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TOPOLOGICAL GAUGE THEORY AND YANG-MILLS THEORY

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Abstract

In these lecture notes we explain a connection between Yang-Mills theory on arbitrary Riemann surfaces and two types of topological field theory, the so called *BF* and cohomological theories. The quantum Yang-Mills theory is solved exactly using path integral techniques. Explicit expressions, in terms of group representation theory, are obtained for the partition function and various correlation functions. In a particular limit the Yang-Mills theory devolves to the topological models and the previously determined correlation functions give topological information about the moduli spaces of flat connections. In particular, the partition function yields the volume of the moduli space for which an explicit expression is derived.

These notes are self contained, with a basic introduction to the various ideas underlying the topological field theories. This includes some relatively new work on handling problems that arise in the presence of reducible connections, which in turn, forms the bridge between the various models under consideration.

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1 Introduction

Topological gauge theories were the first of the topological field theories to be put forward. The two broad types of topological field theory were introduced originally by Schwarz [1], to give a field theoretic description of the Ray-Singer torsion [2], and by Witten [3], to give path integral representations of the Donaldson polynomials [4].

Of the Schwarz type, only the Chern-Simons model of Witten has been extensively analysed [5]. The non-Abelian generalizations of Schwarz's original actions, the so called BF models [6, 7] have been shown to have partition functions which reduce to integrals over the moduli space of flat connections with some power of the Ray-Singer torsion as the measure [6, 8]. Apart from establishing that certain correlators calculate intersection numbers of submanifolds there have been virtually no concrete calculations performed with these theories [6, 9].

The Witten or cohomological type theories have suffered a similar fate. The one exception here being two dimensional topological gravity [10] where a wide range of interesting results have been obtained (see [11] for a recent review). On the formal side, topological gauge theories, of Witten type, can be associated with particular geometric structures on the space of connections \mathcal{A} modulo the group of gauge transformations \mathcal{G} . \mathcal{A}/\mathcal{G} has a natural principle bundle structure (the universal bundle of Atiyah and Singer [12]) and also a natural Riemannian structure [13]-[15]. Hitherto, in cohomological gauge theories, \mathcal{A}/\mathcal{G} has been considered from the principle bundle point of view [16]-[20] and as a Riemannian manifold [21, 22]. For a general reference to, both the Witten and Schwarz, topological theories see [23].

There is one more geometric structure that may be placed on \mathcal{A}/\mathcal{G} , under ideal circumstances, and this particular aspect of the space is a meeting ground for the Schwarz and Witten type theories. Depending on the underlying manifold M it may be possible to induce a symplectic structure on \mathcal{A} . This is indeed possible when M is Kähler (though it need not be). Examples include Riemann surfaces and complex Kähler surfaces.

The topological field theories, that we will be concerned with, are a topological gauge theory of flat connections over Riemann surfaces and a topological gauge theory of instantons over four dimensional manifolds. It turns out that, in order to define the topological theory, one needs to 'regularize' the model to avoid problems with reducible connections. This regularization amounts to considering instead Yang-Mills theory which, in the limit as the gauge coupling e^2 goes to zero, reduces

to the topological theory [24, 25]. We find ourselves in the interesting situation of studying a 'physical' theory in order to extract topological information. Indeed most of the lectures are devoted to an evaluation of the path integrals of Yang-Mills theory on Riemann surfaces.

Now, in their own right, gauge theories in two dimensions have for a long time served as useful laboratories for testing ideas and gaining insight into the properties of field theories in general. While classically Yang-Mills theory on topologically non-trivial surfaces is well understood [26], very little effort had gone into understanding quantum gauge theories on arbitrary Riemann surfaces, the notable exception being in the context of lattice gauge theory [27] which is based on previous work by Migdal [28] (see also [29]). In the continuum quantum Yang-Mills theory on \mathbf{R}^2 was solved in [28, 30] and on the cylinder in [31].

Here we study Yang-Mills theory from the path integral point of view. In particular we will get general and explicit expressions for the partition function and the correlation functions of (contractible and non-contractible) Wilson loops on closed surfaces of any genus as well as for the kernels on surfaces with any number of handles and boundaries. These expressions will yield corresponding results for the topological theory in the limit. We will not be able to fix overall constants in our formulae, these require a more detailed analysis and/or input from another source. The method of calculation is based on published work with Matthias Blau [32]. An analogous, but perhaps more mathematically rigorous, derivation of some of these results may be found in the work of Fine [33]. There are also unpublished lectures by P. Degiovanni [34] where a mixture of canonical quantization and the axiomatic approach to topological field theories is used to get to these results. A derivation in the spirit of [28, 27], was provided by Witten [24].

Perhaps the correct way of deciding on the 'type' of topological field theory one has in hand is with respect to which fixed point theorem applies to it. Atiyah and Jeffrey [35] have shown that the cohomological field theories, as they had been discussed, were naturally understood in terms of the Marhai-Quillen construction [36]. An introductory account of this point of view, explaining how the zero's of a map are singled out, is given in [37]. On the other hand, it had also been known that the path integral formulation of index theorems [38] devolved to calculations of fixed points because of the theorem of Duistermaat and Heckman [39]. It is this aspect of the two dimensional cohomological gauge theory that is stressed in [25]. Unfortunately, there is no time to go into this side of things, except in passing.

I have taken this opportunity to prove some of the technical facts that were

passed over in [32] and also to include some previously unpublished calculations [40]. Notation is by and large explained in appendix A

2 Moduli Space Of Flat Connections And Topological Gauge Theory

Our basic concern in these notes is with the space of flat connections (gauge fields) on a Riemann surface. We define this space on a general manifold M . Pick a connected, compact gauge group G . A connection A on a G bundle over M , or a gauge field on M , is said to be flat when its curvature tensor F_A vanishes,

$$F_A = dA + \frac{1}{2}[A, A] = 0. \quad (2.1)$$

Flatness is preserved under gauge transformations $A \rightarrow A^U$ where

$$A^U = U^{-1}AU + U^{-1}dU, \quad (2.2)$$

as F_A transforms to $U^{-1}F_AU$. The moduli space of flat connections $\mathcal{M}_F(M, G)$ is the space of gauge inequivalent solutions to (2.1). This means that solutions to (2.1) which are not related by a gauge transformation are taken to be different points of $\mathcal{M}_F(M, G)$. On the other hand, if two solutions to (2.1) are related by a gauge transformation they are taken to be the same point in $\mathcal{M}_F(M, G)$, that is $A^U \equiv A$.

There is another description of the moduli space which is useful. This is in terms of representations of the fundamental group $\pi_1(M)$ of the manifold M ,

$$\mathcal{M}_F(M, G) = \text{Hom}(\pi_1, G)/G, \quad (2.3)$$

that is of equivalence classes of homomorphisms

$$\varphi : \pi_1(M) \rightarrow G \quad (2.4)$$

up to conjugation. $\pi_1(M)$ is made up of loops on the manifold M with two loops identified if they can be smoothly deformed into each other. All contractible loops are identified. $\pi_1(M)$ is a group under the composition of loops with the identity element the contractible loops.

We can easily see half of (2.3). Given a flat connection A we can form a map φ_γ by setting $\varphi_\gamma(A)$ to be the holonomy (which is an element of the group G) around a loop γ in M ,

$$\varphi_\gamma(A) = P \exp \int_\gamma A. \quad (2.5)$$

In physicists notation this is a Wilson loop. Recall that P stands for path ordering

$$P \exp \left(\int_0^1 f(t) dt \right) = \sum_{n=0}^{\infty} \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n f(t_n) \dots f(t_1). \quad (2.6)$$

Under a gauge transformation (2.5) goes to

$$\varphi(A^U) = U(0)^{-1} \varphi(A) U(1), \quad (2.7)$$

so, as $U(0) = U(1)$, gauge equivalent A 's give conjugation equivalent $\varphi(A)$'s.

We still need to show that the maps only depend on the homotopy class of the loop γ . This is where flatness comes in; we have not used it yet. Add to γ a small homotopically trivial loop $\delta\gamma = \partial\Gamma$ (it is the boundary of some disc Γ) then

$$\begin{aligned} \varphi_{\gamma+\delta\gamma}(A) - \varphi_\gamma(A) &= \int_0^1 dt P \exp \left(\int_0^t A_\mu \frac{d\gamma^\mu(s)}{ds} ds \right) \delta \left[A_\mu(t) \frac{d\gamma^\mu(t)}{dt} \right] P \exp \left(\int_t^1 A_\mu \frac{d\gamma^\mu(s)}{ds} ds \right) \\ &= \int_0^1 dt P \exp \left(\int_0^t A_\mu \frac{d\gamma^\mu(s)}{ds} ds \right) F_{\nu\mu}(t) \frac{d\gamma^\nu(t)}{dt} \delta\gamma^\mu(t) P \exp \left(\int_t^1 A_\mu \frac{d\gamma^\mu(s)}{ds} ds \right) \\ &= 0. \end{aligned} \quad (2.8)$$

The first equality follows from the variation of the definition of path ordering (2.6), while the second arises on integrating by parts in t . We have just shown that only the homotopy class of the loop γ is involved in the map φ_γ .

This establishes that each point in $\mathcal{M}_F(M, G)$ gives an element in $\text{Hom}(\pi_1(M), G)/G$. The proof of the converse, that each element in $\text{Hom}(\pi_1(M), G)/G$ naturally defines a flat connection, makes use of the notions of covering manifolds and associated bundles.

Dimension Of $\mathcal{M}_F(\Sigma_g, G)$

We now concentrate on compact Riemann surfaces of genus g , $M = \Sigma_g$, and compact gauge group G . In this case it is known that $\mathcal{M}_F(\Sigma_g, G)$ is smooth except at singular points which arise at reducible connections. The reducible connections will be defined shortly; a great deal of the formalism developed is there to get around problems generated by these connections.

Now to a Riemann surface Σ_g there is a standard presentation of π_1 in terms of the $2g$ generators $a_i, b_i, i = 1, \dots, g$. One basis for these 'homology' cycles is displayed in figure 1. However, they are not independent generators. To see this it is easiest to form the cut Riemann surface. One picks a point P on the surface and then cuts from that point along a fundamental cycle back to the point. This

is repeated for the basis of cycles, the final result being a cut Riemann surface. In figure 2 this process is shown for the torus T^2 . Figure 3, shows the homology basis for the genus 2 surface, and how the basis is pulled to the point P and cut is shown in figure 4. Now the path that is defined by the edge of the cut Riemann surface is generated by

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}, \quad (2.9)$$

but this path is contractible to a point in the interior of the cut Riemann surface, so it is the trivial element in π_1 . We have the relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1. \quad (2.10)$$

It turns out that this is the only relation satisfied by the generators on the Riemann surface.

The dimension of the moduli space for $g > 1$ and simple G may be calculated from the information that we have at hand. The *Hom* part of (2.3) asks for the possible assignment of group elements to generators. There are $2g \dim G$ ways of doing this, but we must subtract off the one relation (2.10), that is minus $\dim G$ and also the identification of conjugacy classes implies that we ought to subtract another $\dim G$. We have, therefore,

$$\dim \mathcal{M}_F(\Sigma_g, G) = (2g - 2) \dim G. \quad (2.11)$$

When the manifold is the two sphere, $g = 0$, all loops are contractible so $\pi_1(S^2) = id$ and $\mathcal{M}_F(\Sigma_g, G)$ is one point, the trivial representation. This means that up to gauge equivalence the only flat connection is the trivial connection $A = 0$. For the torus, $g = 1$, the situation changes somewhat. In this case the relation (2.10) is $ab = ba$ so that a and b must commute. The homomorphism must therefore ensure that when mapped into G their images commute. Generically a and b can be represented, in this case, by elements lying in the (same) Maximal torus T of G . The dimension is

$$\dim \mathcal{M}_F(\Sigma_1, G) = 2 \dim T. \quad (2.12)$$

Life is simplified when $G = U(1)$. As everything in sight must commute, the relation (2.10) is automatically satisfied and conjugation acts trivially. We have $\dim \mathcal{M}_F(\Sigma_g, U(1)) = 2g$.

Topological Gauge Theory

We would like to be able to get more information than just the dimension. Different types of topological field theories indeed give different sorts of information about these moduli spaces. Let us define what we mean by a topological field theory.

For the purposes of these lectures a topological field theory is a field theory defined over some manifold M whose partition function is invariant under smooth deformations of any metric one puts on M . In such a theory it is possible to find correlation functions which enjoy the same property. A topological gauge theory is a topological field theory which is also a gauge theory. The correlation functions of interest in this case need to be not only metric independent but also gauge invariant.

In the course of the lectures we will come across two types of topological gauge theory. The first, known as a *BF* model, has a partition function that equals the volume of $\mathcal{M}_F(\Sigma_g, G)$. Due to the singularities of the moduli space, we will need to generalise the discussion somewhat and consider Yang-Mills theory. The partition function for the Yang-mills theory will be determined and in the topological limit we will be able to get a handle on $\text{vol} \mathcal{M}_F(\Sigma_g, G)$. The second type of topological field theory that we come across is known as a cohomological gauge theory. Considerations from this theory show us that the volume we calculate is the symplectic volume of $\mathcal{M}_F(\Sigma_g, G)$. Correlation functions in the cohomological theory may be interpreted in terms of intersection theory on the moduli space. We consider only the dual point of view, that is as integration of differential forms over $\mathcal{M}_F(\Sigma_g, G)$.

3 BF Theory on a Riemann Surface

We are interested in the moduli space of flat connections $\mathcal{M}_F(\Sigma_g, G)$ on a Riemann surface of genus g and compact structure group G . In the previous section we saw that this is the space of gauge inequivalent solutions to the flatness condition

$$F_A = 0. \quad (3.1)$$

A field theory that restricts one to this space is given by the path integral [6, 7]

$$Z(\Sigma_g) = \int D\phi DA \exp \left(\frac{1}{4\pi^2} \int_{\Sigma_g} \text{Tr} \phi F_A \right), \quad (3.2)$$

where ϕ is an adjoint valued field. Traditionally the field ϕ here is denoted by B and a glance at the partition function will explain the reason for the name *BF* theory. We have broken with tradition in order to make a smooth transition to the cohomological model. Formally, at least, on integrating out ϕ the path integral gives the volume of the moduli space of flat connections

$$\begin{aligned} Z(\Sigma_g) &= \int DA \delta(F_A) \\ &= \text{vol} \mathcal{M}_F(\Sigma_g, G). \end{aligned} \quad (3.3)$$

As it stands this formula is far too implicit and we will have need to modify it in making sense of the last equality. But first let us establish some formal aspects of the theory.

(i) Gauge Invariance:

The action in (3.2) is invariant under the gauge transformation (2.2) combined with

$$\phi^U = U^{-1} \phi U. \quad (3.4)$$

The infinitesimal form of the transformations are

$$\delta_A A = d_A \Lambda, \quad \delta_A \phi = [\phi, \Lambda]. \quad (3.5)$$

In order to correctly specify the path integral, we will need to gauge fix. The reason for this is that gauge invariant operators are constant on the orbit of the group of gauge transformations. One gets an infinity as one integrates over each orbit. It is this infinity that needs to be factored out. Rather than integrating over \mathcal{A} one wants to integrate over \mathcal{A}/\mathcal{G} . We use the Fadeev-Popov method to pick the gauge and fix on

$$G(A) = 0 \quad (3.6)$$

where $G(A)$ could be, for example, $d_{A_0} * (A - A_0)$ (where $*$ is the Hodge duality operator with respect to some metric $g_{\mu\nu}$ on Σ_g and A_0 is some preferred connection) or, as we will be mostly working on the disc, $G(A) = A_r$. The partition function is now

$$Z_{\Sigma_g} = \int DAD\phi DcD\bar{c}Db \exp \left(\frac{1}{4\pi^2} \int_{\Sigma_g} \text{Tr} \phi F_A + \int_{\Sigma_g} \text{Tr} (ibG(A) + \bar{c} \frac{\delta G}{\delta A} d_A c) \right). \quad (3.7)$$

The extra contributions to the action may be written as a BRST variation

$$Q \int_{\Sigma_g} \text{Tr} \bar{c} G(A), \quad (3.8)$$

with Q the BRST operator,

$$\begin{aligned} QA &= d_A c, & Qc &= -\frac{1}{2}[c, c], \\ Qc &= ib, & Qb &= 0. \end{aligned} \quad (3.9)$$

As usual one has traded overall gauge invariance for BRST invariance.

This is not quite as much of the group volume that can be factored out. Elements h in G (that is constant maps $h \in \mathcal{G}$) that form the centre of G , $Z(G)$

do not act on A or ϕ . It is also possible factor out the number of elements $\#Z(G)$ so that one should consider

$$\frac{1}{\#Z(G)} Z_{\Sigma_g}, \quad (3.10)$$

but this factor will generally be omitted.

(ii) Metric Independence:

At the level of (3.2), this is manifest, for the metric makes no appearance at all there. However, upon gauge fixing, we have introduced an explicit metric dependence in the action of (3.7). All of the explicit metric dependence rests in $G(A)$, so that, on varying (3.7) with respect to the metric, we find

$$\frac{\delta Z_{\Sigma_g}}{\delta g_{\mu\nu}} = \int_{\Phi} e^{L(\Phi)} Q \text{Tr} \bar{c} \frac{\delta G(A)}{\delta g_{\mu\nu}}, \quad (3.11)$$

where Φ is generic for all the fields and

$$L(\Phi) = \frac{1}{4\pi^2} \int_{\Sigma_g} \text{Tr} \phi F_A + \int_{\Sigma_g} \text{Tr} (ibG(A) + \bar{c} \frac{\delta G}{\delta A} d_A c). \quad (3.12)$$

By the BRST invariance of the theory the right hand side of (3.11) vanishes, whence the metric independence of the partition function is established (an account of how one derives such a Ward identity is given in section 4). This has all been rather formal. A more careful analysis, working with a regularized form of the theory, shows that indeed the theory remains metric independent, substantiating the analysis we have made.

(iii) Relationship to the Ray-Singer Torsion:

Let us suppose that the only flat connection is isolated and call it A_0 . Split the general connection A into $A = A_0 + A_q$ and take as the gauge condition $d_{A_0} * A_q = 0$. The path integral is

$$\begin{aligned} & \int DA_q \delta(F_{A_0+A_q}) \delta(d_{A_0} * A_q) \det(d_{A_0} * d_{A_0+A_q}) \\ &= \int DA_q \delta(d_{A_0} A_q) \delta(d_{A_0} * A_q) \det(d_{A_0} * d_{A_0+A_q}) \\ &= \det(d_{A_0}, d_{A_0} *)^{-1} \det(d_{A_0} * d_{A_0}), \end{aligned} \quad (3.13)$$

where the last equality arises on noting that the two delta functions imply $A_q = 0$, and the inverse determinant comes from extracting the operators out of the delta function with the rule

$$\int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_n \delta(T(x)) = \det(T(0))^{-1}. \quad (3.14)$$

Now the product of determinants is the Ray-Singer torsion. The torsion is unity on an even dimensional manifold, a fact that is easy to prove using field theoretic techniques [6], and we will do so for this case when we consider the trivialising map. Path integral representations of the torsion were introduced by Schwarz [1].

When the flat connections are not isolated the connection A_0 will depend on 'moduli' λ . The path integral must now include an integration over the moduli parameters, but for any λ the product of determinants is still one, so that we are again, formally, left with

$$\int d\lambda = \text{vol } \mathcal{M}_F(\Sigma_g, G). \quad (3.15)$$

Reducible Connections

So far we have concentrated on the flatness equation (2.1) which is one of the equations of motion that is obtained from the action of (3.2). The other equation of motion, obtained on varying the action with respect to the gauge field is,

$$d_A \phi = 0. \quad (3.16)$$

Connections A for which there are non-zero solutions ϕ to (3.16) are called reducible. Thinking of the ϕ as gauge parameters, then (3.16) is the statement that there are some gauge transformations that act trivially on the connection A . This means that $\mathcal{M}_F(\Sigma_g, G)$ is not, in general, a manifold as the quotienting out by the gauge group is not the same at each connection. Generically the connections are irreducible, and there will be isolated reducible connections. $\mathcal{M}(\Sigma_g, G)$ is then an orbifold. Turning this into a bona fide manifold is the process of 'compactification'.

(3.16) clearly holds when the two conditions

$$d\phi = 0, \quad [A, \phi] = 0, \quad (3.17)$$

are fulfilled. As an example consider the $su(2)$ valued gauge field

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (3.18)$$

with the possible form of ϕ being

$$\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad (3.19)$$

with b a constant. The connection (3.18) and the scalar field (3.19) live in a $u(1)$ subalgebra of $su(2)$. The $SU(2)$ gauge field will be flat when a is flat as a $U(1)$ gauge field.

Reducible connections are a source of great difficulty in making sense of topological field theories in general. The problem is that at a reducible connection path integrals of the type (3.2) diverge. The reason for this is that there are integrals to be performed over all the ϕ modes, but those modes which satisfy (3.16) do not appear in the action and hence do not dampen the integrals. For the reducible connection (3.18) there will be the undamped integrals

$$\int_{-\infty}^{+\infty} db = \infty. \quad (3.20)$$

Yang-Mills Connections

In order to overcome the problems associated with reducible connections, Witten has suggested a way of 'thickening' things out [24]. The idea is to spread the delta function (3.2) into a Gaussian in a gauge invariant way. The partition function is taken to be

$$Z_{\Sigma_g}(e^2 A(\Sigma_g)) = \int DAD\phi \exp \left(\frac{1}{4\pi^2} \int_{\Sigma_g} \text{Tr} \phi F_A + \frac{e^2}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right). \quad (3.21)$$

The ghosts and multiplier fields are implicit in this formula.

The dependence on the coupling e^2 and the area of the surface $A(\Sigma_g) = \int *$ in the combination $e^2 A(\Sigma_g)$ may be derived as follows. Scale the metric by $g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}$ then the term

$$\int_{\Sigma_g} \text{Tr} \phi * \phi = \int_{\Sigma_g} \sqrt{\det g} \text{Tr} \phi \phi, \quad (3.22)$$

scales to

$$\lambda \int_{\Sigma_g} \text{Tr} \phi * \phi. \quad (3.23)$$

This factor may be eliminated if we in turn send e^2 to $\lambda^{-1} e^2$. Shortly we will see the existence of a map that guarantees that the metric only enters as a measure (there are no derivatives of it). The invariant combination is then $e^2 A(\Sigma_g)$.

For an arbitrary metric, with no loss of generality, the part of the action $e^2 \int_{\Sigma_g} \text{Tr} \phi * \phi$ may be replaced with $e^2 A(\Sigma_g) \int_{\Sigma_g} \text{Tr} \phi * \phi$ where the metric here has area fixed to unity, $\int_{\Sigma_g} * = 1$. Because of this, we adopt the following convention: in the formulae obtained for the evaluation of the path integral the combination $e^2 A(\Sigma_g)$ will be denoted by ϵ , but also in the action we set $\epsilon = e^2$.

This is a rewriting of the Yang-Mills path integral which makes the relation to the topological theory of flat connections transparent. To see that this is the same as the Yang-Mills partition function, perform the Gaussian integration over the field ϕ^* with

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp(-e^2 x^2/2 + ixy) = \frac{1}{\sqrt{e^2}} \exp(-y^2/2e^2). \quad (3.24)$$

to obtain

$$Z_{\Sigma_g}(\epsilon) = \int DA \exp \left(+ \frac{1}{8\pi^2 \epsilon} \int_{\Sigma_g} \text{Tr} F_A * F_A \right). \quad (3.25)$$

The original theory is obtained in the $\epsilon^2 \rightarrow 0$ limit (or in the limit $A(\Sigma_g) \rightarrow 0$).

The minima of the new action in (3.21) are

$$F_A = -i\epsilon^2 * \phi, \quad d_A \phi = 0, \quad (3.26)$$

on combining the two one obtains the Yang-Mills equations

$$d_A * F_A = 0 \quad (3.27)$$

A gauge field satisfying (3.27) is said to be a Yang-Mills connection. The solutions to (3.26) fall into two distinct classes. The first is that $\phi = 0$ in which case the connections are flat. At this point we see that there is a partial resolution of the problem facing us, for in this sector we now have no ϕ zero-modes to worry about. This does not mean that the flat connections can not be reducible. There may well be non-zero Lie algebra valued functions φ for these connections such that $d_A \varphi = 0$. The equations tell us that these φ are not proportional to F_A . The second class has $\phi \neq 0$ for which the connections are not flat but are certainly reducible. As $\epsilon^2 \rightarrow 0$ these classes merge to give back the complicated situation of flat connections of which most, but not all, are irreducible.

To see that the action “regularizes” the contribution to the path integral of elements of the isotropy subgroup of the group of gauge transformations consider the pair (3.18, 3.19) and take a to be flat (choose the bundles so that this is possible). Inserting these into the path integral (3.21) we see that up to the volume of the flat $u(1)$ connections we are left with an integral

$$\int_{-\infty}^{+\infty} \frac{db}{\sqrt{2\pi}} \exp \left(- \frac{1}{4\pi^2 \epsilon} b^2 \right) = \sqrt{2\pi^2 / \epsilon}, \quad (3.28)$$

thus regularizing the infinity obtained on using the original path integral.

A Trivialising Map

There is no dynamics in pure Yang-Mills theory in two dimensions, even at the quantum level, for there are no physical degrees of freedom associated with the gauge field⁴. Indeed we will present a map which eliminates all (local) reference to the differential operators that are implicit in (3.21). Explicitly the partition

function is now

$$Z_{\Sigma_g}(\epsilon) = \int DAD\phi DcD\bar{c}Db \exp \left(\frac{1}{4\pi^2} \int_{\Sigma} \text{Tr} \phi F_A + \frac{\epsilon}{8\pi^2} \int_{\Sigma} \text{Tr} \phi * \phi + \int_{\Sigma} \text{Tr} (ibG(A) + \bar{c} \frac{\delta G}{\delta A} d_A c) \right). \quad (3.29)$$

The map that we have in mind is $A \rightarrow (\xi, \eta)$ defined by [6, 8]

$$\begin{aligned} \xi(A) &= F_A, \\ \eta(A) &= G(A), \end{aligned} \quad (3.30)$$

which has as its (inverse) Jacobian

$$J^{-1} = \det \frac{\delta(\xi, \eta)}{\delta A} = \det(d_A, \frac{\delta G(A)}{\delta A}). \quad (3.31)$$

Taking $G(A) = d_{A_0} * (A - A_0)$, we need to determine

$$\det(d_A, d_{A_0} *) = \det T. \quad (3.32)$$

Here T may be thought of as the map

$$\begin{aligned} T : \Omega^1(\Sigma_g, \text{Lie} G) &\rightarrow \Omega^0(\Sigma_g, \text{Lie} G) \oplus \Omega^0(\Sigma_g, \text{Lie} G) \\ T(\alpha) &= (*d_A \alpha, *d_{A_0} * \alpha). \end{aligned} \quad (3.33)$$

We can give a path integral representation of this as

$$\det T = \int D\sigma D\bar{\sigma} D\alpha \exp \left(i \text{Tr} \int_{\Sigma_g} \sigma d_A \alpha + \bar{\sigma} d_{A_0} * \alpha \right), \quad (3.34)$$

where σ and $\bar{\sigma}$ are Lie algebra valued Grassmann odd functions, and here α is a Lie algebra valued Grassmann odd one form. In order to get a handle on the determinant we define

$$\begin{aligned} \det_{\epsilon} T &= \int D\sigma D\bar{\sigma} D\alpha \exp \left(\text{Tr} \int_{\Sigma_g} i\sigma u_A \alpha + i\bar{\sigma} d_{A_0} * \alpha + \frac{\epsilon}{2} \alpha \alpha \right) \\ &= \int D(\sigma/\epsilon) D\bar{\sigma} \exp \left(-\text{Tr} \int_{\Sigma_g} \frac{1}{2\epsilon} (d_A \sigma + *d_{A_0} \sigma) * (d_A \sigma + *d_{A_0} \sigma) \right) \\ &= \int D\sigma D\bar{\sigma} \exp \left(-\text{Tr} \int_{\Sigma_g} \frac{1}{2\epsilon} (\epsilon d_A \sigma + *d_{A_0} \bar{\sigma})(\epsilon d_A \sigma + *d_{A_0} \sigma) \right), \end{aligned} \quad (3.35)$$

with

$$\det T \equiv \det_0 T. \quad (3.36)$$

Taking the limit $\epsilon \rightarrow 0$ in (3.35) is straightforward

$$\det T = \int D\sigma D\bar{\sigma} \exp \left(-\text{Tr} \int_{\Sigma_g} \bar{\sigma} d_{A_0} * d_A \sigma \right), \quad (3.37)$$

⁴In d dimensions the number of physical polarizations of a gauge boson is $d - 2$.

As the integrand on the right hand side is gauge invariant, we see that the application of δ once more vanishes, that is the action is BRST invariant. This action seems appropriate for our needs as the integral over B yields a delta function constraint onto the flat connections. That it defines a topological theory is not quite apparent, for the Hodge duality operator appears explicitly. Let \tilde{Z} be the path integral with this action. Then the metric variation δ_g of \tilde{Z} is

$$\begin{aligned}\delta_g \tilde{Z} &= \int_{\Phi} e^L \delta_g L \\ &= \int_{\Phi} e^L \delta \delta_g \int_{\Sigma_g} \text{Tr} (\chi F_A + \bar{\phi} d_A * \psi) \\ &= \int_{\Phi} \delta V_g,\end{aligned}\quad (4.5)$$

the last line being a defining equation for V_g . The order of δ and δ_g is not important as they commute (basically because the transformation rules (4.1) and (4.2) do not involve the metric).

The last line may be shown to vanish in some generality. Consider the vacuum expectation value of any operator \mathcal{O}

$$\int_{\Phi} e^{L(\Phi)} \mathcal{O}(\Phi). \quad (4.6)$$

One may change integration variables $\Phi \rightarrow \Phi + \delta\Phi$ and note that the action satisfies $L(\Phi + \delta\Phi) = L(\Phi)$ while also formally the path integral measure has the same property $\int_{\Phi+\delta\Phi} = \int_{\Phi}$. In terms of the new variables the expectation value is

$$\int_{\Phi} e^{L(\Phi)} \mathcal{O}(\Phi + \delta\Phi) = \int_{\Phi} e^{L(\Phi)} (\mathcal{O}(\Phi) + \delta\mathcal{O}(\Phi)), \quad (4.7)$$

from which we conclude that

$$\int_{\Phi} e^{L(\Phi)} \delta\mathcal{O}(\Phi) = 0. \quad (4.8)$$

This is exactly what is required to set the last line of (4.5) to zero. Indeed, replacing everywhere in this derivation δ with Q gives the Ward identity needed to establish that (3.11) vanishes, as well.

The theory defined by L seems to be just what we want, a topological field theory that lands on $\mathcal{M}_F(\Sigma_g, G)$. However, the partition function \tilde{Z} suffers greatly at the hands of the reducible connections. For, at a reducible connection, there are zero modes for the B , χ , $\bar{\phi}$, η and ϕ fields!

A Formulation In Terms Of The Symplectic Geometry Of \mathcal{A}/\mathcal{G}

Witten has proposed a method for avoiding the problems associated with the reducible connections [15]. The idea is to consider the integral

fields B , χ , $\bar{\phi}$ or η at all. Rather, one begins with the supersymmetric action

$$\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F_A + \frac{1}{2} \psi \psi \right), \quad (4.9)$$

which is a simple generalization of (3.2). Note that the fields ψ have no dynamics at all. Supersymmetry (4.1) fixes the relative coefficient of the two terms. Just as for the action (3.2) there will be ϕ zero modes. One 'thickens' this action out as well to

$$L = \frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F_A + \frac{1}{2} \psi \psi \right) + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi, \quad (4.10)$$

with corresponding path integral

$$Z_{\Sigma_g}(\epsilon) = \int DAD\phi \exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F_A + \frac{1}{2} \psi \psi \right) - \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right). \quad (4.11)$$

The exact relationship between the theories defined by (4.4) and (4.10) will be given at the end of this section.

One of the important properties of the partition function associated with the action (4.9) is that there is a canonical choice of measure. On making a choice for DA we pick the same for $D\psi$; this is supersymmetry preserving and the product $DAD\psi$ does not depend on the choices made. Put another way, if we send $A \rightarrow \lambda A$ then so as not to change the transformation rules (or the relative coefficients in the action) we must also send $\psi \rightarrow \lambda\psi$, and then there is no net effect on the measure, $DAD\psi \rightarrow D(\lambda A)D(\lambda\psi) = DA D\psi$.

Let us now interpret the extra term $\frac{1}{8\pi^2} \int_{\Sigma_g} \text{Tr} \psi \psi$ as a symplectic form on \mathcal{A} . Recall that a symplectic form ω on a $2n$ -dimensional manifold is a non-degenerate two form ($\det \omega \neq 0$) which is closed ($d\omega = 0$). There is a natural symplectic form on \mathcal{A} which is inherited from the two manifold Σ_g . If a and b are tangent vectors in \mathcal{A} , that is $a, b \in \Omega^1(\Sigma_g, \text{Lie} G)$, then one may construct the symplectic form

$$\Omega(a, b) = \frac{1}{8\pi^2} \int_{\Sigma_g} \text{Tr} (a \wedge b). \quad (4.12)$$

That $\Omega(\cdot, \cdot)$ is closed is obvious as it does not depend on the point $A \in \mathcal{A}$ at which it is evaluated. Invertibility is also clear. We see directly that $\frac{1}{8\pi^2} \int_{\Sigma_g} \text{Tr} (\psi \wedge \psi) = \Omega(\psi, \psi)$ represents the symplectic two form of \mathcal{A} .

For a finite dimensional symplectic manifold M , of dimension $2m$, an integral analogous to (4.11)

$$\int d^{2m}x d^{2m}y \exp \left(\frac{1}{2} \omega_{ij} x^i y^j \right)$$

$$\begin{aligned}
&= \int_M d^{2m}x d^{2m}\psi \frac{(\frac{1}{2}\psi\omega\psi)^m}{m!} \\
&= \int_M \frac{\omega^m}{m!},
\end{aligned} \tag{4.13}$$

yields the symplectic volume of M .

We then have the immediate consequence that the partition function of (4.9) (or the $\epsilon^2 \rightarrow 0$ limit of (4.11)) evaluates the symplectic volume of $\mathcal{M}_F(\Sigma_g, G)$.

Observables

There are three 'obvious' conditions that an observable \mathcal{O} (a functional of the fields) should satisfy in a topological gauge theory. These are gauge invariance, BRST invariance and metric independence. The third may be relaxed as we will see later. There is still a fourth condition so as not to get trivial observables. This is that $\mathcal{O} \neq \delta\Theta$ for any globally defined Θ^5 . For if $\mathcal{O} = \delta\Theta$ then by (4.8) its expectation value vanishes. Indeed this tells us that the observables must be BRST equivalence classes of gauge invariant and metric independent functionals of the fields. Two observables \mathcal{O}_1 and \mathcal{O}_2 are BRST equivalent (and have the same expectation value) if $\mathcal{O}_2 = \mathcal{O}_1 + \delta\Theta$ for any globally defined Θ .

On the space $\Sigma_g \times \mathcal{A}/\mathcal{G}$ we have the exterior derivative $d + \delta$ and the curvature form $F + \psi + \phi$. There is also the Bianchi identity

$$(d_A + \delta)(F_A + \psi + \phi) = 0, \tag{4.14}$$

from which we may derive the equations

$$(d + \delta)Tr(F_A + \psi + \phi)^n = 0. \tag{4.15}$$

Let $n = 2$ and write

$$\frac{1}{2}Tr(F_A + \psi + \phi)^2 = \sum_{i=0}^4 \mathcal{O}_i, \tag{4.16}$$

where the \mathcal{O}_i are i -forms with Grassmann grading $(-1)^{(4-i)}$ and are given by

$$\begin{aligned}
\mathcal{O}_0 &= \frac{1}{2}Tr(\phi\phi), & \mathcal{O}_1 &= Tr(\psi\phi), \\
\mathcal{O}_2 &= Tr(F_A\phi + \frac{1}{2}\psi\psi), & \mathcal{O}_3 &= \frac{1}{2}Tr(F_A\psi), \\
\mathcal{O}_4 &= \frac{1}{2}Tr(F_AF_A).
\end{aligned} \tag{4.17}$$

Expand the 'descent' equation (4.15) in terms of form degree and Grassmann grading as

$$\begin{aligned}
\delta\mathcal{O}_0 &= 0, \\
\delta\mathcal{O}_1 &= -d\mathcal{O}_0, \\
\delta\mathcal{O}_2 &= -d\mathcal{O}_1, \\
\delta\mathcal{O}_3 &= -d\mathcal{O}_2, \\
\delta\mathcal{O}_4 &= -d\mathcal{O}_3, \\
0 &= -d\mathcal{O}_4.
\end{aligned} \tag{4.18}$$

The \mathcal{O}_i are clearly gauge invariant and metric independent. The basic observables in the non-Abelian models on Σ_g are built from the \mathcal{O}_i for $i = 0, 1, 2$. The first of these is $\mathcal{O}_0(x) = \frac{1}{2}Tr(\phi(x)\phi(x))$ which is BRST invariant, not BRST exact, but appears to depend on the point x at which it is evaluated. Within the path integral this is not the case,

$$d \int_{\mathcal{Q}} e^L Tr(\phi\phi)(x)/2 = - \int_{\mathcal{Q}} e^L \delta\mathcal{O}_0 = 0. \tag{4.19}$$

Likewise integrating \mathcal{O}_1 over a one cycle γ gives a BRST invariant observable

$$\mathcal{O}_1(\gamma) = \int_{\gamma} Tr(\psi\phi), \tag{4.20}$$

$$\delta \int_{\gamma} Tr(\psi\phi) = - \int_{\gamma} d\mathcal{O}_0 = 0. \tag{4.21}$$

That the expectation value of $\int_{\gamma} Tr(\psi\phi)$ depends only on the homotopy class of γ may be seen as follows. Add to γ a homotopically trivial piece $\delta\gamma = \partial T$, then

$$\begin{aligned}
&\int_{\gamma+\delta\gamma} Tr(\psi\phi) - \int_{\gamma} Tr(\psi\phi) \\
&= \int_{\delta\gamma} Tr(\psi\phi) \\
&= \int_T dTr(\psi\phi) \\
&= -\delta \int_T \mathcal{O}_2.
\end{aligned} \tag{4.22}$$

The third observable is the integral of \mathcal{O}_2 over the Riemann surface,

$$\int_{\Sigma_g} Tr(F_A\phi + \frac{1}{2}\psi\psi), \tag{4.23}$$

with BRST invariance established as for the other observables.

⁵A more detailed account of when observables are trivial or not may be found in [19, 20, 23].

The last observable is the supersymmetric action (4.9) itself! Indeed taking into account that $\frac{1}{2}\text{Tr}(\phi\phi)$ is essentially independent of the point where it is evaluated the thickening term in (4.10) is also essentially an observable,

$$\int_{\Sigma_g} \frac{1}{2} \text{Tr}(\phi * \phi) \sim \mathcal{O}_0. \quad (4.24)$$

Observables In Terms Of The Partition Function

We now show how the expectation values of the observables are determined from the partition function

$$Z_{\Sigma_g}(\epsilon) = \int DA D\psi D\phi \exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F + \frac{1}{2} \psi \psi \right) + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right). \quad (4.25)$$

The first example is afforded by considering powers of \mathcal{O}_0 ,

$$\begin{aligned} \langle \prod_{i=1}^k \frac{1}{4\pi^2} \mathcal{O}_0(x_i) \rangle_\epsilon &= \int DA D\psi D\phi \exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F + \frac{1}{2} \psi \psi \right) \right. \\ &\quad \left. + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right) \cdot \prod_{i=1}^k \frac{1}{8\pi^2} \text{Tr} \phi^2(x_i). \end{aligned} \quad (4.26)$$

In the path integral the position of $\text{Tr} \phi^2(x_k)$ is immaterial so we may replace this with $\int_{\Sigma_g} \text{Tr} \phi * \phi$, using the measure with unit area. We find

$$\begin{aligned} \langle \prod_{i=1}^k \frac{1}{4\pi^2} \mathcal{O}_0(x_i) \rangle_\epsilon &= \int DA D\psi D\phi \exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F + \frac{1}{2} \psi \psi \right) \right. \\ &\quad \left. + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right) \cdot \left(\frac{1}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right)^k \end{aligned} \quad (4.27)$$

which is

$$\frac{\partial^k Z_{\Sigma_g}(\epsilon)}{\partial \epsilon^k}. \quad (4.28)$$

As a second example consider

$$\begin{aligned} \langle \prod_{i=1}^n \frac{1}{4\pi^2} \mathcal{O}_1(\gamma_i) \rangle_\epsilon &= \int DA D\psi D\phi \exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F + \frac{1}{2} \psi \psi \right) \right. \\ &\quad \left. + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right) \cdot \prod_{i=1}^n \frac{1}{4\pi^2} \oint_{\gamma_i} \text{Tr} \phi \psi. \end{aligned} \quad (4.29)$$

Here n must be even or this vanishes. The action is invariant under $\psi \rightarrow -\psi$ while the integrand changes sign if n is odd. A simple way to perform this integral is

to introduce n anti-commuting variables η_i and consider instead the partition function

$$\begin{aligned} Z_{\Sigma_g}(\epsilon, \eta_i) &= \int DA D\psi D\phi \exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F + \frac{1}{2} \psi \psi \right) \right. \\ &\quad \left. + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi + \frac{1}{4\pi^2} \sum_{i=1}^n \eta_i \oint_{\gamma_i} \text{Tr} \phi \psi \right). \end{aligned} \quad (4.30)$$

On differentiating this with respect to each of the η_i (in the order $i = n$ to $i = 1$) and then setting these Grassmann variables to zero one obtains (4.29). Now we introduce De Rham currents J with the following properties

$$\int_{\Sigma_g} J(\gamma_i) \Lambda = \oint_{\gamma_i} \Lambda, \quad dJ = 0 \quad (4.31)$$

for any one form Λ . One completes the square in (4.30) in the ψ field

$$\psi \rightarrow \psi - i \sum_{i=1}^n \eta_i J(\gamma_i) \phi, \quad (4.32)$$

to obtain

$$\begin{aligned} Z_{\Sigma_g}(\epsilon, \eta_i) &= \int DA D\psi D\phi \exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} \left(\phi F + \frac{1}{2} \psi \psi \right) \right. \\ &\quad \left. + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi - \frac{i}{4\pi^2} \sum_{i < j} \eta_i \eta_j \int_{\Sigma_g} J(\gamma_i) J(\gamma_j) \text{Tr} \phi \phi \right) \end{aligned} \quad (4.33)$$

The terms with $i = j$ vanish as $\eta_i^2 = 0$, so that there are no problems with self intersections. The De Rham currents have delta function support onto their associated cycles so that, for any zero form Ψ , ($i \neq j$)

$$\int_{\Sigma_g} J(\gamma_i) J(\gamma_j) \Psi = \sum_{P \in \gamma_i \cap \gamma_j} \sigma(P) \Psi(P), \quad (4.34)$$

with P the points of intersection of γ_i and γ_j and $\sigma(P) (= \pm 1)$ the oriented intersection number of γ_i and γ_j at P . This means that

$$\begin{aligned} &\frac{1}{4\pi^2} \sum_{i < j} \eta_i \eta_j \int_{\Sigma_g} J(\gamma_i) J(\gamma_j) \text{Tr} \phi \phi \\ &= \frac{1}{4\pi^2} \sum_{i < j} \eta_i \eta_j \text{Tr} \phi^2(P) \\ &= \frac{1}{4\pi^2} \sum_{i < j} \eta_i \eta_j \gamma_{ij} \int_{\Sigma_g} \text{Tr} \phi * \phi, \end{aligned} \quad (4.35)$$

where we have used the fact that $Tr\phi^2$ does not depend on the point at which it is evaluated and $\gamma_{ij} = \#(\gamma_i \cap \gamma_j)$ is the matrix of oriented intersection numbers. Putting all the pieces together we arrive at

$$Z_{\Sigma_g}(\epsilon, \eta_i) = Z_{\Sigma_g}(\hat{\epsilon}), \quad (4.36)$$

with

$$\hat{\epsilon} = \epsilon - 2 \sum_{i < j} \eta_i \eta_j \gamma_{ij}. \quad (4.37)$$

For $n = 2$ we obtain

$$\begin{aligned} \langle \frac{1}{4\pi^2} \mathcal{O}_1(\gamma_1) \frac{1}{4\pi^2} \mathcal{O}_1(\gamma_2) \rangle_\epsilon &= \frac{\partial}{\partial \eta_1} \frac{\partial}{\partial \eta_2} Z_{\Sigma_g}(\epsilon - 2\eta_1 \eta_2 \gamma_{12}) \\ &= 2i\gamma_{12} \frac{\partial}{\partial \epsilon} Z_{\Sigma_g}(\epsilon). \end{aligned} \quad (4.38)$$

Likewise for higher values of n the expectation values of $\frac{1}{4\pi^2} \mathcal{O}_1(\gamma_i)$ are obtained on differentiating $Z_{\Sigma_g}(\epsilon)$.

Clearly, expectation values of mixed products

$$\langle \prod_{i=1}^k \frac{1}{4\pi^2} \mathcal{O}_0(x_i) \prod_{j=1}^n \frac{1}{4\pi^2} \mathcal{O}_j(\gamma_j) \rangle_\epsilon, \quad (4.39)$$

are similarly obtained.

Integration On Moduli Space

We have the observables and a way of computing them, at least in principle, but what is lacking, however, is their interpretation. We should think of the ψ 's as one-forms on \mathcal{A}/\mathcal{G} and the ϕ 's as two-forms on \mathcal{A}/\mathcal{G} . When \mathcal{A}/\mathcal{G} is restricted to $\mathcal{M}_F(\Sigma_g, G)$, ψ and ϕ should be thought of as a one-form and a two-form on the moduli space respectively. This means that \mathcal{O}_0 is a four-form on $\mathcal{M}_F(\Sigma_g, G)$ while \mathcal{O}_1 is a three-form there.

On any n dimensional manifold we may integrate an n -form without the need to introduce a metric. The moduli space has dimension (for $g > 1$) $(2g-2)\dim G$ so that any product of the observables as in (4.39) with $4k+3n = (2g-2)\dim G$ is a form that may be integrated on $\mathcal{M}_F(\Sigma_g, G)$. On the other hand, once the constraint that $F_A = 0$ has been imposed, the path integral over \mathcal{A}/\mathcal{G} devolves to an integral over $\mathcal{M}_F(\Sigma_g, G)$. In this way (4.39) is seen to be the integral over $\mathcal{M}_F(\Sigma_g, G)$ of a $(2g-2)\dim G$ -form. Let us denote with a hat the differential form that an observable corresponds to. Then (4.39) takes the more suggestive

form

$$\langle \prod_{i=1}^k \frac{1}{4\pi^2} \mathcal{O}_0(x_i) \prod_{j=1}^n \frac{1}{4\pi^2} \mathcal{O}_j(\gamma_j) \rangle_\epsilon = \int_{\mathcal{M}_F} \prod_{i=1}^k \frac{1}{4\pi^2} \hat{\mathcal{O}}_0(x_i) \prod_{j=1}^n \frac{1}{4\pi^2} \hat{\mathcal{O}}_j(\gamma_j) \exp \Omega. \quad (4.40)$$

When $4k+3n = (2g-2)\dim G$, the symplectic form makes no contribution. However, if $4k+3n = 2m < (2g-2)\dim G$ there will also be contributions from the action to soak up the excess form-degree. On expanding the exponential, the symplectic form $\Omega(\psi, \psi)$ raised to the power $(g-1)\dim G - m$ will survive the Grassmann integration.

We now know that we are calculating integrals of metric independent differential forms on $\mathcal{M}_F(\Sigma_g, G)$. What do such integrals correspond to? They are naturally interpreted as intersection numbers on $\mathcal{M}_F(\Sigma_g, G)$. An explanation of the relationship between the differential form and intersection viewpoints is provided in [23].

Relationship Between The Old And The New

The problem with the presence of B , χ , $\bar{\phi}$ and η zero modes is that the theory, as it stands, is not defined. It is possible, however, to deal directly with these modes. We add to the action a term that is supersymmetric and that damps them. The new action is

$$\begin{aligned} L(t) &= L + t\delta i \int_{\Sigma_g} Tr \chi * \bar{\phi} \\ &= i \int_{\Sigma_g} Tr (BF_A - \chi d_A \psi + \eta d_A * \psi + \bar{\phi} d_A * d_A \phi + \bar{\phi} \{\psi, * \psi\}) \\ &\quad + ti \int_{\Sigma_g} Tr (B * \bar{\phi} - \chi * \eta). \end{aligned} \quad (4.41)$$

By an argument that is similar to the proof of metric independence (4.5)-(4.8), the path integral defined by this action is independent of smooth variations of t . One must be careful, however, as the $t \rightarrow 0$ limit is not the same as taking $t = 0$ directly precisely because of the presence of zero modes.

Integrating out the fields B , $\bar{\phi}$, χ and η generates a new action solely in terms of the geometric fields,

$$L'(t) = \frac{i}{t} \int_{\Sigma_g} Tr (*F_A d_A * d_A \phi + *d_A \psi d_A \psi + F_A \{\psi, * \psi\}). \quad (4.42)$$

The integral over ϕ lands us on the space of solutions to $d_A * d_A * \bar{r}_A = 0$, but this equation is the same as the Yang-Mills equation $d_A * F_A = 0$. One proves this by

considering

$$\begin{aligned} 0 &= \int_{\Sigma_g} \text{Tr} F_A * d_A * d_A * F_A \\ &= \int_{\Sigma_g} \text{Tr} (d_A * F_A) * (d_A * F_A), \end{aligned} \quad (4.43)$$

the last line being the norm of $d_A * F_A$, which vanishes. Hence $d_A * F_A = 0$. In passing to the new action we have moved away from just a description of the flat connections, and find that all the Yang-Mills connections contribute.

Now suppose that we wish to compute the expectation value in this new theory of

$$\exp \left(\frac{i}{4\pi^2} \int_{\Sigma_g} \text{Tr} (\phi F + \frac{1}{2} \psi \psi) + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right). \quad (4.44)$$

The expectation value continues to be independent of t . We may, therefore, set $t = \infty$ as the theory remains well defined for this value. This is the correspondence we were looking for. The partition function of Yang-Mills theory that we have been using (4.11) is, in terms of the original model, the expectation value of (4.44). The expectation values of the observables (4.26)-(4.38) are the same in the original theory as long as it is understood that (4.44) is inserted.

5 U(1) Theory

First Chern Class

$U(1)$ bundles over a Riemann surface Σ_g are classified by their first Chern class

$$c_1 = \frac{1}{2\pi} \int_{\Sigma_g} F_A, \quad (5.1)$$

which is an integer, say k , and for the Abelian theory $F_A = iA$. The most familiar configuration that has a non vanishing first Chern class is the magnetic monopole of Dirac. On S^2 , for example, we may consider a connection A_+ on the northern hemisphere H_+ and A_- on the southern H_- . These 'patch' together, if on the equator, where they overlap, they agree up to a gauge transformation. This means that on the equator there exists a φ such that

$$A_+ = A_- + d\varphi. \quad (5.2)$$

The Chern class may be expressed as

$$\frac{1}{2\pi} \int_{S^2} F_A = \frac{1}{2\pi} \int_{H_+} dA_+ + \int_{H_-} dA_-$$

$$\begin{aligned} &= \frac{1}{2\pi} \oint A_+ - \frac{1}{2\pi} \oint A_- \\ &= \frac{1}{2\pi} \oint d\varphi, \end{aligned} \quad (5.3)$$

with the relative sign appearing due to the opposite orientation of the circle boundary of the northern and southern hemispheres. When, in local coordinates, $\varphi = k\theta$ the first Chern class is k . Such a φ is allowed as it corresponds to a periodic group element, $\exp(ki\theta)$, with $0 \leq \theta < 2\pi$.

On the Torus $T^2 = S^1 \times S^1$ with local (angular) coordinates (σ_1, σ_2) the gauge field

$$A = \frac{m}{2\pi} \sigma_1 d\sigma_2 + \frac{n}{2\pi} \sigma_2 d\sigma_1, \quad (5.4)$$

satisfies

$$\begin{aligned} k &= \frac{1}{2\pi} \int_{T^2} F_A \\ &= \frac{1}{4\pi^2} \int_{T^2} (m - n) d\sigma_1 d\sigma_2 \\ &= m - n. \end{aligned} \quad (5.5)$$

The gauge field is not periodic, but it is periodic up to a gauge transformation. If we send $\sigma_1 \rightarrow \sigma_1 + 2\pi$ then

$$A(\sigma_1 + 2\pi, \sigma_2) = A(\sigma_1, \sigma_2) + e^{im\sigma_2} i d e^{-im\sigma_2}, \quad (5.6)$$

with a similar relationship for $\sigma_2 \rightarrow \sigma_2 + 2\pi$. The gauge group elements are globally defined.

Flat Connections

We have seen that the moduli space of flat $U(1)$ connections on a genus g surface, $\mathcal{M}_F(\Sigma_g, U(1))$, has dimension $2g$. Indeed it is a $2g$ torus $\mathcal{M}_F(\Sigma_g, U(1)) = T^{2g}$. Let us see how it comes out for genus zero and one.

For the sphere ($g = 0$) all loops are contractible and the only flat connection, up to gauge equivalence, is the trivial connection. The moduli space is therefore a point. For the torus ($g = 1$) there are two possible non-trivial holonomies. The corresponding flat gauge field has the form

$$A = \frac{\alpha_1}{2\pi} d\sigma_1 + \frac{\alpha_2}{2\pi} d\sigma_2. \quad (5.7)$$

But what are the ranges of α_1 and α_2 ? Note that we may still perform (single valued) gauge transformations

$$A \rightarrow A + e^{(im_1\sigma_1 + im_2\sigma_2)} i d e^{-(im_1\sigma_1 + im_2\sigma_2)}, \quad (5.8)$$

which corresponds to the shifts

$$\alpha_i \rightarrow \alpha_i + 2\pi m_i, \quad (5.9)$$

for all integers m_i . In other words the gauge inequivalent A have α_i that live on T^2 . This then is the space $\mathcal{M}_F(T^2, U(1))$.

This correspondence between the holonomies of the flat gauge fields and the points of $\mathcal{M}_F(\Sigma_g, U(1))$ can be made even more explicit. The local coordinates of $\mathcal{M}_F(\Sigma_g, U(1)) = T^{2g}$ are simply

$$(\oint_{a_1} A, \dots, \oint_{a_g} A, \oint_{b_1} A, \dots, \oint_{b_g} A). \quad (5.10)$$

In order to establish that the moduli space is a torus, in general, we would need some more notions from the theory of Riemann surfaces, so we forgo this.

Maxwell Connections

These are defined to be the class of connections that satisfy the Maxwell equation

$$d * F_A = 0. \quad (5.11)$$

In terms of the zero-form, $f_A = *F_A$, this equation becomes

$$df_A = 0, \quad (5.12)$$

which has as its solutions the harmonic functions $f_A \in H^0(M, R)$. On a compact manifold these are the constant functions, so that we find

$$F_A = a\omega, \quad (5.13)$$

where a is some constant and ω is a volume form normalised to unity. On a bundle with first Chern class equal to k we have

$$F_A = 2\pi k\omega. \quad (5.14)$$

This last equation is equivalent to the original Maxwell equation, and when $k = 0$, the Maxwell connections are flat. The connection (5.4) on T^2 is a Maxwell connection.

The moduli space of Maxwell connections, $\mathcal{M}_M^k(\Sigma_g, U(1))$ is the same as the moduli space of flat connections, that is

$$\mathcal{M}_M^k(\Sigma_g, U(1)) = \mathcal{M}_F(\Sigma_g, U(1)), \quad (5.15)$$

and as this correspondence holds for any k , we may suppress it. To see that this must be true, let A_k be any Maxwell connection satisfying (5.14). Then all other connections on the $(c_1 = k)$ bundle are of the form

$$A = A_k + X, \quad (5.16)$$

for some one from X . For A to also be a Maxwell connection X must satisfy

$$dX = 0, \quad (5.17)$$

which is the flatness equation and does not depend on k . Gauge inequivalent X 's are the points of the moduli space $\mathcal{M}_M(\Sigma_g, U(1))$, but clearly are also the points of $\mathcal{M}_F(\Sigma_g, U(1))$.

There is a more geometric way of stating this. We have seen that, for flat connections, we can form a map from $\pi_1(M)$ to G and, conversely, that these maps, up to conjugation, characterise the moduli space of flat connections. We can likewise show, that given Maxwell connections on any surface Σ_g , we may form maps from $\pi_1(\Sigma_g)$ to G . Fix a Maxwell connection A_k . Then for any Maxwell connection A , also with first Chern class equal to k , we can form the required map $\hat{\varphi}_\gamma(A) = \varphi_\gamma(A_k)^{-1} \varphi_\gamma(A)$. We see that $\hat{\varphi}_\gamma(A)$ depends only on the homotopy class of γ , for varying the path we get an area contribution from $\varphi_\gamma(A_k)^{-1}$ that cancels that from $\varphi_\gamma(A)$ (the area dependence may be seen in the second last line of (2.8)).

5.1 Maxwell theory on compact closed surfaces

We take the classical action of Maxwell theory on a two-dimensional (orientable) surface to be

$$L = \frac{1}{4\pi^2} \int_{\Sigma_g} i\phi F_A - \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \phi * \phi. \quad (5.18)$$

The partition function of Maxwell theory in the topological sector with monopole charge (first Chern class) k ,

$$\frac{1}{2\pi} \int_{\Sigma_g} F_A = k,$$

is then

$$Z_{\Sigma_g}(k, \epsilon) = \int DA D\phi \exp \left(\frac{1}{4\pi^2} \int_{\Sigma_g} i\phi F_A - \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \phi * \phi \right) \delta \left(\frac{1}{2\pi} \int_{\Sigma_g} F_A - k \right). \quad (5.19)$$

This still needs to be gauge fixed, but we make use of the trivialising map so we forgo the introduction of the fields associated with the gauge fixing procedure and pass directly to the partition function in the form

$$\begin{aligned} Z_{\Sigma_g}(k, \epsilon) &= \int D\xi D\phi \exp \left(\frac{1}{4\pi^2} \int_{\Sigma_g} i\phi\xi - \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \phi * \phi \right) \delta \left(\frac{1}{2\pi} \int_{\Sigma_g} \xi - k \right) \\ &= \int D\xi \exp \left(\frac{-1}{8\pi^2\epsilon} \int_{\Sigma_g} \xi * \xi \right) \delta \left(\frac{1}{2\pi} \int_{\Sigma_g} \xi - k \right). \end{aligned} \quad (5.20)$$

Introducing a multiplier λ to represent the delta-function as

$$\delta \left(\frac{1}{2\pi} \int_{\Sigma_g} \xi - k \right) = \int_{-\infty}^{+\infty} d\lambda \exp \left(i\lambda \left(\int_{\Sigma_g} \xi - 2\pi k \right) \right),$$

the Gaussian integrals over ξ and λ are easily performed to give

$$Z_{\Sigma_g}(k, \epsilon) = \frac{1}{\sqrt{2\pi\epsilon}} \exp \left(\frac{-k^2}{2\epsilon} \right). \quad (5.21)$$

Note that $Z_{\Sigma_g}(k, \epsilon)$ is independent of the genus of Σ_g .

Fixed point theorems

Apart from the universal factor $1/\sqrt{2\pi\epsilon}$, which arises from the reducibility of the connections, the partition function (5.19) is given entirely by the contribution at the Maxwell connection (5.14). That is, (5.21) may be re-written as

$$Z_{\Sigma_g}(k, \epsilon) = \frac{1}{\sqrt{2\pi\epsilon}} \exp(L(A_k)). \quad (5.22)$$

Furthermore, if we sum over the different topological sectors to calculate the overall partition function we find

$$\begin{aligned} Z_{\Sigma_g}(\epsilon) &= \sum_k Z_{\Sigma_g}(k, \epsilon) \\ &= \sum_k \frac{1}{\sqrt{2\pi\epsilon}} \exp(L(A_k)). \end{aligned} \quad (5.23)$$

These results are rather astounding. They tell us that the entire contribution to the path integral comes simply from the values at the critical points of the action. The critical points being the Maxwell connections.

Perhaps the importance of the result is overshadowed by its 'obviousness'. We have only had to perform Gaussian integrals and these are evaluated by their equations of motion; in (3.21) this is $x = y/\epsilon^2$, which, when substituted back into the 'action' yields the exponent on the right hand side. But two facts conspired

to turn the problem into one of simple Gaussian integration. The availability of a trivialising map and the fact that the Maxwell equations become 'algebraic' (5.14). The conspiracy continues unabated in the non-Abelian theory [25].

For integrals over finite dimensional symplectic manifolds such reductions to the fixed point set of the action (exponent) are explained in terms of the fixed point theorems of Duistermaat and Heckman [39]. Witten has generalised these theorems to the non-Abelian case and an infinite dimensional setting (the manifold \mathcal{A}). Quantum Maxwell theory furnishes a very simple example of these ideas.

Reducible Connections And $\text{vol } \mathcal{M}_F(\Sigma_g, G)$

For the Abelian theory, reducible connections do not constitute a real problem as all Abelian connections are reducible and in the same way. This means that we may extract an overall contribution from the constant ϕ field,

$$\int_{-\infty}^{+\infty} d\varphi \exp \left(-\frac{\epsilon}{8\pi^2} \varphi^2 \right) = \frac{2\pi\sqrt{2\pi}}{\sqrt{\epsilon}}. \quad (5.24)$$

Dividing this out of the partition function $Z_{\Sigma_g}(k, \epsilon)$ gives

$$\hat{Z}_{\Sigma_g}(k, \epsilon) = \frac{1}{4\pi^2} \exp \left(\frac{-k^2}{2\epsilon} \right). \quad (5.25)$$

Clearly as $\epsilon \rightarrow 0$ this vanishes for all k except $k = 0$. This is consistent with the fact that we should land on the flat connections in the limit.

For $k = 0$ at $\epsilon = 0$ we have, tentatively,

$$\hat{Z}_{\Sigma_g} = \text{vol } \mathcal{M}_F = \frac{1}{2\pi}. \quad (5.26)$$

This result should not be taken too seriously, as there are many factors that we have not been able to fix uniquely (such as the normalization of the path integral measure). These factors, however, will not be k dependent, and they will have a smooth $\epsilon \rightarrow 0$ limit. Nevertheless we see that it is possible to obtain a finite expression, and, in principle, with a more careful analysis, a correct form for $\text{vol } \mathcal{M}_F$.

Non Contribution Of Harmonic One-Forms

The reason for needing more care is that in one sense we have missed the volume we are looking for altogether! The trivializing map is invertible everywhere in field space (that is in \mathcal{A}/G) outside a finite dimensional set, points of which are in one to one correspondence with the space of flat connections. These are the fields X in (5.16) which satisfy (5.17) and are gauge fixed $d * X = 0$, so that

they are harmonic one-forms. The partition function (5.21) is then still to be multiplied by the $\text{vol } \mathcal{M}_F$. The result for the partition function, up to a standard renormalization (see section 7), that we have derived, is re-obtained in the next section. The standard renormalization implies, that for genus g , the partition function has the form

$$Z_{\Sigma_g}(k, \epsilon) = \frac{\kappa^{(2-2g)}}{\sqrt{2\pi\epsilon}} \exp\left(\frac{-k^2}{2\epsilon}\right), \quad (5.27)$$

for some κ . Clearly a different input is required to fix the constant.

Triviality Of Wilson Loops Along Homology Cycles

The harmonic forms also do not contribute to correlation functions of operators which can be expressed in terms of $\xi (= F_A)$, and the volume of the moduli space of flat connections will consequently drop out of normalized correlation functions in this case as well. One may wonder, however, what happens to correlation functions of operators which are sensitive to the holonomies of the gauge fields along the homology cycles of Σ_g . The gauge invariant observables of interest to us here are Wilson loops

$$\exp\left(i\alpha \oint_{\gamma} A\right)$$

along closed loops γ . If γ is homologically trivial then - by Stokes theorem - the Wilson loop is expressible in terms of ξ and thus falls into the category of operators already dealt with above. One may have some doubts on the validity of Stoke's theorem for connections on non-trivial bundles ($k \neq 0$) but for $k\alpha \in \mathbb{Z}$ Stoke's theorem can indeed still be used in the exponent. This is precisely analogous to the quantization condition in the WZW action, and we will derive this condition below.

This leaves us with Wilson loops for homologically non-trivial γ , which are indeed sensitive to the holonomies of A . In this case α has to be an integer in order to define a gauge invariant operator (under the large gauge transformations). With $\alpha \in \mathbb{Z}$, however, we find that the expectation value $\langle \exp(i\alpha \oint_{\gamma} A) \rangle$, as well as any correlator involving homologically non-trivial loops, is identically zero,

$$\langle \exp\left(i\alpha \oint_{\gamma} A\right) \rangle_k = 0 \quad \text{for } \gamma \neq \partial\Gamma, \alpha \neq 0.$$

Thus the failure of the trivializing map in this case causes no distress. One way of proving the vanishing of this expectation value is to note the fact that the evaluation of the holonomy of one of the X is

$$\int dX \exp\left(i\alpha \oint_{\gamma} X\right) \sim \int_0^1 d\theta e^{2\pi i \alpha \theta} = 0 \quad \alpha \in \mathbb{Z} \quad \alpha \neq 0.$$

as the moduli space is a torus.

Thus in the Abelian case this rules out homologically non-trivial Wilson loops as interesting observables on closed surfaces.

A Quantization Condition And Contractible Loops

We now turn to the computation of correlators of any number of (possibly intersecting and self-intersecting) contractible Wilson loops, starting with the case of a single non-intersecting loop. The first thing to note is that on a closed surface there is an intrinsic ambiguity in trying to write $\oint_{\gamma} A = \int_D \xi$, where A is a connection on a non-trivial bundle and $\partial D = \gamma$, as one could equally well replace D by its complement $\Sigma_g \setminus D = -D'$. Making a particular choice now, we will have to inquire at the end under which circumstances the result is independent of any such choice (and this will, as expected, give rise to the quantization condition on α). Using the same representation for the delta function as above and performing the Gaussian integrals one finds that the normalized expectation value is

$$\begin{aligned} & \langle \exp\left(i\alpha \int_D \xi\right) \rangle_k \\ &= \frac{1}{Z_{\Sigma_g}(k, \epsilon)} \int D\xi \exp\left(-\frac{1}{8\pi^2\epsilon} \int_{\Sigma_g} \xi * \xi + i\alpha \int_D \xi\right) \delta\left(\frac{1}{2\pi} \int_{\Sigma_g} \xi - k\right) \\ &= \exp\left(-2\pi^2\epsilon\alpha^2 \frac{A(D)}{A(\Sigma_g)} \frac{A(D')}{A(\Sigma_g)}\right) \exp\left(2\pi i k \alpha \frac{A(D)}{A(\Sigma_g)}\right). \end{aligned} \quad (5.28)$$

The first term is manifestly symmetric in D and D' and if we compute instead $\langle \exp(-i\alpha \int_{D'} \xi) \rangle_k$ we find that

$$\langle \exp\left(-i\alpha \int_{D'} \xi\right) \rangle_k = \langle \exp\left(i\alpha \int_D \xi\right) \rangle_k \exp(-2\pi i k \alpha),$$

so that $\langle \exp(i\alpha \oint_{\gamma} A) \rangle_k$ can only be defined consistently if $k = 0$ or $\alpha = \frac{n}{k}$, $n \in \mathbb{Z}$. In the first case the structure group of the $U(1)$ bundle can be extended to \mathbb{R} and $\alpha \in \mathbb{R}$ labels the unitary representations of the universal covering group \mathbb{R} of $U(1)$. In the second case α defines a representation of a k -fold covering of $U(1)$. The quantization condition on α has, like that of the WZW model, a natural group theoretic and geometric explanation.

When considering loops with self-intersections or several intersecting loops no substantially new features arise and the calculations can be done in much the same way as that leading to (5.28). The result is that in the general formula for the correlator of n intersecting but non-selfintersecting loops the exponent in (5.28) is replaced by

$$-2\pi^2\epsilon \sum_{j=1}^n \alpha_j^2 \frac{A(D_j)}{A(\Sigma)} \frac{A(D'_j)}{A(\Sigma)} + 2\pi i k \alpha \sum_{j=1}^n \alpha_j \frac{A(D_j)}{A(\Sigma)}$$

$$+4\pi^2\epsilon \sum_{j=1}^n \sum_{i < j} \alpha_j \alpha_i \left(\frac{A(D_j)}{A(\Sigma)} \frac{A(D_i)}{A(\Sigma)} - A(D_j \cap D_i) \right). \quad (5.29)$$

Again it can be checked (with a little bit of algebra) that the result is independent of the choice of D_j or D'_j provided that $k\alpha_j \in \mathbb{Z}$. Moreover, by regarding a self-intersecting loop as two touching but non-selfintersecting loops with opposite orientations, equation (5.29) gives the general result for the correlator of intersecting and self-intersecting loops on a closed surface of any genus.

One curious observation is useful to keep in mind when checking if the result (5.29) is sensible and correct. In flat space the figure eight loop and the figure eight folded into itself give different results [30]. This is of course perfectly reasonable as the folding leads to points in the interior part of the loop being surrounded twice by the loop so that (energy being proportional to the flux squared) these points contribute to the path integral with the four-fold weight of those surrounded by just one loop (something that is also reflected in the quadratic composition law of (5.29)). On the two-sphere, however, these two configurations are indistinguishable, whereas on the torus they are again manifestly different, and one may wonder how the path integral manages to take this into account. As it turns out the path integral automatically gives a sensible answer. Indeed, by staring at a figure eight on the two-sphere one can convince oneself that the process of folding (say) the upper loop into the lower is equivalent to going to the complement of the lower loop, and (5.29) does not depend on whether we choose one interior of a loop or its complement.

Wilson Points

The evaluation of expectation values of Wilson loops is also quite straightforward. The correlator

$$\begin{aligned} & \left\langle \exp \left(\frac{i}{2\pi} \sum_j q_j \phi(x_j) \right) \right\rangle_k \\ &= \frac{1}{Z_{\Sigma_g}(k, \epsilon)} \int D\xi D\phi \exp \left(\frac{1}{4\pi^2} \int_{\Sigma_g} i\xi \phi - \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \phi * \phi \right. \\ & \quad \left. + \frac{i}{2\pi} \sum_j \int_{\Sigma_g} J(x_j) \phi \right) \delta \left(\frac{1}{2\pi} \int_{\Sigma_g} \xi - k \right), \end{aligned} \quad (5.30)$$

is easily evaluated by performing all the Gaussians, but may be obtained directly by redefining ξ . Change variables according to $(J(x_i))$ is a two-form De Rham current that fixes one to the point x_i

$$\xi \rightarrow \xi - 2\pi \sum_j q_j J(x_j), \quad (5.31)$$

which simply acts to get rid of the Wilson points in the exponent and shifts k to $k + \sum_j q_j$ in the delta function, so that the sum must be an integer. We find, therefore, that

$$\begin{aligned} \left\langle \exp \left(\frac{i}{2\pi} \sum_j q_j \phi(x_j) \right) \right\rangle_k &= \frac{1}{\sqrt{2\pi\epsilon}} \exp \left(-\frac{(k + \sum_j q_j)^2}{2\epsilon} \right) \\ &= Z_{\Sigma_g}(k + \sum_j q_j, \epsilon). \end{aligned} \quad (5.32)$$

There are a few points that are worth special mention. The effect of introducing the Wilson points is to make the theory behave as if it is on a bundle of different Chern class. The correlation function does not depend on the location of the Wilson points, a fact which is clear from the cohomological field theory nature of the correlator but not so obvious from the BF point of view. The expectation value of Wilson points with Wilson loops on non-trivial homology cycles will vanish. The expectation value of Wilson points with homologically trivial Wilson loops will reproduce (5.28) but with the shift $k \rightarrow k + \sum_j q_j$. This last result comes from the fact that under the shift (5.31)

$$\exp \left(i\alpha \int_D \xi \right) \rightarrow \exp \left(i\alpha \int_D \xi \right) \exp \left(2\pi i\alpha \sum_j \int_D q_j J(x_j) \right), \quad (5.33)$$

and

$$\int_D J(x_j) = (\pm 1, 0), \quad (5.34)$$

depending on whether the point $x_j \in D$ or not. In either case the second exponential is unity, providing we take account of the fact that both α and $\sum_j q_j$ are integers.

Topological Observables

As far as the Schwarz type topological observables are concerned, only the partition function is non-trivial. We concentrate on the cohomological observables. In the case of $U(1)$ we may also take $n = 1$ in (4.15) so that the integrals of ψ around the homology cycles are observables. Let us set

$$\oint_{a_i} A = \alpha_i, \quad \oint_{b_i} A = \beta_i, \quad (5.35)$$

so that the flat coordinates of the torus T^{2g} are (α_i, β_i) . Then

$$\oint_{a_i} \psi = d\alpha_i, \quad \oint_{b_i} \psi = d\beta_i, \quad (5.36)$$

where the exterior derivative d is that on T^{2g} (it is δ restricted to the torus). While these observables are exact, they are not trivial as the coordinates (α_i, β_i) are not globally defined. We have the following correspondence

$$\langle \prod_{i=1}^g \oint_{a_i} \psi \oint_{b_i} \psi \rangle = \int_{T^{2g}} \prod_{i=1}^g d\alpha_i d\beta_i. \quad (5.37)$$

6 Field Theory On Manifolds With Boundary

The path integral on a manifold M , with boundary $\partial M = B$, requires boundary conditions to be fixed on B . In this way the path integral becomes (for each operator insertion in the interior of M) a functional of the fields on B , which can be regarded as a state in the canonical Hilbert space of the theory on $B \times \mathbb{R}$. While this procedure is of conceptual interest as it sets up the correspondence between the path integral and operator formalisms of field theory, it is generally of little practical use as the path integrals involved are too complicated to be calculated directly. In certain cases symmetry arguments may be invoked to determine the states uniquely (as in string theory [42]) or up to a finite ambiguity (as in Chern-Simons theory [43]).

In the case of topological field theories it is possible to deduce certain general properties of and relations between correlation functions. This is in line with the axiomatic approach to topological field theory as proposed by Atiyah [44]. For two dimensional Yang-Mills theory, which is almost topological, depending only on the measure on the Riemann surface, and is also a gauge theory, it turns out that it is possible to completely determine the states. In turn one may use this information to evaluate the path integral on any surface.

We proceed to explain the underlying ideas and then we reproduce the results we have obtained for the $U(1)$ theory using these techniques.

Boundary Data

When given a path integral to compute on a manifold with boundary, one must specify some boundary configuration of the fields. In equations, we have

$$\Psi_M(\varphi) = \int_{\phi|_B=\varphi} D\phi e^{-S(\phi)}. \quad (6.1)$$

Of course not all boundary data may be specified. There will, depending on the theory at hand, be certain restrictions.

For a gauge theory, it may be possible to demand that Ψ_M be gauge invariant, or at least transform in some well specified way under gauge transformations.

To see this in practice, suppose that the action $S(\phi)$ is invariant under gauge transformations $\phi \rightarrow \phi^g$ that are not the identity on the boundary. Then we have

$$\begin{aligned} \Psi_M(\varphi) &= \int_{\phi|_B=\varphi} D\phi \exp -S(\phi) \\ &= \int_{\phi^g|_B=\varphi} D\phi^g \exp -S(\phi^g) \\ &= \int_{\phi|_B=\varphi} D\phi \exp -S(\phi) \\ &= \Psi_M(\varphi^g|_B^{-1}). \end{aligned} \quad (6.2)$$

The second equality is a change of variables of the dummy ϕ , the third follows from the gauge invariance of the action and the presumed gauge invariance of the path integral measure. In the fourth equality g_B stands for the value of the gauge parameter on the boundary.

This is the behaviour that two dimensional Yang-Mills theory exhibits. An example where (6.2) does not hold is Chern-Simons theory. Here the wavefunctions pick up a phase under gauge transformations and are properly thought of as sections of certain bundles.

For manifolds with more boundary components, the partition function is a functional of the data on each component of the boundary.

Glueing Manifolds Together

We want to see how to get at the partition function of a manifold by glueing together two manifolds. For concreteness and ease of visualisation consider the two sphere and put an imaginary line along the equator. The path integral is an integral over all possible field configurations on the two sphere. Pick some allowed configuration φ on the equator. We can think of performing the path integral on the two sphere by integrating over all configurations which are consistent with φ on the equator and then integrating over all possible φ . As we integrate over the sphere, the path integral on the southern hemisphere gives the partition function of the disc with boundary data φ while the path integral on the northern hemisphere also gives the partition function of the disc (with opposite orientation) and with boundary data φ . We have deduced that

$$Z_S = \int D\varphi \Psi_D(\varphi) \Psi_{-D}(\varphi). \quad (6.3)$$

This generalises directly to arbitrary manifolds. If M is cut into two manifolds M_1 and M_2 along B , then we have

$$Z_M = \int D\varphi \Psi_{M_1}(\varphi) \Psi_{M_2}(\varphi). \quad (6.4)$$

Note that we do not preclude M from having boundary or, equivalently, the M , from having more boundary components than just B .

6.1 Maxwell theory on surfaces with boundary

The Disc

The first thing we have to determine is the allowed boundary conditions. If the resulting state is to be invariant under small gauge transformations (i.e. satisfy Gauss' law) the boundary conditions have to be chosen to be gauge invariant. Now the only gauge invariant degree of freedom of a gauge field on the circle ∂D is its holonomy $\theta \in \mathbf{R}$ defined by

$$\oint_{\partial D} A = 2\pi\theta ,$$

and the only admissible boundary condition is therefore the specification of θ . Computing the path integral with this boundary condition then amounts to inserting $\delta(\oint_{\partial D} A - 2\pi\theta)$ into the path integral, that is

$$\begin{aligned} \Psi_D(\theta, \epsilon) &= \int_{\frac{1}{2\pi} \oint_{\partial D} A = \theta} DA D\phi e^L \\ &\equiv \int DA D\phi e^L \delta\left(\frac{1}{2\pi} \oint_{\partial D} A - \theta\right). \end{aligned} \quad (6.5)$$

This form means that we may use the trivialising map again to simplify matters

$$\Psi(\theta, \epsilon) = \int D\xi D\phi e^L \delta\left(\frac{1}{2\pi} \int_D \xi - \theta\right). \quad (6.6)$$

A question that arises at this point is what type of delta function should appear here? We saw before that there are large gauge transformations on the circle due to $\pi_1(U(1)) = \mathbf{Z}$. These act on θ as $\theta \rightarrow \theta + n$, $n \in \mathbf{Z}$, and we can demand invariance under these transformations which would then render the wave function a periodic function of θ . This is accomplished by inserting the periodic delta function $\delta^P(\int_D \xi - 2\pi\theta)$, defined by

$$\delta^P(x) = \sum_{n \in \mathbf{Z}} \delta(x + 2\pi n) = \sum_{n \in \mathbf{Z}} e^{2\pi i n x}, \quad (6.7)$$

into the path integral.

With these preparatory remarks in mind we calculate, with the standard delta function,

$$\Psi(\theta, \epsilon) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{\theta^2}{2\epsilon}\right), \quad (6.8)$$

and

$$\Psi^P(\theta, \epsilon) = \sum_n \exp(-2\pi^2 \epsilon n^2) \exp(2\pi i n \theta), \quad (6.9)$$

with the periodic delta function. Ψ^P is of the form $\sum_n a_n \chi_n(\theta)$, where χ_n is the character of the unitary irreducible charge n representation of $U(1)$. This is also the general form of the states of Yang-Mills theory on the disc.

The wavefunctions (6.8) and (6.9) are solutions to the heat (Schrödinger) equation on the line and on the circle respectively, with the initial condition that they are delta functions. This is an expected relationship between the path integral with boundary and the Schrödinger equation (see appendix C). There is also an unexpected relationship between (6.8), (6.9) and modular forms which is explained in part in [32].

Twisted States

It is well known that more generally states could carry a non-trivial unitary representation of \mathbf{Z} (i.e. change by a phase under $\theta \rightarrow \theta + n$) labelled by a parameter $e^{2\pi i \vartheta} \in U(1)$. This is the familiar phenomenon of vacuum angles or ϑ -vacua in an embryonic setting.

In the twisted sectors one finds, instead of (6.9), the wave functions

$$\Psi^\vartheta(\theta, \epsilon) = \sum_n \exp(-2\pi^2 \epsilon (n - \vartheta)^2) \exp(2\pi i (n - \vartheta)\theta) \quad (6.10)$$

with the characteristic property

$$\Psi^\vartheta(\theta + m, \epsilon) = \exp(2\pi i m \vartheta) \Psi^\vartheta(\theta, \epsilon).$$

All our calculations could equally well be carried out in one of the twisted sectors of the theory but, as nothing is gained by this, we shall concentrate on the invariant ($\vartheta = 0$) sector in the following.

The Sphere

Write $S^2 = D_1 \cup_{S^1} (-D_2)$ and decompose the delta function appearing in (5.20) as

$$\delta\left(\frac{1}{2\pi} \int_{S^2} \xi - k\right) = \int_{-\infty}^{+\infty} d\theta \delta\left(\frac{1}{2\pi} \int_D \xi - \theta\right) \delta\left(\frac{1}{2\pi} \int_{D'} \xi - (\theta - k)\right). \quad (6.11)$$

The two delta functions give rise to $\Psi_{D_1}(\theta, \epsilon_1)$ and $\Psi_{-D_2}(k - \theta, \epsilon_2)$ respectively (cf. (6.8)), so that the partition function $Z_{S^2}(k, \epsilon)$ (equation (5.21)) can be obtained from the wave functions on the disc by

$$Z_{S^2}(k, \epsilon) = \int_{-\infty}^{+\infty} d\theta \Psi_{D_1}(\theta, \epsilon_1) \Psi_{-D_2}(k - \theta, \epsilon_2) \quad (6.12)$$

as can of course also be checked explicitly, with the help of the Poisson summation formula

$$\sum_n e^{-4\pi^2 n^2 t} e^{2\pi n \theta} = (4\pi t)^{-\frac{1}{2}} \sum_n e^{-(\theta+n)^2/2}. \quad (6.13)$$

One may wonder what the calculation of $\int_0^1 d\theta \Psi_{D_1}^P(\theta, \epsilon_1) \Psi_{-D_2}^P(k - \theta, \epsilon_2)$ results in. Note that $\Psi^P(k - \theta) = \Psi^P(-\theta) = \Psi^P(\theta)$, so that the difference among the topological sectors is washed out in Ψ^P and not unexpectedly one then finds that

$$\int_0^1 d\theta \Psi_{D_1}^P(\theta, \epsilon_1) \Psi_{-D_2}^P(k - \theta, \epsilon_2) = \sum_k Z_{S^2}(k, \epsilon). \quad (6.14)$$

Thus if one is only interested in results summed over all topological sectors (as is frequently the case) Ψ^P is adequate, but to get a handle on the individual sectors we need to use Ψ .

Kernels On $\Sigma_{g,n}$

Denote by $\Sigma_{g,n}$ a genus g Riemann surface with n boundary components. Also we denote the partition function on such a manifold by $K_{\Sigma_{g,n}}$ if we use a standard delta function and by $K_{\Sigma_{g,n}}^P$ when the periodic delta function is used. The symbol Z_{Σ_g} is reserved for the partition function of closed surfaces (boundaryless).

From the derivation of (6.8) and (6.9), it is clear that they are valid not only for the disc but more generally for a disc with an arbitrary number of handles, i.e. for a surface $\Sigma_{g,1}$ surface. We thus have the general result

$$\begin{aligned} K_{\Sigma_{g,1}}(\theta, \epsilon) &= \Psi(\theta, \epsilon), \\ K_{\Sigma_{g,1}}^P(\theta, \epsilon) &= \Psi^P(\theta, \epsilon). \end{aligned} \quad (6.15)$$

The generalization to surfaces $\Sigma_{g,n}$ with $n > 1$ boundaries is also straightforward. In that case we have to specify n holonomies $\theta_1, \dots, \theta_n$. In the path integral, for a manifold with n boundaries, when we change variables from the gauge field to the field strength we find that

$$\int_{\Sigma_{g,n}} \xi = \sum_{i=1}^n \oint_{\gamma_i} A, \quad (6.16)$$

so that one must still integrate over $(n-1)$ gauge fields at the boundaries, the n th being determined by the above relationship. We want to perform the path integral

$$\int DAD\phi e^L \prod_{i=1}^n \delta\left(\frac{1}{2\pi} \oint_{\gamma_i} A - \theta_i\right). \quad (6.17)$$

We may use the trivialising map to pass to the variable ξ but this still leaves the holonomies (6.17) to account for. We may interpret this in the following way.

On the manifold $\Sigma_{g,n}$, the gauge invariant degrees of freedom of the gauge field are represented by the holonomies and the field strength, subject to the one condition (6.16). On using the trivialising map, the path integral measure goes over to

$$\int DA \rightarrow \int D\xi \prod_{i=1}^n D\left(\frac{1}{2\pi} \oint_{\gamma_i} A\right) \delta\left(\frac{1}{2\pi} \int_D \xi - \frac{1}{2\pi} \sum_{i=1}^n \oint_{\gamma_i} A\right). \quad (6.18)$$

Integrating over the holonomies on the boundaries in (6.17) leaves us with

$$\int D\xi e^L \delta\left(\frac{1}{2\pi} \int_{\Sigma_{g,n}} \xi - (\theta_1 + \dots + \theta_n)\right). \quad (6.19)$$

This is easily done and one finds

$$\begin{aligned} K_{\Sigma_{g,n}}(\theta_1, \dots, \theta_n, \epsilon) &= \Psi(\theta_1 + \dots + \theta_n, \epsilon) \\ K_{\Sigma_{g,n}}^P(\theta_1, \dots, \theta_n, \epsilon) &= \Psi^P(\theta_1 + \dots + \theta_n, \epsilon). \end{aligned} \quad (6.20)$$

Wilson Loops

In order to calculate the expectation value of a contractible Wilson loop $\exp i \oint_{\gamma} A$ on a surface $\Sigma_{g,n}$, denoted by

$$K_{\Sigma_{g,n}}(\theta_1, \dots, \theta_n, \epsilon; \alpha), \quad (6.21)$$

we need only know what the expectation value of the Wilson loop on the boundary of a disc is. Let the expectation value of the Wilson loop on the disc be denoted by

$$\Psi(\theta, \epsilon; \alpha). \quad (6.22)$$

Then evidently

$$K_{\Sigma_{g,n}}(\theta_1, \dots, \theta_n, \epsilon; \alpha) = \int_{-\infty}^{+\infty} d\theta_{n+1} K_{\Sigma_{g,n+1}}(\theta_1, \dots, \theta_n, \theta_{n+1}, \epsilon_1) \Psi(\theta_{n+1}, \epsilon_2; \alpha), \quad (6.23)$$

with similar formulae in the case of the periodic kernels. It remains only to determine $\Psi(\theta, \epsilon; \alpha)$. But this requires no calculation, for the boundary data of the disc path integral fixes $\oint_{\gamma} A = 2\pi\theta$, so we have

$$\Psi(\theta, \epsilon; \alpha) = \exp(2\pi i \alpha \theta) \Psi(\theta, \epsilon). \quad (6.24)$$

There are two cases for homologically non-trivial loops of charge $m \in \mathbb{Z}$ on a surface $\Sigma_{g,n}$. The first has to do with such loops that can be pulled 'off'. In this case simply attach a Wilson loop to one of the boundaries, then convolute with

a cylinder on that boundary to move the loop 'inside'. These manipulations give the result

$$\int_{-\infty}^{+\infty} d\theta K_{\Sigma_{g,n},\gamma}(\theta_1, \dots, \theta_{n-1}, \theta, \epsilon_1) \exp(2\pi i \alpha \theta) K_C(-\theta, \theta_n, \epsilon_2). \quad (6.25)$$

The second case is when the non-trivial loop cannot be pulled out of the surface. In this case begin with the surface $\Sigma_{g-1,n+2}$ and put a Wilson loop on one its boundaries, say the $n+1$ 'th. The kernel for this is

$$K_{\Sigma_{g-1,n+2},\gamma}(\theta_1, \dots, \theta_{n+2}, \epsilon; m) = \exp(2\pi i m \theta_{n+1}) K_{\Sigma_{g,n+2}}(\theta_1, \dots, \theta_{n+2}, \epsilon). \quad (6.26)$$

The result we are looking for is obtained by convoluting the $n+1$ boundary component with the $n+2$, which lowers the boundary components by 2 but raises the genus by 1 and at the same time introduces a non-contractible Wilson loop into the surface. We get

$$\begin{aligned} & \int_{-\infty}^{+\infty} d\theta K_{\Sigma_{g-1,n+2},\gamma}(\theta_1, \dots, \theta_n, \theta, -\theta, \epsilon; m) \\ &= \int_{-\infty}^{+\infty} d\theta \exp(2\pi i m \theta_{n+1}) K_{\Sigma_{g,n+2}}(\theta_1, \dots, \theta_n, \theta, -\theta, \epsilon) \\ &= \int_{-\infty}^{+\infty} d\theta \exp(2\pi i m \theta_{n+1}) K_{\Sigma_{g,n}}(\theta_1, \dots, \theta_n, \epsilon) \\ &= \delta_{m,0} K_{\Sigma_{g,n}}(\theta_1, \dots, \theta_n, \epsilon). \end{aligned} \quad (6.27)$$

This generalises the result that, for closed manifolds, non-trivial Wilson loops have trivial expectation values (5.1).

Wilson Points

It is clear that the expectation value of Wilson points on an arbitrary surface is obtained by convoluting surfaces with more boundaries with discs that have the Wilson points in them. So for us the expectation value of some Wilson points on the disc is adequate. The calculation is exactly the same as for the closed surfaces in the previous section. We get for n such points with charges q_i

$$\Psi\left(\theta + \sum_{i=1}^n q_i, \epsilon\right). \quad (6.28)$$

This result may be understood from the canonical quantization point of view. ϕ is the canonical conjugate momentum to A , so its action on θ is by differentiation, that is

$$i\phi\theta = 2\pi. \quad (6.29)$$

In terms of operators the expectation value of the Wilson points on the disc takes the form

$$\begin{aligned} & \exp\left(\sum_{i=1}^n \frac{i}{2\pi} q_i \phi(x_i)\right) \Psi(\theta, \epsilon) \\ &= \exp\left(\sum_{i=1}^n q_i \frac{\partial}{\partial \theta}\right) \Psi(\theta, \epsilon) \\ &= \Psi\left(\theta + \sum_{i=1}^n q_i, \epsilon\right). \end{aligned} \quad (6.30)$$

Consistency Checks

The next thing we check is the proper behaviour of the kernels $K_{\Sigma_{g,n}}$ under the operation of glueing surfaces along boundaries. Again, in view of (6.15) and (6.20), it is quite sufficient to check this in the particular case of two cylinders: C_1 and C_2 glued along a common boundary γ_1 to form a cylinder C with $A(C) = A(C_1) + A(C_2)$, or $\epsilon = \epsilon_1 + \epsilon_2$. Writing $\partial C_1 = \gamma_0 + \gamma_1$, $\partial C_2 = \gamma_1 + \gamma_2$ one has $\partial C = \gamma_0 + \gamma_2 = \gamma_0 + \gamma_1 - \gamma_1 + \gamma_2$, so that we expect K_C to be given by

$$K_C(\theta_1, \theta_2, \epsilon) = \int_{-\infty}^{+\infty} d\theta K_{C_1}(\theta_1, \theta, \epsilon_1) K_{C_2}(-\theta, \theta_2, \epsilon_2), \quad (6.31)$$

and using (6.9) and (6.20) this can easily be verified explicitly. Mutatis mutandis (6.31) is valid for the glueing of any two surfaces to form a surface with $n > 0$ boundaries. It is also possible to consider the joining of two boundaries of a surface, $\Sigma_{g,n+2} \rightarrow \Sigma_{g+1,n}$. In general this is described by a formula similar to (6.31) (which we used in the calculation of expectation values), but due to the linear and additive way in which the holonomies enter into (6.20) in the Abelian case, this simply results in

$$K_{\Sigma_{g+1,n}}(\theta_1, \dots, \theta_n, \epsilon) = K_{\Sigma_{g,n+2}}(0, 0, \theta_1, \dots, \theta_n, \epsilon), \quad (6.32)$$

(and analogously for K^P).

Any other more complicated calculation can be reduced to a combination of the three examples discussed above.

7 Partition Function In Yang-Mills Theory

For $U(1)$ gauge theory, we were able to evaluate the partition function on an arbitrary genus surface directly. The global information about the trivializing map was straightforward to encode. For trivial bundles, this amounted to the

observation that $\int_{\Sigma_g} F_A = \int_{\Sigma_g} dA = 0$, so that $\int_{\Sigma_g} \xi = 0$. This is a gauge invariant condition. For $SU(n)$ bundles (or trivial $U(n)$ bundles) we would also expect a condition of the form $\int_{\Sigma_g} dA^a = 0$ but this is clearly not gauge invariant and it is far from obvious what one should take to be the non-Abelian generalization of $\int_{\Sigma_g} F_A = 0$.

On manifolds with boundary, however, all the information that was required of the trivializing map, for the $U(1)$ theory, had to do with gauge invariant boundary data. By glueing manifolds with boundary together, it was possible to arrive at the results for the compact closed manifolds. There is a direct non-Abelian generalization of this. Indeed it is enough to know the result for the disc so as to generate the results on arbitrary Riemann surfaces, with or without boundary. Recall that identifying the sides of a cut Riemann surface gives back the original Riemann surface. A cut Riemann surface is just a disc.

In this section we restrict our attention to Lie groups which are compact, connected and simply connected. All of the results obtained will be in terms of group representation theory. The set of equivalence classes of irreducible unitary representations of G is denoted by \hat{G} . For $\lambda \in \hat{G}$, we denote by $d(\lambda)$ the dimension of the representation, χ_λ the character (normalised by $\chi_\lambda(1) = d(\lambda)$) and by $c_2(\lambda)$ the quadratic Casimir invariant of λ . We use various properties of the characters that are treated in detail in, for example, [45, 46].

The Wave Function On The Disc

Let $\gamma = \partial D$ be the boundary of a disc D ($\gamma \sim S^1$). Just as for the $U(1)$ theory, the only gauge invariant degree of freedom of a gauge field on the circle is its holonomy. Choosing the boundary condition to be $P \exp \oint_\gamma A = g_1 \in G$ (modulo conjugation, i.e. gauge transformations of A), our task is to compute the path integral

$$K_D(g_1, A(D)) \equiv \int_{\phi} e^L \delta(P e^{\oint_\gamma A}, g_1). \quad (7.1)$$

We need to specify the delta function that appears in (7.1) and which is some delta function on the group G . There are two possibilities. The first is to use the delta function of $L^2(G)$, given in the spectral representation by

$$\delta(g, h) = \sum_{\lambda \in \hat{G}} d(\lambda) \chi_\lambda(g^{-1}h). \quad (7.2)$$

With this choice of delta function, the path integral (7.1) is not manifestly conjugation invariant but, as the result turns out to be, use of (7.2) is sufficient for our present purposes. We can however build in conjugation invariance from the outset

by using the delta function $\delta(g, h)$ on the space $L^2(G)^G$ of conjugation invariant functions (class functions),

$$\delta(g, h) = \sum_{\lambda \in \hat{G}} \chi_\lambda(g^{-1}) \chi_\lambda(h), \quad (7.3)$$

related to $\tilde{\delta}(g, h)$ by

$$\delta(g, h) = \int_G dg' \tilde{\delta}(g, g' h g'^{-1}), \quad (7.4)$$

as a consequence of the relation

$$\int dg \chi_\lambda(x g y g^{-1}) = d(\lambda)^{-1} \chi_\lambda(x) \chi_\lambda(y). \quad (7.5)$$

The group measure is normalised here so that the group volume is one

$$\int_G dg = 1. \quad (7.6)$$

Some consequences of changing this are explored later in this section.

In the case of surfaces with more than one boundary component, the use of (7.3) actually becomes mandatory if one wants to work with the gauge fixed path integral and retain conjugation invariance, as explained in [32]. So here we will use the second alternative, though for the disc both delta functions lead to the same results.

The boundary data is given in terms of the gauge potential. We need to specify it in terms of the field strength. Using the non-Abelian Stokes' theorem, this is possible in general on the disc and is explained in appendix B. The part we need is that the Schwinger-Fock gauge

$$x^\mu A_\mu^a = 0, \quad (7.7)$$

allows us to express the gauge field in terms of the field strength

$$A_\mu^a(x) = \int_0^1 ds s x^\nu F_{\nu\mu}^a(sx). \quad (7.8)$$

The trivializing map is available, with $G^a(A) = x^\mu A_\mu^a$, so that we obtain

$$K_D(g_1, \epsilon) = \int D\xi D\phi \exp \left(\frac{1}{4\pi^2} \int_{\Sigma_g} i \text{Tr} \phi \xi + \frac{\epsilon}{8\pi^2} \int_{\Sigma_g} \text{Tr} \phi * \phi \right) \sum_{\lambda \in \hat{G}} \chi_\lambda(g^{-1}) \chi_\lambda(g_1), \quad (7.9)$$

with $g = P \exp \left(\int_\gamma A \right)$ and A expressed in terms of ξ through (7.8).

Let us now assemble the techniques that will go into computing (7.1). It will be convenient to replace the path ordered exponential in (7.1) by a quantum

mechanics amplitude, namely

$$\begin{aligned} & \chi_\lambda \left(P \exp \oint_\gamma A \right) \\ &= \int D\eta D\bar{\eta} \exp \left(i \int_0^1 dt [\bar{\eta}^i(t) \dot{\eta}^i(t) \right. \\ & \quad \left. - i A_\mu^a(\gamma(t)) \dot{\gamma}^\mu(t) (\bar{\eta} \lambda^a \eta)(t) \right] \bar{\eta}^i(1) \eta^k(0) \delta_{ik}, \end{aligned} \quad (7.10)$$

where η^k and $\bar{\eta}^k$, $k = 1, \dots, d(\lambda)$ are Grassmann variables (with the obvious generalization to traces of the form $\chi_\lambda(Pe^{\oint_\gamma A} g)$, $g \in G$) and where $(\bar{\eta} \lambda^a \eta)(t) = \bar{\eta}^i(t) \lambda_{ik}^a \eta^k(t)$. A short proof of (7.10) uses the fact that the fermion propagator in one dimension is

$$\int D\eta D\bar{\eta} \exp \left(i \int_0^1 dt \bar{\eta}^i(t) \dot{\eta}^i(t) \right) \bar{\eta}^i(s) \eta^j(0) = \delta^{ij} \theta(s).$$

Together with the change of variables

$$\begin{aligned} \eta_i(t) &\rightarrow \left[P \exp \int_0^t A_\mu^a \lambda^a \dot{\gamma}^\mu ds \right]_{ij} \eta^j(t), \\ \bar{\eta}_i(t) &\rightarrow \bar{\eta}^j(t) \left[P \exp \int_t^0 A_\mu^a \lambda^a \dot{\gamma}^\mu ds \right]_{ji}, \end{aligned}$$

(path ordering is done from the lower end of the integral to the upper regardless of which is greater) this can be seen to imply (7.10).

Using the Schwinger-Fock gauge allows us to write

$$\begin{aligned} & \int_0^1 dt A_\mu^a(\gamma(t)) \frac{\dot{\gamma}^\mu(t)}{dt} (\bar{\eta} \lambda^a \eta)(t) \\ &= \int_0^1 dt \int_0^1 ds \xi_{\nu\mu}^a s \gamma^\nu \frac{d\gamma^\mu(t)}{dt} (\bar{\eta} \lambda^a \eta)(t) \\ &= \int_0^1 dt \int_0^1 ds \xi_{\nu\mu}^a \frac{d(s\gamma^\mu)}{dt} \frac{d(s\gamma^\nu)}{ds} (\bar{\eta} \lambda^a \eta)(t) \\ &= \int_D \xi^a (\bar{\eta} \lambda^a \eta)(t), \end{aligned} \quad (7.11)$$

with the local polar coordinates on the disc given by $s\gamma(t)$. To evaluate (7.9), we first perform the Gaussian integrals,

$$\begin{aligned} & \int D\xi D\phi \exp \left(\frac{1}{4\pi^2} \int_D i \text{Tr} \phi \xi + \frac{\epsilon}{8\pi^2} \int_D \text{Tr} \phi * \phi - i \int_D \xi (\bar{\eta} \lambda^a \eta)(t) \right) \\ &= \exp \left(-\frac{\epsilon}{8\pi^2} \int_0^1 dt (\bar{\eta} \lambda^a \eta)(t) (\bar{\eta} \lambda^a \eta)(t) \right), \end{aligned} \quad (7.12)$$

which then leaves us with the task of evaluating the fermionic integral

$$\int D\eta D\bar{\eta} \exp \left(i \int_0^1 dt \bar{\eta}^k(t) \dot{\eta}^k(t) - \frac{\epsilon}{8\pi^2} \int_0^1 dt (\bar{\eta} \lambda^a \eta)(t) (\bar{\eta} \lambda^a \eta)(t) \right) \bar{\eta}^i(1) \eta^k(0).$$

In a direct calculation for the cylinder [32], one encounters a slight generalization of this, namely

$$\begin{aligned} & \int D\eta D\bar{\eta} \exp \left(\int_0^1 dt [i \bar{\eta}^k(t) \dot{\eta}^k(t) + \rho^a(t) (\bar{\eta} \lambda^a \eta)(t)] \right. \\ & \quad \left. - \frac{\epsilon}{8\pi^2} \int_0^1 dt (\bar{\eta} \lambda^a \eta)(t) (\bar{\eta} \lambda^a \eta)(t) \right) \bar{\eta}^i(1) \eta^k(0). \end{aligned} \quad (7.13)$$

These integrals can be calculated order by order in perturbation theory, but in appendix D we have given a simple derivation of the result

$$(7.13) = \exp \left(-\frac{\epsilon}{8\pi^2} c_2(\lambda) \right) [P \exp \int_0^1 dt \rho(t)]^{ik}. \quad (7.14)$$

Combining this with (7.10) and (7.2) or (7.3), we finally arrive at the equation for the kernel (wave function) on the disc (7.9),

$$K_D(g_1, \epsilon) = \sum_{\lambda \in G} d(\lambda) \chi_\lambda(g_1) \exp \left(-\frac{\epsilon}{8\pi^2} c_2(\lambda) \right). \quad (7.15)$$

The Two Sphere

K_D can be used to compute the partition function of Yang-Mills theory on S^2 as well as expectation values of Wilson loops. Considering S^2 as the union of two discs,

$$S^2 = D_1 \cup_\gamma D_2, \quad \partial D_1 = \partial(-D_2) = \gamma,$$

we see that we can write Z_{S^2} as

$$Z_{S^2}(\epsilon) = \int_G dg K_{D_1}(g, \epsilon_1) K_{D_2}(g^{-1}, \epsilon_2), \quad (7.16)$$

(the inverse g^{-1} being due to the opposite orientation of ∂D_2). That dg , up to some overall constant, is the correct measure to use can be seen from the change of variables $A \rightarrow P \exp \oint_\gamma A$ on γ . Using the orthonormality

$$\int_G dg \chi_\lambda(g) \overline{\chi_\mu(g)} = \int_G dg \chi_\lambda(g) \chi_\mu(g^{-1}) = \delta_{\lambda\mu} \quad (7.17)$$

of the characters this becomes

$$Z_{S^2}(\epsilon) = \sum_{\lambda \in G} d(\lambda)^2 \exp \left(-\frac{\epsilon}{8\pi^2} c_2(\lambda) \right), \quad (7.18)$$

where $\epsilon = \epsilon_1 + \epsilon_2$ (this is the statement that $A(S^2) = A(D) + A(D')$).

We are now in a position to determine the correct constraint on the field ξ that is needed so as to define the trivializing map directly on the sphere. The required constraint can be deduced from inserting the definition (7.1) of the kernel on the

disc into (7.16). Doing this, one finds (the connection A should again be thought of as being expressed in terms of ξ via (3.30))

$$Z_{S^2}(\epsilon) = \int D\xi \exp\left(+\frac{1}{8\pi^2\epsilon} \int_{S^2} \text{Tr} \xi \star \xi\right) \delta\left(P \exp\left(-\oint_{\partial D} A\right), P \exp\left(-\oint_{\partial D'} A\right)\right). \quad (7.19)$$

This constraint expresses the requirement that the holonomies of A along ∂D_1 and $\partial(-D_2)$ are equal up to conjugation.

We should like to express this in a form which makes transparent what the condition on ξ is. Via the non-Abelian Stokes' theorem (details may be found in appendix B), the path ordered exponential entering (7.19) can be written as

$$P \exp - \oint_{\partial D_1} A = \mathcal{P} \exp - \int_{D_1} \xi. \quad (7.20)$$

We therefore obtain

$$Z_{S^2}(\epsilon) = \int D\xi \exp\left(+\frac{1}{8\pi^2\epsilon} \int_{S^2} \text{Tr} \xi \star \xi\right) \delta\left(\mathcal{P} \exp\left(-\int_{D_1} \xi\right) \mathcal{P} \exp\left(-\int_{D_2} \xi\right)\right). \quad (7.21)$$

The splitting of S^2 into D_1 and D_2 is arbitrary here and for any other choice of disc and complement this formula remains correct.

The Cylinder

With this example we come to the heart of the matter. It is possible to quite straightforwardly, following closely the analysis for the disc, derive from scratch the partition function for the cylinder C [32]. However, such a direct approach is difficult to implement in the case of higher genus surfaces, or for surfaces with more boundary components. For that reason we will now give an evaluation of the partition function on the cylinder which is based on nothing but the kernel for the disc (7.15) and the fact that K_C can depend only on the holonomies along the boundaries and the area $A(C)$. These considerations will be seen to generalize directly to any surface.

We deform the disc to a rectangle with the same area with edges a, b, c and d , that is, we view it as the cut surface of the torus or of the cylinder. This is as in figure 2, where c is the a^{-1} cycle and d is taken to be the b^{-1} cycle. Write the holonomy g_1 around the boundary of D as $g_1 = g_a g_b g_c g_d$ (this is possible as the holonomy is a path ordered exponential and can therefore be written as the product of the group elements obtained from going along a , then along b , etc.). Identifying the edges a and c (with opposite orientation) now amounts to setting $g_c = g_a^{-1}$ and integrating. Figures 5a and 5b give two ways of visualising this. Using (7.5) and (7.15), we find

$$I_C(g_b, g_a, \epsilon) = \int_G dg_a K_D(g_a g_b g_a^{-1} g_d, \epsilon)$$

$$\begin{aligned} &= \int_G dg_a \sum_{\lambda \in \hat{G}} d(\lambda) \chi_\lambda(g_a g_b g_a^{-1} g_d) \exp\left(-\frac{\epsilon}{8\pi^2} c_2(\lambda)\right) \\ &= \sum_{\lambda \in \hat{G}} \chi_\lambda(g_b) \chi_\lambda(g_d) \exp\left(-\frac{\epsilon}{8\pi^2} c_2(\lambda)\right), \end{aligned} \quad (7.22)$$

which agrees with the more simple minded approach in [32]. We should emphasise that this procedure works, as Yang-Mills theory in two dimensions is invariant under area preserving deformations.

There are a number of checks that can be made on this result. One we mention here has to do with the axiomatic approach to topological field theory. It is always possible to think of a genus g Riemann surface as a genus g_1 disc glued to one end of a cylinder and a genus g_2 disc glued at the other end, with $g = g_1 + g_2$. Think of the discs as generating states in the physical Hilbert space. The cylinder then has the interpretation of an inner product between the 'incoming' genus g_1 state and the 'outgoing' genus g_2 state. In the topological limit $\epsilon \rightarrow 0$, (7.22) becomes

$$\sum_{\lambda \in \hat{G}} \chi_\lambda(g_b) \chi_\lambda(g_d), \quad (7.23)$$

which is what we would expect. This simply says that the holonomies on the left and right discs have to match up to conjugation. If we Fourier transform, this is clear

$$\begin{aligned} C_{nm} &= \int_G dg_b \int_G dg_d \bar{\chi}_n(g_b) \bar{\chi}_m(g_d) \sum_{\lambda \in \hat{G}} \chi_\lambda(g_b) \chi_\lambda(g_d) \\ &= \delta_{nm}. \end{aligned} \quad (7.24)$$

Another direct check of the method is to glue a disc to the d end of the cylinder, which yields a disc, and to see if this reproduces the kernel for the disc (7.15). This indeed occurs

$$\begin{aligned} &\int_G dg_d \sum_{\mu} \chi_\mu(g_d^{-1}) \exp\left(-\frac{\epsilon_1}{8\pi^2} c_2(\mu)\right) \sum_{\lambda \in \hat{G}} \chi_\lambda(g_b) \chi_\lambda(g_d) \exp\left(-\frac{\epsilon_2}{8\pi^2} c_2(\lambda)\right) \\ &= \sum_{\lambda \in \hat{G}} \chi_\lambda(g_b) \exp\left(-\frac{\epsilon}{8\pi^2} c_2(\lambda)\right), \end{aligned} \quad (7.25)$$

with $\epsilon = \epsilon_1 + \epsilon_2$. Other tests may be found in [32].

The Pants

Does the same method allow us to calculate the kernel for the 'pair of pants' $\Sigma_{0,3}$? Indeed it does. In figure 6 we have exhibited one possible cut Riemann surface of the pants. Once more express the holonomy around the boundary of

the disc as the product of the holonomies of the eight edges. From the figure we see that it is enough, once we visualize the cylinder as a rectangle (disc) with a hole as in figure 5, to identify the marked edges, call them a and a^{-1} , as above. Thus, to obtain $K_{\Sigma_{0,3}}$, all we have to do is calculate

$$\int dg_a K_C(g_a g_1 g_a^{-1} g_2, g_3, \epsilon), \quad (7.26)$$

and, using (7.22) and (7.5), this becomes

$$K_{\Sigma_{0,3}}(g_1, g_2, g_3, \epsilon) = \sum_{\lambda \in G} d(\lambda)^{-1} \chi_\lambda(g_1) \chi_\lambda(g_2) \chi_\lambda(g_3) \exp\left(-\frac{\epsilon}{8\pi^2} c_2(\lambda)\right). \quad (7.27)$$

Again one may check that glueing a disc to any of the ends reproduces the kernel for the cylinder.

Extension To $\Sigma_{g,n}$

Knowing the kernel of the 'pants' and the rules for joining boundaries and glueing surfaces it is now a simple matter to deduce from (7.27) the general formulae

$$K_{\Sigma_{g,n}}(g_1, \dots, g_n, \epsilon) = \sum_{\lambda \in G} d(\lambda)^{2-2g-n} \chi_\lambda(g_1) \dots \chi_\lambda(g_n) \exp\left(-\frac{\epsilon}{8\pi^2} c_2(\lambda)\right), \quad (7.28)$$

and

$$Z_{\Sigma_g}(\epsilon) = \sum_{\lambda \in G} d(\lambda)^{2-2g} \exp\left(-\frac{\epsilon}{8\pi^2} c_2(\lambda)\right). \quad (7.29)$$

It is rather remarkable that, in a sense, the basic building block of Yang-Mills theory in two dimensions is not the kernel (7.27) of the 'pants' but rather that of the disc (7.15). This can be understood as a consequence of the fact that the theory is not only almost topological in the above sense but also a gauge theory.

Note that in (7.28, 7.29) the power of $d(\lambda)$ is always the Euler number $2-2g-n$ of $\Sigma_{g,n}$. That it is precisely this function of g and n which appears is of course no coincidence. Compatibility of (7.28, 7.29) with the operations of joining $2b$ boundaries of a surface $\Sigma_{g,n}$,

$$\Sigma_{g,n} \rightarrow \Sigma_{g+b, n-2b},$$

and of glueing two surfaces $\Sigma_{g,n}$ and $\Sigma_{g',n'}$ along b boundaries,

$$(\Sigma_{g,n}, \Sigma_{g',n'}) \rightarrow \Sigma_{g+g'+b-1, n+n'-2b},$$

requires the putative power $p(g, n)$ of $d(\lambda)$ to satisfy

$$\begin{aligned} p(g, n) &= p(g+b, n-2b) \\ p(g, n) + p(g', n') &= p(g+g'+b-1, n+n'-2b) \end{aligned} \quad (7.30)$$

and this fixes $p(g, n)$ uniquely (up to a scale) to be $p(g, n) = 2 - 2g - n$. The scale can then be determined by computing e.g. the kernel on the disc (7.15) or the partition function of the two-sphere (7.18).

vol $\mathcal{M}(\Sigma_g, SU(2))$

We specialise to the case that the structure group is $SU(2)$. Setting $\epsilon = 0$, in the partition function (7.29), we get

$$\text{vol } \mathcal{M}(\Sigma_g, SU(2)) \sim \sum_{\lambda \in SU(2)} d(\lambda)^{2-2g}. \quad (7.31)$$

The irreducible representations of $SU(2)$ are labeled by the positive integers n and the dimension of the n 'th unitary irreducible representation is $n+1$. There is a simple formula for

$$\sum_{n=0}^{\infty} (n+1)^{2-2g}, \quad (7.32)$$

obtained by passing to the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re}(s) > 0. \quad (7.33)$$

Thus we are interested in $\zeta(2g-2)$, which is related to the Bernoulli polynomial B_{2g-2} by

$$\zeta(2g-2) = \frac{(2\pi)^{2g-2}}{2(2g-2)!} |B_{2g-2}|, \quad (7.34)$$

our tentative expression for the volume being

$$\begin{aligned} \text{vol } \mathcal{M}(\Sigma_g, SU(2)) &\sim \zeta(2g-2) \\ &= \frac{(2\pi)^{2g-2}}{2(2g-2)!} |B_{2g-2}|. \end{aligned} \quad (7.35)$$

This result is quite good. The volume of the moduli space is known, for example by making use of Verlinde's formula, and is

$$\text{vol } \mathcal{M}(\Sigma_g, SU(2)) = 2 \frac{\zeta(2g-2)}{(2\pi^2)^{g-1}}. \quad (7.36)$$

The factor of 2 discrepancy between (7.35) and (7.36) is accounted for by noting that the centre of $SU(2)$ is Z_2 which has order 2, this being one of the factors we had previously mentioned but omitted to carry around. The factor $(2\pi^2)^{g-1}$ has a partial explanation in terms of our inability to fix the normalisation of the group integral used in the glueing rule. The exponent $g-1$ is determined in this way but why it is $2\pi^2$ and not some other constant is difficult to ascertain.

Standard Renormalizations

There are two obvious source of arbitrariness in our calculations, thus far, which we would now like to control. They have the same source, namely, that we are not sure of the normalisation of the path integral measure. The first is that the wavefunction or kernel on the disc should be multiplied by an arbitrary constant κ . The second is that we are also unable to fix the correct group measure in our glueing rules, so let it be ρ times the one thus far used.

We derive some consistency rules. If we glue two discs together, D_1 and D_2 , along one common edge to reproduce a new disc D , then the convolution of the kernels on the two discs should give the kernel on the new disc. This is the case for the kernel (7.15) with group volume normalised to unity. With the new normalisations we would have,

$$\kappa_D K_D(\epsilon) = \kappa_{D_1} \kappa_{D_2} \rho K_D(\epsilon), \quad (7.37)$$

which serves to fix the dependence of κ and ρ on the parameters that are in the theory. If κ is area dependent, then it must be exponential, so set $\kappa_D = \exp(v + bA(D))$. Consistency is achieved if $\rho = \exp -v$. If we demand the scaling symmetry that relates the coupling constant to the area, then $bA(D)$ should be replaced by $u\epsilon$ for some u . The net effect of this factor is to multiply all of the previously derived kernels and partition functions by $\exp u\epsilon$. In the topological limit this term plays no role.

We would like to work out the dependence on $\exp v$ for arbitrary kernels. The way we do this is to begin with the kernel on a surface of genus zero with n boundaries, glue on a disc and demand it yields the kernel on the zero genus surface with $n - 1$ boundaries. It is not difficult to see that all of these kernels are given by (7.28) with $g = 0$ multiplied by $\exp v$. Higher genus surfaces are obtained in the normal manner. Indeed for a closed manifold the result depends on the genus only and is $\exp v(2 - 2g)$ times our previous result (7.29).

The ability to redefine the theory, by the introduction of the parameters u and v , may be viewed as the normal ambiguity one faces in using different regularizations in any field theory. Changing the values of u and v amounts to renormalization and Witten has dubbed these variations, 'standard renormalizations'. The factor $(2\pi^2)^{1-g}$, for example, may be obtained on setting $v = \frac{1}{2} \ln(2\pi^2)$. These considerations show, that if we know the volume of the moduli space of flat connections for one surface (with $g \geq 2$), then all the factors may be fixed. The Torus will not do, as $\exp v(2 - 2g) = 1$, while for the sphere we run into $\zeta(-2) = 0$.

7.1 Expectation values of Observables

For the case of the cohomological theories, with non-Abelian groups, we saw that it is enough to know the partition function in order to evaluate the observables of interest. So here we concentrate on the observables that are intrinsic to the BF theory, namely Wilson loops and Wilson points. The situation is quite unlike the Abelian case as here the Wilson loops are most certainly not trivial.

With the general formula (7.28) for $K(\Sigma_{g,n})$ and the rules for glueing surfaces and joining boundaries at our disposal, it is rather straightforward to compute expectation values of Wilson loops (the generalization to correlation functions of several non-intersecting loops being immediate). There are three different types of non-selfintersecting loops to consider, contractible (homotopically trivial) loops, non-contractible homologically trivial loops, and homologically non-trivial loops. As it is really homology and not homotopy that matters, the first is actually a special case of the second type, but for simplicity we will treat them separately.

Contractible Loops

Expectation values of contractible loops on a surface Σ_g can be computed by glueing a disc $\Sigma_{0,1}$ and a $\Sigma_{g,1}$ with a Wilson loop on the boundary. We do the calculation for a Wilson loop on the two-sphere. We want to compute the expectation value

$$\langle \chi_\mu(P \exp \oint_\gamma A) \rangle_{S^2}. \quad (7.38)$$

We split S^2 along γ into two discs D_1 and D_2 and put a Wilson loop on the boundary of D_1 before glueing D_1 and D_2 together again. In equations this amounts to computing

$$\langle \chi_\mu(P \exp \oint_\gamma A) \rangle_{S^2, \epsilon} = \int_G dg K_{D_1}(g, \epsilon_1) \chi_\mu(g) K_{D_2}(g^{-1}, \epsilon_2). \quad (7.39)$$

In order to calculate this we make use of one further property of characters, namely that

$$\chi_\lambda(g) \chi_\mu(g) = \chi_{\lambda \otimes \mu}(g) \equiv \sum_{\rho \in \lambda \otimes \mu, \rho \in \hat{G}} \chi_\rho(g). \quad (7.40)$$

Then we find

$$\langle \chi_\mu(P \exp \oint_\gamma A) \rangle_{S^2, \epsilon} = \sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda) d(\rho) \exp \left(-\frac{1}{8\pi^2} (c(\lambda)\epsilon_1 + c(\rho)\epsilon_2) \right) \quad (7.41)$$

for the unnormalized expectation value of a Wilson loop on S^2 .

To get the result on a general surface one replaces K_D in (7.39) by $K_{\Sigma_{g,1}}(\epsilon)$. Using the multiplicative property (7.40) of characters we thus find

$$\begin{aligned} \langle \chi_\mu(P \exp \oint_\gamma A) \rangle_{\Sigma_{g,\epsilon}} &= \int_G dg K_D(g, \epsilon_D) \chi_\mu(g) K_{\Sigma_{g,1}}(g^{-1}, \epsilon_{\Sigma_{g,1}}) \\ &= \sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda) d(\rho)^{1-2g} \exp \left(-\frac{1}{8\pi^2} (c(\lambda)\epsilon_D + c(\rho)\epsilon_{\Sigma_{g,1}}) \right) \end{aligned}$$

In the topological limit this is

$$\sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda) d(\rho)^{1-2g}, \quad (7.43)$$

which because of the flatness condition should be $d(\mu)$ times the partition function.

Non-contractible Homologically Trivial Loops

These types of loops exist on surfaces of genus > 1 and cut a surface $\Sigma_{g'+g}$ into a $\Sigma_{g',1}$ and a $\Sigma_{g,1}$. Thus the only difference to the example above is that we have to replace D in (7.42) by $\Sigma_{g',1}$. This gives the result

$$\begin{aligned} \langle \chi_\mu(P \exp \oint_\gamma A) \rangle_{\Sigma_{g'+g}} &= \sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda)^{1-2g'} d(\rho)^{1-2g} \exp \left(-\frac{1}{8\pi^2} (c_2(\lambda)\epsilon_{\Sigma_{g',1}} + c_2(\rho)\epsilon_{\Sigma_{g,1}}) \right) \end{aligned} \quad (7.44)$$

which reduces to (7.42) for $g' = 0$.

Homologically Non-Trivial Loops

Not unexpectedly the formulae in this case turn out to be slightly more complicated than (7.42, 7.44). The required operation is now not that of glueing two surfaces together but rather that of joining the two boundaries of a $\Sigma_{g-1,2}$ with a loop in between. In equations this amounts to calculating

$$\begin{aligned} \langle \chi_\mu(P \exp \oint_\gamma A) \rangle_{\Sigma_{g,\epsilon}} &= \int_G dg \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g} \chi_\lambda(g) \chi_\mu(g) \chi_\lambda(g^{-1}) \exp \left(-\frac{\epsilon}{8\pi^2} c_2(\lambda) \right) \\ &= \sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda)^{2-2g} \delta_{\lambda, \rho} \exp \left(-\frac{\epsilon}{8\pi^2} c_2(\lambda) \right). \end{aligned} \quad (7.45)$$

This means that a representation $\lambda \in \hat{G}$ will only contribute to the sum if it appears again in the decomposition of $\lambda \otimes \mu$. Let $m_\mu(\lambda)$ denote the multiplicity of λ in $\lambda \otimes \mu$. Then

$$\langle \chi_\mu(P \exp \oint_\gamma A) \rangle_{\Sigma_{g,\epsilon}} = \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g} m_\mu(\lambda) \exp \left(-\frac{\epsilon}{8\pi^2} c_2(\lambda) \right). \quad (7.46)$$

For $SU(2)$, two extreme cases are represented by choosing μ to be a half-integer spin representation or the spin one representation. If μ is half-integer, then for no value of λ will λ reappear in $\lambda \otimes \mu$, so that we have the general result that for a homologically non-trivial loop γ

$$\langle \chi_{n+\frac{1}{2}}(P \exp \oint_\gamma A) \rangle_{\Sigma_{g,\epsilon}} = 0.$$

On the other hand if $\mu = 1$, then $m_\mu(\lambda) = 1 \forall \lambda \in \hat{G}$ and thus all representations will contribute to the sum in (7.46),

$$\begin{aligned} \langle \chi_{\mu=1}(P \exp \oint_\gamma A) \rangle_{\Sigma_{g,\epsilon}} &= \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g} \exp \left(-\frac{\epsilon}{8\pi^2} c_2(\lambda) \right) \\ &= Z_{\Sigma_g}(\epsilon). \end{aligned} \quad (7.47)$$

In the topological limit this gives back the volume of the moduli space.

The results of this section can of course also be used to calculate correlation functions of several non-intersecting Wilson loops. The intersecting case is more difficult but can be dealt with at the level of the fermionic path integral representation (7.10).

Wilson Points

Let us work directly in the topological limit. In this case the position of the Wilson points makes no difference to the result, so on the genus g Riemann surface we may as well consider all Wilson points to lie in a disc. We proceed in the by now familiar fashion. We calculate the insertion of the Wilson points on the disc and then we glue this to $\Sigma_{g,1}$ to recover the result on Σ_g . We content ourselves with one insertion. The general case is a simple extension.

We wish to calculate

$$\int D\xi D\phi \exp \left(\frac{1}{4\pi^2} \int_D i\phi \xi \right) Tr_R \exp(iq\phi) \sum_{\lambda \in \hat{G}} \chi_\lambda(g_1) \chi_\lambda(P \exp \oint A), \quad (7.48)$$

on the disc and to do it we use the fermionic representations of the Wilson points (3.46). The integral over ϕ restricts ξ to satisfy (3.49), which tells us that $\chi_\lambda(P \exp \oint A)$ does not depend on the loop. The path integral reduces to

$$\begin{aligned} &< 0 | \bar{\eta}^i \sum_{\lambda \in \hat{G}} \chi_\lambda(g_1) \chi_\lambda(\exp q\eta \cdot R \cdot \bar{\eta}) \eta_i | 0 > \\ &= \sum_{\lambda \in \hat{G}} \chi_\lambda(g_1) Tr_R Tr_\lambda \exp(qR^a \otimes \lambda^a), \end{aligned} \quad (7.49)$$

and I leave it to the reader to disentangle this.

8 Instantons on complex Kähler surfaces

There are natural generalizations of the two dimensional theories that we have considered. These involve the moduli spaces of Einstein-Hermitian structures [47] and of semi-stable holomorphic bundles [48]. Here we will content ourselves with a brief application to the moduli space of instantons over four dimensional Kähler manifolds (complex Kähler surfaces).

Any two form Φ on an orientable four manifold may be decomposed into its self-dual and anti-self-dual pieces

$$\Phi^+ = \frac{1}{2}(1 + *)\Phi, \quad \Phi^- = \frac{1}{2}(1 - *)\Phi, \quad (8.1)$$

by virtue of the fact that $*^2 = 1$ so that $\frac{1}{2}(1 \pm *)$ are projection operators. Thus the space of two forms $\Omega^2(M)$ decomposes as

$$\Omega^2(M) = \Omega_+^2(M) \oplus \Omega_-^2(M). \quad (8.2)$$

Extending this to Lie algebra valued forms we have the decomposition

$$\Omega^2(M, \text{Lie}G) = \Omega_+^2(M, \text{Lie}G) \oplus \Omega_-^2(M, \text{Lie}G). \quad (8.3)$$

In terms of this decomposition, the curvature tensor F_A of a connection on a bundle over M may likewise be split and a connection is said to be anti-self-dual (ASD) if

$$F_A^+ = 0. \quad (8.4)$$

An ASD connection is an anti-instanton.

On a complex manifold there is a second decomposition of $\Omega^2(M, \text{Lie}G)$ available,

$$\Omega^2(M, \text{Lie}G) = \Omega^{(2,0)}(M, \text{Lie}G) \oplus \Omega^{(1,1)}(M, \text{Lie}G) \oplus \Omega^{(0,2)}(M, \text{Lie}G). \quad (8.5)$$

The grading (i, j) refers to the holomorphic and anti-holomorphic degrees so that, for example,

$$\begin{aligned} F_{z_1 z_2} dz_1 dz_2 &\in \Omega^{(2,0)}(M, \text{Lie}G), \\ F_{z_1 \bar{z}_2} dz_1 d\bar{z}_2 &\in \Omega^{(1,1)}(M, \text{Lie}G), \\ F_{z_1 \bar{z}_2} dz_1 d\bar{z}_2 &\in \Omega^{(1,1)}(M, \text{Lie}G). \end{aligned} \quad (8.6)$$

Furthermore, if the manifold M is Kähler then it comes complete with a non-degenerate closed two form ω of type $(1, 1)$ so that there is a further refinement

$$\Omega^{(1,1)}(M, \text{Lie}G) = \Omega_0^{(1,1)}(M, \text{Lie}G) \oplus \Omega^0(M, \text{Lie}G)\omega, \quad (8.7)$$

where $\Omega_0^{(1,1)}(M, \text{Lie}G)$ consists of the $(1, 1)$ two forms that are pointwise orthogonal to ω . $\Omega^0(M, \text{Lie}G)\omega$ is meant to indicate the component of the two form along the Kähler form times the Kähler form.

The complex decomposition and the duality decomposition of $\Omega^2(M, \text{Lie}G)$ are related by

$$\begin{aligned} \Omega_+^2(M, \text{Lie}G) &= \Omega^{(2,0)}(M, \text{Lie}G) \oplus \Omega^0(M, \text{Lie}G)\omega \oplus \Omega^{(0,2)}(M, \text{Lie}G), \\ \Omega_-^2(M, \text{Lie}G) &= \Omega_0^{(1,1)}(M, \text{Lie}G). \end{aligned} \quad (8.8)$$

Anti-Instanton Moduli Space

The ASD connections using the Riemannian structure of the four manifold are, as we have seen, compactly written in terms of one equation

$$F_A^+ = 0. \quad (8.9)$$

In terms of the Kähler structure of the manifold, the ASD connections are specified by three equations, the first two being

$$F_A^{(2,0)} = 0, \quad F_A^{(0,2)} = 0, \quad (8.10)$$

while the third $(F_A, \omega) = 0$, is neatly written as

$$F_A \omega = 0 = \frac{1}{2}(F_A, \omega)\omega^2. \quad (8.11)$$

The space of connections that satisfy the equations (8.10) is denoted $\mathcal{A}^{(1,1)}$

Topological Field Theory For ASD Instantons

As $F_A \omega$ is a four-form, we may integrate it over the manifold. This suggests that we take as an action the obvious generalization of the two dimensional action namely

$$\frac{i}{4\pi^2} \int_M \text{Tr} \left(\phi F_A \omega + \frac{1}{2} \psi \psi \omega \right) + \frac{\epsilon}{8\pi^2} \int_M \text{Tr} \phi^2 \omega^2, \quad (8.12)$$

where we have already included a term to take care of possible ϕ zero modes. In order to impose the other two conditions (8.10), we need to introduce two more Grassmann even fields $B^{(2,0)} \in \Omega^{(2,0)}(M, \text{Lie}G)$, $B^{(0,2)} \in \Omega^{(0,2)}(M, \text{Lie}G)$ and two Grassmann odd fields $\chi^{(2,0)} \in \Omega^{(2,0)}(M, \text{Lie}G)$, $\chi^{(0,2)} \in \Omega^{(0,2)}(M, \text{Lie}G)$. These are given the following transformation rules

$$\begin{aligned} \delta \chi^{(2,0)} &= B^{(2,0)}, \quad \delta B^{(2,0)} = [\chi^{(2,0)}, \phi], \\ \delta \chi^{(0,2)} &= B^{(0,2)}, \quad \delta B^{(0,2)} = [\chi^{(0,2)}, \phi], \end{aligned} \quad (8.13)$$

so that δ continues to enjoy the property $\delta^2 \Phi = \mathcal{L}_\phi \Phi$.

We add to the action the following δ exact term

$$\begin{aligned} \delta \int_M \text{Tr} \left(\chi^{(2,0)} F_A^{(0,2)} + \chi^{(0,2)} F_A^{(2,0)} \right) \\ = \int_M \text{Tr} \left(B^{(2,0)} F_A^{(0,2)} + B^{(0,2)} F_A^{(2,0)} \right. \\ \left. - \chi^{(2,0)} (d_A \psi)^{(0,2)} + \chi^{(0,2)} (d_A \psi)^{(2,0)} \right). \end{aligned} \quad (8.14)$$

Integration over the fields $B^{(2,0)}$ and $B^{(0,2)}$ forces the gauge fields to satisfy (8.10) so that the path integral over \mathcal{A} is restricted to $\mathcal{A}^{(1,1)}$. Likewise integration over the χ forces the ψ to be tangents to $\mathcal{A}^{(1,1)}$.

The path integral that we have is then an interesting mixture of two types of topological field theories. These correspond to the two types of fixed point theorems that are available. The first part of the action (8.12), just as in the two dimensional theories, is a Duistermaat-Heckman type action. The second term (8.14) is of the Mathtai-Quillen form.

The analogy with the two dimensional version may be pushed further. The critical points of (8.12), taking into account (8.10) are Yang-Mills connections. We can see this by noting that on $\mathcal{A}^{1,1}$, $F_A \omega$ is essentially F_A^+ , so that on eliminating ϕ we produce an action which is, up to an additive constant, the Yang-Mills action.

One more point worthy of note is the relationship between the theory presented here and Donaldson theory (in Kähler form). These are related to each other in the same way as the old and new versions of the cohomological field theory in two dimensions as described at the end of section 4.

Observables

The simplest observable is the path integral, with action the sum of (8.12) and (8.14). For simplicity, consider the case where $H_A^2 = 0$, that is where there are no zero modes at all of B , χ or ϕ . The $B^{(2,0)}$ and $\chi^{(2,0)}$ integrals give us

$$\delta(F_A^{(2,0)}) \delta((d_A \psi)^{(2,0)}), \quad (8.15)$$

which, off the zero set and around a preferred connection $A_0 \in \mathcal{A}^{(1,1)}$, $A = A_0 + a$, becomes

$$\delta((d_{A_0} a)^{(2,0)}) \delta((d_{A_0} \psi)^{(2,0)}) = \delta(a^{(2,0)}) \delta(\psi^{(2,0)}). \quad (8.16)$$

The determinants will exactly cancel (at the end of the day), up to a sign which is not indicated. The sign, however, is irrelevant, for when we take into account the (0,2) contributions we will obtain the same sign which squares to unity. The

path integral is now over the (co-)tangent bundle of $\mathcal{A}^{(1,1)}$, with the action given by (8.12).

We may set $\epsilon = 0$ with impunity and we do so. The partition function is then equal to the symplectic volume of the moduli space. This establishes that the simplest of Donaldson's invariants is not zero (indeed is positive).

In the remainder we relax slightly the condition that $H_A^2 = 0$ and allow for the "obstruction" space H_A^2 to be made up entirely of $H_A^2 \omega$. In other words, we allow for ϕ zero modes (reducible connections) but not for B or χ zero modes. The B and χ fields may be integrated out as before and the partition function of interest is

$$Z_M(\epsilon) = \int_{T\mathcal{A}^{(1,1)}} D\phi \exp \left(\frac{i}{4\pi^2} \int_M \text{Tr}(\phi F \omega + \frac{1}{2} \psi \psi \omega) + \frac{\epsilon}{8\pi^2} \int_M \text{Tr} \phi^2 \omega^2 \right). \quad (8.17)$$

We do not evaluate this partition function, but rather can express other observables in terms of it. The easiest examples are the expectation values of products of \mathcal{O}_0 . One may follow line for line the steps in (4.26) and (4.27) to obtain a formula in terms of the differentiation of the partition function with respect to ϵ .

If we could follow the steps of the two dimensional theory to calculate the partition function, we would be able to go a long way towards evaluating many of the Donaldson invariants of these moduli spaces. Unfortunately our technology at the moment seems to be not up to this task. The boundaries of four-manifolds being three-manifolds makes the specification of the boundary data rather more involved. In this context the work of Donaldson on the boundary value problem for Yang-Mills fields may be helpful [49].

A Conventions

Lie Algebra Valued Fields

When, in the text, a field ϕ is said to be Lie algebra valued this means

$$\phi = \phi^a T_a, \quad (A.1)$$

where the (anti-hermitian) T_a are generators of the Lie algebra. Commutators are Lie algebra brackets,

$$[T_a, T_b] = f^c_{ab} T_c, \quad (A.2)$$

so that

$$d_A \lambda = d\lambda + [A, \lambda] = (d\lambda^a + f^a_{bc} A^b \lambda^c) T_a, \quad (A.3)$$

and

$$F_A = dA + \frac{1}{2}[A, A] = (dA^a + \frac{1}{2}f^a_{bc}A^bA^c)T_a. \quad (A.4)$$

Local Coordinate Expressions

In the text differential form notation has been used. For those that prefer explicit index notation, we give the correspondences here.

A zero-form is a function. Any one-form A has as a local expression

$$A = A_\mu dx^\mu, \quad (A.5)$$

while a two form B is

$$B = B_{\mu\nu} dx^\mu dx^\nu. \quad (A.6)$$

The differentials dx^μ anti-commute amongst themselves, so that only the anti-symmetric part of $B_{\mu\nu}$ appears in (A.6). The differential d is

$$d = dx^\mu \partial_\mu, \quad (A.7)$$

and squares to zero. With these rules we have

$$\begin{aligned} F_A &= dA + \frac{1}{2}[A, A] \\ &= (\partial_\mu A_\nu + \frac{1}{2}[A_\mu, A_\nu])dx^\mu dx^\nu \\ &= \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu. \end{aligned} \quad (A.8)$$

The local gauge transformation, for the gauge field, becomes

$$\delta_\lambda A = d\lambda + [A, \lambda] = D_\mu \lambda dx^\mu, \quad (A.9)$$

with

$$D_\mu = \partial_\mu + [A_\mu, \cdot]. \quad (A.10)$$

Given a metric $g_{\mu\nu}$ on the manifold we also have the Hodge $*$ operator that in n dimensions maps p -forms to $(n-p)$ -forms. Its action is defined by

$$*(dx^{\mu_1} \dots dx^{\mu_p}) = \frac{\sqrt{\det g}}{(n-p)!} \epsilon^{\mu_1 \dots \mu_p}_{\mu_{p+1} \dots \mu_n} dx^{\mu_{p+1}} \dots dx^{\mu_n}, \quad (A.11)$$

where $\det g \equiv \det g_{\mu\nu}$ and the epsilon symbol with all the labels down is the antisymmetric matrix density with entries $(0, 1, -1)$ when any labels are repeated, or they are in an even permutation, or an odd permutation, respectively. For

example, in two dimensions $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = 1$. One raises the labels with the metric tensor, so that

$$\epsilon^{\mu_1 \dots \mu_n} = \det g \epsilon^{\mu_1 \dots \mu_n}. \quad (A.12)$$

The invariant volume element is $\sqrt{\det g} dx^{\mu_1} \dots dx^{\mu_n}$ which is often written as $\sqrt{\det g} d^n x$.

The following are now easily derived

$$\begin{aligned} \int_{\Sigma_g} \phi * \phi &= \int_{\Sigma_g} \sqrt{\det g} d^2 x \phi^2(x) \\ \int_{\Sigma_g} \phi F_A &= \int_{\Sigma_g} d^2 x \phi F_{12}. \end{aligned} \quad (A.13)$$

We also have, in two dimensions

$$*dA_0 * A_0 = \nabla_\mu A^\mu, \quad (A.14)$$

where ∇_μ is the covariant derivative in the metric sense, and also covariant with respect to A_0 , while the Yang-Mills equations read

$$dA * F_A = \frac{1}{2} \nabla_\mu F^\mu{}_\nu dx^\nu. \quad (A.15)$$

Instantons And The Symplectic Form

Let us fix on R^4 the standard coordinates x^μ , and on the complex 2-plane the coordinates $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$. The $(2, 0)$ and $(0, 2)$ forms are spanned by

$$\begin{aligned} dz_1 dz_2 &= (dx^1 dx^3 - dx^2 dx^4) + i(dx^1 dx^4 + dx^2 dx^3), \\ d\bar{z}_1 d\bar{z}_2 &= (dx^1 dx^3 - dx^2 dx^4) - i(dx^1 dx^4 + dx^2 dx^3). \end{aligned} \quad (A.16)$$

The symplectic 2-form is

$$\omega = \frac{i}{2} dz_1 d\bar{z}_1 + \frac{i}{2} dz_2 d\bar{z}_2 = dx^1 dx^2 + dx^3 dx^4. \quad (A.17)$$

The self-dual two forms Φ satisfy

$$\Phi_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \Phi^{\gamma\delta}, \quad (A.18)$$

so that

$$\begin{aligned} \Phi &= 2\Phi_{12}(dx^1 dx^2 + dx^3 dx^4) + 2\Phi_{13}(dx^1 dx^3 - dx^2 dx^4) + 2\Phi_{14}(dx^1 dx^4 + dx^2 dx^3) \\ &= i\Phi_{12}\omega + (\Phi_{13} - i\Phi_{14})dz_1 dz_2 + (\Phi_{13} + i\Phi_{14})d\bar{z}_1 d\bar{z}_2. \end{aligned} \quad (A.19)$$

This is the decomposition advertised in the text.

B Non-Abelian Stokes' Theorem

In the following we are working on a contractible manifold M of dimension m or, equivalently, consider what follows to be performed on a single coordinate neighbourhood, which is diffeomorphic to an open set in \mathbb{R}^m . Given any gauge field (connection) A on a principle G bundle over M , there is a gauge transformed connection $A^U = U^{-1}AU + U^{-1}dU$ such that

$$x.A^U = 0. \quad (B.1)$$

This (Schwinger-Fock) gauge allows us to represent the gauge field A in terms of the field strength (curvature) F_A and the group element U . Equation (B.1) serves as a definition of U . It is in terms of these quantities that the non-Abelian Stokes' theorem is stated.

Abelian Stokes' Theorem

The integral around a loop γ of a connection A is, by Stokes' theorem, equated with the integral over any surface Γ with boundary $\partial\Gamma = \gamma$ of the curvature $F_A = dA$,

$$\int_{\gamma} A = \int_{\Gamma} F_A. \quad (B.2)$$

Alternatively for Wilson loops this is

$$\exp\left(\int_{\gamma} A\right) = \exp\left(\int_{\Gamma} F_A\right), \quad (B.3)$$

and it is this formula that generalises, in a gauge invariant way, to the non-Abelian case.

Non-Abelian Stokes' theorem

Before turning to this let us make one observation. Within M the surface Γ may be quite contorted. However by a suitable choice of local coordinate functions it may be taken to be the unit disc in an \mathbb{R}^2 plane of \mathbb{R}^m centred at the origin. We work with these local coordinates.

As it is not, perhaps, apparent that one may specify any connection in terms of its curvature and a group element (corresponding to the usual gauge freedom) we show this first. We express A^U in terms of $F_A^U = F(A^U) = U^{-1}F_AU$,

$$\begin{aligned} A_{\mu}^U &= \int_0^1 ds \frac{d}{ds} [A_{\mu}^U(sx)s] \\ &= \int_0^1 ds [sx^{\nu} \partial A_{\mu}^U(sx) / \partial (sx^{\nu}) + A_{\mu}^U(sx)] \end{aligned}$$

$$\begin{aligned} &= \int_0^1 ds [sx^{\nu} \partial A_{\mu}^U(sx) / \partial (sx^{\nu}) - sx^{\mu} \partial A_{\nu}^U(sx) / \partial (sx^{\mu})] \\ &= \int_0^1 ds sx^{\nu} F_{\nu\mu}^U(sx) \end{aligned} \quad (B.4)$$

where

$$F_{\nu\mu}^U(sx) = \frac{\partial}{\partial sx^{\nu}} A_{\mu}^U(sx) - \frac{\partial}{\partial sx^{\mu}} A_{\nu}^U(sx) + [A_{\nu}^U(sx), A_{\mu}^U(sx)]. \quad (B.5)$$

The third line in (B.4) follows by differentiation of (B.1) at the point sx , that is, $A_{\mu}^U(sx) + sx^{\nu} \partial A_{\nu}^U(sx) / \partial (sx^{\mu}) = 0$, while the last line follows from the fact that $sx^{\mu} [A_{\mu}^U(sx), A_{\nu}^U(sx)] = 0$. In terms of the original field we have field

$$A_{\mu}(x) = U(x) \int_0^1 ds sx^{\nu} U^{-1}(sx) F_{\nu\mu}(sx) U(sx) U^{-1}(x) - \partial_{\mu} U(x) U^{-1}(x), \quad (B.6)$$

though this may be unifying.

More interesting for us is the application of these ideas to the path ordered exponential

$$P \exp\left(\oint_{\gamma_x} A\right), \quad (B.7)$$

around a (necessarily) contractible loop γ with preferred point x . The path ordered exponentials for A and A^U are related by

$$P \exp\left(\oint_{\gamma_x} A^U\right) = U^{-1}(x) P \exp\left(\oint_{\gamma_x} A\right) U(x), \quad (B.8)$$

or

$$\begin{aligned} &P \exp\left(\oint_{\gamma_x} A\right) \\ &= U(x) P \exp\left(\oint_{\gamma_x} A^U\right) U^{-1}(x) \\ &= U(x) P \exp\left(\oint_{\gamma_x} \int_0^1 s \gamma^{\nu} U^{-1}(s\gamma) F_{\nu\mu}(s\gamma) U(s\gamma) ds d\gamma^{\mu}\right) U^{-1}(x). \end{aligned} \quad (B.9)$$

The last equality is known as the non-Abelian Stokes' theorem. This terminology is justified by noting that in the Abelian case this equivalence reduces to the usual Stokes' theorem. Let us parameterise the boundary curve (unit circle) by t . The local coordinates x^{μ} restricted to the disc are given in terms of s and t by $x^{\mu} = s\gamma^{\mu}(t)$. The s coordinate is 'radial' while t is 'angular'. In this way we see that for Abelian groups, where path ordering is irrelevant (all the matrices commute so their order is immaterial), the exponents appearing in (B.9) may be equated as

$$\int_{\gamma} A = \oint A_{\mu}(\gamma(t)) d\gamma(t)$$

$$\begin{aligned}
&= \oint \int_0^1 F_{\nu\mu}(s\gamma(t)) s\gamma^\nu(t) \frac{d}{dt} \gamma^\mu(t) ds dt \\
&= \oint \int_0^1 F_{\nu\mu}(s\gamma(t)) \frac{d}{dt} (s\gamma^\mu(t)) \frac{d}{ds} (s\gamma^\nu(t)) ds dt \\
&= \int_\Gamma F_A.
\end{aligned} \tag{B.10}$$

The gauge invariant version of the non-Abelian Stokes' theorem is obtained by taking the trace on both sides of (B.9)

$$\begin{aligned}
\text{Tr} P \exp \left(\oint_{\gamma_x} A \right) &= \text{Tr} P \exp \left(\oint_{\gamma_x} \int_0^1 s\gamma^\nu U^{-1}(s\gamma) F_{\nu\mu}(s\gamma) U(s\gamma) ds d\gamma^\mu \right) \\
&= \text{Tr} P \exp \int_\Gamma F_A^U,
\end{aligned} \tag{B.11}$$

with the second line defining what is meant by the surface ordered exponential.

Alternative derivations may be found in [50, 51].

C Laplacian on G and the Schrödinger equation on the disc

The Schrödinger equation that we are interested in is

$$\left[\frac{\partial}{\partial t} + \oint \frac{\delta}{\delta A} \cdot \frac{\delta}{\delta A} \right] \Psi = 0, \tag{C.1}$$

where Ψ is gauge invariant and depends on A only through its holonomy on the boundary of the disc and t represents some 'evolution' from the centre of the disc. We want to relate the solutions of (C.1) to the eigenfunctions of the Laplacian on the group G . These are the characters of G ,

$$\Delta_G \chi_\lambda(g) = c_2(\lambda) \chi_\lambda(g), \tag{C.2}$$

where c_2 is the quadratic Casimir of the representation.

By the Peter-Weyl theorem Ψ must have the form

$$\Psi(g) = \sum_\lambda a_\lambda \chi_\lambda(g), \tag{C.3}$$

where the a_λ are functions of t and

$$g = P \exp \oint A. \tag{C.4}$$

The A represent tangent space variables to the group elements, and, in particular, for those group elements of the form (C.4), the Laplacian at g is just the flat (tangent space) Laplacian so that

$$\oint \frac{\delta}{\delta A} \cdot \frac{\delta}{\delta A} \chi_\lambda(g) = c_2(\lambda) \chi_\lambda(g). \tag{C.5}$$

This may be obtained explicitly by noting

$$\frac{\delta}{\delta A^a(x)} \text{Tr} P \exp \left(\oint A \right) = \text{Tr} P \lambda^a \exp \left(\oint_x A \right), \tag{C.6}$$

with the notation \oint_x indicating that the path begins at x .

The Schrödinger equation becomes (by orthogonality of the characters)

$$\left[\frac{\partial}{\partial t} + c_2(\lambda) \right] a_\lambda = 0, \tag{C.7}$$

so that the most general solution is

$$\Psi(t, g) = \sum_\lambda c_\lambda \chi_\lambda(g) \exp(-t c_2(\lambda)), \tag{C.8}$$

with the c_λ constants.

D Path integral Representation of Wilson loops

In the text we introduced a path integral representation for the character of the holonomy of the gauge field in a given representation λ of the structure group. The solution of the following path integral was needed at various points of the analysis

$$\begin{aligned}
[\Psi_\lambda(\rho, \epsilon)]^{ij} &= \int D\eta D\bar{\eta} \exp \left(\int_0^1 dt [i\bar{\eta}^k(t) \dot{\eta}^k(t) + \rho^a(t) (\bar{\eta} \lambda^a \eta)(t)] \right. \\
&\quad \left. - \frac{1}{2} \epsilon \int_0^1 dt (\bar{\eta} \lambda^a \eta)(t) (\bar{\eta} \lambda^a \eta)(t) \right) \bar{\eta}^i(1) \eta^j(0).
\end{aligned} \tag{D.1}$$

Consider first the trace of this $\Psi_\lambda(\rho, \epsilon) = \Psi_\lambda(\rho, \epsilon)^{ij} \delta_{ij}$. We do not need to evaluate the trace of (D.1). Rather we note that it satisfies the Schrödinger equation (C.1) with the initial condition that it is the character

$$\Psi_\lambda(\rho, 0) = \chi_\lambda(P \exp \oint \rho). \tag{D.2}$$

The solution, following our previous analysis, is thus

$$\Psi_\lambda(\rho, \epsilon) = \chi_\lambda(P \exp \oint \rho) \exp\left(-\frac{\epsilon}{2} c_2(\lambda)\right). \tag{D.3}$$

To see that $\Psi_\lambda(\rho, \epsilon)$ satisfies (C.1) (with $t = \epsilon/2$) firstly differentiate (D.1) with respect to $\epsilon/2$ to obtain

$$\begin{aligned} \frac{\partial}{\partial \epsilon/2} \Psi_\lambda(\rho, \epsilon) &= \langle \int_0^1 dt (\bar{\eta} \lambda^a \eta)(t) (\bar{\eta} \lambda^a \eta)(t) \rangle \\ &= \oint \frac{\delta^2}{\delta \rho^2} \Psi_\lambda(\rho, \epsilon), \end{aligned} \quad (D.4)$$

where $\langle Z \rangle$ means the insertion of the field Z in the path integral on the right hand side of (D.1).

When considering the matrix elements $\Psi_\lambda(\rho, \epsilon)^{ij}$, we use the same argument with the initial condition that at $\epsilon = 0$ this is

$$\Psi_\lambda(\rho, 0)^{ij} = [P \exp \oint \rho]^{ij} \quad (D.5)$$

to arrive at

$$\Psi_\lambda(\rho, \epsilon)^{ij} = [P \exp \oint \rho]^{ij} \exp \left(-\frac{\epsilon}{2} c_2(\lambda) \right). \quad (D.6)$$

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Figure Captions

Fig. 1 A genus g surface with a particular choice of the homology basis.

Fig. 2 The homology basis for the Torus with its associated cut surface.

Fig. 3 Homology basis for a genus 2 surface.

Fig. 4 The cut Riemann surface associated to the genus 2 surface.

Fig. 5a,b Two ways of seeing the identification of edges of a disc to form a cylinder.

Fig. 6 A possible identification of the edges of a disc to obtain the pants.

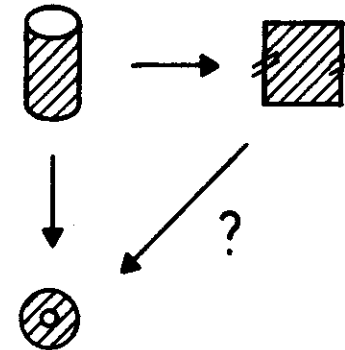


Fig. 5a

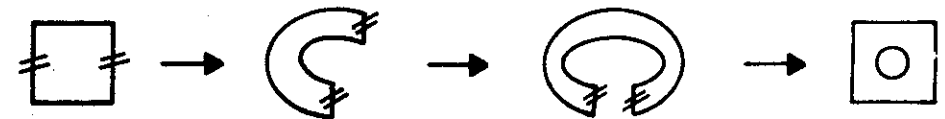


Fig. 5b

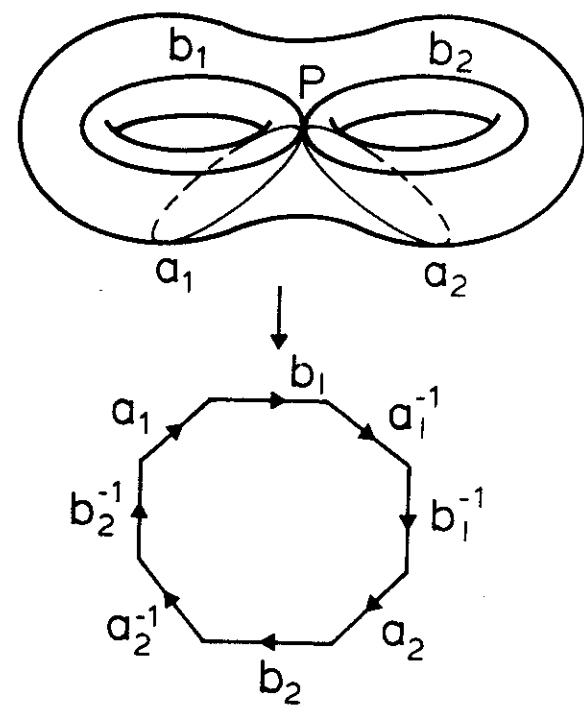
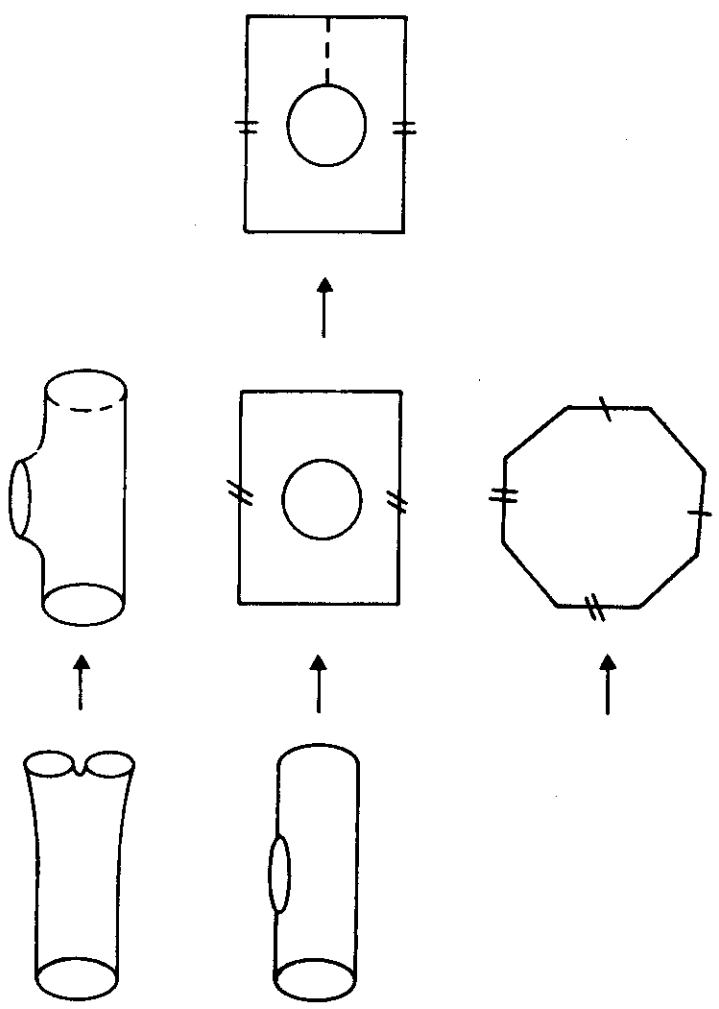
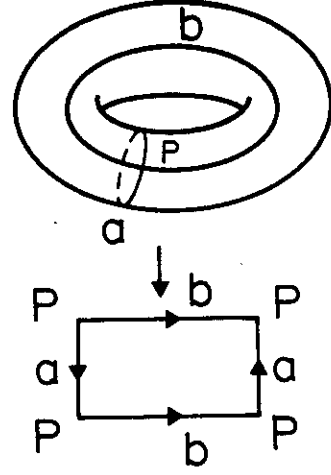
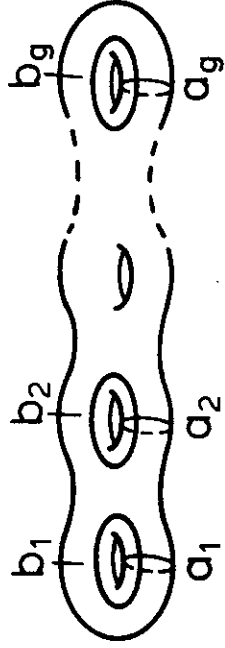


Fig 1



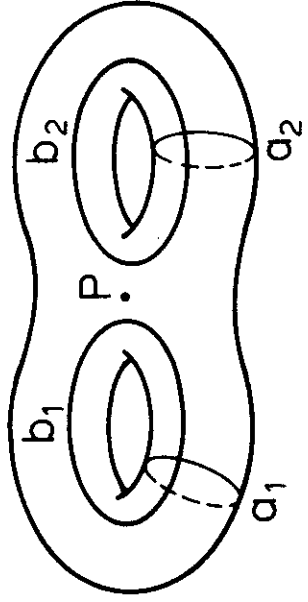


Fig 3

