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CLASSICAL AND QUANTUM *W*-GRAVITY

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ABSTRACT

Classical $\mathcal{W}$-gravities and the corresponding quantum theories are reviewed. $\mathcal{W}$-gravities are higher-spin gauge theories in two dimensions whose gauge algebras are $\mathcal{W}$-algebras. The geometrical structure of classical $\mathcal{W}$-gravity is investigated, leading to surprising connections with self-dual geometry. The anomalies that arise in quantum $\mathcal{W}$-gravity are discussed, with particular attention to the new types of anomalies that arise for non-linearly realised symmetries and to the relation between path-integral anomalies and non-closure of the quantum current algebra. Models in which all anomalies are cancelled by ghost contributions lead to new generalisations of string theories.

1. Introduction

There has been considerable interest in the infinite-dimensional algebras that can arise as symmetries of two-dimensional field theories and there is an intimate relationship between such algebras and two-dimensional gauge theories or string theories. Perhaps the most important example is the Virasoro algebra, which is a symmetry algebra of any two dimensional conformal field theory. This infinite-dimensional rigid symmetry can be promoted to a local symmetry (two-dimensional diffeomorphisms) by coupling the two-dimensional field theory to gravity, resulting in a theory that is Weyl-invariant as well as diffeomorphism invariant. The two-dimensional metric enters the theory as a Lagrange multiplier imposing constraints which satisfy the Virasoro algebra, and the Virasoro algebra also emerges as the residual symmetry that remains after choosing a conformal gauge. The quantisation of such a system of matter coupled to gravity defines a string theory and if the matter system is chosen such that the world-sheet metric $h_{\mu\nu}$ decouples from the quantum theory (*i.e.* if the matter central charge is $c = 26$), the string theory is said to be critical [1]. Remarkably, the introduction of gravity on the world-sheet leads to a critical string theory which leads to gravity in space-time.

The situation is similar for each of the cases in the following table.

Algebra	Spins	2-D Gauge Theory	Gauge Fields	String Theory
Virasoro Algebra	2	Gravity	$h_{\mu\nu}$	Bosonic String
Super-Virasoro	$2, \frac{3}{2}$	Supergravity	$h_{\mu\nu}, \psi_\mu$	Superstring
$N = 2$ Super-Virasoro	$2, \frac{3}{2}, \frac{3}{2}, 1$	$N = 2$ Supergravity	$h_{\mu\nu}, \psi_\mu, \bar{\psi}_\mu, A_\mu$	$N = 2$ Superstring
Topological Virasoro	$2, 2, 1, 1$	Topological Gravity	$h_{\mu\nu}, g_{\mu\nu}, A_\mu, \psi_\mu$	Topological String
$\mathcal{W}$-Algebra	$2, 3, \dots$	$\mathcal{W}$-Gravity	$h_{\mu\nu}, B_{\mu\nu\rho}, \dots$	$\mathcal{W}$-Strings

In the first column are extended conformal algebras, *i.e.* infinite dimensional algebras that contain the Virasoro algebra. Each algebra is generated by a set of currents, whose spins are labelled in the second column. Each algebra can arise as the symmetry algebra of a particular class of conformal field theories

e.g. the super-Virasoro algebra is a symmetry of super-conformal field theories while the topological Virasoro algebra is a symmetry of topological conformal field theories. For such theories, the infinite-dimensional rigid symmetry of the matter system can be promoted to a local symmetry by coupling to the gauge theory listed in the third column. In this coupling, the currents generating the extended conformal algebra are coupled to the corresponding gauge fields in the fourth column. In each case, the gauge fields enter as Lagrange multipliers and the constraints that they impose satisfy the algebra given in the first column. Finally, integration over the matter and gauge fields defines a generalisation of string theory which is listed in the last column. In general, the gauge fields will become dynamical in the quantum theory, but for special choices of conformal matter system (e.g. $c = 26$ systems for the bosonic string, $c = 0$ systems for the topological string or $c = 15$ systems for the $N = 1$ superstring), the string theory will be 'critical' and the gauge fields will decouple from the theory. A row can be added to the table corresponding to any two-dimensional extended conformal algebra.

The subject of this review is the set of models corresponding to the last row of the table. A $\mathcal{W}$-algebra might be defined as any extended conformal algebra, *i.e.* a closed algebra that satisfies the Jacobi identities, contains the Virasoro algebra as a subalgebra and is generated by a (possibly infinite) set of chiral currents. Often the definition of $\mathcal{W}$-algebra is restricted to those algebras for which at least one of the generating currents has spin greater than 2, but relaxing this condition allows the definition to include all the algebras in the table and almost all of the results to be reviewed here apply with this more general definition. However, many (but not all) interesting $\mathcal{W}$-algebras contain a spin-three current, and for this reason 3 is included as a typical higher spin in the $\mathcal{W}$-algebra entry in the table.

The simplest $\mathcal{W}$-algebras are those that are Lie algebras, with the generators t_a (labelled by an index a which will in general have an infinite range) satisfying

commutation relations of the form

$$[t_a, t_b] = f_{ab}{}^c t_c + c_{ab} \tag{1.1}$$

for some structure constants $f_{ab}{}^c$ and constants c_{ab}, which define a central extension of the algebra. However, for many $\mathcal{W}$-algebras, the commutation relations give a result non-linear in the generators

$$[t_a, t_b] = f_{ab}{}^c t_c + c_{ab} + g_{ab}{}^{cd} t_c t_d + \ldots = F_{ab}(t_c) \tag{1.2}$$

and the algebra can be said to close in the sense that the right-hand-side is a function of the generators. Most of the $\mathcal{W}$-algebras that are generated by a finite number of currents, with at least one current of spin greater than two, are non-linear algebras of this type. Classical $\mathcal{W}$-algebras for which the bracket in (1.2) is a Poisson bracket are straightforward to define, as the non-linear terms on the right-hand-side can be taken to be a product of classical charges. For quantum $\mathcal{W}$-algebras, however, the bracket is realised as a commutator of quantum operators and the definition of the right-hand-side requires some normal-ordering prescription. The complications associated with the normal-ordering mean that there are classical $\mathcal{W}$-algebras for which there is no corresponding quantum $\mathcal{W}$-algebra that satisfies the Jacobi identities [4]. At first sight, it appears that there might be a problem in trying to realise a non-linear algebra in a field theory, as symmetry algebras are usually Lie algebras. However, as will be seen, a non-linear algebra can be realised as a symmetry algebra for which the structure 'constants' are replaced by field-dependent quantities.

Consider a field theory in flat Minkowski space with metric $\eta_{\mu\nu}$ and coordinates x^0, x^1. The stress-energy tensor is a symmetric tensor $T_{\mu\nu}$ which, for a translation-invariant theory, satisfies the conservation law

$$\partial^\mu T_{\mu\nu} = 0 \tag{1.3}$$

A spin-s current in flat two-dimensionsal space is a rank-s symmetric tensor

$W_{\mu_1\mu_2...\mu_s}$* and will be conserved if

$$\partial^{\mu_1} W_{\mu_1\mu_2...\mu_s} = 0 \qquad (1.4)$$

A theory is conformally invariant if the stress tensor is traceless, $T_\mu{}^\mu = 0$. Introducing null coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, the tracelessness condition becomes $T_{+-} = 0$ and (1.3) then implies that the remaining components $T_{\pm\pm}$ satisfy

$$\partial_+ T_{--} = 0, \qquad \partial_- T_{++} = 0 \qquad (1.5)$$

If a spin-s current $W_{\mu_1\mu_2...\mu_s}$ is traceless, it will have only two non-vanishing components, $W_{++...+}$ and $W_{--...-}$. The conservation condition (1.4) then implies that

$$\partial_- W_{++...+} = 0, \qquad \partial_+ W_{--...-} = 0 \qquad (1.6)$$

so that $W_{++...+} = W_{++...+}(x^+)$ and $W_{--...-} = W_{--...-}(x^-)$ are right- and left-moving chiral currents, respectively. For a given conformal field theory, the set of all right-moving chiral currents generate a closed current algebra, the right-moving chiral algebra, and similarly for left-movers. The right and left chiral algebras are examples of $\mathcal{W}$-algebras but are often too large to be useful. In studying conformal field theories, it is often useful to restrict attention to all theories whose chiral algebras contain a particular $\mathcal{W}$-algebra; the representation theory of that $\mathcal{W}$-algebra then gives a great deal of useful information about the spectrum, modular invariants etc of those theories and may lead to a classification.

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* Recall that, in two dimensions, any tensor can be decomposed into a set of symmetric tensors, e.g. $V_{\mu\nu} = V_{(\mu\nu)} + V\epsilon_{\mu\nu}$ where $V = \frac{1}{2}\epsilon^{\mu\nu}V_{\mu\nu}$. Thus without loss of generality, all the conserved currents of a given theory can be taken to be symmetric tensors. A rank-s symmetric tensor transforms as the spin-s representation of the two-dimensional Lorentz group.

A field theory with action S_0 and symmetric tensor conserved currents $T_{\mu\nu}, W^A_{\mu_1\mu_2...\mu_{s_A}}$ (where $A = 1, 2, ...$ labels the currents, which have spin s_A) will be invariant under rigid symmetries with constant parameters $k^\mu, \lambda_A^{\mu_1\mu_2...\mu_{s_A-1}}$ (translations and '$\mathcal{W}$-translations') generated by the Noether charges $P_\mu, Q^A_{\mu_1\mu_2...\mu_{s_A-1}}$ (momentum and '$\mathcal{W}$-momentum') given by $P_\mu = \int dx^0 T_{0\mu}$ and $Q^A_{\mu_1\mu_2...\mu_{s_A-1}} = \int dx^0 W^A_{\mu_1\mu_2...\mu_{s_A-1}0}$. This is true of non-conformal theories (e.g. affine Toda theories) as well as conformal ones. However, if the currents are traceless, then the theory is in fact invariant under an infinite dimensional extended conformal symmetry. The parameters $\lambda_A^{\mu_1\mu_2...\mu_{s_A-1}}$ are then traceless and the corresponding transformations will be symmetries if the parameters are not constant but satisfy the conditions that the trace-free parts of $\partial^{(\nu}k^{\mu)}, \partial^{(\nu}\lambda_A^{\mu_1\mu_2...\mu_{s_A-1})}$ are zero. This implies that $\partial_\mp k^\pm = 0$ and $\partial_\mp \lambda_A^{\pm\pm...\pm} = 0$ so that the parameters are 'semi-local', $k^\pm = k^\pm(x^\pm)$ and $\lambda_A^{\pm\pm...\pm} = \lambda_A^{\pm\pm...\pm}(x^\pm)$ and these are the parameters of conformal and '$\mathcal{W}$-conformal' transformations.

The rigid symmetries corresponding to the currents $T_{\mu\nu}, W^A_{\mu_1\mu_2...\mu_{s_A}}$ can be promoted to local ones by coupling to the $\mathcal{W}$-gravity gauge fields $h^{\mu\nu}, B_A^{\mu_1\mu_2...\mu_{s_A}}$ which are symmetric tensors transforming as

$$\delta h^{\mu\nu} = \partial^{(\nu}k^{\mu)} + ..., \qquad \delta B_A^{\mu_1\mu_2...\mu_s} = \partial^{(\nu}\lambda_A^{\mu_1\mu_2...\mu_{s-1})} + ..., \qquad (1.7)$$

to lowest order in the gauge fields. The action is given by the Noether coupling

$$S = S_0 + \int d^2x \left(h^{\mu\nu}T_{\mu\nu} + B_A^{\mu_1\mu_2...\mu_{s_A}} W^A_{\mu_1\mu_2...\mu_{s_A}} \right) + \cdots \qquad (1.8)$$

plus terms non-linear in the gauge fields. If the currents $T_{\mu\nu}, W^A_{\mu_1\mu_2...\mu_s}$ are traceless, i.e. if there is extended conformal symmetry, then the traces of the gauge fields decouple and the theory is invariant under Weyl and '$\mathcal{W}$-Weyl' transformations given to lowest order in the gauge fields by

$$\delta h^{\mu\nu} = \Omega \eta^{\mu\nu} + ..., \qquad \delta B_A^{\mu_1\mu_2...\mu_s} = \Omega_A^{(\mu_1\mu_2...\mu_{s-2}}\eta^{\mu_{s-1}\mu_s)} + ... \qquad (1.9)$$

where $\Omega(x^\nu), \Omega_A^{\mu_1\mu_2...\mu_{s-2}}(x^\nu)$ are the local parameters. This defines the linearised coupling to $\mathcal{W}$-gravity. The full non-linear theory is in general non-polynomial in coupling to $\mathcal{W}$-gravity.

the gauge fields of spins 2 and higher. The non-linear theory can be constructed to any given order using the Noether method, but to obtain the full non-linear structure requires a deeper understanding of the geometry underlying $\mathcal{W}$-gravity.

This review falls into three main parts. In the first part (chapters 2-5) classical $\mathcal{W}$-algebras and $\mathcal{W}$-gravity will be discussed. In the second part, (chapter 6), a geometric approach will be used to obtain the full non-linear structure of a particular $\mathcal{W}$-gravity theory. In the final part, (chapters 7-10) the quantisation of $\mathcal{W}$-gravity coupled to $\mathcal{W}$-matter will be investigated and the anomaly structure described. These results will then be used to consider the construction of string theories based on $\mathcal{W}$-algebras.

2. Classical $\mathcal{W}$-Algebras

Consider a set $\mathcal{S}$ of classical right-moving chiral currents $T(x^+) = T_{++}(x^+), W(x^+), \ldots$ of spins $2, s_W, \ldots$. As the currents are classical, there is no problem in defining products of currents. Suppose there is also a (graded) anti-symmetric bracket product $[A, B]$ defined on the set of currents. The main example that will be of interest here is that in which the currents arise from some classical field theory and the bracket is the Poisson bracket in a canonical formalism in which x^- is regarded as the time variable. The current T satisfies the conformal algebra if

$$[T(x^+), T(y^+)] = -\delta'(x^+ - y^+)[T(x^+) + T(y^+)] + \frac{c}{12}\delta'''(x^+ - y^+) \qquad (2.1)$$

in which case its modes L_n generate the Virasoro algebra with central charge c. A current W is said to be primary of spin s_W if

$$[T(x^+), W(y^+)] = -\delta'(x^+ - y^+)[W(x^+) + (s_W - 1)W(y^+)] \qquad (2.2)$$

The set $\mathcal{S}$ of currents will generate a $\mathcal{W}$-algebra if the bracket of any two currents gives a function of currents in $\mathcal{S}$ and if the bracket satisfies the Jacobi identities.

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Consider first the case in which there are just two currents, T and W, where W is primary of spin $s = s_W$. For simplicity, suppose that $c = 0$ (at least classically) and that the $[W, W]$ bracket takes the form

$$[W(x^+), W(y^+)] = -2\kappa\delta'(x^+ - y^+)[\Lambda(x^+) + \Lambda(y^+)] \qquad (2.3)$$

for some Λ, where κ is a constant. If the algebra is to close, the current Λ must be a function of the currents T, W and their derivatives. If $s = 3$, then Λ is a spin-four current and the Jacobi identities are satisfied if

$$\Lambda = TT \qquad (2.4)$$

The algebra then closes non-linearly to give a certain classical limit of the W_3 algebra of Zamolodchikov [13]. (In the limit $\kappa \to 0$, this contracts to a linear algebra [4, 17].)

For $s > 3$, the algebra will again close and satisfy the Jacobi identities if Λ depends on T, W but not on their derivatives. If s is even, the most general such Λ is of the form

$$\Lambda = \alpha T^{s-1} + \beta W T^{s/2-1} \qquad (2.5)$$

for some constants α, β, while if s is odd, such a Λ must be of the form (2.5) with $\beta = 0$. The algebra given by (2.1),(2.2),(2.3) and (2.5) is the algebra $W_{s/s-2}$ of ref. [4]. For $s = 3$, it is a classical limit of the W_3 algebra while for $s = 4$ and $s = 6$ it is a classical limit of the quantum algebras conjectured to exist by Bouwknegt [18] and constructed in [19,20]. For all other integer values of $s(> 4)$, the argument of Bouwknegt [18] shows that there can be no quantum operator algebra that satisfies the Jacobi identities for generic values of the Virasoro central charge and for which (2.1),(2.2),(2.3),(2.5) is a classical limit.

A large number of $\mathcal{W}$-algebras are now known. The $\mathcal{W}_N$ algebra [15,16] has currents of spins $2, 3, \ldots, N$ (so that $\mathcal{W}_2$ is the Virasoro algebra), the $\mathcal{W}_\infty$ [21,22]

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algebra has currents of spins $2, 3, \ldots, \infty$ while the $\mathcal{W}_{1+\infty}$ algebra [22] has currents of spins $1, 2, 3, \ldots, \infty$. Each of these algebras is a classical limit of a quantum algebra. The classical algebra $\mathcal{W}_{N/M}$ has currents of spins $2, 2 + M, 2 + 2M, 2 + 3M, \ldots, N$ and $\mathcal{W}_{\infty/M}$ has currents of spins $2, 2 + M, 2 + 2M, 2 + 3M, \ldots, \infty$ [4] and these algebras are not limits of quantum $\mathcal{W}$-algebras in general. There is an algebra $\mathcal{W}G$ associated with any Lie group G and the algebra associated with $SU(N)$ is the $\mathcal{W}_N$ algebra refered to above [15,16].

3. Field Theory Realisations of $\mathcal{W}$-Algebras

Consider a theory of D free scalar fields ϕ^i ($i = 1, \ldots, D$) with action

$$S_0 = \int d^2x \; \partial_+\phi^i \partial_-\phi^i \tag{3.1}$$

where the two-dimensional space-time has null coordinates $x^\mu = (x^+, x^-)$ which are related to the usual Cartesian coordinates by $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$. The stress-energy tensor

$$T_{++} = \frac{1}{2}\partial_+\phi^i \partial_+\phi^i \tag{3.2}$$

is conserved, $\partial_- T_{++} = 0$, and generates the Poisson bracket algebra (2.1) with $c = 0$ (in a canonical treatment regarding x^- as time [23]) which is the conformal algebra with vanishing central charge. For any rank-s constant symmetric tensor $d_{i_1 i_2 \ldots i_s}$ one can construct a current

$$W_{++\ldots+} = \frac{1}{s}d_{i_1 i_2 \ldots i_s}\partial_+\phi^{i_1}\partial_+\phi^{i_2}\ldots\partial_+\phi^{i_s} \tag{3.3}$$

which is conserved, $\partial_- W = 0$, and which is a spin-s classical primary field, $i.e.$ its Poisson bracket with T is given by (2.2). The Poisson bracket of two W's is (2.3), where Λ is given by

$$\Lambda = \frac{1}{4\kappa}d_{i\ldots j}{}^m d_{k\ldots lm}\partial_+\phi^i \ldots \partial_+\phi^j \partial_+\phi^k \ldots \partial_+\phi^l \tag{3.4}$$

(the indices $i, j, \ldots$ are raised and lowered with the flat metric δ_{ij}).

Consider first the case $s = 3$. In general, closing the algebra generated by T, W will lead to an infinite sequence of currents $(T, W, \Lambda, \ldots)$. However, if $\Lambda = T^2$, for some constant κ, then the algebra closes non-linearly on T and W, to give the classical $\mathcal{W}_3$-algebra of the last chapter.

In [2], it was shown that for any number D of bosons, the necessary and sufficient condition for (2.4) to be satisfied and hence for the classical $\mathcal{W}_3$ algebra to be generated is that the 'structure constants' d_{ijk} satisfy

$$d_{(ij}{}^m d_{k)lm} = \kappa\delta_{(ij}\delta_{k)l}, \tag{3.5}$$

This rather striking algebraic constraint has an interesting algebraic interpretation.[*] It implies that the d_{ijk} are the structure constants for a Jordan algebra (of degree 3) [27] which is a commutative algebra for which (3.5) plays the role of the Jacobi identities. Moreover, the set of all such algebras has been classified [28], allowing one to write down the general solution to (3.5) [27]. In particular, (3.5) has a solution for any number D of bosons. Examples of solutions to (3.5) are given for $D = 1$ by $d_{111} = \kappa$ and for $D = 8$ by taking d_{ijk} proportional to the d-symbol for the group $SU(3)$ [2]. For $D = 2$, the construction of [14] gives a solution of (3.5) in which the only non-vanishing components of d_{ijk} are given by $d_{112} = -\kappa$ and $d_{222} = \kappa$, together with those related to these by symmetry.

The conserved currents T, W correspond to the invariance of the free action S_0 under the conformal symmetries

$$\delta\phi^i = k_-\partial_+\phi^i + \lambda_{--}d^i{}_{jk}\partial_+\phi^j\partial_+\phi^k \tag{3.6}$$

where the parameters satisfy

$$\partial_- k_- = 0, \qquad \partial_- \lambda_{--} = 0. \tag{3.7}$$

Symmetries of this kind whose parameters are only functions of x^+ (or only of

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[*] This identity has in fact occurred at least once before in the physics literature, in the study of five-dimensional supergravity theories [26].

x^-) will be referred to here as semi-local. The symmetry algebra closes to give

$$[\delta_{k_1} + \delta_{\lambda_1}, \delta_{k_2} + \delta_{\lambda_2}] = \delta_{k_3} + \delta_{\lambda_3} \tag{3.8}$$

where

$$
\begin{aligned}
k_3 &= [k_2 \partial_+ k_1 + 4\kappa(\lambda_2 \partial_+ \lambda_1)T_{++}] \ - \ (1 \leftrightarrow 2) \\
\lambda_3 &= [2\lambda_2 \partial_+ k_1 + k_2 \partial_+ \lambda_1] \ - \ (1 \leftrightarrow 2)
\end{aligned}
\tag{3.9}
$$

In particular, the commutator of two λ transformations is a field-dependent k-transformation, which is precisely the transformation generated by the spin four current $\Lambda = TT$. The gauge algebra structure 'constants' are not constant but depend on the fields ϕ through the current T, reflecting the TT term in the current algebra.

The situation is similar for $s > 3$. The algebra will close, *i.e.* (2.5) will be satisfied, if the d-tensor in (3.3) satisfies a quadratic constraint [4] and again this constraint has an algebraic interpretation [4]. The k and λ-transformations become

$$\delta \phi^i = k\partial_+ \phi^i + \lambda d^i{}_{i_1 \ldots i_{s-1}} \partial_+ \phi^{i_1} \ldots \partial_+ \phi^{i_{s-1}} \tag{3.10}$$

where the parameters satisfy $\partial_- k = 0$, $\partial_- \lambda = 0$. The symmetry algebra again has field dependent structure 'constants'.

More generally, any set of constant symmetric tensors $d^A_{ij \ldots k}$ labelled by some index A can be used to construct a set of conserved currents

$$W^A_{++\ldots+} = d^A_{ij \ldots k} \partial_+ \phi^i \partial_+ \phi^j \ldots \partial_+ \phi^k \tag{3.11}$$

which are classical primary fields, *i.e.* their Poisson bracket with T is (2.2). The current algebra will close if the d^A tensors satisfy certain algebraic constraints and the Jacobi identities will automatically be satisfied as the algebra occurs as a symmetry algebra. In this way, a large class of classical $\mathcal{W}$-algebras can be constructed by seeking d^A-tensors satisfying the appropriate constraints. D boson

realisations of the $\mathcal{W}_N$ algebras were constructed in this way in ref. [4], where it was shown that the $\mathcal{W}_N$ d-tensor constraints had an interpretation in terms of Jordan algebras of degree N, and this again allowed the explicit construction of solutions to the d-tensor constraints. These realisations of classical $c = 0$ algebras can be generalised to ones with $c > 0$ by introducing a background charge a_i, so that the stress tensor becomes $T = \partial_+ \phi^i \partial_+ \phi^i + a_i \partial_+^2 \phi^i$ and adding appropriate higher derivative terms (*i.e.* ones involving $\partial_+^m \phi^i$ for $m > 1$) to the other currents. The clasical central charge becomes $c = a^2/12$, and for the $N-1$ boson realisation of $\mathcal{W}_N$, the structure of the higher derivative terms in the currents W_n can be derived using Miura transform methods [16,15].

Another important realisation of classical $\mathcal{W}$-algebras is as the Casimir algebra of Wess-Zumino-Witten (WZW) models [5]. For the WZW model corresponding to a group G, the Lie-algebra valued currents $J_+ = g^{-1} \partial_+ g$ generate a Kac-Moody algebra and are (classical) primary with respect to the Sugawara stress-tensor $T = \frac{1}{2} tr(J_+ J_+)$. Similarly, the higher order Casimirs allow a generalised Sugawara construction of higher spin currents $tr(J_+^n)$. For example, for $G = SU(N)$ the set of currents $W_n = \frac{1}{n} tr(J_+^n)$ for $n = 2, 3, \ldots, N$ generate a closed algebra which is a classical $\mathcal{W}_N$ algebra [5]; similar results hold for other groups. Quantum mechanically, however, the Sugawara expressions for the currents need normal ordering and must be rescaled [25,24]. For example, in the case of $SU(3)$, the quantum Casimir algebra leads to a closed $\mathcal{W}$-algebra (after a certain truncation) only in the case in which the Kac-Moody algebra is of level one [24].

$\mathcal{W}$-algebras also arise as symmetry algebras of many other field theories, including Toda-theories [16], free-fermion theories [5] and non-linear sigma-models [2,29], giving corresponding realisations of $\mathcal{W}$-algebras.

4. Gauging $\mathcal{W}$-Algebras

In this chapter, the gauging of $\mathcal{W}$-algebras will be discussed, that is the coupling of a $\mathcal{W}$-conformal field theory to $\mathcal{W}$-gravity gauge fields so that the extended conformal symmetry is promoted to a local '$\mathcal{W}$-diffeomorphism' symmetry. Consider first the simple example of the free scalar field theory with action S_0 (3.1) which is invariant under the chiral $\mathcal{W}_3$ transformations (3.6) generated by the currents $T_{++} = \frac{1}{2}\partial_+\phi^i\partial_+\phi^i$ and $W_{+++} = \frac{1}{3}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k$, where the tensor d_{ijk} satisfies (3.5) so that the symmetry algebra closes to give (3.8),(3.9). If the symmetry parameters k_-, λ_{--} are taken to be local, *i.e.* the conditions (3.7) are dropped, then the action (3.1) varies under (3.6) to give

$$\delta S_0 = \int d^2x \, (T_{++}\partial_-k_- + W_{+++}\partial_-\lambda_{--}) \tag{4.1}$$

and this can clearly be cancelled by introducing gauge fields $h = h_{--}, B = B_{---}$ transforming as

$$\delta h_{--} = \partial_-k_- + \ldots, \qquad \delta B_{---} = \partial_-\lambda_{--} + \ldots \tag{4.2}$$

and adding to the action (3.1) the Noether coupling [2]

$$S_1 = -\int d^2x \, (h_{--}T_{++} + B_{---}W_{+++}). \tag{4.3}$$

The action is then invariant to lowest order in the gauge fields. Remarkably, terms of higher order in the gauge fields can be added to the transformations (4.2) in such a way that the linear action $S_0 + S_1$ is fully gauge invariant; surprisingly, no non-linear terms are needed in the action [2]. The total action $S = S_0 + S_1$ is invariant under (3.6) together with

$$
\begin{aligned}
\delta h_{--} =& \partial_-k_- + k_-\partial_+h_{--} - h_{--}\partial_+k_- \\
&+ 2\kappa\left(\lambda_{--}\partial_+B_{---} - B_{---}\partial_+\lambda_{--}\right)T_{++} \\
\delta B_{---} =& \partial_-\lambda_{--} + 2\lambda_{--}\partial_+h_{--} - h_{--}\partial_+\lambda_{--} \\
&- 2B_{---}\partial_+k_- + k_-\partial_+B_{---}.
\end{aligned}
\tag{4.4}
$$

This is the action for the coupling of the free scalar field theory to chiral $\mathcal{W}_3$

gravity [2]. The gauge fields are Lagrange multipliers imposing the constraints $T = 0$, $W = 0$ and these constraints satisfy the algebra (2.1),(2.2), (2.3), (2.4). The gauge algebra is still given by (3.8) and (3.9), up to terms that vanish when the classical equations of motion are satisfied. However, as $T = 0$ is the h_{--} equation of motion, the symmetry algebra (3.8) is equivalent on-shell to the simpler one obtained by setting $T = 0$ in (3.8), so that two λ transformations commute up to equations of motion. This simpler algebra corresponds to the commutation relations (2.1),(2.2) and

$$[W(x^+), W(y^+)] = 0 \tag{4.5}$$

While the previous algebra given by (2.1),(2.2), (2.3), (2.4) only had a maximal finite subalgebra of $SL(2,\mathbb{R})$ (generated by the modes $L_0, L_{\pm 1}$), the algebra in which (2.3) is replaced by (4.5) has an interesting finite subgroup of $ISL(2,\mathbb{R})$, a group contraction of $SL(3,\mathbb{R})$ (generated by the modes $L_0, L_{\pm 1}, W_0, W_{\pm 1}, W_{\pm 2}$).

So far only a right-handed $\mathcal{W}$-algebra has been gauged. Gauging the left-handed reparameterisations generated by $T_{--} = \frac{1}{2}\partial_-\phi^i\partial_-\phi^i$ by introducing a gauge field h_{++} leads to a 'heterotic' model in which a right-moving $\mathcal{W}_3$ algebra and a left-handed Virasoro algebra are gauged [2]. To gauge both a left-handed and a right-handed $\mathcal{W}_3$ algebra requires a further spin-three gauge field B_{+++} corresponding to the current $W_{---} = \frac{1}{3}d_{ijk}\partial_-\phi^i\partial_-\phi^j\partial_-\phi^k$. The action is given by $S = S_0 + S_1 + \ldots$ plus terms of quadratic order and higher in the gauge fields, where S_1 is the Noether coupling

$$S_1 = -\int d^2x \, (h_{--}T_{++} + h_{++}T_{--} + B_{---}W_{+++} + B_{+++}W_{---}) \tag{4.6}$$

The full action can then be constructed iteratively using the Noether method. However, this requires an infinite number of steps as the action is non-polynomial in $h_{\pm\pm}$ and in $B_{\pm\pm\pm}\partial_\mp\phi^i$, both of which have zero (world-sheet) dimension (as $B_{\pm\pm\pm}$ has dimension -1 and $\partial\phi$ has dimension $+1$) [2].

The gauge fields $h^{\pm\pm}, B^{\pm\pm\pm}$ are the components of traceless gauge fields $h^{\mu\nu}, B^{\mu\nu\rho}$ and the action given by adding (4.6) to S_0 can be rewritten as (1.8). The tracelessness condition on the gauge fields can be dropped and traces $h^{+-}, B^{\pm+-}$ formally introduced. However, they were not needed for the linearised action, which is consequently invariant under the linearised $\mathcal{W}$-Weyl transformations (1.9), or equivalently $\delta h^{+-} = \Omega, \delta B^{\pm+-} = \Omega^{\pm}$. Indeed, the traces are not needed at all in the classical theory, $i.e.$ there is a non-linear form of the $\mathcal{W}$-Weyl symmetry. However, if there are anomalies, this need not be the case in the quantum theory.

This example is typical of the general case. Consider any classical conformal field theory with a $\mathcal{W}$-algebra symmetry, (e.g. a free boson or fermion model, a non-linear sigma model, a Wess-Zumino-Witten model, a coset model, a Toda model,...). Let the classical action be S_0 and the currents generating the $\mathcal{W}$-algebra be W_{+A}, W_{-A} labelled by some index A and satisfying the chiral conservation laws $\partial_- W_{+A} = 0, \partial_+ W_{-A} = 0$. To gauge a chiral $\mathcal{W}$-algebra, say the right-handed one generated by the currents W_{+A} alone, a gauge field h^{+A} is introduced for each current and adding the Noether coupling gives the linearised action

$$S = S_0 + \sum_A \int d^2x \; h^{+A} W_{+A} \tag{4.7}$$

It was seen in the case of chiral $\mathcal{W}_3$ that no higher order terms are needed and that this action was fully gauge invariant. Remarkably, it can be shown that this is the case in general and that for any classical conformal field theory and any chiral $\mathcal{W}$-algebra, the full gauge-invariant action is linear in the gauge fields and given simply by the Noether coupling [5]. To gauge both left and right-handed $\mathcal{W}$-algebras, one introduces gauge fields h^{+A} and h^{-A}. In this case, adding the Noether coupling does not give a gauge-invariant theory in general and the full theory is non-polynomial in the gauge fields. To lowest order it is given by

$$S = S_0 + \sum_A \int d^2x \; \left[h^{+A} W_{+A} + h^{-A} W_{-A} + O(h^2) \right] \tag{4.8}$$

and the higher order corrections can be calculated to any given order in the gauge fields using the Noether method. Nevertheless, it will be seen in the next chapter, that the non-chiral gaugings can be written in a type of canonical formalism in which the gauged action is given by a Noether coupling. However, the canonical momenta cannot be eliminated in closed form so that this approach only gives an implicit form of the action.

5. Canonical Construction of Non-Chiral $\mathcal{W}$-Gravity

We now turn to the full non-linear structure of the non-chiral $\mathcal{W}$-gravity coupling, and again the example of the free boson realisation of $\mathcal{W}_3$ will be used. There are two canonical-style approaches which give a simple but implicit form of the action. The first is a conventional Hamiltonian treatment [11]. The free action (3.1) can be written in first-order form as

$$S_0 = \int d^2x \; \left[\pi_i \partial_\tau \phi^i - \frac{1}{2}\pi_i \pi^i - \frac{1}{2} \partial_\sigma \phi^i \partial_\sigma \phi^i \right] \tag{5.1}$$

with $\tau = x^0, \sigma = x^1$. The momentum π^i is an auxiliary field that can be eliminated using its equation of motion $\pi^i = \partial_\tau \phi^i$ to recover (3.1). The currents defined by

$$T_{\pm\pm}(\Pi)=\frac{1}{2}\Pi_\pm^i\Pi_\pm^i, \qquad W_{\pm\pm\pm}(\Pi)=\frac{1}{3}d_{ijk}\Pi_\pm^i\Pi_\pm^j\Pi_\pm^k \tag{5.2}$$

where

$$\Pi_\pm^i=\frac{1}{\sqrt{2}}\left(\pi^i \pm \partial_\sigma \phi^i\right) \tag{5.3}$$

generate a Poisson bracket algebra consisting of two copies of the classical $\mathcal{W}_3$ algebra given in chapter 2. The first-class constraints

$$T_{\pm\pm}(\Pi) \sim 0 \;, \qquad W_{\pm\pm\pm}(\Pi) \sim 0 \tag{5.4}$$

can then be imposed using Lagrange multipliers $h^{\pm\pm}, B^{\pm\pm\pm}$, so that the action

is given by

$$S = S_0 - \int d^2x \left[h^{++}T_{++}(\Pi) + h^{--}T_{--}(\Pi) + B^{+++}W_{+++}(\Pi) + B^{---}W_{---}(\Pi) \right] \tag{5.5}$$

Here, S_0 is the free action given by (5.1). After shifting the fields $h^{\pm\pm} \to h^{\pm\pm} + 1$,

$$S = \int d^2x \left[\pi_i \partial_\tau \phi^i - h^{++}T_{++}(\Pi) - h^{--}T_{--}(\Pi) \right. \\ \left. - B^{+++}W_{+++}(\Pi) - B^{---}W_{---}(\Pi) \right] \tag{5.6}$$

This gives the complete coupling of scalar fields to non-chiral $\mathcal{W}_3$ gravity in Hamiltonian form. However, although the field equations for the momenta π^i are still algebraic, they are difficult to solve in closed form, so that eliminating the momenta is problematic. However, they can be solved to any given order in the gauge fields, and the solution can then be used to find the Lagrangian form of the action to that order in the gauge fields.

The action given by (5.5),(5.6) is an example of a first class action of the form

$$S = \int d\tau \left[p_a \partial_\tau q^a - \lambda^A G_A \right] \tag{5.7}$$

where the coordinates $q^a(\tau)$ and momenta $p_a(\tau)$ correspond to $\phi^i(\sigma, \tau)$ and $\pi_i(\sigma, \tau)$ respectively, with the index a representing both the discrete index i and the continuous variable σ, so that summation over a corresponds to summation over i and integration over σ. The Lagrange multipliers $\lambda^A(\tau)$ correspond to $h^{\pm\pm}(\sigma), B^{\pm\pm\pm}(\sigma)$ and impose the constraints $G_A \sim 0$, corrsponding to (5.4). Suppose the Poisson bracket algebra generated by the constraints closes to give

$$\{G_A, G_B\} = f_{AB}{}^C G_C \tag{5.8}$$

for some $f_{AB}{}^C$, which may depend on the phase space variables. In our example, (5.8) is just the classical $\mathcal{W}_3$ algebra.

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Any action of the form (5.7) where the constraints satisfy (5.8) is invariant under the following local symmetries with parameter $\alpha^A(\tau)$

$$\delta p_a = \alpha^A \{G_A, p_a\}, \qquad \delta q^a = \alpha^A \{G_A, q^a\} \tag{5.9}$$

$$\delta \lambda^A = \partial_\tau \alpha^A - f_{BC}{}^A \lambda^B \alpha^C \tag{5.10}$$

These then give the $\mathcal{W}_3$ gravity symmetries of the action (5.6), which are given explicitly in [11].

The Hamiltonian approach has the disadvantage that two-dimensional Lorentz covariance is not manifest. A related method which does maintain covariance was found by Schoutens, Sevrin and van Nieuwenhuizen in [3]. This method, which was found before the Hamiltonian one described above, was motivated by a careful study of the terms that occur in the order-by-order construction of the action [3]. Instead of a single momentum π^i conjugate to each ϕ^i, a covariant vector π^i_μ is introduced. The free action S_0 can then be written as

$$S = \int d^2x \left(-\frac{1}{2}\partial_+\phi^i\partial_-\phi^i - \pi^i_+\pi^i_- + \pi^i_+\partial_-\phi^i + \pi^i_-\partial_+\phi^i \right) \tag{5.11}$$

The π field equation is algebraic:

$$\frac{\delta S_0}{\delta \pi^i_\pm} = 0 \Rightarrow \pi^i_\pm = \partial_\pm \phi^i \tag{5.12}$$

so that the π^i_μ are auxiliary fields. Using (5.12) to eliminate the auxiliary fields (*i.e.* setting $\pi^i_\pm = \partial_\pm\phi^i$), the action reduces to the usual free form (3.1). The next step is to introduce 'π-currents' $T(\pi), W(\pi)$

$$T_{\pm\pm}(\pi) = \frac{1}{2}\pi^i_\pm\pi^i_\pm \qquad W_{\pm\pm\pm}(\pi) = \frac{1}{3}d_{ijk}\pi^i_\pm\pi^j_\pm\pi^k_\pm \tag{5.13}$$

Adding the Noether coupling

$$S_1 = -\int d^2x \left[h_{--}T_{++}(\pi) + h_{++}T_{--}(\pi) + B_{---}W_{+++}(\pi) + B_{+++}W_{---}(\pi) \right] \tag{5.14}$$

to (5.11) imposes the constraints that the π-currents vanish.

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This Noether coupling turns out to be all that is needed to give a gauge-invariant theory. The complete action is then [3]

$$S = \int d^2x \left(-\frac{1}{2}\partial_+\phi^i\partial_-\phi^i - \pi_+^i\pi_-^i + \pi_+^i\partial_-\phi^i + \pi_-^i\partial_+\phi^i \right.$$
$$\left. -\frac{1}{2}h_{--}\pi_+^i\pi_+^i - \frac{1}{2}h_{++}\pi_-^i\pi_-^i \right. \tag{5.15}$$
$$\left. -\frac{1}{3}B_{+++}d_{ijk}\pi_-^i\pi_-^j\pi_-^k - \frac{1}{3}B_{---}d_{ijk}\pi_+^i\pi_+^j\pi_+^k \right)$$

(This form of the action is a generalisation of one given in [8] and is related to the action of [3] by field redefinitions.) This action is invariant under the diffeomorphisms and λ-transformations

$$\delta\phi^i = k_-\pi_+^i + \lambda_{--}d^i{}_{jk}\pi_+^j\pi_+^k + (+ \leftrightarrow -)$$
$$\delta h_{\pm\pm} = \partial_\pm k_\pm + k_\pm\partial_\mp h_{\pm\pm} - h_{\pm\pm}\partial_\mp k_\pm$$
$$\qquad + 2\kappa T_{\mp\mp}(\Pi)\left(\lambda_{\pm\pm}\partial_\mp B_{\pm\pm\pm} - B_{\pm\pm\pm}\partial_\mp\lambda_{\pm\pm}\right)$$
$$\delta B_{\pm\pm\pm} = \partial_\pm\lambda_{\pm\pm} + 2\lambda_{\pm\pm}\partial_\mp h_{\pm\pm} - h_{\pm\pm}\partial_\mp\lambda_{\pm\pm} \tag{5.16}$$
$$\qquad - 2B_{\pm\pm\pm}\partial_\mp k_\pm + k_\pm\partial_\mp B_{\pm\pm\pm}$$
$$\delta\pi_\pm^i = \partial_\pm\left(k_\mp\pi_\pm^i + \lambda_{\mp\mp}d^i{}_{jk}\pi_\pm^j\pi_\pm^k\right)$$

where $T_{\pm\pm}(\Pi) = \frac{1}{2}\pi_\pm^i\pi_\pm^i$. The field equation for the auxiliary fields is algebraic

$$\pi_\pm^i = \partial_\pm\phi^i - h_{\pm\pm}\pi_\mp^i - B_{\pm\pm\pm}d^i{}_{jk}\pi_\mp^j\pi_\mp^k \tag{5.17}$$

but difficult to solve for π in closed form, so that again there is not a closed form for the action without π's. Nevertheless, one can solve for π to any given order in the gauge fields and the result agrees with that obtained by using the Noether method to find the corrections to (3.1),(4.6) to that order in the gauge fields.

To obtain a better understanding of these actions, it may be useful to consider setting the spin-three gauge-fields to zero in the actions considered above to obtain the coupling to pure (spin-two) gravity, which can then be compared with the conventional minimal coupling to gravity. The Noether coupling approach gives the action

$$S_n = \int d^2x \left(\partial_+\phi^i\partial_-\phi^i - \frac{1}{2}h_{++}\partial_-\phi^i\partial_-\phi^i - \frac{1}{2}h_{--}\partial_+\phi^i\partial_+\phi^i + O(h^2) \right) \tag{5.18}$$

Although one could calculate some of the higher order corrections to this and attempt to guess the general form, this is clearly not the best way of finding the coupling of a scalar field to gravity. The approach of [3] gives

$$S = \int d^2x \left(-\frac{1}{2}\partial_+\phi^i\partial_-\phi^i - \pi_+^i\pi_-^i + \pi_+^i\partial_-\phi^i + \pi_-^i\partial_+\phi^i \right.$$
$$\left. -\frac{1}{2}h_{--}\pi_+^i\pi_+^i - \frac{1}{2}h_{++}\pi_-^i\pi_-^i \right) \tag{5.19}$$

and the π field equation is

$$\pi_\pm^i = \partial_\pm\phi^i - h_{\pm\pm}\pi_\mp^i \tag{5.20}$$

which can be solved explicitly to give

$$\pi_\pm^i = \frac{\partial_\pm\phi^i - h_{\pm\pm}\partial_\mp\phi^i}{1 - h_{--}h_{++}} \tag{5.21}$$

Substituting (5.21) into (5.19) gives the complete non-polynomial form of the action

$$S = \int d^2x \frac{1}{1 - h_{++}h_{--}}\left[(1 + h_{++}h_{--})\partial_+\phi^i\partial_-\phi^i - h_{--}T_{++} - h_{++}T_{--}\right] \tag{5.22}$$

This gives the full non-linear corrections to (5.18).

Similarly, the Hamiltonian approach gives

$$S = \int d^2x \left[\pi_i\partial_\tau\phi^i - h^{++}T_{++}(\Pi) - h^{--}T_{--}(\Pi)\right] \tag{5.23}$$

and the momenta can again be eliminated explicitly to give a non-polynomial action similar to (5.22).

Of course, most people would prefer to use a little geometry and write down the standard minimal coupling to a metric $g_{\mu\nu}$

$$S = \frac{1}{2} \int d^2x \, \sqrt{-g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i \qquad (5.24)$$

If one chooses to parameterise the metric as

$$g_{\mu\nu} = \Omega \begin{pmatrix} 2h_{++} & 1 + h_{++}h_{--} \\ 1 + h_{++}h_{--} & 2h_{--} \end{pmatrix} \qquad (5.25)$$

then Ω drops out of the action (5.24) as a consequence of classical Weyl invariance and (5.24) becomes precisely (5.22). Note that, contrary to claims sometimes made, (5.25) does not correspond to partial gauge-fixing; (5.25) is simply a convenient parameterisation of a *general* metric $g_{\mu\nu}$.

However, for non-zero B, (5.20) gives an equation for π which is difficult to solve in closed form, although it is straightforward to solve order by order in B; substituting the perturbative solution to (5.20) back into the action recovers the results of the Noether method. However, just as the non-linearity in $g_{\mu\nu}$ is best understood in terms of Riemannian geometry, it seems likely that the non-linearity in B can also be best understood in terms of some higher spin geometry, which would also allow the coupling of W-gravity to more general matter systems.

The two canonical approaches described here for W_3-gravity also work for other W-algebras and other matter systems. The Hamiltonian approach clearly works quite generally; one writes the matter action in first order form and replaces time derivatives of fields in the W-currents by the corresponding momenta. The full action is given by adding the Noether coupling of these currents to Lagrange multiplier gauge fields. The resulting action is of the form (5.7) and so invariant under the transformations (5.9), (5.10). The covariant canonical approach of [3] has been generalised to w_∞ [8], W_N [4] and indeed free boson realisations of *any* W-algebra [4]. It has also been applied to non-linear sigma-models, free-fermion models [4] and supersymmetric models [30,10] and probably again applies quite generally.

6. The Geometry of W-Gravity

The non-polynomial structure of gravity is best understood in terms of Riemannian geometry and this suggests that the key to the non-linear structure of W-gravity might be found in some higher spin generalisation of Riemannian geometry. The approaches of [3] and [11] describe W-gravity in an implicit form, but an understanding of the non-linear structure of the theory without auxiliary fields seems desirable. A 'covariantisation' of the approach of [3] is given in [6,9], but the resulting theories still have auxiliary fields. Other approaches to the geometry of W-gravity are presented in [31,32,33]. In [12], a geometric formulation of w_∞ gravity was derived and this will now be briefly reviewed.

Riemannian geometry is based on a line element $ds = (g_{ij}d\phi^i d\phi^j)^{1/2}$, while a spin-$n$ gauge field could be used to define a geometry based on a line element $ds = (g_{i_1 i_2 \ldots i_n} d\phi^{i_1} d\phi^{i_2} \ldots d\phi^{i_n})^{1/n}$ (first considered by Riemann [34]). A further generalisation is to consider a line element $ds = N(\phi, d\phi)$ where N is some function which is required to satisfy the homogeneity condition $N(\phi, \lambda d\phi) = \lambda N(\phi, d\phi)$. This defines a Finsler geometry [35] and generalises the spin-n Riemannian line element. To describe W-gravity, it seems appropriate to generalise still further and drop the homogeneity condition on N, so that the line element can be written as

$$ds^2 = \mathcal{N}[\phi^i, d\phi^i] = g_{ij}(\phi)d\phi^i d\phi^j + d_{ijk}(\phi)d\phi^i d\phi^j d\phi^k + \ldots \qquad (6.1)$$

One can then contemplate the extension of the diffeomorphism symmetry $\delta\phi^i = \Lambda^i(\phi)$ to the action of a much larger group, consisting of transformations of the form $\delta\phi^i = \Lambda^i(\phi, d\phi)$. Given a scalar field $\phi^i(x^\mu)$ taking values in this space, the line element can be pulled back to the 'world-sheet' to give the world-sheet line-element

$$\begin{aligned} ds^2 = \mathcal{N}^*[\phi^i(x), \partial_\mu\phi^i dx^\mu] = {} & g_{ij}(\phi(x)) \, \partial_\mu\phi^i\partial_\nu\phi^j \, dx^\mu dx^\nu \\ & + d_{ijk}(\phi(x)) \, \partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k \, dx^\mu dx^\nu dx^\rho + \ldots \end{aligned} \qquad (6.2)$$

and the transformations of ϕ now take the form

$$\delta\phi^i = \Lambda^i(\phi, \partial_\mu\phi) \qquad (6.3)$$

In fact, these transformations are too general and as will be seen, it is necessary to restrict to a subgroup of these. To use the line element (6.2) to construct an action, it is now necessary to introduce world-sheet gauge fields $g^{\mu\nu}, B^{\mu\nu\rho}, \ldots$. This is best done by introducing a generating function

$$F(x^\mu, y_\mu) = g^{\mu\nu}(x)y_\mu y_\nu + B^{\mu\nu\rho}(x)y_\mu y_\nu y_\rho + \ldots \qquad (6.4)$$

where y_μ is some vector on the world-sheet M. Note that (6.4) is a generalisation of the inverse metric on M in the same sense that (6.1) is a generalisation of the metric on the target space. Note that x^μ, y_μ can be thought of as coordinates on the cotangent bundle of the world-sheet, T^*M, and if F is required to be a function on T^*M, then the gauge fields $g^{\mu\nu}, B^{\mu\nu\rho}, \ldots$ are tensor fields on M. The function F can be used to define some $\tilde{F}$ given by

$$\tilde{F}(x^\mu, y_\mu) = \tilde{g}_{(2)}^{\mu\nu}(x)y_\mu y_\nu + \tilde{g}_{(3)}^{\mu\nu\rho}(x)y_\mu y_\nu y_\rho + \ldots \qquad (6.5)$$

where

$$\tilde{g}_{(2)}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}, \qquad \tilde{g}_{(3)}^{\mu\nu\rho} = \sqrt{-g}(B^{\mu\nu\rho} - \frac{3}{2}g^{(\mu\nu}B^{\rho)\sigma\tau}g_{\sigma\tau}), \ldots \qquad (6.6)$$

are tensor densities on M. Then the W-gravity action is given by the natural product

$$\begin{aligned} S &= \int d^2x \; \tilde{F} \cdot \mathcal{N}^* \\ &= \int d^2x \; \left[g_{ij}(\phi(x))\partial_\mu\phi^i\partial_\nu\phi^j\tilde{g}_{(2)}^{\mu\nu} + d_{ijk}(\phi(x))\partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\tilde{g}_{(3)}^{\mu\nu\rho} + \ldots \right] \end{aligned} \qquad (6.7)$$

To see how these structures arise from a slightly different viewpoint, consider the simple example of the realisation of the w_∞ in terms of a free single boson ϕ

[8]. No d-tensors are needed, and the infinite set of currents

$$W_{+n} = \frac{1}{n}(\partial_+\phi)^n, \qquad n = 2, 3, \ldots \qquad (6.8)$$

generate the algebra w_∞ [21] (a certain $c = 0$ limit of the $\mathcal{W}_\infty$ algebra of [22]). These currents generate the infinitesimal transformations

$$\delta\phi = \sum_{n=2}^{\infty} \lambda^{+n}(x^+)(\partial_+\phi)^{n-1} \equiv \Lambda(x^+, \partial_+\phi) \qquad (6.9)$$

Here W_{+2} is the stress-tensor and $\lambda^{+2}(x^+)$ is the parameter of conformal transformations. The symmetry algebra is

$$[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{\{\Lambda, \Lambda'\}} \qquad (6.10)$$

where

$$\{\Lambda, \Lambda'\} = \frac{\partial\Lambda}{\partial x^+}\frac{\partial\Lambda'}{\partial y_+} - \frac{\partial\Lambda'}{\partial x^+}\frac{\partial\Lambda}{\partial y_+} \qquad (6.11)$$

is the Poisson bracket on the phase space with coordinates $x^+, y_+ \equiv \partial_+\phi$. If the world-sheet is cylindrical, then x^+ is the coordinate for a circle and x^+, y_+ are coordinates for the cotangent bundle $T^*S^1 \sim S^1 \times \mathbb{R}$. The symmetry algebra is then isomorphic to the Poisson bracket algebra on T^*S^1, which is also the algebra of symplectic diffeomorphisms of T^*S^1, i.e. those diffeomorphisms of the two-dimensional space T^*S^1 that preserve the symplectic structure $dx^+_\wedge dy_+$.

The currents $W_{-n} = \frac{1}{n}(\partial_-\phi)^n$ generate a second commuting copy of w_∞ so that the symmetry algebra is given by two copies of the symplectic diffeomorphisms, $Diff_0(T^*S^1) \times Diff_0(T^*S^1)$. The symmetry algebra is a subalgebra of the more general set of transformations

$$\delta\phi = \sum_{n=2}^{\infty} \lambda_{(n)}^{\mu_1\mu_2\ldots\mu_{n-1}}(x^\nu)\partial_{\mu_1}\phi\partial_{\mu_2}\phi\ldots\partial_{\mu_{n-1}}\phi \equiv \Lambda(x^\mu, y_\mu) \qquad (6.12)$$

where $y_\mu = \partial_\mu\phi$ and the $\lambda_{(n)}^{\mu_1\mu_2\ldots\mu_{n-1}}(x^\nu)$ $(n = 2, 3, \ldots)$ are infinitesimal parameters which are symmetric tensor fields on M. These transformations satisfy the

algebra (6.10), where the Poisson brackets are those for the phase space with coordinates x^μ, y_μ, which is the cotangent bundle of the world-sheet, T^*M. Thus (6.12) is a field-theoretic realisation of the symplectic diffeomorphism algebra, $Diff_0(T^*M)$. It has been suggested [31] that this might be the symmetry algebra for the corresponding W-gravity theory, but as will be seen, it is a sub-algebra of this that is in fact needed.

Before proceding to the full theory, it will be useful to consider first linearised w_∞ gravity. The currents $W_{\pm n}$ are the only non-vanishing components of the symmetric tensor current given by

$$W^n_{\mu_1\mu_2\dots\mu_n} = \frac{1}{n}\partial_{\mu_1}\phi\partial_{\mu_2}\phi\dots\partial_{\mu_n}\phi - \ traces \qquad (6.13)$$

which is conserved, $\partial^{\mu_1}W^n_{\mu_1\mu_2\dots\mu_n} = 0$ and traceless, $\eta^{\mu_1\mu_2}W^n_{\mu_1\mu_2\dots\mu_n} = 0$. From chapter 1, the linearised W-gravity action is given by the Noether coupling. The action (4.3) can be rewritten as

$$S = \int d^2x\left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \sum_{n=2}^{\infty}\frac{1}{n}\tilde{h}^{\mu_1\dots\mu_n}_{(n)}\partial_{\mu_1}\phi\dots\partial_{\mu_n}\phi + O(h^2)\right] \qquad (6.14)$$

where the $\tilde{h}^{\mu_1\mu_2\dots\mu_n}_{(n)}$ are symmetric tensor gauge fields satisfying

$$\eta_{\mu\nu}\tilde{h}^{\mu\nu\rho\dots\sigma}_{(n)} = 0 + O(h^2) \qquad (6.15)$$

at least to lowest order in the gauge fields. The transformation of the scalar fields is of the form (6.12) with the parameters satisfying the tracelessness condition

$$\eta_{\mu\nu}\lambda^{\mu\nu\rho\dots\sigma}_{(n)} = 0 + O(h^2) \qquad (6.16)$$

at least to lowest order in the gauge fields.

Let us turn now to the full non-linear theory. The action depends on $\partial_\mu\phi$ but not on higher order derivatives and so can be written as

$$S = \int_M d^2x\tilde{F}(x,\partial\phi) \qquad (6.17)$$

for some $\tilde{F}$, which has the following expansion in $y_\mu = \partial_\mu\phi$:

$$\tilde{F}(x,y) = \sum_{n=2}^{\infty}\frac{1}{n}\tilde{g}^{\mu_1\mu_2\dots\mu_n}_{(n)}(x)y_{\mu_1}\,y_{\mu_2}\dots y_{\mu_n} \qquad (6.18)$$

where $\tilde{g}^{\mu_1\mu_2\dots\mu_n}_{(n)}(x)$ are gauge fields which are tensor densities on M. The $\tilde{g}^{\mu_1\dots\mu_n}_{(n)}$ are non-polynomial functions of the $\tilde{h}^{\mu_1\dots\mu_n}_{(n)}$ (e.g. $\tilde{g}^{\mu\nu}_{(2)} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu}_{(2)} + O(\tilde{h}^2)$) but it will be convenient to work with the $\tilde{g}$ gauge fields, in terms of which the action is linear, rather than the $\tilde{h}$ gauge fields. The transformation of ϕ takes the form (6.12). From the linearised analysis, it is seen that Λ and $\tilde{F}$ must be restricted to satisfy equations which take the following form to lowest order in the gauge fields

$$\eta_{\mu\nu}\frac{\partial^2\Lambda}{\partial y_\mu\partial y_\nu} = 0 + \dots, \qquad \eta_{\mu\nu}\frac{\partial^2\tilde{F}}{\partial y_\mu\partial y_\nu} = -2 + \dots \qquad (6.19)$$

These impose the tracelessness of the parameters (6.16) and of the linearised gauge fields, to lowest order in the gauge fields. The full theory should have non-linear constraints that generalise these and which do not involve any background metric $\eta^{\mu\nu}$. This is indeed the case, and the full constraints take a strikingly simple form:

$$det\left(\frac{\partial^2\tilde{F}(x,y)}{\partial y_\mu\partial y_\nu}\right) = -1 \qquad (6.20)$$

and

$$det\left(\frac{\partial^2}{\partial y_\mu\partial y_\nu}[\tilde{F}+\Lambda](x,y)\right) = -1 \qquad (6.21)$$

Expanding (6.20) in y_μ gives an infinite number of algebraic constraints on the

density gauge fields $\tilde{g}^{\mu\nu\cdots}_{(n)}$

$$det\left(\tilde{g}^{\mu\nu}_{(2)}(x)\right) = -1, \qquad \tilde{g}^{\mu\nu\rho}_{(3)}\tilde{g}_{(2)\mu\nu} = 0,\ldots \qquad (6.22)$$

and these can be solved in terms of unconstrained gauge fields $g^{\mu\nu}, B^{\mu\nu\rho}, \ldots$ to give (6.6). The theory written in terms of these unconstrained gauge fields is invariant under an infinite set of local symmetries generalising Weyl symmetry:

$$\delta g^{\mu\nu} = \sigma_{(2)}g^{\mu\nu}, \qquad \delta B^{\mu\nu\rho}_{(3)} = \sigma_{(2)}B^{\mu\nu\rho}_{(3)} + \frac{3}{2}\sigma^{(\mu}_{(3)}g^{\nu\rho)}, \ldots \qquad (6.23)$$

The transformations can be written in terms of the generating function (6.4) as $\delta F(x,y) = \sigma(x,y)F(x,y)$ where $\sigma(x,y) = \sigma_{(2)}(x) + \sigma^{\mu}_{(3)}(x)y_{\mu} + \ldots$ These transformations can be used to remove all traces from the gauge fields, leaving only traceless gauge fields, as in [8,6].

Similarly, expanding (6.21) in y_{μ} gives an infinite number of algebraic constraints on the parameters of the form $\tilde{g}_{(2)\mu\nu}\lambda^{\mu\nu\cdots}_{(n)} = 0 + \ldots$ and these can again be solved in terms of unconstrained quantities [12].

To summarise, the full non-linear action for w_{∞} gravity coupled to a single boson is given by (6.17),(6.18) where the gauge fields $\tilde{g}^{\mu_1\mu_2\cdots\mu_n}_{(n)}(x)$ are required to satisfy the algebraic constraints given by expanding (6.20). These constraints can be solved in terms of unconstrained gauge fields as described above, but it is not necessary to do so. The action is invariant (up to a surface term) under w_{∞} gravity transformations under which the scalar fields transform as (6.12), where the infinitesimal parameters are required to satisfy the constraint (6.21), and this constraint implies that the symmetry algebra is a subalgebra of $Diff_0(T^*M)$. The gauge fields transform as

$$\delta\,\tilde{g}^{\mu_1\mu_2\cdots\mu_p}_{(p)} = \sum_{m,n=2}^{\infty} \delta_{m+n,p+2}\left[(m-1)\lambda^{(\mu_1\mu_2\cdots}_{(m)}\partial_{\nu}\,\tilde{g}^{\cdots\mu_p)\nu}_{(n)} - (n-1)\tilde{g}^{\nu(\mu_1\mu_2\cdots}_{(n)}\partial_{\nu}\lambda^{\cdots\mu_p)}_{(m)}\right.$$
$$\left. + \frac{(m-1)(n-1)}{p-1}\partial_{\nu}\left\{\lambda^{\nu(\mu_1\mu_2\cdots}_{(m)}\,\tilde{g}^{\cdots\mu_p)}_{(n)} - \tilde{g}^{\nu(\mu_1\mu_2\cdots}_{(n)}\,\lambda^{\cdots\mu_p)}_{(m)}\right\}\right] \qquad (6.24)$$

From (6.24), the $\tilde{g}_{(s)}$ transform as tensor densities under reparameterisations of M (i.e. $\lambda_{(2)}$ transformations), as expected.

The equation (6.20) is of a type that plays an important role in geometry. Let $\zeta_{\mu}, \bar{\zeta}_{\bar{\mu}}$ ($\mu = 1,2$) be complex coordinates on $\mathbb{R}^4$. Then, for each x^{μ}, a solution $\tilde{F}(x,y)$ of (6.20) can be used to define a function $K_x(\zeta,\bar{\zeta})$ on $\mathbb{R}^4$ by

$$K_x(\zeta,\bar{\zeta}) = \tilde{F}(x^{\mu},\zeta_{\mu} + \bar{\zeta}_{\mu}) \qquad (6.25)$$

For each x, K_x can be viewed as the Kahler potential for a Kahler metric $G_{\mu\bar{\mu}} = \partial_{\mu}\partial_{\bar{\mu}}K_x$ of signature $(2,2)$ on $\mathbb{R}^4$. As a result of (6.20), each K_x satisfies the equation $det(\partial_{\mu}\partial_{\bar{\mu}}K_x) = -1$ and which is often referred to as the Monge-Ampere equation or one of Plebanski's equations. The Ricci tensor for the Kahler space is $R_{\nu\bar{\nu}} = \partial_{\nu}\partial_{\bar{\nu}}log\,|det(G_{\mu\bar{\mu}})|$ and this vanishes if (6.20) holds, so that the metric is Kahler and Ricci-flat, which implies that the curvature tensor is either self-dual or anti-self-dual. (For Euclidean world-sheets, a similar analysis goes through and leads to a Kahler Ricci-flat space with Euclidean signature [12].)

As the Kahler potential is independent of the imaginary part of ζ_{μ}, the metric has two commuting (triholomorphic) Killing vectors, given by $i(\partial/\partial\zeta_{\mu} - \partial/\partial\bar{\zeta}_{\bar{\mu}})$. Thus the lagrangian $\tilde{F}(x,y)$ corresponds to a two-parameter family of Kahler potentials $K_{x^{\mu}}$ for self-dual geometries on $\mathbb{R}^4$ with two Killing vectors. The parameter constraint (6.21) implies that $\tilde{F} + \Lambda$ is also a Kahler potential for a hyperkahler metric with two killing vectors, so that for each x, Λ represents an infinitesimal deformation of the hyperkahler geometry. Other relations between w_{∞} and self-dual geometry have been discussed in [21,38].

Techniques for solving the Monge-Ampere equation can be used to solve (6.20). The general solution of the Monge-Ampere equation can be given implicitly by Penrose's twistor transform construction [36]. For solutions with one (triholomorphic) Killing vector, the Penrose transform reduces to a Legendre transform solution which was found first in the context of supersymmetric non-linear sigma-models [37]. Substituting the Legendre transform solution of (6.20) in the action (6.17),(6.18) gives precisely the Hamiltonian form of the w_{∞} action

(*i.e.* the w_∞ generalisation of (5.6)). In particular, the twistor space approach of [37] gives a twistor interpretation to the auxilairy fields π.

For self-dual spaces with two Killing vectors, it is possible to write down a new solution of the Monge-Ampere equation, using a generalisation of the Legendre transform solution that involves transforming with respect to both components of y_μ. Any $\tilde{F}(x, y)$ can be written as a transform of a function H as follows:

$$\tilde{F}(x^\mu, y_\nu) = 2\pi^\mu y_\mu - \frac{1}{2}\eta^{\mu\nu} y_\mu y_\nu - 2H(x, \pi) \tag{6.26}$$

where the equation

$$y_\mu = \frac{\partial H}{\partial \pi^\mu} \tag{6.27}$$

implicitly determines $\pi_\mu = \pi_\mu(x^\nu, y_\rho)$. Then $\tilde{F}$ will satisfy (6.20) if and only if its transform H satisfies

$$\frac{1}{2}\eta^{\mu\nu}\frac{\partial^2 H}{\partial\pi_\mu \partial\pi_\nu} = \frac{\partial^2 H}{\partial\pi_+ \partial\pi_-} = 1 \tag{6.28}$$

The general solution of this is

$$H = \pi_+\pi_- + f(x, \pi_+) + \bar{f}(x, \pi_-) \tag{6.29}$$

This solution can be used to write the action

$$S = \int d^2x \left(2\pi^\mu y_\mu - \eta_{\mu\nu}\pi^\mu\pi^\nu - \frac{1}{2}\eta^{\mu\nu}y_\mu y_\nu - 2f(x, \pi_+) - 2\bar{f}(x, \pi_-)\right) \tag{6.30}$$

The field equation for π^μ is (6.27), and using this to substitute for π gives the action (6.17) subject to the constraint (6.20). Alternatively, expanding the functions $f, \bar{f}$ as $f = \sum s^{-1}h_s(x)(\pi_+)^s$, $\bar{f} = \sum s^{-1}\bar{h}_s(x)(\pi_-)^s$ gives precisely the form of the action given in [8]. The parameter constraint (6.21) is solved similarly, and the solutions can be used to write the symmetries of (6.30) in the form given in [8].

7. Quantum $\mathcal{W}$-Algebras

Classical $\mathcal{W}$-algebras were considered in chapter 2. For those $\mathcal{W}$-algebras that are Lie algebras, it is usually straightforward to obtain a corresponding quantum algebra by replacing the classical currents with quantum field operators, replacing the Poisson brackets with operator commutators and inserting factors of $i\hbar$ in accordance with the minimal Dirac prescription. Then the fact that the classical algebra satisfies the Jacobi identities usually implies that the quantum algebra does also.

For non-linear algebras, however, the situation is more complicated as the non-linear terms occurring on the right-hand-sides of the commutation relations involve the products of quantum field operators at the same point and so some regularisation prescription is necessary. For example, in the classical W_3 algebra, the commutator of two spin-three currents gives rise to the spin-four current $\Lambda = TT$, whose modes are $\Lambda_n = \sum_m L_{n-m}L_m$, where L_n are the modes of the stress-tensor T. In the quantum theory, $T(z)T(z)$ is singular and so to define Λ it is necessary to introduce a regularisation. The following normal ordering prescription is the most convenient:

$$\substack{\times\\\times}L_n L_m\substack{\times\\\times} = \begin{cases} L_n L_m & \text{if } m > n \\ L_m L_n & \text{if } n \leq m \end{cases} \tag{7.1}$$

Using this, the $[W, W]$ commutator can be written in terms of the spin-four current with modes

$$\Lambda_m = \sum_n \substack{\times\\\times}L_{m-n}L_n\substack{\times\\\times} - \frac{3}{10}(m + 3)(m + 2)L_m \tag{7.2}$$

(The term linear in L_n is added to make Λ_n quasi-primary [13].) The regularisation (7.1) corresponds to subtracting the singular terms in the operator product expansion of $T(z)T(w)$ and then taking the limit $z \to w$ and can be generalised to define the product of any two currents in the $\mathcal{W}$-algebra.

The full quantum W_3 algebra with central charge c [13] consists of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}.$$ (7.3)

the relation

$$[L_m, W_n] = [2m - n]W_{m+n}$$ (7.4)

so that the spin-three current W (with modes W_n) is primary, and

$$[W_m, W_n] = b^2(m-n)\Lambda_{m+n} + \frac{1}{15}(m-n)\left\{(m+n)^2 - \frac{5}{2}mn - 4\right\}L_{m+n}$$
$$+ \frac{c}{360}(m^3 - m)(m^2 - 4)\delta_{m+n}$$ (7.5)

where $b^2 = \frac{16}{22+5c}$. The coefficients are fixed by requiring the algebra to satisfy the Jacobi identities [13].

7.1 BRST Charges

In [40], Thierry-Mieg constructed a BRST charge for the W_3 algebra and this was extended in [41] to a construction of a BRST charge for any quadratically non-linear W-algebra; it seems likely that this can be generalised to any W-algebra. For W_3, one introduces the usual conformal ghost system b_{++}, c_- corresponding to T_{++} and a ghost system u_{+++}, v_{--} corresponding to W_{+++}. The BRST charge takes the form

$$Q = \int dx^+ \left[c_-(T_{++} - \alpha) + v_{--}(W_{+++} - \beta) + \ldots\right]$$ (7.6)

and is nilpotent if and only if T, W satisfy the W_3 algebra with central charge $c = 100$ and the intercepts are $\alpha = 4$ and $\beta = 0$ [40]. If the matter system has $c = 100$, then this central charge is cancelled by a contribution of -26 from the ghosts b_{++}, c_- and a contribution of -74 from the ghosts u_{+++}, v_{--}.

For a general W-algebra, one introduces a spin-s anti-ghost u^A and a spin $1-s$ ghost v^A corresponding to each spin-s generator W^A. The BRST charge takes the form $Q \sim \int dx^+(\sum_A v^A W^A + \ldots)$ and this can only be nilpotent if the matter central charge cancels against the total ghost contribution to the central charge, although this condition need not be sufficient in general. The contribution to the central charge from the ghosts for a spin s generator is $c_s = -2(6s^2 - 6s + 1)$ so that for W_4 the total ghost contribution is -246 while for W_∞ the total ghost contribution is given by the divergent sum $-\sum_{s=2}^{\infty} c_s = -26 - 74 - 146 - \ldots$. This series can formally be summed using ζ-function regularisation to give the value 2 [49], but it is not clear at present what the significance of this result might be.

8. Free Boson Realisations of Quantum Algebras

In this chapter we will introduce a number of examples of free boson realisations which will be useful when we discuss W-gravity anomalies. In each case we start with a free boson model with a set of currents that satisfy a Poisson bracket W-algebra. We then quantize the boson system and investigate the quantum algebra generated by the W-currents. One might expect the quantum algebra to be obtained from the classical one by using the standard Dirac quantisation prescription of introducing factors of $i\hbar$ into the classical relations. In general, however, the quantum algebra corresponds to the Dirac-quantized algebra plus extra corrections of order $\hbar^2$ or higher, and these corrections can either be central extensions of the algebra or can involve currents constructed from the boson fields. Each higher order correction to the algebra corresponds to a W-gravity anomaly, as will be seen in the next chapter.

There is a crucial difference between linear and non-linear realisations of a classical current algebra. For linear realisations, the quantum current algebra is given by the Dirac-quantized algebra plus central extension terms of order $\hbar^2$ and there are no higher order terms. For non-linear realisations, the situation is altogether more complicated as new currents can arise in the quantum algebra

which were not present in the classical $\mathcal{W}$-algebra and these reflect a new kind of anomaly which arises for such models [42].

8.1 LINEAR REALISATION OF THE VIRASORO ALGEBRA

The stress-tensor for D free bosons given by

$$T = \frac{1}{2}\partial_+\phi^i\partial_+\phi^i \tag{8.1}$$

(with $i = 1, \ldots, D$) generates the classical Poisson bracket algebra given by (2.1) with $c = 0$, which can be written schematically as

$$[T, T] \sim T \tag{8.2}$$

On quantisation, the modes α_n^i of $\partial_+\phi^i$ are taken to satisfy harmonic oscillator commutation relations

$$[\alpha_n^i, \alpha_m^j] = \hbar\delta^{ij}\delta_{m+n} \tag{8.3}$$

It is necessary to regularise (8.1), and it is convenient to define $T =: \frac{1}{2}\partial_+\phi^i\partial_+\phi^i :$ so that $L_n = \frac{1}{2}\sum_m : \alpha_{n+m}^i\alpha_{-m}^i :$, where colons denote the normal ordering with respect to the modes α_n^i:

$$: \alpha_n^i\alpha_m^j := \begin{cases} \alpha_n^i\alpha_m^j & \text{if } m > n \\ \alpha_m^j\alpha_n^i & \text{if } n \le m \end{cases} \tag{8.4}$$

Then the commutator algebra generated by the quantum operator T is given by the Virasoro algebra (2.1) with central charge $c = D$. It will be convenient to suppress delta-functions and numerical factors and present such algebras schematically as

$$[T, T] \sim i\hbar T + \hbar^2 c \tag{8.5}$$

The term linear in $\hbar$ is what one would expect from applying the Dirac prescription to (8.2), but there is in addition a central charge term of order $\hbar^2$, which corresponds to an anomaly, as we shall see.

The stress-tensor (8.1) can be modified by adding a non-minimal 'background charge' term, $T \to T' = T + a_i\partial_+^2\phi^i$ where a_i is some constant vector. The classical Poisson bracket algebra then has a central charge, $[T', T'] \sim T' + c_0$ where $c_0 = \frac{1}{24}a_ia_i$, and the quantum algebra again takes the form (8.5), but with $c = D + c_0/\hbar$. This factor of $\hbar^{-1}$ can be absorbed into a rescaling of the background charge $a \to \sqrt{\hbar}a$, so that the stress-tensor becomes

$$T' = \frac{1}{2}\partial_+\phi^i\partial_+\phi^i + \sqrt{\hbar}a_i\partial_+^2\phi^i \tag{8.6}$$

and the quantum central charge becomes $c = D + a^2/24$.

8.2 LINEAR REALISATION OF W_∞

Let ϕ^i be D complex free bosons. Then the currents [39]

$$W_n(x^+) = \sum_{r=1}^{n-1}\beta_{n,r}\partial_+^r\phi^i\partial_+^{n-r}\bar{\phi}^i, \qquad n = 2, 3, \ldots \tag{8.7}$$

for suitably chosen constants $\beta_{n,r}$ generate a Poisson bracket algebra which is a certain classical limit of the W_∞ algebra of [22], (note that this algebra is not the w_∞ algerba of [21]) which has the generic form [22, 39]

$$[W_n, W_m] \sim W_{n+m-2} + W_{n+m-4} + W_{n+m-6} + W_{n+m-8} + \ldots \tag{8.8}$$

This is a linear realisation of a classical limit of W_∞, as these currents generate variations of ϕ^i which are linear in ϕ^i. The quantum currents given by normal-ordering (8.7) generate an algebra of the general structure (for some constants c_n)

$$[W_n, W_m] \sim c_n\hbar^2\delta_{n,m}i\hbar W_{n+m-2} + i\hbar W_{n+m-4} + i\hbar W_{n+m-6} + \ldots \tag{8.9}$$

which consists of the Dirac quantisation of the algebra (8.8), plus central extension terms of order $\hbar^2$. This quantum algebra contains the Virasoro algebra with central extension $2D$ and is the W_∞ algebra of [22] with $c = 2D$ [42], generalising the $c = 2$ construction of [39].

8.3 Non-Linear Realisation of W_3

For D free real bosons, as was seen in chapter 3, the currents given by (8.1) and $W = \frac{1}{3}d_{ijk}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k$ generate a classical W_3 algebra provided the constants d_{ijk} satisfy (3.5). The corresponding transformations of ϕ^i are given by (3.6) and are non-linear for spin 3. The classical algebra (2.1),(2.2),(2.3) can be written schematicaly as (8.2) together with

$$[T,W] \sim W, \qquad [W,W] \sim \Lambda, \qquad \Lambda \equiv TT \qquad (8.10)$$

In the quantum theory, the currents can be defined using the normal ordering (8.4). The spin-two currents again generate the Virasoro algebra (8.5) with $c = D$. The $[T,W]$ commutator now takes the form

$$[T,W] \sim i\hbar W + \hbar^2 J, \qquad J \equiv d^i{}_{ij}\partial_+\phi^j \qquad (8.11)$$

consisting of the Dirac prescription term, plus a term of order $\hbar^2$ which involves a new spin one current, J [2]. Thus although the classical algebra closes, the quantum one does not unless the d-tensor is traceless, $d^i{}_{ij} = 0$. Even if it is traceless, the $[W,W]$ commutator gives

$$[W,W] \sim i\hbar : TT : +(D+2)\hbar^2(T^{2,0} + T) + \hbar^3 c', \qquad T^{2,0} \equiv\, : \partial_+\phi^i\partial_+^3\phi^i : \quad (8.12)$$

which consists of the Dirac term, a central charge term of order $\hbar^3$ proportional to $c' \equiv d_{ijk}d^{ijk}$ and a new spin-four current $T^{2,0}$ [2,43]. If $d^i{}_{ij} \neq 0$ there are extra terms in (8.12) involving J [43]. Since the coefficient of the spin-four current $T^{2,0}$ is non-zero for any $D > 0$, the quantum algebra never closes when the normal ordering prescription (8.4) is used, as the right hand side of (8.12) cannot be written entirely in terms of T,W and composites constructed using (8.4), such as $: TT :, : TW :$ etc. To close this algebra, it is necessary to introduce $J, T^{2,0}$ as generators, and then to introduce further generators, such as $T^{n,0} = \partial_+^{n+1}\phi^i\partial_+\phi^i, W^{m,n,0} = d_{ijk}\partial_+^m\phi^i\partial_+^n\phi^j\partial_+\phi^k, \ldots$ [43].

However, instead of defining the composite operator $: TT :$ using the prescrition (8.4), it could instead be defined as ${}^\times_\times TT{}^\times_\times$ using (7.1). The two definitions are related by [2]

$${}^\times_\times TT{}^\times_\times =\, : TT : -\hbar T^{2,0} \qquad (8.13)$$

so that they differ by a finite term of order $\hbar$. Using this, the algebra (8.12) can be rewritten as

$$[W,W] \sim i\hbar\,{}^\times_\times TT{}^\times_\times + (D-2)\hbar^2(T^{2,0} + T) + \hbar^3 c', \qquad T^{2,0} \equiv \partial_+\phi^i\partial_+^3\phi^i \quad (8.14)$$

so that the coefficient of the spin-four current $T^{2,0}$ is now $D-2$ instead of $D+2$, with the result that the algebra closes non-linearly on $T, W, {}^\times_\times TT{}^\times_\times$ if and only if $d^i{}_{ij} = 0$ and $D = 2$ [2]. For $D = 2$, the solution of (3.5) given by $d_{112} = -\kappa$ and $d_{222} = \kappa$ gives a traceless d-tensor and hence a closed algebra which becomes preciesly the W_3 algebra (7.3),(7.4),(7.5) after rescaling the currents. This is the two boson realisation of the $c = 2$ W_3 algebra given in [14].

This model is closely related to the Casimir construction of W_3 [25,24]. The classical W_3 algebra is realised by the Sugawara currents $T = \frac{1}{2}tr(J_+J_+)$ and $W = \frac{1}{3}tr(J_+J_+J_+)$, where J_+ is an $SU(3)$ Kac-Moody current [5]. In the quantum theory, these currents (after normal ordering) no longer generate a closed algebra, as the commutator $[W,W]$ gives rise to the spin-four current $T^{2,0} = tr(J_+\partial_+^2 J_+)$ [25,24], which is similar to the current $T^{2,0}$ that arose in the free boson model. In the case in which the Kac-Moody algebra is of level one, it is possible to perform a truncation to a realisation of the quantum W_3 algebra, and this is related to the fact that the level one Kac-Moody algebra can be constructed from the two boson model discussed above [24].

8.4 W_3 Realisations with Background Charges

As in the Virasoro case, one can consider adding higher derivative terms to the currents to give

$$T' = T + \sqrt{\hbar} a_i \partial_+^2 \phi^i, \qquad W' = W + \sqrt{\hbar} e_{ij} \partial_+^2 \phi^i \partial_+ \phi^j + \hbar f_i \partial_+^3 \phi^i \qquad (8.15)$$

for some constants a_i, e_{ij}, f_i. The current algebra will close on $T, W, {}_\times^\times TT {}_\times^\times$ to give the W_3 algebra (7.3),(7.4),(7.5) provided the tensors a_i, e_{ij}, f_i satisfy certain constraints [2]. For the $D = 2$ model these constraints are satisfied by choosing the only non-vanishing components to be given in terms of a free parameter a by $a_1 = a, e_{12} = a, e_{21} = 3a, f_2 = 6a^2$, giving a model with central charge $c = 2 + 24a^2$ [14]. For real a one obtains models with any value of $c \geq 2$, while for imaginary a, a unitary theory can only be defined if the background charge a is chosen to take the discrete values $a^2 = -[p(p+1)]^{-1}$ for $p = 4, 5, 6, \dots$ giving the minimal series of representations of the $\mathcal{W}$-algebra with central charge $c_p = 2 - 24[p(p+1)]^{-1}$ [14]. For $D \neq 2$, the constraints on the coefficients in (8.15) were solved in [27], giving a realisation of the W_3 algebra in terms of any number D of bosons, with arbitrary central charge $c = D + a^2/24$. In particular, it is possible to construct in this way realisations of W_3 with central charge $c = 100$ for which there is a nilpotent BRST operator.

One can instead ask whether modifying the currents as in (8.15) can give an algebra that closes using the normal ordering (8.4), i.e. whether an algebra can be found in which $[W, W]$ can be written entirely in terms of T, W and $: TT :$ instead of T, W and ${}_\times^\times TT {}_\times^\times$. It was shown in [53] that this is only possible if there are precisely $D = 2$ bosons, in which case $a_i = f_i = 0$ and e_{ij} is proportional to ϵ_{ij}, so that the Virasoro subalgebra has $c = 2$.

8.5 One Boson Realisation of W_∞

For one boson ϕ, the currents $W_n = \frac{1}{n}(\partial_+ \phi)^n$ for $n = 2, 3, \dots$ generate the w_∞ algebra classically [8]. The w_∞ algebra takes the schematic form

$$[W_n, W_m] \sim W_{n+m-2} \qquad (8.16)$$

All the currents except the Virasoro current W_2 generate non-linear transformations and this non-linearity leads to new currents on the right-hand-sides of the quantum commutation relations e.g. $[W_2, W_3]$ gives rise to the current $J = \partial_+ \phi$ while $[W_3, W_3]$ gives rise to $T^{2,0} =: \partial_+ \phi \partial_+^3 \phi :$. However, one can again consider adding higher derivative terms to the currents W_n (as in (8.15)) to modify the algebra. It was shown in [56] that the coefficients can be chosen in such a way as to close the algebra, giving rise to the W_∞ algebra of [22], which is of the schematic form (8.8), with central charge $c = -2$. This is precisely what is needed to cancel the contribution of $+2$ coming from using a ζ-function to sum the infinite number of ghost contributions [56].

9. $\mathcal{W}$-Gravity Anomalies from Matter Integration

There has been a great deal of recent work on the quantisation of $\mathcal{W}$-gravity and the anomalies that arise [42-59]. The classical coupling of a free boson system to $\mathcal{W}$-gravity is described by an action of the form (4.8), where S_0 is the free boson action (3.1) and the $h^{\pm A}$ are gauge fields. In this chapter we will consider the integration over the matter fields ϕ^i only, regarding the gauge fields as external sources, and study the anomalies that arise in the Ward identities corresponding to the classical $\mathcal{W}$-gravity symmetries. No gauge fixing is needed as the gauge fields are not being integrated over, and the scalar fields have a well-defined propagator. In the next chapter, we will gauge fix all the local symmetries, introduce the appropriate ghost fields and discuss the integration over all fields.

For simplicity, consider a chiral $\mathcal{W}$-gravity, so that the complete action is of the form (4.7), consisting of the free boson action (3.1) plus the linear Lagrange multiplier terms. Since this is just a free system plus constraints, the normal ordering prescription (8.4) is sufficient to subtract all divergences. It is convenient to use the background field method, writing ϕ^i as the sum $\bar{\phi}^i + \varphi^i$ of a background field $\bar{\phi}^i$ and a quantum field φ^i and then integrating over φ^i to obtain a (renormalised) effective action $\Gamma[\bar{\phi}^i, h^{+A}]$, which is given by the classical action (4.7), plus quantum corrections $\Gamma[\bar{\phi}^i, h^{+A}] = S[\bar{\phi}^i, h^{+A}] + O(\hbar)$. The classical action is invariant under the classical $\mathcal{W}$-symmetries and these lead to Ward identities for Γ. These are of the form $\delta\Gamma + \ldots = \Delta \cdot \Gamma$ where $\delta\Gamma$ is the gauge variation of the effective action, Δ is a local operator and $\Delta \cdot \Gamma$ denotes all 1PI graphs with precisely one insertion of Δ. The dots denote other terms in the Ward identity, the details of which are discussed in [63,64] and which will not be important here. For non-anomalous theories, Δ is zero or can be cancelled by adding finite local counterterms to the action.

We will now present the anomalies for the models given by gauging the bosonic realisations of $\mathcal{W}$-algebras described in the previous chapter.

9.1 GRAVITATIONAL ANOMALY

Consider first the free boson model coupled to chiral gravity with action $S = S_0 - \int d^2x\, h_{--} T_{++}$ with T_{++} given by (8.1). The effective action takes the form

$$\Gamma = S + \frac{i}{2\hbar} \int d^2x\, d^2y\, h_{--}(x) h_{--}(y) < T_{++}(x) T_{++}(y) > + \ldots \qquad (9.1)$$

and varying this under a chiral diffeomorphism with parameter k_- leads to the standard chiral gravitational anomaly (to linearised order in the gauge field h_{--}) [1]

$$\Delta_{gravity} = D \frac{i\hbar}{24\pi} \int d^2x\, k_- \partial_+^3 h_{--} \qquad (9.2)$$

This anomaly, which is proportional to D, corresponds to the central charge

term in (8.5), which is also proportional to D. The relation between the two can be seen by noting that the central term in the commutator $[T, T]$ leads to a term proportional to $D p_+^4 / p^2$ in the momentum space correlation function $< T_{++}(p) T_{++}(-p) >$. Substituting this in (9.1) gives

$$\Gamma = S_0 + D \frac{i\hbar}{24\pi} \int d^2x\, h_{--} \frac{\partial_+^4}{\Box} h_{--} + \ldots \qquad (9.3)$$

and varying this using $\delta h_{--} = \partial_- k_- + \ldots$ leads to the anomaly (9.2). If the stress-tensor is modified to become (8.6), then the anomaly is again given by (9.2), but with D replaced by $D + a^2/24$.

9.2 LINEAR W_∞ GRAVITY

Gauging the linear realisation of chiral W_∞ [39] reviewed in section 8.2 gives a theory with spin-n gauge fields h_n and symmetries with parameters λ_n of spin $n - 1$, with transformations of the form $\delta h_n = \partial_- \lambda_n + \ldots$, for $n = 2, 3, \ldots$. The anomaly takes the form [42]

$$\Delta = \sum_n c_n \hbar \int d^2x\, \lambda_n \partial_+^{2n-1} h_n \qquad (9.4)$$

for some coefficients c_n and these terms correspond precisely to the $O(\hbar^2)$ central extension terms in (8.9). The $n = 2$ term is the gravitational anomaly (9.2).

In this and the previous example, the anomalies depend only on the gauge fields and not on the background matter fields $\bar{\phi}^i$ and correspond to central charge terms in the current algebra. All anomalies that occur in linearly realised symmetries are of this type and we shall refer to such anomalies as *universal anomalies* [42].

9.3 ANOMALIES OF CHIRAL W_3 GRAVITY

Chiral W_3-gravity has an action given by the sum of (3.1) and (4.3), with gauge fields h_{--}, B_{---}. The linearised anomaly for the k_- and λ_{--} symmetries is the sum of the following terms [2,4,43]: the gravitational anomaly (9.2), the mixed spin-2/spin-3 anomaly

$$\Delta_{mixed} = -\frac{i\hbar}{12\pi}\int d^2x\ \bar{J}_+(B_{---}\partial_+^3 k_- + \lambda_{--}\partial_+^3 h_{--}),\qquad(9.5)$$

the universal spin-3 anomaly [46]

$$\Delta_{W-univ} = -c'\frac{\hbar^2}{1440\pi^2}\int d^2x\ \lambda_{--}\partial_+^5 B_{---},\qquad(9.6)$$

and finally the remainder of the spin-3 anomaly

$$\Delta_{mat} = -\frac{i\hbar(D+2)}{24\pi}\int d^2x\ \left[3\bar{T}^{2,0}(\lambda\partial_+B - B\partial_+\lambda) + \bar{T}(-3\partial_+\lambda\partial_+^2 B + \ldots) + O(\bar{J})\right],$$
$$(9.7)$$

Here

$$\bar{J} \equiv d^i{}_{ij}\partial_+\bar{\phi}^j,\qquad \bar{T}^{2,0} \equiv \partial_+\bar{\phi}^i\partial_+^3\bar{\phi}^i,\qquad \bar{T} = \frac{1}{2}\partial_+\bar{\phi}^i\partial_+\bar{\phi}^i\qquad(9.8)$$

are the currents $J, T, T^{2,0}$ of section 8.3 for the background fields ϕ^i. These results were confirmed and extended to higher orders in the gauge fields in [51].

The universal spin-3 anomaly (9.6) corresponds to the central term in (8.12) but arises at two loops instead of one-loop, as a result of the non-linearity of the symmetry. The mixed anomaly (9.5) depends on the scalar fields through the current $\bar{J}$ and corresponds to the J-dependent term in (8.11). Similarly, the anomaly (9.7) depends on the scalar fields through the currents $\bar{T}, \bar{T}^{2,0}$ corresponding to the $T, T^{2,0}$ terms in (8.12) which are also proportional to $D+2$, and through the current $\bar{J}$ corresponding to the $O(J)$ terms which were suppressed in (8.12).

It should be stressed that the theory is regularised using the normal ordering prescription (8.4), and that the prescription (7.1) would not be sufficient to remove all the divergences from the path integral. For this reason, the anomaly corresponds to the algebra (8.12) written using the normal ordering (8.4) and so has a coefficient $D+2$ as in (8.12), instead of the $D-2$ that occurs in the algebra (8.14) that uses (7.1).

For obvious reasons, the non-universal anomalies depending on the matter currents will be referred to as *matter-dependent anomalies* [42]. They do not arise for linearly realised symmetries but usually occur for non-linearly realised ones. The part of the anomaly (9.7) which is proportional to $\bar{T}$ can be cancelled by modifying the λ-transformation of the field h_{--} by a term proportional to $\hbar(D+2)(-3\partial_+\lambda\partial_+^2 B + \ldots)$ but the terms proportional to $\bar{T}^{2,0}$ and $\bar{J}$ appear to be non-trivial anomalies.

To check whether the anomalies are non-trivial, it is necessary to investigate whether they can be cancelled by adding finite local counterterms. Consider adding counterterms of the form [54, 59]

$$S_{count} = -\int d^2x\ \left[\sqrt{\hbar}a_i\partial_+^2\phi^i h_{--} + (\sqrt{\hbar}e_{ij}\partial_+^2\phi^i\partial_+\phi^j + \hbar f_i\partial_+^3\phi^i)B_{---}\right]\quad(9.9)$$

to the W_3 gravity action so that the gauge field term is now proportional to $hT' + BW'$ where T', W' are the modified currents in (8.15). These appear to be the most general terms that can be added without introducing new fields or dimensionful couplings and which might affect the linearised anomaly. There are values of the coefficients a_i, e_{ij}, f_i that lead to a cancellation of all the matter-dependent anomalies (after appropriate modifications of the transformation rules) if and only if the number of bosons is $D = 2$, in which case $a_i = f_i = 0$ and e_{ij} is proportional to ϵ_{ij}. These were precisely the values that led to the current algebra which closes on $T', W', : T'T' :$ [53]. Thus the conditions that the theory obtained by gauging the algebra generated by T', W' be free of matter-dependent anomalies are precisely the conditions that T', W' generate a closed W-algebra (with the

normal ordering prescription (8.4)). However, even in this case, there remain non-trivial universal anomalies. For $D \neq 2$, the matter-dependent anomalies are non-trivial.

9.4 Non-Linear w_∞ Gravity

Chiral w_∞ gravity is given by introducing the Noether coupling $\sum_n h^n W_n$ of the gauge fields h^n to the currents (6.8) [8]. Integrating out the scalar field ϕ gives contributions to the effective action of the form

$$\sum_{m,n} a_{m,n} \int d^2x d^2y \; h^m(x) h^n(y) < W_m(x) W_n(y) >$$

(for some coefficients $a_{m,n}$) and varying this under $\delta h^n = \partial_- \lambda^n + \dots$ gives an infinite set of anomalies similar to those described for W_3. There are matter-dependent anomalies for all spins higher than two, as all the higher spin symmetries are non-linearly realised. It was argued in [56] that all the matter-dependent anomalies can be cancelled by adding higher derivative counterterms similar to (9.9). The action with the counterterms now involves the coupling $\sum_n h^n W'_n$ where W'_n are modified currents which are precisely the currents that generate the quantum algebra W_∞ with $c = -2$ [56]. There remain non-trivial universal anomalies (one for each spin), including a gravitational anomaly proportional to $c = -2$. These can cancel against ghost contributions, however, as will be seen in the next chapter. Unfortunately, this cancellation of anomalies does not seem to be possible for the realisations of w_∞ in terms of more than one boson [54].

9.5 Anomalies and Current Algebras

The previous examples are sufficient to illustrate the general principle that non-trivial anomalies arise if the corresponding current algebra closes classically but not quantum mechanically. In general the quantum current algebra is given to lowest order in $\hbar$ by applying the Dirac prescription to the classical algebra.

However, there will in general be extra terms of order $\hbar^2$ or higher and each of these terms corresponds to a term in the anomaly.

The matter-dependent terms in the current algebra of order $\hbar^2$ or higher fall into two classes, depending on whether they involve only the currents W_A that occurred in the classical algebra (e.g. $\{W_A\} = \{T, W\}$ for W_3), or whether they introduce new currents into the algebra (e.g. $J, T^{2,0}$ for W_3). Those which involve only the currents W_A do not affect the closure of the algebra but represent an $\hbar$-dependent deformation of the classical W-algebra. The corresponding terms in the anomaly depend on the currents W_A. The classical action (4.7) involves the term $\sum_A h^A W_A$ and so these W_A-dependent anomalies can be cancelled by modifying the gauge transformations of the gauge fields h^A by $\hbar$-dependent terms. An example of this kind of term is the term proportional to T in (8.12), which leads to the term proportional to $\bar{T}$ in the anomaly (9.7); this term was cancelled by modifying the transformation of h_{--}.

Matter-dependent anomalies of this type were first found in [50] in models with a non-linearly realised local supersymmetry. For these models, there were universal anomalies as usual, together with a matter-dependent supersymmetry anomaly involving the stress-tensor T_{++}. The matter-dependent anomaly could then be cancelled by modifying the supersymmetry transformation rule for the graviton h_{--} by appropriate $\hbar$-dependent terms. This corresponded to a deformation of the classical symmetry algebra, which was of the form $Q^2 = 0$, where Q was the classical 'supercharge', to a quantum $(1,0)$ supersymmetry algebra of the form $Q^2 = \hbar P$, with P the chiral momentum [50].

If there are matter-dependent terms in the current algebra that involve new currents not in the classical algebra, then the quantum algebra does not close on the set of classical currents $\{W_A\}$. For example, the presence of $T^{2,0}$ in (8.12) means that the algebra does not close on the classical currents T, W. There are corresponding terms in the anomaly involving the new currents (e.g. the $\bar{T}^{2,0}$ term in (9.7)) and these cannot be cancelled by modifying the transformation

rules.

One way of cancelling such matter-dependent anomalies is as follows [43]. The set of currents W_A generate an algebra which closes classically but not quantum mechanically. One can introduce a (possibly infinite) set of new currents into the algebra until one obtains a quantum algebra (with generators Z^m, say) which is closed up to central charge terms. For example, in the case of W_3, one introduces $T^{2,0}$ as a generator, and then finds that the algebra generated by $T, W, T^{2,0}$ still does not close, and so one is led to introduce an infinite set of currents $T^{n,0} = \partial_+^{n+1} \phi^i \partial_+ \phi^i, W^{m,n,0} = d_{ijk} \partial_+^m \phi^i \partial_+^n \phi^j \partial_+ \phi^k, \ldots$ which do generate a closed quantum algebra. If instead of gauging the original W-algebra, one gauges the its quantum closure by introducing a gauge field corresponding to each of the generators Z^m, then there will be no non-trivial matter-dependent anomalies and only the universal anomalies will remain [43].

Central charge terms in the algebra correspond to universal anomalies and these are non-trivial (*i.e.* anomalies which cannot be cancelled by local counterterms) if and only if the central extension is a non-trivial cocycle (*i.e.* one which cannot be absorbed into redefinitions of the generators). If the classical algebra had no central charges, the quantum generation of such terms means that the quantum algebra does not close on the original set of generators but requires the addition of a central charge generator. In this case, one can follow the approach outlined above and introduce a spin-zero gauge field which couples to the generator of central charge gauge transformations [7], so that one is gauging an algebra which closes properly at the quantum level, not just up to central charge terms. However, it is more conventional not to do this, but instead to try and cancel the central charge terms against ghost contributions.

9.6 Quantum W-Gravity

The effective action for chiral gravity was given to lowest order by (9.1). For the non-chiral case, it is given instead by

$$\Gamma = S_0 + D \frac{i\hbar}{24\pi} \int d^2x \left[h_{--} \frac{\partial_+^4}{\Box} h_{--} + h_{++} \frac{\partial_-^4}{\Box} h_{++} + \ldots \right] \qquad (9.10)$$

to lowest order in the gauge fields. This is not gauge invariant, but can be made so by introducing the trace of the metric, h_{+-}, which transforms as $\delta h_{+-} = \partial_+ k_- + \partial_- k_+ + \ldots$, and adding a finite local counterterm to (9.10), so that it becomes [1]

$$\Gamma = S_0 + D \frac{i\hbar}{24\pi} \int d^2x \left[R^{(2)} \frac{1}{\Box} R^{(2)} \right] \qquad (9.11)$$

where $R^{(2)}$ is the curvature scalar. This is gravitationally covariant, but not Weyl-invariant since it depends on the trace of the metric. Thus the gravitational anomaly has been cancelled at the expense of introducing a Weyl anomaly. Note that the integration over ghost fields shifts the coefficient D in these formulae to $D - 26$.

If $D \neq 26$, the action $R^{(2)} \frac{1}{\Box} R^{(2)}$ can be thought of as a non-local kinetic term for the two-dimensional metric, which becomes a dynamical field. In the conformal gauge, the metric is given in terms of the conformal mode $\phi^{(2)} = h_{+-}$ and the action takes the local form $\phi^{(2)} \Box \phi^{(2)}$ [1]. In the chiral gauge, the action takes the non-local form $h_{--} \partial_+^4 \Box^{-1} h_{--}$ [61].

For W_3-gravity, the universal spin-3 anomaly gives an extra contribution to the effective action, which is given to lowest order by

$$\int d^2x \left[B_{---} \frac{\partial_+^6}{\Box} B_{---} + B_{+++} \frac{\partial_-^6}{\Box} B_{+++} + \ldots \right] \qquad (9.12)$$

In this case, there are several ways to introduce new variables and add finite local counterterms to cancel the W-gravity anomaly at the expense of introducing a

W-Weyl anomaly [42,57]. As in chapter 6, one can introduce the traces $B_{\pm+-}$ for the spin-3 field $B_{\mu\nu\rho}$, which can then be used to define the invariant curvature given to linearised order by [42]

$$R^{(3)} = \epsilon^{\mu\alpha}\epsilon^{\nu\beta}\epsilon^{\rho\gamma}\partial_\mu\partial_\nu\partial_\rho B_{\alpha\beta\gamma} \qquad (9.13)$$

Then adding a finite local counterterm brings the action (9.12) to the form $R^{(3)}\frac{1}{\Box}R^{(3)}$ which is invariant under the W-transformation $\delta B_{\alpha\beta\gamma} = \partial_{(\alpha}\lambda_{\beta\gamma)} + \cdots$ but is anomalous under the W-Weyl transformations corresponding to shifts of the traces, $\delta B_{\pm+-} = \Omega_\pm$. In a generalisation of the conformal gauge, the traces $B_{\pm+-}$ are written in terms of a scalar $\phi^{(3)}$ and the W_3-gravity action takes the higher-derivative form $\phi^{(2)}\Box\phi^{(2)} + \phi^{(3)}\Box^2\phi^{(3)}$[42]. In a chiral gauge the action takes the form $B_{---}\partial_+^6\Box^{-1}B_{---}$ to lowest order in the gauge fields [42]. Further details, including non-linear corrections in the chiral gauge, are discussed in [51,55].

Instead of introducing the vector $B_{\mu+-}$, one can introduce a scalar χ and define an invariant curvature $\bar{R}^{(3)} = \partial_+^3 B_{---} + \partial_-^3 B_{+++} + \Box\chi$ [42]. This can be used to add finite local counterterms so that the action again takes the covariant form $\bar{R}^{(3)}\frac{1}{\Box}\bar{R}^{(3)}$. This is not invariant under the W-Weyl transformation corresponding to shifts in χ. In this case, the linearised conformal gauge W_3 action takes the form $\phi^{(2)}\Box\phi^{(2)} + \chi\Box\chi$ which does not involve any higher derivative terms. Thus apparently inequivalent versions of quantum W-gravity can be obtained by using different ways of introducing conformal modes into the covariant theory [42]. For further discussion of quantum W-gravity, see [44,45,51,55].

10. Quantum W-Gravity and W-Strings

Consider first chiral W_3 gravity. The gauge symmetries (4.4) can be used to gauge away the gauge fields and impose the gauge conditions $h_{--} = B_{---} = 0$, or to impose a background gauge $h_{--} = \bar{h}_{--}$, $B_{---} = \bar{B}_{---}$ for some fixed background gauge fields $\bar{h}_{--}, \bar{B}_{---}$.[*] In doing so, it is necessary to introduce the usual gravitational conformal ghosts b_{++}, c_- together with their spin-three counterparts u_{+++}, v_{--}. The Faddeev-Popov method cannot be used as the gauge algebra does not close off-shell, but the more general methods of Batalin and Vilkovisky [62] can, and yield the following gauge-fixed action [2]

$$S = S_0 - \frac{1}{2}\int d^2x \left[\bar{h}_{--}(T + T^{gh}) + \bar{B}_{---}(W + W^{gh})\right] \qquad (10.1)$$

where the ghost spin two and three currents are

$$T^{gh}_{++} = b_{++}\partial_+c_- + \partial_+\left(b_{++}\,c_-\right) + u_{+++}\partial_+v_{--} + 2\partial_+\left(u_{+++}\,v_{--}\right)$$
$$W^{gh}_{+++} = 2u_{+++}\partial_+c_- + \partial_+\left(u_{+++}\,c_-\right) + \left(b_{++}\,\partial_+v_{--}\right)T_{++} + \partial_+(b_{++}v_{--}T_{++})$$
$$(10.2)$$

Integration over the ghost fields as well as the matter fields gives further contributions to the anomalies of section 9.3, and also gives new matter-dependent anomalies which depend on the ghosts instead of the matter fields [52,54]. The ghosts change the coefficient of the gravitational anomaly (9.2) from D to $D-100$ and change the coefficient of the $T^{2,0}$ term in (9.7) from $D + 2$ to $D - 2$ [52]. This latter change is particularly striking, as it is similar to the change from the algebra (8.12) to the algebra (8.14).

The next step is to attempt to cancel the anomalies by adding finite local counterterms. The current W^{gh} is primary with respect to T^{gh} but is only primary with respect to the total energy-momentum $T + T^{gh}$ if it is modified to give

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* Strictly speaking, the gauge fields cannot be gauged away completely in general, but can be set equal to quadratic and cubic differentials respectively. The integration over the gauge fields then reduces to an integration over these spaces of differentials, which constitute the moduli space for W_3 gravity.

[2]

$$\hat{W}^{gh} = W^{gh} + \frac{\hbar c_m}{192}[-2v\partial_+^3 b - 9\partial_+ v\partial_+^2 b + 15\partial_+ b\partial_+^2 v + 10b\partial_+^3 v].$$ (10.3)

The relation between current algebras and anomalies described in the last section suggests replacing W^{gh} with $\hat{W}^{gh}$ in (10.1) in order to avoid mixed anomalies. The analysis of the last chapter suggests adding background charges so that the matter currents T, W are replaced by the currents (8.15). It then remains to see whether the free coefficients can be chosen so that the anomalies cancel [54]. Remarkably, all the anomalies are cancelled by ghost contributions and the effective action satisfies the BRST Ward-identities if and only if the currents T', W' are chosen so that they generate Zamolodchikov's W_3 algebra with $c = 100$ [59] and for these theories the nilpotent BRST charge is precisely the W_3 BRST charge given in [40]. This analysis can be extended to non-chiral W_3 gravity, with the result that the anomalies cancel if the currents T_{++}, W_{+++} and T_{--}, W_{---} in the action (4.6) generate two copies of the $c = 100$ W_3 algebra [59]. Thus the $c = 100$ realisations of W_3 that were constructed in [27] can be used to define critical W_3 strings for which all the anomalies cancel and the integration over gauge fields reduces to an integration over moduli.

It seems likely that similar critical strings can be defined for other W-algebras. For the linearly realised W_∞ gravity of section 9.2, the quantum gauge algebra is W_∞ with $c = 2D$, while for the non-linearly realised W_∞ gravity of section 9.4, the quantum gauge algebra is W_∞ with $c = -2$ after the cancellation of the matter-dependent anomalies. In both cases, the integration over the ghost fields does not introduce any new matter-dependent anomalies, so the theories will be anomaly free if and only if the ghost contributions to the universal anomalies cancel the matter contributions. The ghost contributions to the anomalies are given by a divergent sum, which can be regularised and summed using the zeta-function technique [49] and the anomalies will then cancel only if the matter central charge is $c = -2$. Thus for the linear realisation, the anomalies do not

cancel for any value of the number of bosons D, but for the one-boson non-linear realisation, all the anomalies cancel [56] (if the zeta-function trick is used) and a critical W_∞ string appears to emerge.

Much work remains to be done to see whether these W-string theories can be extended to fully interacting models with a sensible space-time interpretation; some preliminary discussion of the spectrum is given in [47,58,60]. Nevertheless, it is indeed a striking result that the standard string theory, which is based on the Virasoro algebra, has a generalisation to a critical W-string theory based on the W_3 algebra. It seems reasonable to expect that these results will extend to other W-algebras, giving an infinite set of new string theories. Whether or not they will have important physical implications is not yet clear, but it seems likely that the mathematical structure of these theories will continue to fascinate researchers for some time to come.

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