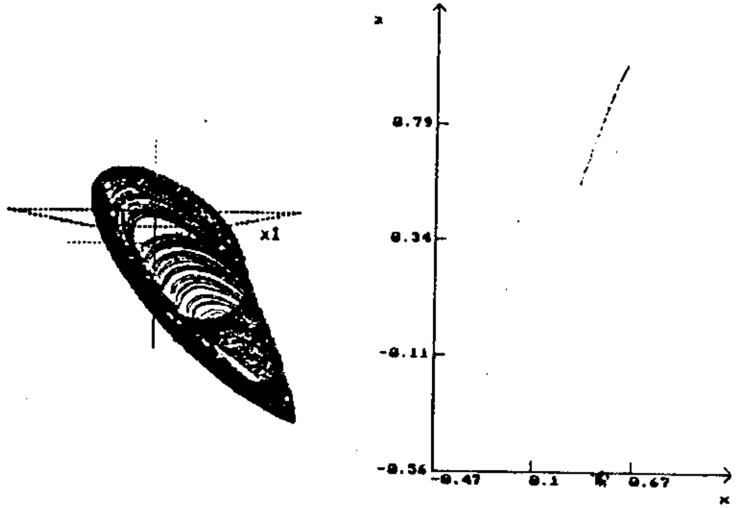

Fig. 7. The first return map.

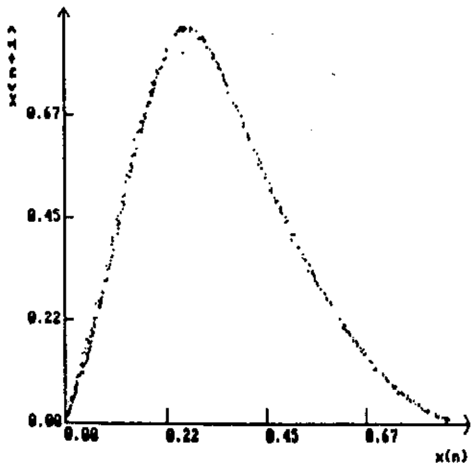

Fig. 8. The reconstructed 1-D map.

Notice an important point here. We have been able to show that an essential characteristic of the dynamics of our (continuous, multidimensional system) can be captured by a 1-D discrete formalism. However, this is *not* equivalent to finding a constant time delay between each two successive values of the representative variable. On the contrary, the time interval between each two successive intersections of the 'reconstructed' Poincare map will in general be different. This basic difference between the '1-D Map' and the fixed delay map makes the former a more flexible and more satisfactory representation of those economic mechanisms (the great majority, we believe) for which no 'natural period' may be properly defined.

6. CONCLUSIONS

Altogether our findings seem to confirm the possibility of chaotic dynamics in continuous models of economics, without the need for recourse to unduly strict and unrealistic assumptions concerning the lag structure.

Two considerations in particular seem to be prompted by our analysis. First of all, for the class of models under investigation, it is the combination of non-monotonic non-linearities of the one-hump type *and* a certain structure of the lags that produces chaos. The extreme assumptions implicit in 1-D, fixed delay models appear unnecessary. Multiple exponential lags possess a much greater flexibility, they permit one to model a large variety of economic situations, and their two controlling parameters, T and n can be estimated econometrically.

On the other hand, the results of this paper indicate that a sufficiently high order of the exponential lag, which corresponds to a sufficiently low variance in the distribution of individual reaction times, is a necessary (not sufficient) condition for chaos to occur. A hint for understanding this fact (although by no means a rigorous explanation) may be given by considering that a peaked 'weighting function' corresponds to a low variance of the individual reaction-times around the mean, i.e. it corresponds to a system which, in this respect, is homogeneous and tends to respond 'rigidly' to stimuli. It is known that, other things being equal, such a dynamical rigidity makes the dampening of impulses more difficult and resonance phenomena, and the related fluctuating behaviour, more likely.

It would be extremely interesting to be able to specify more completely and rigorously the relationship between the pattern of the lags and the insurgence of chaotic dynamics. However, this we shall leave for future research.

As a final critical note, we may observe that the comment, sometimes found in the literature, that continuous counterparts of discrete models yield qualitative results sharply different from the original ones, is ill-conceived. When the approximation is performed correctly, i.e. when it is fully understood that a discrete model is essentially infinite-dimensional, it becomes possible to reproduce qualitatively in a continuous-time setting the same results as those obtained in a discrete-time representation.

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