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INTRODUCTION TO SUPERSYMMETRY

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Please note: These are preliminary notes intended for internal distribution only.

7. Superspace

Usually, irreducible representations of symmetries have several components (or degrees of freedom). When the symmetry is realized linearly, it is useful to introduce a single object which has all the degrees of freedom as its components. For example, in the case of isospin symmetry we can introduce a ~~two~~ nucleon field which has the proton and the neutron as components. The action of the isospin generators is then realized through matrix multiplication. The advantage of this formulation is that many manipulations can be done using the abstract algebraic properties of the isospin generators, and there is no need to write explicitly ~~all~~ components of the nucleon field or of the isospin generators.

After the introduction of auxiliary fields, the action of the SUSY generators is realized linearly for the Wess Zumino model, and it forms a closed algebra. We can therefore try to build a single object which will contain both ϕ, ψ and F as components. This is achieved by constructing <u>superfields</u> in <u>superspace</u>.

There are two main differences between the above example of an internal symmetry and SUSY. First, ~~as~~ a representation of the SUSY algebra contains fields of different dimension and spin. The <u>superfield</u> we want to construct ~~of~~ out of them should have definite dimension and statistics. We therefore have to introduce <u>anticommuting</u> objects which have ~~non-zero~~ dimension $-\frac{1}{2}$.

Second, the result of two SUSY transformations is a translation (recall (5.11)). Therefore, the action of the SUSY generator cannot be represented ~~just~~ by matrix multiplication – the $\frac{\partial}{\partial x_\mu}$ operator should somehow be contained in the operator which represents the SUSY generator in <u>superspace</u>.

What is ~~this~~ superspace? In addition to the four ordinary coordinates x_μ, $\mu = 0,1,2,3$, we introduce four anticommuting variables

$$\theta_\alpha \, , \, \bar{\theta}^{\dot\alpha} \, , \qquad \alpha, \dot\alpha = 1,2 \quad . \tag{7.1}$$

The anticommutativity we postulate means e.g.

$$\theta_1 \theta_2 = -\theta_2 \theta_1 \qquad \left(= \tfrac{1}{2} \theta^T \epsilon \theta \equiv \tfrac{1}{2} \theta\theta \right) \tag{7.2}$$

$$\theta_1^2 = \theta_2^2 = 0 \tag{7.3}$$

We consider the θ-s as additional variables on which a <u>superfield</u> can depend. Thus, formally, our coordinates are now

$$(x_\mu , \theta^\alpha , \bar{\theta}_{\dot\alpha}) \tag{7.4}$$

and a superfield is, by definition, an object which depends on all these coordinates

$$\Phi = \Phi (x_\mu , \theta^\alpha , \bar{\theta}_{\dot\alpha}) \tag{7.5}$$

By analogy to ~~and~~ $\frac{\partial}{\partial x_\mu}$ we define also $\frac{\partial}{\partial \theta_\alpha}$ by postulating

$$\frac{\partial}{\partial \theta_\alpha} \theta^\beta = \delta_\alpha{}^\beta \qquad \frac{\partial}{\partial \theta^\alpha} \bar{\theta}^{\dot\alpha} = 0 \tag{7.6}$$

We further postulate that $\frac{\partial}{\partial \theta}$-s anticommute among themselves and with θ-s, e.g.

$$\frac{\partial}{\partial \theta_1} \theta_2 \theta_1 = - \theta_2 \frac{\partial}{\partial \theta_1} \theta_1 = -\theta_2 \tag{7.7}$$

In ordinary space one can expand a function ~~along~~ as power series

$$\phi(x) = \phi(0) + x \left(\frac{\partial}{\partial x} \phi \right) \Big|_{x=0} + \ldots \tag{7.8}$$

Similarly, we can expand a superfield in terms of its θ-coordinates. The basic property of this expantion is that it ~~automatically~~ contains a <u>finite</u> <u>number</u> of terms

$$\Phi(x, \theta, \bar\theta) = c(x) + \theta \chi_1(x) + \bar\theta \bar\chi_2(x) + \theta\theta\, m_1(x) + \bar\theta\bar\theta\, \bar m_2(x) + \theta\sigma^\mu\bar\theta\, v_\mu(x)$$
$$+ (\theta\theta)\, \bar\theta \bar\lambda_1(x) + \bar\theta\bar\theta\, \theta \lambda_2(x) + (\theta\theta)(\bar\theta\bar\theta)\, d(x) \tag{7.9}$$

any polynomial in θ and $\bar\theta$ which is not included on the r.h.s. of (7.9) will contain at least three θ-s or three $\bar\theta$-s and hence it vanishes identically — the highest non-vanishing polynomial contains every anticommuting variable precisely once.

The expansion (7.9) contains 8 bosonic components $c, m_1, \bar m_2, v_\mu, d$ and 8 fermionic components $\chi_1, \bar\chi_2, \lambda_1, \bar\lambda_2$ (each one is, a priori, a complex field). Ordinary boson or fermion fields will be identified with those components. (Notice that the number of θ-s determines the dimension of component fields.)

The finiteness of the θ-expansion implies a big qualitative difference between the ~~bosonic~~ "ordinary directions" x_μ and the "fermionic directions" θ and $\bar\theta$. The real axis contains an infinite number of points, and in order to define a function one has in principle to give an infinite list — the values of the function at every point. On the other hand, for a fixed point x^0_μ in space, one only has to give 16 complex numbers in order to define ~~all~~ $\Phi(x^0, \theta, \bar\theta)$ for all $\theta, \bar\theta$. What we really do when we go from ordinary space to superspace is to introduce 16 copies of ordinary space with a set of rules for going from one copy to another. Superspace should be considered as an ~~set of rules which~~ algebraic construction which simplifies calculations, and not as additional dimensions of the real world.

Our first task is to represent the SUSY generator by a differential operator in superspace. We define

$$Q = \frac{\partial}{\partial\theta} + i\sigma^\mu\bar\theta\frac{\partial}{\partial x^\mu} \qquad (7.10)$$
$$\bar Q = -\left(\frac{\partial}{\partial\bar\theta} + i\theta\sigma^\mu\frac{\partial}{\partial x_\mu}\right)$$

The reader can check, using the definition of $\frac{\partial}{\partial\theta}$ & $\frac{\partial}{\partial\bar\theta}$ that $Q, \bar Q$ of (7.10) satisfy

$$\{Q\ Q\} = \{\bar Q\ \bar Q\} = 0$$
$$\{\bar Q\ Q\} = -2i\sigma^\mu\partial_\mu \qquad (7.11)$$

A SUSY transformation of a superfield is defined by

$$\Phi'(x,\theta,\bar\theta) = \delta_\xi \Phi(x,\theta,\bar\theta) = (\xi\theta + \bar\xi\bar Q)\,\Phi(x,\theta,\bar\theta) \tag{7.12}$$

where $Q, \bar Q$ are given by (7.10). To see how component fields transform, we compare the θ-expansion of $\Phi'(x,\theta,\bar\theta)$ with the r.h.s. of (7.12). For instances, the $\theta=\bar\theta=0$ component of the r.h.s. of (7.12) is $\xi\chi_1 + \bar\xi\bar\chi_2$. Hence

$$c'(x) \equiv \delta_\xi c(x) = \xi\chi_1(x) + \bar\xi\bar\chi_2(x) \tag{7.13}$$

By construction, the superfield $\Phi(x,\theta,\bar\theta)$ of eq (7.9) forms a representation of the SUSY algebra. However, this representation is not irreducible. The standard way to construct irreducible representation is to impose <u>invariant constraints</u> on $\Phi(x,\theta,\bar\theta)$. The two ~~reps~~ irreducible representations that occur in renormalizable theories are

(a) Vector Superfield. It is defined by imposing a reality condition

$$V(x,\theta\,\bar\theta) = V^*(x,\theta,\bar\theta)\,. \tag{7.14}$$

In terms of the expansion (7.9), this implies that c, v_μ, d are real ~~and~~ $\bar m_2 = (m_1)^*$ and $\chi_1 = \chi_2$, $\lambda_1 = \lambda_2$. Given V which satisfies (7.14), $\delta_\xi V$ will also satisfy the reality condition since

$$[(\xi Q + \bar\xi\bar Q)V]^* = (\xi Q + \bar\xi\bar Q)^* V^* = (\xi Q + \bar\xi\bar Q)V \tag{7.15}$$

The constraint is invariant under SUSY. This means that we can write down the transformation rules for the components of $V(x,\theta,\bar\theta)$ such that each component transforms into other components, but not into objects not contained in the supermultiplet. The vector supermultiplet will be used in the next section to construct SUSY gauge theories.

Due to the presence of both θ-s and $\bar\theta$-s, another constraint one could think of is to require that a superfield will depend only on θ but not on $\bar\theta$:

$$\frac{\partial}{\partial\bar\theta}\,\Phi(x,\theta,\bar\theta) = 0 \qquad (7.16)$$

This would imply that Φ contains only θ-s in its expansion but no $\bar\theta$. However, the constraint (7.16) is not invariant under SUSY. The reason is the $i\sigma^\mu\bar\theta\frac{\partial}{\partial x_\mu}$ term in Q. Starting from Φ which satisfies (7.16), we would find

$$\frac{\partial}{\partial\bar\theta}(\xi Q + \bar\xi\bar Q)\,\Phi \neq 0 \qquad (7.17)$$

In other words, the component fields which have only θ-s (but no $\bar\theta$-s in front of them, do not transform among themselves.

To remedy this problem we introduce covariant derivatives

$$\begin{aligned} D &= \frac{\partial}{\partial\theta} - i\sigma^\mu\bar\theta\frac{\partial}{\partial x_\mu} \\ \bar D &= -\frac{\partial}{\partial\bar\theta} + i\theta\sigma^\mu\frac{\partial}{\partial x_\mu} \end{aligned} \qquad (7.18)$$

The important property of the D-s is that they anticommutes with the Q-s.

$$\{D, Q\} = \{\bar D, Q\} = \{D\ \bar Q\} = \{\bar D\ \bar Q\} = 0 \qquad (7.19)$$

Among themselves, the D-s satisfy

$$\begin{aligned} \{D\ D\} &= \{\bar D\ \bar D\} = 0 \\ \{D\ \bar D\} &= 2i\sigma^\mu\partial_\mu \end{aligned} \qquad (7.20)$$

This is very similar to the algebra of the Q-s, except for the sign on the r.h.s. of the $\{D\bar D\}$ anticommutator. It is tempting to say that we have found another independent representation for the SUSY generators. This is not true. Because of the wrong sign, using the D-s as SUSY generators would lead to a hamiltonian which is negative and unbounded from below.

Using the D-s we can construct an irreducible representation.

(b) Chiral superfield. It is defined by

$$\bar{D}_{\dot\alpha}\Phi = 0 \qquad (7.21)$$

Acting with the SUSY generators we find

$$\bar{D}_{\dot\alpha}\Phi' = \bar{D}_{\dot\alpha}(\xi Q + \bar\xi\bar Q)\Phi = -(\xi Q + \bar\xi \bar Q)\bar{D}_{\dot\alpha}\Phi = 0 \qquad (7.22)$$

Thus, the constraint is ~~co~~variant, and the independent components of the chiral superfield transform among themselves.

To find the ~~ness~~ independent components of the chiral superfield it is convenient to change variables $(x_\mu, \theta, \bar\theta) \to (y_\mu, \theta, \bar\theta)$ where

$$y_\mu = x_\mu + i\theta\sigma_\mu\bar\theta \qquad (7.23)$$

In the new variables, the covariant derivatives take the form (using the chain rule)

$$D_\alpha = \frac{\partial}{\partial\theta} - 2i\sigma^\mu\bar\theta\frac{\partial}{\partial y_\mu} \qquad (7.24)$$
$$\bar D = -\frac{\partial}{\partial\bar\theta}$$

and the Q-s are

$$Q = \frac{\partial}{\partial\theta} \qquad (7.25)$$
$$\bar Q = -\frac{\partial}{\partial\bar\theta} - 2i\,\theta\sigma^\mu\frac{\partial}{\partial y_\mu}$$

The reader can verify explicitly that the D-s and Q-s given here satisfy the same algebra as the original D-s and Q-s (without using the relation (7.23) between x_μ and y_μ).

In terms of the new variables, the constraint (7.21) takes the form

$$\frac{\partial}{\partial\bar\theta}\Phi(y,\theta,\bar\theta) = 0 \qquad (7.26)$$

(Thus, (7.16) was not such a bad guess, after all). Hence, ~~$\Phi(y,\theta,\bar\theta)$~~

$$\Phi(y,\theta,\bar\theta) = \Phi(y,\theta) \qquad (7.27)$$

The expansion of the chiral superfield is therefore

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \qquad (7.28$$

The use of (ϕ,ψ,F) as components is no coincidence – we can ~~now~~ identif the components of a chiral superfield with the ~~fields of~~ physical (ϕ,ψ) and auxiliary (F) fields of the Wess-Zumino model. The transformation law for the components (ϕ,ψ,F) is obtained by comparing the coefficients in the θ-expansion of the transformation law:

$$\begin{aligned}\Phi'(y,\theta) &= \phi'(y) + \sqrt{2}\theta\psi'(y) + \theta\theta F'(y) \\ &= (\xi Q + \bar{\xi}\bar{Q})(\phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)) \qquad (7.29\end{aligned}$$

where $Q, \bar{Q}$ are given by (7.25). An explicit calculation shows that the resulting transformation law is (6.1).

8. Superspace lagrangian for the Wess-Zumino model

So far, we have introduced (a) the "fermionic coordinates" $\theta, \bar{\theta}$ (b) superfields and (c) differential operators (Q-s and D-s) in superspace. With all this machinary, the SUSY transformations of the Wess-Zumino model are ~~represented~~ very compactly contained in the definition (7.21) of a chiral superfield and in the chiral representation of the Q-s (7.25).

We would now like to reformulate the Wess-Zumino model entirely in terms of ~~superf~~ chiral superfields in superspace. In particular, we ~~have~~ want to construct a superspace lagrangian and to derive superspace Feynman rules. We will not be able to describe the superspace Feynman rules here, and we refer the reader to the literature on the subject. But we will discuss the construction of a superspace lagrangian.

In ordinary space, the action is the spacetime integral of a lagrangian density $\mathcal{L}(x)$. By analogy, we would like to define the superspace action as a superspace <u>integral</u>

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\; \mathcal{L}(x,\theta,\bar\theta) \qquad (8.1)$$

We will have to <u>define</u> the $\int d\theta$ integrals. Before we do so, let us try to find what is a suitable superspace lagrangian.

The lagrangian (6.4) is nonlinear in the fields of the Wess-Zumino model. We will therefore have to study products of superfields.

We consider first the product of two ~~super~~ chiral superfields. The product $\Phi_1(y,\theta)\,\Phi_2(y,\theta)$ is defined simply by multiplying the θ-expansion of Φ_1 and Φ_2 in terms of their components. In doing so, one uses the "multiplication rules" (7.2), (7.3) for the θs and <u>Fierz identities</u>, e.g. (Notice : $\bar D_{\dot\alpha}\Phi_1\Phi_2 = 0$)

$$(\theta\psi_1)(\theta\psi_2) = -\tfrac{1}{2}(\theta\theta)(\psi_1\psi_2) \qquad (8.2)$$

Another useful Fierz identity is

$$(\theta\psi)\,\bar\chi = +\tfrac{1}{2}(\psi\sigma^\mu\bar\chi)\,\theta\sigma_\mu \qquad (8.3)$$

One obtains

$$\Phi_1\Phi_2 = \phi_1\phi_2 + \sqrt{2}\,\theta(\psi_1\phi_2 + \psi_2\phi_1) + \theta\theta\,(\phi_1F_2 + \phi_2F_1 - \psi_1\psi_2) \qquad (8.4)$$

(Notice that $\quad \psi_1\psi_2 = \psi_2\psi_1 = \tfrac{1}{2}(\psi_1\psi_2 + \psi_2\psi_1)$ $\qquad (8.5)$

Substituting the explicit form of $W(\phi)$ eq. (5.2) in the lagrangian (6.4), we find that the term which depends on m is $(\phi F - \tfrac{1}{2}\psi\psi)$. It is therefore equal to the $\theta\theta$ component of the product $\tfrac{1}{2}\Phi^2(y,\theta)$. Likewise, one can check that the cubic terms in (6.4) are equal to $\Phi^3(y,\theta)\big|_{\theta\theta}$. (We use $X|_{\theta\theta}$ to denote the $\theta\theta$-component of X).

Thus, the mass and interaction terms of the Wess-Zumino model are given by

$$\begin{aligned}\mathcal{L}_{m,g} &= \int d^4y\,\left(\tfrac{m}{2}\,\Phi^2(y,\theta) + \tfrac{g}{3}\,\Phi^3(y,\theta)\right)\Big|_{\theta\theta} + \text{h.c.} \\ &= \int d^4y\; W(\Phi(y,\theta))\big|_{\theta\theta} + \text{h.c.}\end{aligned} \qquad (8.6)$$

Next, we must express the kinetic part of the lagrangian in terms of superfields. The kinetic part is real, and so we will have to use both $\Phi(y,\theta)$ and $(\Phi(y,\theta))^*$ for it. We first recall the relation (7.23) between x_μ and y_μ and express $\Phi(y,\theta)$ in terms of $\Phi(x,\theta,\bar\theta)$. We find

$$\Phi(y,\theta) = \Phi(x_\mu + i\theta\sigma_\mu\bar\theta , \theta) =$$
$$= \phi(x) - i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi(x) - \tfrac{1}{4}\,\theta\theta\,\bar\theta\bar\theta\,\Box\phi(x) \qquad (8$$
$$+ \sqrt{2}\,\theta\psi(x) + \frac{i}{\sqrt{2}}\,\theta\theta\,\partial_\mu\psi(x)\,\sigma^\mu\bar\theta$$
$$+ \theta\theta F$$

In (8.7), we wrote on each row the terms which follows from the expansion of a single term in (7.28). To expree Φ^*, use the rule $(\theta)^* = \bar\theta$. The reader can now check that the kinetic part of the lagrangian is equal to the highest component of $\Phi^*\Phi$.

$$\mathcal{L}_K = \int d^4x\,(\Phi^*(x,\theta,\bar\theta)\,\Phi(x,\theta,\bar\theta))\big|_{\theta\theta\bar\theta\bar\theta} \qquad (8.8$$

Notice that both $\mathcal{L}_K$ and $\mathcal{L}_{m,g}$ are equal to the highest component of some (composite) superfield. The reason is the following. Since the superspace operators Q and $\bar Q$ are linear, a given component field can transform into a component which has one more power of θ, or into the derivative of a lower component with one less θ. For instance, for the chiral superfield Φ of eq. (7.28), ϕ can transform only into ψ, ψ can transform either into F or into $\partial\phi$, and F can transform only into $\partial\psi$.

More generally, highest components of superfield always transform into space-time derivative of something else. Consequently, using a highest component of some composite superfield as a lagrangian density (in ordinary space) will always lead to an invariant action.

The last task left for us is to <u>invent</u> rules for $\int d\theta$ integration which will resemble as much as possible ordinary integration and which reproduce the lagrangians (8.6) and (8.8).

We define:

$$\int d\theta = \int d\theta \cdot 1 = 0 \qquad\qquad (8.9$$
$$\int d\theta \; \theta = 1$$

These rules are sufficient to calculate all $\int d\theta$ integral over arbitrary superfields. For example,

$$\int d\theta \; (a + b\theta) = b$$
$$\int d\theta_1 d\theta_2 \,(a + b_1\theta_1 + b_2\theta_2 + c\; \theta_2\theta_1) = c \qquad\qquad (8.10$$

Notice that we have to decide on a given ordering, since $\int d\theta_1 d\theta_2 \;\theta_2\theta_1 = 1$ but $\int d\theta_1 d\theta_2 \;\theta_2\theta_1 = \int d\theta_2 d\theta_1 \;\theta_2\theta_1 = -1$. The rules (8.9) imply in particular

$$\int d\theta \; \frac{\partial}{\partial\theta} (a + b\theta) = \int d\theta \; b = 0 \qquad\qquad (8.11$$

The θ-integral of a total derivative vanishes, and so one can freely integrate by parts if the integrand contains superfield $\frac{\partial}{\partial\theta}$ derivatives.

With the definition (8.9) for integration, one has

$$\mathcal{L}_K = \int d^4x \; d^2\theta \, d^2\bar\theta \quad \Phi^*(x,\theta,\bar\theta)\,\Phi(x,\theta,\bar\theta) \qquad\qquad (8.12$$
$$\mathcal{L}_{m,g} = \int d^4y \; d^2\theta \quad W(\Phi(y,\theta)) \; + h.c. \qquad\qquad (8.13$$

These are equal to the previous expressions because the $\int d^2\theta$ and $\int d^4\theta$ integrals pick the highest components of a chiral and real (vector) multiplet respectively.

Finally, we note that the definition (8.9) implies

$$b = \int d\theta \,(a + b\theta) = \frac{\partial}{\partial\theta} (a + b\theta) \qquad\qquad (8.14$$

Our definitions imply that differentiation and integration are the same thing for the θ-s. also, $\int d\theta \; \theta \,(a + b\theta) = a$, hence one can say that $\delta(\theta) = \theta$ since we would like to have $\int d\theta \, \delta(\theta)\, (a+b\theta) = a$. These properties are useful in ~~constructing~~ obtaining superspace Feynman rules.

9. Supersymmetric gauge theories

So far we discussed the construction of SUSY theories with only scalars and fermions. For SUSY to be relevant to the real world, one have to construct SUSY gauge theories.

In introducing the Wess-Zumino model, we started with the on-shell formalism, next we introduced the auxiliary field, and finally we reformulate everything in superspace. We note that the superspace formulation had precisely the same content as the off-shell component formulation. We merely wrote the same formulas in a more compact notation.

In the present section we will follow a somewhat different path. We will first introduce the off-shell vector multiplet and write the lagrangian of the gauge sector. Next we will couple the vector multiplet to supersymmetric matter (i.e. chiral multiplets and obtain the most general renormalizable SUSY theory in both the off-shell and on-shell formulation. As expected, the on-shell SUSY Transformations will be model dependent.

We will then introduce the superspace version of SUSY gauge theories. Unlike the Wess-Zumino model, this will turn out to be more than re-writing the same thing in different notation. In particular, we will find that the physical and auxiliary fields of the vector multiplet do not form a superfield. In order to build a superfield we will have to introduce additional field of a new kind - they will be compensating fields. We will then discuss briefly the non-trivial differences between the superspace formulation and the components formulation of SUSY gauge theories.

We begin by giving the field content of a vector supermultiplet. First we have the gauge field B^a_μ, next we have a Weyl fermion λ^a_α called the gaugino, finally we have an auxiliary field D^a. The index a runs over the adjoint representation of $SU(N)$ for Yang-Mills theories. For $U(1)$ theories it is absent. We will usually drop this index below. The SUSY ~~#~~ transformation rules are

$$\delta_\xi B_\mu = i(\xi \sigma_\mu \bar\lambda + \bar\xi \bar\sigma_\mu \lambda) \qquad (9.1a)$$

$$\delta_\xi \lambda = -\tfrac{i}{2}\sigma^{\mu\nu} \xi G_{\mu\nu} - i\xi D \qquad (9.1b)$$

$$\delta_\xi D = \xi \sigma^\mu D_\mu \bar\lambda - \bar\xi \bar\sigma^\mu D_\mu \lambda \qquad (9.1c)$$

where

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig[B_\mu, B_\nu] \qquad (9.2)$$

is the ~~nonabelian~~ field strength. The invariant lagrangian is

$$\mathcal{L}_G = -\tfrac{1}{4} G_{\mu\nu}G^{\mu\nu} + \tfrac{i}{2}\bar\lambda \bar\sigma^\mu \overleftrightarrow{D}_\mu \lambda + \tfrac{1}{2}D^2 \qquad (9.3)$$

All derivatives in (9.1) and (9.3) are covariant

$$D_\mu = \partial_\mu + igB^a_\mu T^a \qquad (9.4)$$

where T^a is the generator in the appropriate representation. (For the gaugino, $(iT^a)_{bc} = f_{abc}$). Notice that the equation of motion for D is $D=0$, and that D transforms into the gaugino field equation (see (9.1c)). Therefore, the on-shell theory is obtained by setting $D=0$ in the lagrangian and the transformation rules. In the case of a $U(1)$ symmetry the gaugino is a singlet, and the lagrangian (9.3) describes a <u>free</u> theory consisting of a massless photon field and a massless neutral Weyl fermion - the photino. Notice that, since the adjoint representation is real, both λ and $\bar\lambda$ transforms the same way under gauge transformations. One sometimes says that λ is a Majorana fermion.

We next give the coupling of a vector supermultiplet to SUSY matter i.e. to a chiral multiplet. In a SUSY gauge theory, the transformation law for a chiral multiplet is modified: one has to replace ordinary derivatives by covariant ~~mass~~ ones (9.4) in the transformation law (6.1.

The mass and interaction terms (second row of (6.4)) of the matter lagrangians remain unchanged. The kinetic part is replaced by

$$\mathcal{L}_K^{cov} = |D_\mu \phi|^2 + \frac{i}{2}\bar{\Psi}\bar{\sigma}^\mu \overleftrightarrow{D}_\mu \Psi + |F|^2$$
$$+ ig\sqrt{2}\,(\phi^\dagger T^a \Psi \lambda^a - \bar{\lambda}^a \bar{\Psi} T^a \phi) + g D^a (\phi^\dagger T^a \phi) \qquad (9.5$$

Notice that the terms on the first row correspond to the kinetic part of (6.4), the only difference ~~being~~ being that ordinary derivatives have been replaced by covariant ones. The terms on the second row are new. In the presence of several matter supermultiplet one sums (9.5) over all of them.

An important class of SUSY gauge theories is when the matter multiplets come in ~~cons~~ pairs of complex conjugate representations. For instance, in SU(N) SUSY gauge theories we ~~have~~ take an equal number of multiplets in the fundamental representation N and in the complex conjugate one $\bar{N}$. These theories are called "Supersymmetr QCD"; because the matter fermions can be combined into <u>Dirac</u> fermions (with two complex scalars per every Dirac fermions). Otherwise, the theory is chiral.

The on-shell theory is obtained as usual by solving the field equations of the auxiliary fields. For F(x), this equation is (6.2) as before. For D(x), ~~As~~ one has

$$D^a = g \sum_i \phi_i^\dagger T_i \phi_i \qquad (9.6$$

where we have explicity shown the summation over the ~~vari~~ all multiplets.

We finally note that, for an abelian symmerty, the following lagrangian

$$\mathcal{L}_D = m^2 D \tag{9.7}$$

is both gauge invariant and supersymmetric, and so (9.7) can be added to the previous langangian with an arbitrary coefficient. This is the so-called D-Term.

Let us now examine the SUSY transformations of a gauge theory more closely. We notice that for a nonabelian group, the transformati (9.1) is non-linear. The same is true for the chiral multiplet ~~of the~~ after the replacement of ordinary derivatives by covariant ones.

The non-linearity of the transformation implies that we will not be able to obtain it from the action of a superspace Q_α on a superfield, because this action is always linear. A similar problem is found by calculating by anticommutator of two SUSY Transformations. Instead of (5.11) one finds

$$\delta_\eta \delta_\xi - \delta_\xi \delta_\eta = \epsilon^\mu \partial_\mu + ig\,\delta_G(\epsilon^\mu B_\mu) \tag{9.8}$$

where

$$\epsilon^\mu = 2i\,(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta) \tag{9.9}$$

Comparing to (5.11), we see that the second term on the r.h.s. of (9.8) is new. Here, ~~the~~ we use $\delta_G(\omega)$ to denote a <u>gauge</u> transformatio with parameter ω. Thus, $\delta_G(\epsilon^\mu B_\mu)$ is a gauge transformation with ~~parameter~~ ~~$\epsilon^\mu B_\mu$~~ <u>field dependent</u> parameter $\epsilon^\mu B_\mu$. Recall that

$$\delta_G(\omega)\, B_\mu = \tfrac{1}{g}(D_\mu\,\omega)^a \tag{9.10}$$

$$\delta_G(\omega)\,\phi = i\omega T^a \phi \tag{9.11}$$

where ϕ denotes here all fields except the ~~ξ~~ gauge field itself. Thus for all fields except the gauge field, the r.h.s. of (9.8) is $\epsilon^\mu D_\mu$. For the gauge field, the ~~anticommutator~~ role of the second term is to replace $\epsilon^\mu\partial_\mu A_\nu$ by the covariant expression $\epsilon^\mu G_{\mu\nu}$. The appearance of a covariant derivative in the algebra should not be a surprise. After all, we have replaced all ordinary derivatives by covariant ones in the transformation laws. Once more, the occurence of a field-dependent gauge transformation in the SUSY algebra implies that we will not be able to obtain it as the action of a linear differential operator in superspace.

The inability to write down a linear SUSY transformation rule can be ~~traced~~ traced to the counting of off-shell degrees of freedom in the vector multiplet. If one simply counts all compoents off-shell, then B_μ has 4 ~~compo~~ degrees of freedom, ~~λ~~ λ has four and $D(x)$ has one, i.e. the balance between the number of bosonic and fermionic degrees of freedom is violated. On the other hand, if one identifies field configuration related by a gauge transformation ~~then~~ the situation changes. While ~~the~~ $\lambda(x)$ and $D(x)$ have the same number of degrees of freedom as before, the gauge field $B_\mu(x)$ has only three now: the longitudal piece $\partial^\mu B_\mu$ can be set to zero by a gauge transformation. Thus, equality of the number of degrees of freedom is maintained only if a linear SUSY transformation is ~~followed by a non-linear gauge~~ "corrected" by a piece which looks like a gauge transformation with a field dependent parameter.

A similar problem exists in the on-shell formalism. At the level of the equations of motion, one uses the gauge symmetry twice to find that the photon field contains $4-2=2$ ~~de~~ physical degrees of freedom. For instance, ~~in the A_0~~ we first choose the gauge $A_0=0$ and then we use Gauss' law to eliminate another component of the gauge field. However, when quantizing in the gauge $A_0=0$, Gauss' law holds only on the physical subspace, and so equality of the nuber of bosonic and fermionic degrees of freedom holds only on this subspace. This is reflected in the fact that $[Q\ H]\neq 0$. Rather, $[Q\ H]$ is proportional to the integral of $\lambda(D_k E_k - \rho)$ i.e. to Gauss' law!

We now turn to the superspace formulation of <u>SUSY</u> gauge theories. In order to build a superfield, we must have an equal number of bosonic and fermionic degrees of freedom <u>without</u> <u>relying</u> <u>on</u> <u>the</u> <u>gauge invariance</u>. So far, we have 4 fermionic degrees of freedom (λ) but five bosonic (B_μ and D). We have to add at least one Weyl fermion - and this brings the number of fermion degrees of freedom to 8. By adding 3 more bosonic degrees of freedom we will reach an equality. This is indeed the content of the <u>vector</u> superfield. Its expansion is

$$
\begin{aligned}
V(x,\theta,\bar\theta) = {}& C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \tfrac{i}{2}\theta\theta M(x) - \tfrac{i}{2}\bar\theta\bar\theta M^*(x) \\
&+ \theta\sigma^\mu\bar\theta B_\mu(x) + i\theta\theta\bar\theta\left(\bar\lambda(x) - \tfrac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right) \\
&- i\bar\theta\bar\theta\theta\left(\lambda(x) - \tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right) + \tfrac{1}{2}\theta\theta\bar\theta\bar\theta\left(D(x) - \tfrac{1}{2}\Box C(x)\right) \qquad (9.1
\end{aligned}
$$

We have defined the expansion such that (B_μ, λ, D) can be identified with the physical and auxiliary ~~com~~ fields off the components formulation.

The action of a ~~SUSY~~ superspace SUSY transformation on $V(x\,\theta\,\bar\theta)$ is given by the Q-s of eq. (7.10). Under this transformation, the new fields (C, M, χ) mix with the old ones (B_μ, λ, D) and so they are crucial to obtain a representation of the linear transformatio (7.10).

We should now ask ourselves what is the role of the new fields. This will become clear when we consider the meaning of local gauge invariance in superspace. In ordinary space, local symmetry implies an invariance under transformations parametrized by a <u>function</u> $w(x)$.

In superspace, ordinary fuctions do not exist as independent object but only as components of superfields. Thus, in superspac one must have a superlocal symmetry which is parametrized by a superfield. ~~For reaso~~ We would like the transformation law to resemble that of ordinary space. For a chiral (matter) superfield, we would like to have

$$\Phi'(x,\theta,\bar\theta) = e^{-\Lambda(x,\theta,\bar\theta)}\,\Phi(x,\theta,\bar\theta) \qquad (9.1$$

In order to keep the number of components unchanged $\Phi'(x,\theta,\bar\theta)$ must also be chiral. Hence, the superparameter $\Lambda(x,\theta,\bar\theta)$ must itself be a chiral superfield. We now have to define the action of a superlocal transformation on the ~~gauge~~ vector multiplet. We will give here the answer for ~~and a U(1) symmetry~~ a $U(1)$ symmetry (The non-abelian case is technically more complicated but do not contain any new ingredients and so ~~we~~ we will restrict ourselves from now on to the $U(1)$ case).

We define

$$V' = V + \Lambda + \Lambda^* \qquad (9.1$$

Notice that the reality character of V is preserved. Let us write explicitly what (9.14) means in terms of components. ~~We have~~ denoting $\Lambda(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$ we have

$$C' = C + 2\,\mathrm{Re}\,\phi \qquad (9.15$$

$$\chi' = \chi - i\sqrt{2}\,\psi \qquad (9.15$$

$$M' = M - 2iF \qquad (9.15$$

$$B'_\mu = B_\mu + 2\partial_\mu \,\mathrm{Im}\,\phi \qquad (9.15$$

$$\lambda' = \lambda \qquad (9.15$$

$$D' = D \qquad (9.15$$

Consider first the fields ~~and~~ of the components formulation. We see that λ and D are invariant, as they should for an abelian symmetry. The transformation rule for B_μ means that we should identify local parameter $w(x)$ of ordinary space with $\mathrm{Im}\,\phi = \mathrm{Im}\,\Lambda|_{\theta=\bar\theta=0}$

Consider now the transformation of the new components. What is remarkable about (9.15 a – c) is that (C, M, χ) transform by a <u>shift</u> i.e. there are no derivatives in the transformation law. In particular we can set (C, M, χ) to zero by going to a special gauge – the <u>Wess – Zumino gauge</u>. To this end, we simply choose

$$\begin{aligned} \mathrm{Re}\,\Phi &= -\tfrac{1}{2} C \\ \psi &= \tfrac{i}{\sqrt{2}}\, \chi \\ F &= \tfrac{i}{2} M \end{aligned} \qquad (9.16)$$

and we leave $\mathrm{Im}\,\Phi$ unspecified. ~~[illegible]~~ Making a superlocal gauge transformation with the components of $\Lambda(x, \theta)$ given by (9.16) we get

$$C = \chi = M = 0 \qquad (9.17)$$

Thus, the new components can be eliminated <u>algebraically</u>. We can now understand the relation between the ordinary and supe space formulation. In going to superspace, we invent at the same time new local symmetries (parametrized by $\mathrm{Re}\,\Phi$, ψ and F) and new fields (C, M, ψ) transforming in a pecliar way (by a shift) under the new local symmetries. By going to the unitary gauge (9.17) we get rid of both the new symmetries and the new fields.

However – the unitary Wess-Zumino gauge will not be preserve by a SUSY transformation (7.10). To restore it, the SUSY transformation must be ~~restored~~ followed by a superlocal gauge transformation with a field dependent parameter. The SUSY transformation of the component formulation is the sum of the above two transformations.

To proceed, we would like to construct the analog of the field strength $G_{\mu\nu}$ in superspace. More precisely, we would like to find a superfield containing $G_{\mu\nu}$ as component. This

Turns out to be the <u>chiral</u> superfield

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \qquad (9.18)$$

(Eq. (9.18) is valid only for the U(1) case. For the definition of W_α in the non-abelian case see e.g. Wess & Bagger). W_α contains λ_α, $G_{\mu\nu}$ and D as components. It can be shown that the lagrangian (9.3) is equal to

$$\mathcal{L}_G = \frac{1}{4}\int d^4x\, d^2\theta\, W^\alpha W_\alpha + \text{h.c.} \qquad (9.19$$

The gauge invariant superspace lagrangian for the matter fields is

$$\mathcal{L}^S_K = \int d^4x\, d^2\theta\, d^2\bar{\theta}\ \ \Phi^* e^V \Phi \qquad (9.2($$

We conclude with a few comments regarding the differences between the superspace formulation and the components formulation upon <u>quantization</u>.

The first difference can be seen already at the level of the classical lagrangian. Comparing $\mathcal{L}^S_K$ (9.20) with $\mathcal{L}_K$ (9.5) we find that, while $\mathcal{L}_K$ is <u>manifestly</u> <u>renormalizable</u>, $\mathcal{L}^S_K$ is not: for instance, all terms in (9.5) appear in (9.20) multiplied by $e^{C(x)}$ (recall, ~~...~~ C(x) is the lowest component of V). Hence $\mathcal{L}^S_K$ contains non-polynomial interactions.

The second important difference appears upon quantization. In the components formulation this is done using the usual Faddeev-Popov procedure. But the resulting gauge fixing and ghost terms <u>break</u> <u>SUSY</u> <u>explicitly</u>. Of course, as long as the theory is anomaly free this is not a true breaking, and one can prove that <u>physical</u> <u>quantities</u> i.e. expectation values of gauge invariant operators between physical states are supersymmetric.

In superspace, we have to ~~gauge fix~~ fix a supergauge for the superlocal symmetry parametrized by the Λ superfield. This can be done in a manifestly SUSY way. Hence, the superspace quantum action (including gauge fixing and ghost terms) is manifestly SUSY. This, however, has a price. In gauge theories, only gauge invariant operators ~~have physical~~ are physical observables. In an ordinary gauge theory (and in the component formulation of SUSY gauge theories) one can construct gauge invariant operators using only physical field, e.g. $\phi^*\phi$, $\phi^* D_\mu \phi$ etc. In ~~ϕ~~ superspace these operators are not gauge invariant. For instance, the gauge invariant operator that correspond to $\phi^*\phi$ is $e^{c}\,\phi^*\phi$. The reader is invited to construct the ~~gauge~~ supergauge invariant version of other operators.

We see that the compensating component play a special role in constructing supergauge invariant quantities. A related property is that the superspace formulation has a bigger number of field equations. In addition to the field equantions for physical and auxiliary fields, there are field equations for compensating fields. One can show that these equations usually give rise to on-shell constraints on the operators of the theory, and it is necessary to verify that these constraints are physically sensible.

The last comment is related to the issue of regularization. For the superspace formulation to be truly useful, one has to calculate using a regularization which preserve SUSY. In the case of ~~scalar~~ the Wess-Zumino model this can be done using a SUSY version of the Pauli-Villars method. In non-abelian theories, however, one has to use dimensional regularization which does not preserve SUSY. A SUSY version of dimensional regularization (also known as dimensional reduction is not a consistent regularization. For instance, it fails to predict correctly the chiral anomaly. ~~Although~~ Although one can justify its use aposteriori in many case, but not in all of them. Thus, for instance, one cannot rely on dimensional reduction in order to prove that SUSY is not anomalous