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A New Supersymmetric Index

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We show that $\mathrm{Tr}(-1)^F F e^{-\beta H}$ is an index for $N=2$ supersymmetric theories in two dimensions, in the sense that it is independent of almost all deformations of the theory. This index is related to the geometry of the vacua (Berry's curvature) and satisfies an exact differential equation as a function of β. For integrable theories we can also compute the index thermodynamically, using the exact S-matrix. The equivalence of these two results implies a highly non-trivial equivalence of a set of coupled integral equations with these differential equations, among them Painleve III and the affine Toda equations.

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1. Introduction

There has been much progress in understanding supersymmetric quantum field theories in the last decade. Supersymmetry turns out to be a strong symmetry principle which allows one to get a firm grip on certain aspects of these theories. For example, Witten's index $\mathrm{Tr}(-1)^F e^{-\beta H}$ [1] is an effective tool in addressing questions of supersymmetry breaking. It is natural to ask if there are other 'index-like' objects which can be computed exactly and provide further insight into the structure of supersymmetric theories. The aim of this paper is to show in two-dimensional $N=2$ supersymmetric theories, there is such an object: $\mathrm{Tr}(-1)^F F e^{-\beta H}$. We call this an index because it is independent of almost all deformations of the action. It, however, *does depend* on a finite set of (relevant or marginal) perturbations in a way which can be computed exactly.

Supersymmetric theories in two dimensions are among the simplest quantum field theories. Two-dimensional conformal theories with $N=2$ supersymmetry can be used to construct string vacua and have thus been studied extensively recently. All $N=2$ theories in two dimensions, whether or not they are conformal, have a set of observables, the (supersymmetric) chiral fields, which form a ring under operator product. This is called the chiral ring [2] (for a review see [3]). This ring can be computed *exactly* using the techniques of topological field theories [4] (see also [5,6]) as all the $N=2$ theories have a topological counterpart (called the 'twisted version'). The study of chiral rings turn out to be a very powerful tool in unravelling the geometry of the vacua of the supersymmetric theory. In particular by joining the topological and anti-topological versions of $N=2$ theories, one can derive (integrable) differential equations (tt^* equations) to compute the Berry's curvature for the vacuum bundle of the supersymmetric theory as one perturbs the $N=2$ theory [3] (see also [7,8]). It was observed in [3] that the solutions of these equations resemble a kind of partition function for kinks of the theory. This, however, remained a somewhat mysterious connection to be explained. In this paper we will see that these computations are related to the new index $\mathrm{Tr}(-1)^F F e^{-\beta H}$ which encodes aspects of the spectrum and the interactions of the kinks. In particular, the tt^* equations provide an exact differential equation in β for the new index for any $N=2$ theory.

These somewhat formal derivations can be checked very explicitly in many special cases. In particular when the $N=2$ theory is integrable, the existence of infinitely-many conserved charges allows one to construct the S-matrix (more or less uniquely)[9]. In such cases, one can use the exact S-matrix to find integral equations for the non-perturbative partition function $\mathrm{Tr}\ e^{-\beta H}$. This powerful method is known as the thermodynamic Bethe ansatz (TBA)[10]. In particular, the TBA analysis for a large class of $N=2$ integrable theories in two dimensions was carried out in [11,12], confirming the conjectured S-matrices as in particular reproducing the correct central charges in the UV limit. One can extend the usual TBA analysis by allowing arbitrary chemical potentials, and in particular one can compute objects such as $\mathrm{Tr} e^{i\alpha F} e^{-\beta H}$. This allows us, as a special case, to compute $\mathrm{Tr}(-1)^F F e^{-\beta H}$ in these theories in terms of integral equations.

Thus for integrable theories we seem to have two inequivalent methods to compute the new index: one in terms of *differential equations* characterizing the geometry of the vacuum bundle, and the other in terms of *coupled integral equations* coming from TBA. It is a highly non-trivial check on all these ideas that the *solutions* to these equations are the same. We have checked this using numerical solutions to both systems of equations. Due to the non-linearity of our differential equation and complexity of the coupled integral equations, we have not been able to show directly (i.e., analytically) that these are the same. In fact, turning things around, physics has predicted a surprising equivalence between coupled integral equations and certain differential equations (such as radial affine toda equations), a result which is yet to be proven mathematically!

The organization of this paper is as follows:

In section 2 we introduce the new index, and discuss in what sense it is an index (i.e., we see that it is independent of D-term perturbations). The derivation of this result in this section is very simple but unfortunately requires a certain formal manipulation which is not always easy to rigorously justify. In section 3 we discuss the geometry of vacua and review the results of [3]. Here we show how to rephrase our new index as a computation in the geometry of vacua. In particular we show why our index depends only on F-terms, thus giving a more rigorous derivation of the results of section 2. Moreover, this allows us to effectively compute the new index in terms of solutions of certain non-linear differential equations.

In section 4 we discuss the infra-red expansion of the index. We show in particular that at least the leading term (the one-particle contribution) and the next leading term (the two-particle contribution) are universal. This means that they just depend on the mass and the central term of the supersymmetry algebra and the allowed soliton configurations. In section 5 we review briefly the results of [11] and discuss how the new index can be computed for integrable theories using the TBA. In section 6 we consider a number of examples including $N{=}2$ sine-Gordon and minimal $N{=}2$ theories perturbed by least and most relevant perturbations. We write down the differential equations and the integral equations which are presumably equivalent. We explicitly check this for some of the examples numerically. Moreover in this section we use the TBA to compute the more general object $\mathrm{Tr}(-1)^F F^l e^{-\beta H}$ and show that, for $l > 1$ it is *not* an index and it does depend on the choice of D-terms, as expected.

In section 7 we present our conclusions. Finally in appendix A the tt^* equations are rederived, in a quick but somewhat non-rigorous way in the same spirit as the arguments in section 2.

2. Tr $(-1)^F\ F\ e^{-\beta \mathrm{H}}$

In this section we discuss the existence of a new supersymmetric 'index' for $N{=}2$ supersymmetric quantum field theories in two dimensions. Our emphasis in this section is just on formulating what this index is; in the following sections we show how it may be computed.

Let us start with Witten's index $\mathrm{Tr}(-1)^{\mathrm{F}}\mathrm{e}^{-\beta\mathrm{H}}$ [1]. This index is defined for $N \geq 1$ supersymmetric theories in any dimension. It is an index because it is independent of finite perturbations of the theory, provided the space is compact and does not break supersymmetry (e.g., a d-dimensional torus). The idea is simply that there are two types of states in the Hilbert space: states which come in pairs $|s\rangle, Q|s\rangle$ where Q is the supersymmetry charge with $Q^2 = H$, and states which come isolated, i.e., the ones which are annihilated by Q and are ground states of the theory with $H = 0$. The pairs, which necessarily have $H \neq 0$, do not contribute to the Witten index as they have opposite $(-1)^F$. This follows from the fact that $\{(-1)^F, Q\} = 0$. Therefore this index simply counts the ground states of the theory weighted with ± 1, depending on the parity of $(-1)^F$. Any finite perturbation of the theory does not change this index: if massive states become ground states they must do so in pairs, so one adds a $+1$ and a -1 to the index. Similarly, the only way a ground state can become massive is for ground states with opposite $(-1)^F$ to pair up, so again the net contribution to the index is zero. This index has been a powerful object in probing questions of supersymmetry breaking in supersymmetric theories.

It is well known that the above argument does not apply for non-compact spaces. Consider a Hilbert space based on R^d. The above argument breaks down in this case because the eigenvalues of H are typically continuous. In particular it may not be true that the density of states for $|s\rangle$ and $Q|s\rangle$ are equal. The contribution to the index may be written as

$$(-1)^f \int dE(g_+(E) - g_-(E))e^{-\beta E}, \tag{2.1}$$

where $g_\pm(E)$ are the density of states distributions for $|s\rangle$ and $Q|s\rangle$ and these states contribute $\pm(-1)^f$ to the index. If $g_+ - g_-$ is nonzero for $E \neq 0$, the contribution of massive states to the index does not vanish and thus as we change the parameters in the theory the index changes. In particular it does depend on β. Examples of this phenomena have been found in some simple quantum-mechanical systems with $N{=}1$ supersymmetry [13] where it can be computed exactly using the Callias-Bott-Seeley index theorem [14].

Let us consider this situation for the $N{=}2$ supersymmetric theories in $d = 2$ where we take space to be the real line. The $N{=}2$ supersymmetry algebra on the real line can be written as

$$Q^{+2} = Q^{-2} = \overline{Q}^{+2} = \overline{Q}^{-2} = \{Q^+, \overline{Q}^-\} = \{Q^-, \overline{Q}^+\} = 0$$

$$[F, Q^\pm] = \pm Q^\pm \quad [F, \overline{Q}^\pm] = \mp \overline{Q}^\pm$$

$$[Q^{\pm}, H_{L,R}] = [\overline{Q}^{\pm}, H_{L,R}] = [F, H_{L,R}] = 0$$

$$\{Q^+, Q^-\} = H_L \qquad \{\overline{Q}^+, \overline{Q}^-\} = H_R$$

$$\{Q^+, \overline{Q}^+\} = \Delta \qquad \{Q^-, \overline{Q}^-\} = \overline{\Delta} \tag{2.2}$$

where $H_{L,R} = H \pm P$, and Δ is a c–number which is the central term of the supersymmetry algebra. The fermion number F is the charge corresponding to the global $O(2)$ symmetry of N=2 supersymmetric theories. We also have $(Q^+)^\dagger = Q^-$ and $(\overline{Q}^+)^\dagger = \overline{Q}^-$. Defining $Q_\pm = (1/\sqrt{2})(Q^\pm + \overline{Q}^\pm)$ we have

$$\{Q_-, Q_+\} = H \qquad [Q_\pm, H] = 0. \tag{2.3}$$

It is well known that on a non-compact space in general the central term Δ can be non-zero [15]. Δ depends on the boundary conditions at spatial infinity: with multiple vacua, we have the freedom of having different boundary conditions at left-right spatial infinity and thus different central terms. Let us denote the vacuum at left spatial infinity by a and the one on the right by b, so that the central term in the above algebra may be labeled by Δ_{ab}. Having multiple vacua allows kinks which interpolate from one vacuum at left spatial infinity to another one at right spatial infinity. The kinks we will denote by k_{ab}. In general, such kinks (even the lowest-energy configuration in each sector) may be stable or unstable. We can, however, derive a lower bound on the mass of any kink. In the ab sector the positivity of $\{A, A^\dagger\}$ where $A = (H_R\overline{\Delta})^{1/2}Q^+ - (H_L\Delta)^{1/2}\overline{Q}^-$ implies the Bogomolnyi bound $E^2 - P^2 \geq |\Delta_{ab}|^2$. A kink k_{ab} or, more generally, any state in the ab sector must therefore have mass $m \geq |\Delta_{ab}|$.

We now ask whether Witten's index is a good index for the N=2 case on open space. Consider varying the Hamiltonian of the theory, respecting N=2 supersymmetry. We wish to compute

$$\delta \mathrm{Tr}(-1)^F e^{-\beta \mathrm{H}} = -\beta \mathrm{Tr}(-1)^F \delta \mathrm{H} e^{-\beta H}$$

Using (2.3) we can write this as[1]

$$-\beta \mathrm{Tr}(-1)^F \delta\{Q_+, Q_-\} e^{-\beta \mathrm{H}}$$

$$= -\beta \mathrm{Tr}(-1)^F (\{\delta Q_+, Q_-\} + \{Q_+, \delta Q_-\}) e^{-\beta H}$$

Each of the above terms vanishes. To see this note that whenever we are computing

$$\mathrm{Tr}(-1)^F \{A, B\}\, O$$

where A and B are fermionic and where at least one of them commutes with O, we formally get zero. Suppose A commutes with O. Then for the AB term contributing to the above trace, we can take A around the trace, because the trace is cyclic. The term picks up a minus sign because A anticommutes with $(-1)^F$. This leaves $-BA$, which cancels $+BA$ from the other term in the anti-commutator. The same argument works if A and B are bosonic operators, and we replace anti-commutators with commutators. We shall refer to this as the AB argument. For this formal argument to be actually valid one needs to put restrictions on the nature of the operators A and B, which we assume to be satisfied in our case[2] [16]. That the AB argument is valid in our case is confirmed in the next section where we derive the results of this section without making use of this assumption.

Applying these general statements to the above variation of the index, where in one term Q_- and in the other term Q_+ plays the role of A in the AB argument, we find that the variation of Witten's index is zero for N=2 theories, and it is thus a good index even for non-compact space. This in particular means that in sectors where the left and right vacua are not the same $\mathrm{Tr}(-1)^F e^{-\beta H}$ vanishes: we are free to take β large because it is an index, and since the ground

[1] We have been somewhat cavalier with regard to boundary conditions at spatial infinity in taking the variations. This point is elaborated upon in the appendix.

[2] In the supersymmetric quantum-mechanical version of this statement, this can be explicitly checked to be true, where A (after being dressed by O) is a trace-class operator and B (after being dressed by O) is bounded.

state in this sector has non-zero energy (because it interpolates between two distinct vacua) we get zero.

For an N=1 supersymmetric theory in two dimensions the fermion number F is only defined mod 2. However, in a two-dimensional N=2 theory there is a $U(1)$ fermion-number charge, because the fermions are complex. Given the power of Witten's index in understanding the structure of supersymmetric quantum field theories, it is thus natural to ask what kinds of objects may be of interest when we have this additional charge. The most natural thing to consider would be

$$Z(\alpha,\beta) = \mathrm{Tr}\; e^{i\alpha F} e^{-\beta H}. \tag{2.4}$$

At $\alpha=\pi$ this is just Witten's index. For $\alpha=0$ it is just the standard partition function of the theory, so we expect $Z(0,\beta)$ to be the extreme opposite to an 'index', as it should depend on ***every little detail*** of the theory. So let us go back to the point $\alpha=\pi$ and just move ***slightly away***. In other words, consider

$$I_l(\beta) = \frac{\partial^l Z(\alpha,\beta)}{\partial (i\alpha)^l}\Big|_{\alpha=\pi} = \mathrm{Tr}(-1)^F F^l e^{-\beta H}.$$

Needless to say, we should not expect all I_l to be indices as that would enable us to reconstruct Z itself. But maybe some of them are! In particular, consider $I_1 = \mathrm{Tr}(-1)^F F e^{-\beta H}$. Among all I_l with $l \geq 1$, we will show that this and only this is a new 'index'.

To define what we mean by 'index', we must recall that there are two distinct ways to perturb an N=2 supersymmetric theory in two dimensions [17]: $D-$terms and $F-$terms. In general, the $D-$terms can be written as integrations of superfields over the full superspace $d^4\theta$ and the $F-$terms which are integration of chiral and anti-chiral fields over half the superspace $d^2\theta^+$ and $d^2\theta^-$ respectively. Chiral fields commute with Q^+ and $\overline{Q}^+$ and anti-chiral fields commute with Q^- and $\overline{Q}^-$. The Tr $(-1)^F$ F $e^{-\beta \mathrm{H}}$ is ***independent*** of the D-terms and in this sense it is an index. It however, ***does depend*** on the $F-$terms. In order to explain why we use the word 'index' when it does depend on F-terms it is convenient to consider the following interesting class of examples of N=2 supersymmetric QFT's. Consider 2d supersymmetric sigma models with

target space being a Kahler manifold M. This gives an N=2 supersymmetric theory [18]. Any variation of the metric of M respecting the complex structure of M and the Kahler class of the metric (i.e., leaving the integral of the Kahler form on the two-cycles unchanged) can be written as a $D-$term and so does not affect our index. In fact this includes essentially ***all*** possible perturbations of the manifold, modulo variations of complex structure and Kahler structure which usually form a finite dimensional space of perturbations. So it is with this kind of example in mind that we call the above object an 'index'. Another interesting class of N=2 supersymmetric theories is provided by Landau-Ginzburg theories. In these cases the superpotential W is the F-term and it has only a finite number of perturbations which do not change the behavior of potential at infinity in field space. These turn out to be the relevant (and marginal) perturbations. The index depends only on W.

Here we show that $\mathrm{Tr}F(-1)^F e^{-\beta H}$ does not depend on the $D-$terms. The variation of the $D-$term can be written as inserting $\{Q^+,[\overline{Q}^-,\Lambda(x)]\}$ in the path integral where Λ itself can be written as $\{Q^-,[\overline{Q}^+,K]\}$. This follows from the fact that the $D-$term comes from integration over all four Grassman coordinates. The path-integral is over an infinite cylinder of perimeter β with the above term inserted at all points x and integrated over the cylinder. Let us denote by Λ the integral of $\Lambda(x)$ over space. Since F commutes with both Λ and the Hamiltonian we find that the integration of Λ over the perimeter simply introduces an irrelevant factor of β which can be ignored. So we can write the variation of our index as (proportional to)

$$\delta = \mathrm{Tr}(-1)^F\; F\{Q^+,[\overline{Q}^-,\Lambda]\}e^{-\beta \mathrm{H}}$$

We are almost ready to apply the AB argument, using Q^+ as our A. This works fine, except for the fact that as we try to take Q^+ around the trace, since it does not commute with the F, we pick up a commutator term

$$\delta = \mathrm{Tr}(-1)^F[F,Q^+][\overline{Q}^-,\Lambda]e^{-\beta \mathrm{H}} = \mathrm{Tr}(-1)^F Q^+[\overline{Q}^-,\Lambda]e^{-\beta \mathrm{H}}$$

Now we can apply an argument similar to the AB argument, by taking $\overline{Q}^-$ in the term $\overline{Q}^-\Lambda$ around the trace. Here we pick up two minus signs, and so we get back $\Lambda\overline{Q}^{-}$ which thus cancels the second term in the commutator and we get zero, as was to be shown.

The AB argument does not allow us to show that any of the other I_l are independent of D-term perturbations[3]. In fact, in a free massive $N = 2$ theory, it is easy to compute I_l; the I_{2l} are non-vanishing and for $l \neq 0$ they do depend on the mass of the particle, which in turn depends on the D-term. In section 6 we consider other examples for which all I_l are non-vanishing and all of them for $l > 1$ depend on the D-term. Therefore I_1 is the only additional index that exists other than Witten's index. From now on we will refer to the new index simply as I, dropping the subscript 1.

Our index is actually a matrix, because we have to fix the boundary condition at spatial infinities to be vacua of the theory. If we choose the left vacuum to be a and the right one to be b, we have the index I as a matrix

$$I_{ab} = \mathrm{Tr}_{ab}(-1)^F F\ e^{-\beta H}.$$

One has to be careful about what we mean by $(-1)^F$. In general all that is required from this operator is that it anti-commute with fermionic fields. In our case, as we have mentioned before, since F is in fact well defined as an operator, one can just define $(-1)^F = e^{i\pi F}$. Note, however this operator no longer squares to one. The reason for this is that in the (ab) sector the vacuum will in general have a non-integral fermion number f_{ab}. This phenomenon is well known [19,20,21]. Only the fermion number relative to that of the vacuum is integral. Using this fact and the hermiticity of H and F we can write

$$I_{ab} = \pm e^{i\pi f_{ab}}|I_{ab}| \tag{2.5}$$

CPT invariance puts constraints on our index. CPT takes a state in the (ab) sector to one in (ba) sector, and it takes fermion number F to $-F$. In particular, CPT invariance requires $f_{ab} = -f_{ba}$, and therefore

$$I_{ab} = -I^*_{ba} \tag{2.6}$$

[3] One can show using the AB argument that the D-terms will not affect the one-particle contribution of Bogomolnyi-saturated states to I_1, in accordance with the fact that the mass and fermion number of these states are independent of D-terms.

There is no fractional fermion number in (aa) sector: the fermion number is additive, i.e., $f_{ac} = f_{ab} + f_{bc}$, implying that $f_{aa} = f_{ab} + f_{ba} = 0$. We see from (2.5) that I_{aa} is real, and from (2.6) it follows that

$$I_{aa} = 0. \tag{2.7}$$

Note that if we had defined $(-1)^F = e^{(2n+1)i\pi F}$ then the index would have changed by a phase $I_{ab} \to e^{2in\pi f_{ab}} I_{ab}$. However, under this ambiguity, the *eigenvalues* of I are unambiguous. Because of the additivity of the fractional part of the fermion number, and since $f_{ab} = -f_{ba}$, we can write $f_{ab} = f_a - f_b$; a change of basis $b \to e^{2in\pi f_b} b$ gets rid of the phase without changing the eigenvalues.

In fact, we can do better; we can get rid of all the phases (modulo $\pm$) of our matrix by changing the basis $b \to e^{i\pi f_b} b$. In this way we find that I is a purely real matrix, and the condition (2.6) implies that it is anti-symmetric. Thus its eigenvalues are either zero, or (purely imaginary) complex-conjugate pairs. To make the eigenvalues real, we define the Q-index to be

$$Q_{ab} = \frac{i\beta}{L}\mathrm{Tr}_{ab}(-1)^F F\ e^{-\beta H}, \tag{2.8}$$

where L is the volume of the space. With this definition, Q is a hermitian matrix with real eigenvalues, such that non-zero eigenvalues come in pairs of opposite sign. To see the reason we divided by L in the definition of Q, consider $Z(\alpha, \beta)$ (from (2.4)) with boundary conditions at infinity corresponding to a normalized eigenstate of Q_{ab}. Because it is an extensive thermodynamic quantity, $\ln Z(\alpha,\beta) = \ln Tr(e^{i\alpha F} e^{-\beta H})$ is proportional to L as $L \to \infty$. Therefore

$$\frac{\beta}{L}\partial_\alpha \log \mathrm{Tr}(e^{i\alpha F} e^{-\beta H})\big|_{\alpha=\pi} = \frac{\frac{i\beta}{L}\mathrm{Tr} F(-1)^F e^{-\beta H}}{\mathrm{Tr}(-1)^F e^{-\beta H}} = Q, \tag{2.9}$$

where the denominator is not proportional to L, because it is Witten's index, and can be chosen to be 1 in an orthonormal basis of eigenstates of Q. Thus we see that Q as defined above is well-defined as $L \to \infty$.

Usually the contribution of n kinks (particles) to a partition function is proportional to L^n. One may incorrectly conclude from this that only the one-kink states contribute to Q. In fact we will see in later sections that the n-kink contributions to I do not generally vanish and are proportional to L. The contribution comes from regions where all the kinks are near each other and the factor of L is associated with the center of mass. It can be seen that any configuration where one of the kinks is very far from the rest does not contribute to Q: the contribution factorizes and at least one piece will simply be the contribution to Witten's index from massive kinks, which vanishes. One can also see this from the path-integral computation where the exact fermion zero modes associated to each kink when they are far away cannot be absorbed by one F.

In the next section we will see that Q is the same as the matrix element of the chiral fermion number:

$$Q_{ab} = \langle a|Q^5|b\rangle.$$

Using this expression along with the hermiticity of Q^5, and noting that CPT changes the sign of Q^5, we again see that the eigenvalues of Q are real and symmetrically located relative to zero.

For the remainder of this section, we will discuss the kind of states in the Hilbert space which contribute to our index. In general, there are three types of irreducible representations of the supersymmetry algebra (2.2). The generic irreducible representation of (2.2) is four-dimensional, with a definite eigenvalue for E and P (as $H_{L,R}$ commute with everything). This follows from the fact that the four supersymmetry charges which generate the algebra are pairwise adjoint of one another and have c-number anti-commutators. We can generate the representation by taking Q^+ and $\overline{Q}^-$ as 'creation' operators acting on a state which is annihilated by the 'annihilation' operators Q^- and $\overline{Q}^+$:

$$|s\rangle \quad Q^+|s\rangle \quad \overline{Q}^-|s\rangle \quad Q^+\overline{Q}^-|s\rangle \tag{2.10}$$

When $E^2 - P^2 = \Delta\overline{\Delta}$, i.e., if the state saturates the Bogomolnyi bound, then it is well known [15] that this representation is reducible: $A = (H_R\overline{\Delta})^{1/2}Q^+ - (H_L\Delta)^{1/2})\overline{Q}^-$ and its adjoint anticommute, and so both must annihilate $|s\rangle$. This leaves us with the reduced supersymmetry multiplet

$$|s\rangle \qquad Q^+|s\rangle \tag{2.11}$$

Finally, for $E = P = \Delta = 0$ this representation is further reduced to the trivial representation. This representation only appears for the (aa) sectors, and are the only states which contribute to Witten's index in this sector. However, because $I_{aa} = 0$ these states are not relevant for the new index I.

At first glance, one might think that only the reduced multiplets contribute to our index. If $|s\rangle$ has fermion number f, a non-reduced multiplet (2.10) naively contributes (up to an overall phase) $(f - 2(f+1) + (f+2))e^{-\beta E} = 0$, whereas a reduced multiplet (2.11) contributes (up to an overall phase) $(f - (f+1))e^{-\beta E} = -e^{-\beta E}$. Thus it appears that I receives contributions only from Bogomolnyi-saturated states, which are simply the one-soliton subsectors. This argument is incorrect, for the same reason that the naive argument which states that $\mathrm{Tr}(-1)^F e^{-\beta H}$ is independent of β is not in general valid when H has a continuous spectrum, as is the case in non-compact spaces. Formally, we have deduced the vanishing of the contribution of the non-reduced multiplets only when the spectrum of the Hamiltonian is discrete. When it is continuous, as with a model on a real line, we have to deal with the *density* of states of the non-reduced multiplets; they are not necessarily equal and do not necessarily cancel in computing I. We may wish to regularize the theory by putting it in a box of size L and then take $L \to \infty$. In order to recover the soliton sector ab, the field configurations on the left and the right of the box (in this case just a line interval) cannot be the same. Thus we *cannot* impose periodic boundary conditions. We must compute the object in finite but not periodic box, and this breaks the supersymmetry. The spectrum is discrete in this case, but without supersymmetry the naive argument no longer holds. Thus for a finite box we may get contribution from non-reduced supersymmetry multiplets to the index Q in the sector ab with $a \neq b$. This may persist even when the size $L \to \infty$ [4].

[4] This is indeed one way that the β-dependence of $\mathrm{Tr}(-1)^F e^{-\beta H}$ has been computed in the supersymmetric quantum mechanics examples [22], which is related to the Callias-Bott-Seeley index.

Thus we are computing a kind of 'anomaly', which remains after the regulator is removed.

Let $g_f(E)$ be the density of states for $|s\rangle$, $2g_{f+1}(E)$ be the density of states for states spanned by $Q^+|s\rangle$ and $\overline{Q}^-|s\rangle$, and $g_{f+2}(E)$ be the density of states for states $Q^+\overline{Q}^-|s\rangle$. We should thus not expect the continuum densities $g_f(E)$, $g_{f+1}(E)$, and $g_{f+2}(E)$ to be equal in the (a,b) sector of the theory with $a \neq b$. Recall, though, that we proved using N=2 supersymmetry that the contribution to Witten's index from these states must cancel. This means that we must have

$$g_f(E) + g_{f+2}(E) = 2g_{f+1}(E), \tag{2.12}$$

since the states on the two sides make opposite contributions to Witten's index. Using (2.12) we see that the contribution of the four dimensional representation to the index I in the (a,b) sector is of the form

$$e^{i\pi f}\int dE(g_{f+2}(E) - g_f(E))e^{-\beta E}. \tag{2.13}$$

We will see explicitly how this is generically nonzero in the following sections.

3. Geometry of Ground States and the New Supersymmetric Index

In this section we review some aspects of the work done in [3] which are useful for the considerations of this paper. In particular, we show why the 'Q'-matrix discussed there is in fact the new supersymmetric index given by $\mathrm{Tr}(-1)^F Fe^{-\beta H}$ discussed in the previous section. It is convenient to exchange the role of space and time (i.e., do a 'modular transformation') and take the space to be a circle (with perimeter β to correspond to the index computation), with periodic boundary conditions. Time is now a line of length L.

Consider an arbitrary N=2 supersymmetric quantum field theory in two dimensions. From (2.3) and the positivity of the inner product together with the fact that $Q_- = Q_+^\dagger$, it is easy to show that the ground states of the theory are characterized by

$$H|a\rangle = 0 \leftrightarrow Q_\pm|a\rangle = 0.$$

There is thus a one-to-one correspondence between the ground states of the theory and the Q_+ or Q_- cohomology. This cohomology is definable because each of these operators squares to zero (note that on a compact space (circle) the supersymmetry algebra has no central term and we get $Q_+^2 = Q_-^2 = 0$). The analogy to keep in mind is that Q_+ is like a d operator acting on the differential forms on a manifold, Q_- is like the adjoint operator $d^\dagger$ and the ground states $|a\rangle$ are like the harmonic representative of d or $d^\dagger$ cohomology.

In correspondence with the ground states in the Hilbert space, there are chiral operators ϕ_i in the theory defined by the condition that

$$[Q_+, \phi_i] = 0$$

and similarly there are anti-chiral operators $\overline{\phi}_i$ which commute with Q_-. Acting on a vacuum by a chiral operator, we get another state which is Q_+ closed, another Q_+ cohomology element. In this way the chiral fields, modulo the fields that are trivially chiral, i.e. modulo fields which are themselves Q_+ (anti-)commutator, are in one-to-one correspondence with the Q_+ cohomology elements and thus the ground states. If we pick a canonical ground state (to be defined below) denoted by $|0\rangle$, this can be stated as

$$\phi_i|0\rangle = |i\rangle + Q_+|\Lambda\rangle$$

where $|i\rangle$ denotes another ground state. Similarly we can label the ground states using the anti-chiral fields $\overline{\phi}_i$ which leads to the states $|\bar{i}\rangle$. The chiral fields form a ring among themselves, called the chiral ring, which is defined by

$$\phi_i\phi_j = C_{ij}^k\phi_k + [Q^+, \Lambda]$$

$$\phi_i|j\rangle = C_{ij}^k|k\rangle.$$

The matrix $(C_i)_j^k = C_{ij}^k$ denotes the action of the chiral field ϕ_i on the ground states (once we ignore the components orthogonal to ground states). Similar statements apply to anti-chiral fields with C_{ij}^k replaced by the complex conjugate quantity $(C_{ij}^k)^*$.

We can define a symmetric metric η and a hermitian metric g among the ground states by

$$\eta_{ij} = \langle i|j\rangle \qquad g_{i\bar{j}} = \langle \bar{j}|i\rangle$$

Note that the metric g is the usual metric in the Hilbert space of the N=2 theory and η, which is not hermitian, is a kind of 'topological' metric. As discussed in the previous section there are two ways to perturb the action: the 'D-terms' (denoted by $K(X,\overline{X})$ below) and the 'F-terms' which are the chiral fields (now viewed as superfields) and integrated over half of the superspace:

$$S \to S + \int d^2z d^4\theta K(X,\overline{X}) + \int d^2z d^2\theta^+ \; t_i\phi_i + \int d^2z d^2\theta^- \; \bar{t}_i\overline{\phi}_i.$$

Then it is possible to show (see [3]) that the chiral ring and the metrics η and g *depend only on the* F-*terms*, i.e. they depend only on $t_i, \bar{t}_i$ and are independent of K. The flavor of the argument is very similar to the argument in the previous section in showing that our index is independent of D–terms, but it has the advantage of being rigorous.

The ring matrices C_i and the metric η can also be related to computations of correlation functions in a topological theory [4] corresponding to 'twisting' the N=2 quantum field theory and can thus be easily computed exactly [4,5,23,6]. Basically the topological theory is the same as the ordinary N=2 theory on flat manifolds but differs from it when the two-dimensional manifold is not flat, in such a way that the charge Q_+ transforms as a scalar, and is thus a symmetry even if the space is not flat. The way this is accomplished is by introducing a background gauge field set equal to half the spin connection of the manifold, and coupling it to the fermion number current. Thus a field which previously had spin s and fermion charge q will now have spin $s - \frac{1}{2}q$. This in particular makes Q_+ which had spin 1/2 and fermion number +1, a scalar. If S denotes the action of ordinary N=2 theory, S_t denotes the action for the topological theory, j denotes the fermion number current, and ω_μ denotes the $U(1)$ spin connection we have

$$S_t = S + \frac{i}{2}\int j_\mu \omega^\mu. \tag{3.1}$$

An important property of the topological action is that the energy-momentum of the topological theory is itself Q_+ trivial:

$$T^t_{\mu\nu} = T_{\mu\nu} + \frac{1}{2}\epsilon_{\alpha(\mu}\partial_{\nu)}j^\alpha = \{Q_+, \Lambda\} \tag{3.2}$$

implying that the correlation functions for chiral fields are independent of the metric. By translating the computation of the N=2 topological theory into the language of the ordinary N=2 theory, this provides exactly the quantities η and the ring matrices C. The basic observation is that if we consider a hemisphere and do the path-integral in the topological theory we get a state (on the boundary circle) which is annihilated by the symmetry charge Q_+. Moreover because the energy momentum tensor is Q_+-trivial, any local variation of the data (such as the variation of the metric on the hemisphere) does not change the Q_+ cohomology class of the state, and so the path integral of the topological theory leads to a well-defined state in the Q_+-cohomology, and thus to a ground state of the ordinary N=2 theory. In particular the state that we called the vacuum $|0\rangle$ corresponds to the state we get when we do the path-integral with no insertion of any fields on the hemisphere. Simple arguments show that C and η depend only on t_i and not on $\bar{t}_i$. In other words, they are holomorphic.

Similarly, we can consider the *anti-topological* theory, which is obtained when we make Q_- a scalar. This is done simply by changing the sign of the background field, which shifts the spins by $s \to s + \frac{1}{2}q$. So the action for the anti-topological theory S_{t*} is

$$S_{t*} = S - \frac{i}{2}\int j_\mu \omega^\mu$$

From the anti-topological theory we can easily compute $\eta_{\bar{i}\bar{j}}$ and $\overline{C}$ which are simply the complex conjugate of the corresponding topological quantities η and C.

The computation of the hermitian ground-state metric g as a function of perturbation parameters $(t_i, \bar{t}_i)$ is more difficult. It turns out that by fusing the topological theory on one hemisphere with the anti-topological theory on the other hemisphere, we can find equations which characterize it [3]. This we shall call topological-anti-topological fusion, or tt^* for brevity. One simply introduces a gauge connection such that the variation of ground states are orthogonal to the ground states themselves:

$$D_i|a\rangle = \partial_i - A_i|a\rangle \qquad \overline{D}_i|a\rangle = \overline{\partial}_i - \overline{A}_i|a\rangle.$$

This in particular means that the metric g is covariantly constant

$$D_i g = \overline{D}_i g = 0,$$

and one finds the equations

$$[D_i, D_j] = [\overline{D}_i, \overline{D}_j] = 0$$

$$[D_i, \overline{D}_j] = -\beta^2 [C_i, \overline{C}_j] \tag{3.3}$$

(and some other equations which we will not need here). The perimeter of the space (circle) is β. We will give a quick (but not rigorous) derivation of the above equations in the spirit of the AB argument of previous section in the appendix.

The first equation (3.3) shows that we can choose a holomorphic gauge with $\overline{A}_i = 0$. This turns out to be the natural gauge in the topological theory. In more mathematical terminology we can say that the topological path-integral automatically gives holomorphic sections of the vacuum bundle. Using the covariant constancy of the metric we can write the metric g as

$$A_i = -g\partial_i g^{-1}$$

and so the second equation in (3.3) becomes

$$\overline{\partial}_j (g\partial_i g^{-1}) = \beta^2 [C_i, gC_j^\dagger g^{-1}]$$

In many examples these equations turn out to be among the celebrated equations of mathematical physics. For the N=2 sine-Gordon theory the above equation as a function of the scale turns out to correspond to radial solutions of the sinh-Gordon differential equation, which is a special case of Painleve III. These differential equations are always integrable, being related to a tau function. The integrability of these equations has been recently elaborated upon in [24]. Explicit numerical computations have been done for flows among conformal theories and also flows under generic perturbations away from conformal theories [25].

Among the perturbations of the N=2 theory, there is a special one corresponding to renormalization group flow. In particular if we denote the perimeter of the circle on which we base our Hilbert space as e^τ, then changing τ should be equivalent to changing the coupling in the theory in some particular way. In the case of Landau-Ginzburg theories, this has the same effect on the F-terms as multiplying it by e^τ. From the definition of connection it follows that

$$\partial_\tau |a\rangle = A_\tau |a\rangle$$

On the other hand it was shown in [3] that the variation of the ground states with respect to the perimeter is related to the action of the chiral fermion number charge Q^5 on the ground states by

$$\partial_\tau |a\rangle = \frac{1}{2}(Q^5 + n)|a\rangle \tag{3.4}$$

where the above equality holds as long as we project both sides back to the ground states. Here n is a number which measures the chiral anomaly of the theory (equal to the number of chiral fields in the LG theory). So we see that as far as the ground state action is concerned, in a holomorphic (topological) basis[5]

$$\frac{1}{2}(Q^5 + n)|i\rangle = A_{\tau\ i}^{\ j} |j\rangle = (-g\partial_\tau g^{-1})_i^{\ j} |j\rangle \tag{3.5}$$

The equation (3.4) was derived in [3] in the context of Landau-Ginzburg theories. Since this is an important equation for us in this paper, we will now present a more general derivation of it.

[5] More precisely, in a topological basis $|i\rangle$ obtained by inserting in the topological integral chiral operators ϕ_i with $\partial_\tau \phi_i = 0$. Two such bases are related by a τ independent 'gauge transformation'. Under such changes of bases $g\partial_\tau g^{-1}$ transforms as a tensor.

It is convenient to work in the topological basis. Then a state $|i\rangle$ can be obtained as a result of topological path-integral on hemisphere, with insertion of the chiral field ϕ_i. In view of the fact that the energy momentum tensor of the topological theory is Q_+ trivial, it sounds contradictory to expect $\partial_\tau|i\rangle$ not to be zero (i.e., Q_+ trivial). The way this comes about is by a subtle boundary term, as we will now see.

Let us denote the metric on the hemisphere by $h = e^{2\phi}dzd\bar{z}$. In terms of ϕ the spin connection is $\omega = *d\phi$ and so the topological action (3.1) is

$$S_t = S + \frac{i}{2}\int j \wedge d\phi.$$

Now we are interested in the variation of this action on the right hemisphere as we change ϕ by a constant. Varying the metric by an overall scale $\phi \to \phi + \epsilon$ has the effect of changing the perimeter by shifting $\tau \to \tau + \epsilon$. It is convenient to first do a partial integration on the second term above and write it ass

$$\int_{S_R} j \wedge d\phi = \oint_{S^1} j\phi - \int_{S_R} \phi dj$$

where S_R denotes the right hemisphere and S^1, the boundary circle of S_R, is where we base our Hilbert space. Shifting ϕ brings down from S the trace of the energy momentum tensor, and from the topological addition the divergence of the axial current plus the variation of the boundary term, i.e.,

$$\delta S = \int_{S_R}(T^\mu_\mu + \frac{1}{2}D_\mu j^{5\mu}) + \frac{i}{2}\oint_{S^1} j$$

where we have used that in *two*-dimensional Euclidean field theory $j^5_\mu = i\epsilon_{\mu\nu}j^\nu$. The term integrated over the right hemisphere appears to be Q_+ trivial because it is the trace of the energy momentum tensor of the topological theory. This statement is almost true, except for the fact that there is a well-known anomaly in the divergence of axial current which contributes $n/2$ (in the LG theory n is the number of fields). But now we see that the boundary term is also present, and is equivalent to the action of $Q^5/2$ at the boundary (as follows from $j^5 = i(*j)$). So the net effect on a state of the change of τ is given by

$$\partial_\tau|a\rangle = \frac{1}{2}(Q^5 + n)|a\rangle$$

(as long as we compute the matrix element of both sides of the above equation among ground states). This is the equation (3.4) we wished to derive.

We have seen that the matrix elements of Q^5 among ground states of the supersymmetric theory are possible to compute, if we know g (from (3.5)). Note that even though the fermion number is always conserved the chiral fermion number is conserved only at conformal points.

Since we are considering both massive and massless theories it may seem strange to see that the matrix elements of a non-conserved charge are somehow 'interesting' and related to RG-variations of ground states. Let us rephrase this by using a modular transformation. Consider the theory on a very long cylinder of length L and circumference β. Let us put a ground state $|b\rangle$ at one end of the cylinder and another ground state $\langle a|$ at the other end. We denote the coordinates along the cylinder by x and that along the circumference by t. The matrix elements of Q^5 can then be written as

$$\langle a|Q^5|b\rangle = \langle a|i\oint_{S^1} j_t(0,t)dt|b\rangle$$

where S^1 is a circle wrapped around the middle of the long cylinder. We have to take the limit $L \to \infty$ at the end in order to project onto the ground states in a natural way. It clearly does not matter where we insert the circle. So let us put the circle at any x, integrate over all x and divide by L, i.e.,

$$\langle a|Q^5|b\rangle = \frac{i}{L}\langle a|\int j_t(x,t)dxdt|b\rangle \tag{3.6}$$

Now viewing x as space, and t as time, we see that $\int j_t(x,t)dx$ is the definition of the fermion number F on the Hilbert space which is along the cylinder. Since fermion number is conserved, integrating along t will just introduce an additional factor of the circumference of the cylinder β. In other words we have

$$\int j_t(x,t)dxdt = \beta F$$

So we have finally

$$Q_{ab} = \langle a|Q^5|b\rangle = \frac{i\beta}{L}\mathrm{Tr}_{ab}(-1)^F F\ e^{-\beta H} \tag{3.7}$$

where the Tr_{ab} means that we are taking the boundary conditions on the left and right to correspond to $\langle a|$ and $|b\rangle$ vacua. This is the new index discussed in the previous section! What is surprising is that the index can be computed exactly in terms of g, and g is determined exactly by the differential equations (3.3). In particular we see from (3.5) that the index is given by

$$Q_{ab} = \frac{i\beta}{L}\ \mathrm{Tr}_{ab}(-1)^F F\ e^{-\beta H} = -(\beta g\partial_\beta g^{-1} + n)_{ab}, \tag{3.8}$$

where we have used $2g\partial_\tau g^{-1} = \beta g\partial_\beta g^{-1}$ which follows because, by scaling, we can set $\beta = e^{\tau/2+\tau^*/2}$. Often it is difficult to compare the topological basis for ground states with the path-integral choice emphasized in the previous section and more natural from the viewpoint of kinks. In such cases it is convenient to compare the *eigenvalues* of the Q matrix on both sides of the above equation.

Note that the matrix Q, since it can be written solely in terms of g, depends only on the knowledge of the $F-$term and is independent of the D-terms in accord with our proof in the previous section. Our final formula, (3.8), expresses the new supersymmetric index in terms of the geometry of supersymmetric ground states. Because the curvature of this space is determined simply from the chiral ring structure constants using (3.3), the index will be an exact solution of a differential equation whose form is determined simply by the chiral ring. In other words, though our index is not purely topological, its flow in β is determined using only topological data, namely the chiral ring.

At the conformal point, where chiral fermion number is conserved, Q measures the chiral charge of Ramond vacua, i.e., the left-moving fermion number plus the right moving fermion number. In this case the state with highest charge has $Q = \hat{c}$ where $\hat{c}$ is the central charge of the N=2 superconformal theory [2,26]. So off criticality each eigenvalue of the Q matrix, and in particular the highest one, is a kind of a generalization of a c-function [7](which has no direct relationship with Zamolodchikov's definition [27], as discussed in [3]).

4. The infra-red expansion of $\mathrm{Tr}F(-1)^F e^{-\beta H}$

In section 2 we discussed which states in the Hilbert space contribute to $\mathrm{Tr}F(-1)^F e^{-\beta H}$. In this section we show how to calculate the one- and two-particle contributions. These are the leading terms in infra-red limit where $\beta >> 1$. We will see the simple but non-trivial nature of our index. These results must be the leading infra red behavior of the tt^* differential equations of the previous section.

Let us start with the contribution of one-particle (kink) states to the index. In order to calculate the density of states, we put the system in a box of length L with the a and b boundary conditions at the end of the box To obtain a non-vanishing contribution to the index, recalling (2.7), we take $a \neq b$; in particular, we do not want periodic boundary conditions. The allowed momenta of a particle in a box are quantized as $p = n\pi/L$, where n is a positive integer. Thus the density of states for each component of a supersymmetry multiplet is the same and given by $g(E)dE = Ldp/\pi$. From relation (2.13), we see that one-particle states in four-dimensional multiplets do not contribute to the index; a single particle contributes if and only if it is part of a reduced supersymmetry multiplet. This in particular means that its mass should saturate the Bogomolnyi bound $m_{ab} = |\Delta_{ab}|$. So the one-particle contribution to $Q_{ab} = \frac{i\beta}{L}\mathrm{Tr}_{ab}(-1)^F F e^{-\beta H}$ from a kink multiplet with fermion number $(f_{ab}, f_{ab}+1)$ is given by

$$\begin{aligned} &i\beta(f_{ab} - (f_{ab}+1))e^{i\pi f_{ab}} \int_0^\infty \frac{dp}{\pi} e^{-\beta\sqrt{p^2+m_{ab}^2}} \\ &= -i|\Delta_{ab}|\beta e^{i\pi f_{ab}} \frac{1}{\pi} K_1(|\Delta_{ab}|\beta), \end{aligned} \tag{4.1}$$

where K_1 is a Bessel function. This simple statement explains and makes precise the observation made in [3] that in the infra-red the Q-matrix is a kind of partition function of the solitons of the theory. The fact that the leading term in the infra-red limit is proportional to K_1 follows easily from the tt^* equations (see appendix B of [3]).

The next-leading contribution in the infrared to the index comes from the two-particle states. A two-particle state generally forms one or more four-dimensional non-reduced supersymmetry multiplets. This is true even if both particles are individually reduced-multiplet, unless $m_1 + m_2 = |\Delta_1 + \Delta_2|$. Thus the two-particle state generally *does not* saturate the Bogomolnyi bound. This is the first case where we can check whether we get contributions of the form (2.13) from four-dimensional representations. We will see that the two-particle contribution is very simple and general, and often not zero, for the case where both particles are part of reduced multiplets.

Computing the two-particle contribution is easy if one knows the two-particle S-matrix; the S-matrix encodes the density of states [28]. In a large box, the particles spend a negligible amount of phase space near each other, so the exact details of the interaction are unnecessary. The S-matrix allows one to match the free-particle solution of the equation with $x_1 >> x_2$ with the one for $x_1 << x_2$.

Consider a two-particle state $|i(p_1, p_2)\rangle$ which scatters entirely into another state $|j(p_1', p_2')\rangle$ with S-matrix element $S_{ij}(p_1, p_2)$. [6] Relativistic invariance ensures that S_{ij} actually depends only on $s \equiv (E_1 + E_2)^2 - (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_1 m_2 \cosh(\theta_1 - \theta_2)$, where we define rapidities via $p = m \sinh \theta$. Generically, $|i\rangle \neq |j\rangle$ for solitons even in elastic forward scattering, where the individual Δ change. Consider a two-particle wavefunction connecting vacua a and b at the box ends. Properly-matched plane-wave states satisfy

$$\psi(x_1, x_2) = \begin{cases} e^{ip_1 x_1 + ip_2 x_2} & x_1 < x_2, \\ e^{ip_1' x_1 + ip_2' x_2} S_{ij}(\theta_2 - \theta_1) & x_1 > x_2. \end{cases} \tag{4.2}$$

The momenta can change in the collision; the relations $p_1 + p_2 = p_1' + p_2'$ and $E_1 + E_2 = E_1' + E_2'$ give us the final momenta in terms of the initial. Thus we can write $p_1' = p_1'(p_1, p_2)$.

Since our system is in a box of length L, an allowed wavefunction must vanish at the walls. This quantizes the momenta just as in the free case, but

[6] We neglect processes which take two particles to more than two (which should be a good assumption in the infrared limit).

here the two quantization relations are coupled. Requiring the wavefunction vanish at $x_1 = 0$ means making a standing wave by subtracting the solution with opposite p_1. When making it vanish at $x_1 = L$, we use the second relation in (4.2), and it follows that[7]

$$e^{ip_1'(p_1,p_2)L} S_{ij}(\theta_2 - \theta_1) = e^{ip_1'(-p_1,p_2)L} S_{ij}(\theta_2 + \theta_1), \tag{4.3}$$

where we note that $p_1'(-p_1, p_2)$ is not necessarily equal to $-p_1'(p_1, p_2)$. Requiring the vanishing at $x_2 = 0$ and $x_2 = L$ gives another equation:

$$e^{-ip_2'(p_1,p_2)L} S_{ij}(\theta_2 - \theta_1) = e^{-ip_2'(p_1,-p_2)L} S_{ij}(-\theta_2 - \theta_1). \tag{4.4}$$

Taking the log of (4.3) gives

$$2n\pi = k_1 L + Im \ln \frac{S_{ij}(\theta_2 - \theta_1)}{S_{ij}(\theta_2 + \theta_1)}, \tag{4.5}$$

where n is an integer, and we define the kinematic factors

$$k_1 = p_1'(p_1, p_2) - p_1'(-p_1, p_2) \qquad k_2 = p_2'(p_1, p_2) - p_2'(p_1, -p_2)$$

(notice that for forward elastic scattering, $k_i = 2p_i$). Taking the log of (4.4) gives another relation:

$$2\overline{n}\pi = k_2 L - Im \ln \frac{S_{ij}(\theta_2 - \theta_1)}{S_{ij}(-\theta_2 - \theta_1)}, \tag{4.6}$$

The contribution to the index from the two-particle state i comes from summing over all integers n and $\overline{n}$, so that p_1 and p_2 are greater than zero. Since we have free on-shell states, the energy is just the free-particle energy. The levels are close together because the box is large, so we replace these sums

[7] We also need to define the states so that when $|i(p_1, p_2)\rangle$ scatters only into $|j(p_1', p_2')\rangle$, then $|j(p_1'(-p_1, p_2), p_2'(-p_1, p_2))\rangle$ also scatters only into $|i(-p_1, p_2)\rangle$. In other words, the scattering remains diagonal even after a particle bounces off the wall.

with integrals. We also make the integral over θ_1 and θ_2, so that we must multiply by the density of states g_{f_i}, which is the Jacobian

$$g_{f_i} = \frac{\partial n}{\partial\theta_1}\frac{\partial\overline{n}}{\partial\theta_2} - \frac{\partial n}{\partial\theta_2}\frac{\partial\overline{n}}{\partial\theta_1}. \tag{4.7}$$

The relations (4.5) and (4.6) give $\partial n/\partial\theta_i$ and $\partial\overline{n}/\partial\theta_i$ each as the sum of two terms, one proportional to L (the "free" piece) and the other involving the S-matrix. Thus g_{f_i} has a piece proportional to L^2. This results in the two-free-particle contribution to the index. However, summing over each four-dimensional representation gives a contribution of the form (2.13), and the L^2 piece vanishes in $g_{f+2} - g_f$. We know this must happen, from the discussion following (2.9). The contribution proportional to L from a state with fermion number f_i is

$$f_i e^{i\pi f_i}\frac{L}{2(2\pi)^2}\int\int d\theta_1 d\theta_2\left(\left(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}\right)(k_1+k_2)\right)$$
$$\frac{\partial}{\partial\theta_1} Im \ln S_{ij}(\theta_2-\theta_1)e^{-\beta(m_1\cosh\theta_1+m_2\cosh\theta_2)}. \tag{4.8}$$

To simplify the expression, we have rewritten the rapidity integrals to go from $-\infty$ to ∞ by using the fact that $S(-\theta) = S^*(\theta) = 1/S(\theta)$, which follows from analyticity and unitarity of the S-matrix.

We now specialize to the case where both particles are in reduced multiplets[8]. The result simplifies remarkably, and only depends on the individual Δ's of the particles and not on any details of the S-matrix. We decompose the initial states into four-dimensional supersymmetry representations i with fermion numbers $(f_i, f_i+1, f_i+1, f_i+2)$ as in (2.10). Denoting a reduced multiplet by (d,u), a two-particle state with fermion number f_i+2 is of the form $|u_1u_2\rangle_i$, while the one with fermion number f_i is $|d_1d_2\rangle_i$. By fermion-number conservation and supersymmetry, $|u_1u_2\rangle_i$ must scatter only into a state $|u_1'u_2'\rangle_j$ and likewise $|d_1d_2\rangle_i$ must scatter only into a state $|d_1'd_2'\rangle_j$. We denote the corresponding S-matrix elements by a_{ij} and $\tilde{a}_{ij}$ respectively.

[8] It is perfectly conceivable that only configurations comprised of particles belonging to reduced multiplets contribute to the index.

The relation (2.13) means that we do not need to calculate all of the densities: we only need the difference $g_{f_i+2} - g_{f_i}$. Looking at (4.8), we see that the index thus depends only on the ratio of S-matrix elements $a_{ij}/\tilde{a}_{ij}$. The striking fact is that this ratio can be found without knowing the full S-matrix; it follows from the supersymmetry alone. We know that $Q^+\overline{Q}^-|f_i\rangle = \lambda_i|(f+2)_i\rangle$, where λ_i depends on the details of the representation i. The S-matrix commutes with the supersymmetry generators, which means that the diagram

$$\begin{array}{ccc} |d_1d_2\rangle & \xrightarrow{\tilde{a}} & |d_1'd_2'\rangle \\ {\scriptstyle Q^+\overline{Q}^-}\downarrow & & \downarrow{\scriptstyle Q^+\overline{Q}^-} \\ |u_1u_2\rangle & \xrightarrow{a} & |u_1'u_2'\rangle \end{array}$$

must commute. This implies that

$$\frac{a_{ij}}{\tilde{a}_{ij}} = \frac{\lambda_j}{\lambda_i}. \tag{4.9}$$

We can find the λ_i from one-particle information. The supersymmetry is represented on a doublet with $m = |\Delta|$ as

$$\begin{aligned} Q^-|u(\theta)\rangle &= \sqrt{m}e^{\theta/2}|d(\theta)\rangle & \qquad \overline{Q}^+|u(\theta)\rangle &= \omega\sqrt{m}e^{-\theta/2}|d(\theta)\rangle \\ Q^+|d(\theta)\rangle &= \sqrt{m}e^{\theta/2}|u(\theta)\rangle & \qquad \overline{Q}^-|d(\theta)\rangle &= \omega^*\sqrt{m}e^{-\theta/2}|u(\theta)\rangle, \end{aligned} \tag{4.10}$$

where $\omega = \Delta/|\Delta|$. All other actions annihilate the states. The supersymmetry is defined on multi-particle states in the usual manner. Since Q is fermionic, one picks up phases when Q is brought through a particle with fermion number. For example, bringing Q through a fermion results in a minus sign. Since we have fractional charges, we must generalize this notion, so that the action of $Q^\pm$ on the tensor product of two states is

$$Q^\pm \otimes 1 + e^{\pm i\pi F}\otimes Q^\pm. \tag{4.11}$$

The charges $\overline{Q}^\mp$ act with the same phases as $Q^\pm$. In our two-particle case of interest, we have

$$\begin{aligned} &Q^+\overline{Q}^-|d_1(\theta_1)d_2(\theta_2)\rangle \\ &= e^{i\pi f_1}\sqrt{m_1m_2}\left(\omega_1^* e^{(\theta_2-\theta_1)/2} - \omega_2^* e^{(\theta_1-\theta_2)/2}\right)|u_1(\theta_1)u_2(\theta_2)\rangle. \end{aligned} \tag{4.12}$$

The quantity of relevance in (4.8) is thus

$$\frac{\partial}{\partial \theta_1} \ln \frac{a_{ij}}{\bar{a}_{ij}} = \frac{\partial}{\partial \theta_1} \ln \frac{\sinh(\frac{\theta_2 - \theta_1}{2} + \mu)}{\sinh(\frac{\theta_2' - \theta_1'}{2} + \mu')}, \tag{4.13}$$

where $\mu = \frac{1}{2} \ln \omega_2 \omega_1^*$. The contribution of two reduced multiplets to the index Q thus depends only on Δ_1, Δ_2, Δ_1' and Δ_2' (the masses and hence the θ' follow from this because $m = |\Delta|$), and is

$$e^{i\pi f_i} \frac{i\beta}{2(2\pi)^2} \int\int d\theta_1 d\theta_2 \left((\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})(k_1 + k_2) \right) \frac{\partial}{\partial \theta_1} Im \ln \frac{\sinh(\frac{\theta_2 - \theta_1}{2} + \mu)}{\sinh(\frac{\theta_2' - \theta_1'}{2} + \mu')} e^{-\beta(m_1 \cosh \theta_1 + m_2 \cosh \theta_2)}. \tag{4.14}$$

The result simplifies in the case of elastic scattering, where the masses of the particles do not change. (Forward elastic scattering is the only allowed process in integrable theories.) For forward elastic scattering, the kinematic prefactor in (4.14) is $2m_1 \cosh \theta_1 + 2m_2 \cosh \theta_2$. Moreover, for forward or backward elastic scattering, one has

$$\theta_1 - \theta_2 = \theta_1' - \theta_2', \qquad \mu' = -\mu,$$

showing clearly that the elastic two-particle contribution to the index vanishes only when $\mu = 0, i\pi/2$.

For the two-particle contribution from all the reduced multiplets, we sum (4.14) over all pairs. We have thus seen that two-kink contribution to the index from kinks belonging to reduced supersymmetry multiplets is non-vanishing and easily computable.

The results in this section can be compared with the infra-red limit of the index as obtained from the tt^* differential equations discussed in the previous section. It is non-trivial that they agree, but they must. This is being investigated numerically in some examples of N=2 theories which are not integrable [25]. In the next sections we will focus on integrable theories since then we can obtain, via the exact S matrix and thermodynamics, exact integral equations for the index which can be compared with the tt^* equations along the entire renormalization group flow.

5. $\mathrm{Tr}F(-1)^F e^{-\beta H}$ in integrable theories

In this section we will show how to compute the index for an integrable theory, using the exact S-matrix. Integrable theories in two dimensions have been under intensive investigation recently. These theories are characterized by the existence of infinitely many conserved charges, which essentially allows one to solve these theories explicitly. In particular the scatterings are purely elastic; the particles behave as if they are free particles and as they pass through each other they just pick up phases (modulo change of internal indices). The multi-particle S-matrix factorizes into two-body S-matrices and these are often completely determined by the symmetries of the theory (plus some minimality assumptions which can be verified [10]). The factorizability and elasticity of the S-matrix in an integrable theory implies that we can assign rapidities (momenta) to individual particles even in multi-particle configurations. In particular the total energy of multi-particle configurations is simply the sum of the individual ones. The only effect of the interaction is to shift the density of allowed states. This is an ideal situation for computation of our index; the non-trivial part of our index (2.13) results precisely from a discrepancy between densities of states within a non-reduced supersymmetry multiplet.

It is clear that we can in principle continue the analysis of the previous section, using the exact S-matrix of an integrable theory, to calculate higher-order corrections in the infra-red expansion. In fact, we can do *much* better. There is a trick (known as the *Thermodynamic Bethe Ansatz* [10]) which allows us to compute the index exactly along the entire renormalization group flow, even in the *ultra-violet* limit! The idea is to not fix the number of solitons, but to consider a thermodynamic ensemble of them. We then minimize the free energy in the ensemble. As we will review, this allows us to calculate exactly, i.e. non-perturbatively, $\mathrm{Tr}F(-1)^F e^{-\beta H}$ in an integrable theory with a known exact S-matrix. In fact it is no more difficult to compute the more general quantity $\log \mathrm{Tr}(e^{i\alpha F} e^{-\beta H})$. This allows us to test our claim that while this quantity depends on the D-terms its first derivative with respect to α at $\alpha = \pi$ is independent of the D-terms. Even more generally, let us consider the free energy $\mathcal{F}_{\mu_a}(\beta)$ with chemical potentials μ_a for the various conserved species labels

$$-\beta\mathcal{F}_{\mu_a}(\beta) = \ln \mathrm{Tr}(e^{\beta\sum_a \mu_a N_a} e^{-\beta H}), \tag{5.1}$$

where N_a is the number operator for conserved species a. Using the exact S-matrix we can obtain an exact expression for $\beta\mathcal{F}_{\mu_a}(\beta)$ by finding the minimum value of $\beta E - S - \beta\sum_a \mu_a N_a$ in the space of all states, where S is the entropy. Choosing the chemical potentials in (5.1) such that $\beta\sum_a \mu_a N_a = i\alpha F$, we thereby obtain $\mathrm{Tr}(e^{i\alpha F}e^{-\beta H})$.

Since the S-matrix of an integrable theory preserves rapidities and some set of species labels a, a general, multi-soliton state can be characterized by a collection of distributions $\rho_a(\theta)$ of rapidities occupied by the various solitons in the multi-soliton state. In particular, the energy of this state is given by

$$E = \sum_a \int d\theta \rho_a(\theta) m_a \cosh\theta. \tag{5.2}$$

To do thermodynamics we need to calculate the entropy S and so we need to know the distributions $P_a(\theta)$ of available levels as well as the above distributions $\rho_a(\theta)$ of occupied levels. In particular

$$S = \sum_a \int d\theta \; P_a \log P_a - \rho_a \log \rho_a - (P_a - \rho_a)\log(P_a - \rho_a), \tag{5.3}$$

corresponding to one particle allowed per level.

Using the exact, factorizable S-matrix it can be found that the distributions of available levels are given in terms of the distributions of occupied levels in the general manner:

$$2\pi P_a(\theta) = m_a L \cosh\theta + \sum_b \int d\theta' \rho_b(\theta')\phi_{ab}(\theta - \theta'). \tag{5.4}$$

The $m_a L\cosh\theta$ term in (5.4) is the usual density of available states for a free particle, the ϕ_{ab} reflect the interaction with the other particles, as given by the exact S-matrix. If the S-matrix is diagonal, with species a and b scattering with the phase shift S_{ab}, the interaction is seen to be given by $\phi_{ab}(\theta) = -i\partial \log S_{ab}(\theta)/\partial\theta$. For non-diagonal S-matrices such as our N=2 S-matrices, it is generally difficult to obtain the ϕ_{ab} from the S-matrix; one needs to find the eigenvalues of the multi-particle transfer matrices.

Now we minimize $\beta E - S - i\beta\sum_a \mu_a N_a$, expressed in terms of the above distributions, with respect to the $\rho_a(\theta)$ subject to the constraints (5.4). Defining the quantities $\epsilon_a(\theta)$ by

$$\frac{\rho_a(\theta)}{P_a(\theta)} = \frac{e^{\beta\mu_a - \epsilon_a(\theta)}}{1 + e^{\beta\mu_a - \epsilon_a(\theta)}}, \tag{5.5}$$

it is seen that the free energy is given by

$$\log \mathrm{Tr}(e^{\beta\sum_a \mu_a N_a} e^{-\beta H}) = -\sum_a m_a L \int \frac{d\theta}{2\pi} \cosh\theta \ln(1 + e^{\beta\mu_a - \epsilon_a(\theta)}), \tag{5.6}$$

where the $\epsilon_a(\theta)$ are obtained as the solutions to the coupled integral equations:

$$\epsilon_a(\theta) = m_a\beta\cosh(\theta) - \sum_b \int \frac{d\theta'}{2\pi}\phi_{ab}(\theta - \theta')\ln(1 + e^{\beta\mu_b - \epsilon_b(\theta')}). \tag{5.7}$$

These are the thermodynamic Bethe ansatz [10] integral equations with chemical potentials [29].[9] Our interest is in the case where the chemical potentials are chosen such that $\beta\sum_a \mu_a N_a = i\alpha F$. The expression (5.6) was obtained by summing over all boundary conditions at spatial infinity. The different eigenvalues of our matrix trace can be obtained from this expression by inserting appropriate additional chemical potentials. Examples will be discussed in the following section.

6. Examples

A generic N=2 theory will not, of course, be integrable. Nevertheless, our index for such a theory can be obtained by solving the differential equations discussed in the tt^* section of this paper. We would like, however, to compare

[9] We note that it is straightforward to rederive the TBA equations with our fixed boundary conditions instead of the usual periodic one. The result is the same.

the computation of the index from the tt^* differential equations with the computation from the thermodynamic integral equations. We will thus restrict our examples to integrable theories for which the exact S-matrix is known (or conjectured). Examples of such theories are discussed in [11,12].

We will focus on integrable theories with spontaneously broken $\mathbf{Z}_n$ symmetry. For every n there are a wide variety of such examples, including perturbations of $N=2$ minimal models and Kazama-Suzuki models, supersymmetric CP^{n-1} sigma models, and $N=2$ affine toda theories. For a given value of n, the tt^* differential equations and the TBA integral equations for all these $\mathbf{Z}_n$ integrable theories are found to be essentially the same, the only variation being in the boundary conditions. We will first consider several examples of $\mathbf{Z}_2$ theories, namely ordinary $N=0$ sine-Gordon at the particular coupling where it is $N=2$ supersymmetric, $N=2$ sine-Gordon, $N=2$ minimal models perturbed by th least relevant perturbation, and the supersymmetric CP^1 sigma model. The indices for all of these theories are obtained from the same differential equation, Painleve III. They span *all* the possible regular boundary conditions. The TBA integral equations also exhibit this fact in a non-trivial guise. We next discuss the more general $\mathbf{Z}_n$-type integrable theories starting with the simplest such theory, the A_n $N=2$ minimal model perturbed by the most relevant operator. We finally discuss how to modify the equations in order to determine the index for the other $\mathbf{Z}_n$-type theories.

For the most part, we will consider $N=2$ theories which can be described by a Landau-Ginzburg action[10]. Such an action is of the form [26]

$$\int K(X_i,\overline{X}_i) + \int W(X_i) + \int \overline{W}(\overline{X}_i)$$

where the superfields X_i are chiral (in supersymmetric sense, i.e. annihilated by $D^+,\overline{D}^+$) fields, W is the superpotential and is integrated over half the superspace, and K gives the kinetic terms (the D-term) and is integrated over the full superspace. Using topological techniques one can prove that the chiral

[10] The existence of Landau-Ginzburg description seems to apply also to non-supersymmetric and $N=1$ supersymmetric theories [30].

ring of this theory is exactly characterized by W [6]. In particular the chiral fields of the theory are all products of the fields X_i modulo setting to zero $\partial_i W$. The chiral ring structure constants entering in (3.3) are obtained by simply multiplying the various products of X_i together and imposing the relations $\partial_i W=0$.

The physical potential for the theory is

$$V = K^{i\bar{j}}\partial_i W \partial_{\bar{j}}\overline{W}$$

where $K^{i\bar{j}}$ is the inverse of the Kahler metric $\partial_i\partial_{\bar{j}}K$. The vacua a of the theory are thus in one-to-one correspondence with the critical points of W. The kinks k_{ab} are the finite energy solutions to the equations of motion connecting the a and b vacua: $X(\sigma=-\infty)=X^{(a)}$, $X(\sigma=+\infty)=X^{(b)}$ (as discussed in [31], not all such kinks are to be regarded as fundamental solitons). The central term in the $N=2$ algebra (2.2) is given simply in terms of the superpotential by $\Delta = 2\Delta W \equiv 2[W(X(\sigma=+\infty)) - W(X(\sigma=-\infty))]$. The mass of the (u,d) soliton doublet representation is thus given simply in terms of the superpotential by $m = 2|\Delta W|$.

The fractional fermion number in the soliton sector is also given simply in terms of the superpotential by a spectral flow argument [12] or by adiabatic or index theorem techniques [20,21]. The result is that the (u,d) soliton doublet has the fermion numbers $(f, f-1)$ where

$$f = -\frac{1}{2\pi}(Im \ln\det(\partial_i\partial_j W(X)))\Big|_{\sigma=-\infty}^{\sigma=+\infty}. \qquad (6.1)$$

In all of the integrable theories we consider, the entire spectrum consists of such soliton doublets saturating the bound. There are other integrable $N=2$ theories, for example the theories with superpotential $W = x^{n+1}/(n+1) - \lambda x^2$ [31,32] , for which this is not the case; exact S-matrices for these and many other integrable $N=2$ theories have been recently discussed in [33].

6.1. $N=0$ sine-Gordon at the $N=2$ point

Ordinary $N=0$ sine-Gordon theory is $N=2$ supersymmetric at a particular coupling [34]. In a manifestly $N=2$ supersymmetric setup, this point is described by a Landau-Ginzburg superpotential

$$W=\lambda(\frac{X^3}{3}-X), \tag{6.2}$$

with some suitable choice of K [26]. The vacua $|a\rangle$ are at $X=\pm1$. Our matrix index Q has eigenvalues $Q(z)$ and $-Q(z)$, where $z=m\beta$ and $m=2|W(X=1)-W(X=-1)|=\frac{8\lambda}{3}$. We will first use the tt^* equations to find an exact expression for our index in terms of a famous differential equation.

The tt^* analysis of this theory was discussed at length in [3]. The result is that the metric on the space of ground states, in the basis spanned by $1, X$, is given by $g=e^{\sigma_3 u(z)/2}$. Using (3.3) with the chiral ring $X^2=1$, it follows that $u(z)$ satisfies the radial sinh-Gordon equation

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}=\sinh u. \tag{6.3}$$

The radial sinh-Gordon equation is a special case of Painleve III. From relation (3.8) it follows that the index $Q(z)$ is given by

$$Q(z)=\frac{1}{2}z\frac{d}{dz}u(z). \tag{6.4}$$

If we wish, we could eliminate $u(z)$ from these equations and write a differential equation for $Q(z)$

$$Q''-z^{-1}Q'=Q\sqrt{4z^{-2}Q'^2+1}. \tag{6.5}$$

The solutions $u(z)$ to (6.3) behave for $z\to0$, i.e. the ultra-violet or conformal limit, as

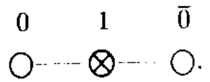
$$\begin{aligned} u(z;r) &\sim r\log\frac{z}{2}+s+O(z^{2-|r|}) \qquad \text{with } |r|<2 \\ &\sim \pm\log\frac{z}{2}\pm\log[-(\log(\frac{z}{4}+C)]+O(z^4\log^2 z) \qquad \text{for } |r|=2 \end{aligned} \tag{6.6}$$

where

$$e^{s/2}=\frac{\Gamma(\frac{1}{2}-\frac{r}{4})}{2^r\Gamma(\frac{1}{2}+\frac{r}{4})}$$

and where C is the Euler constant, and r is just a parameter to label the boundary condition. For our theory regularity requires $r=2/3$ [3]. From (6.6) it follows that $\pm Q(z=0)=\pm1/3$. This is to be expected, since in this limit the eigenvalues of the Q matrix index are the left plus the right $U(1)$ charge of the Ramond ground states.

We will now obtain integral equations for the function $Q(z)$ [11], using the exact S-matrix obtained in [9,35] . The one particle spectrum of this theory consists of a soliton reduced multiplet (u,d), with mass $m=2|\Delta W|$, and fermion numbers obtained from (6.1) to be $(1/2,-1/2)$. The soliton connects either vacuum with the other one. A multi-soliton state can be characterized by a distribution $\rho_1(\theta)$ of rapidities occupied by the solitons. Because the u and d solitons do not scatter diagonally, we can not assign individual distributions for u type and d type solitons. Instead, there are two additional distributions $\rho_l(\theta)$, $l=0,\bar{0}$, which encode the way in which the solitons are distributed as u or d solitons. The distributions $\rho_l(\theta)$ arose in [11] in obtaining the eigenvectors and eigenvalues of the multi-particle transfer matrices. We correspondingly have distributions $P_1(\theta)$ and $P_l(\theta)$ for the density of states. As discussed in [11], these densities satisfy relations of the type (5.4), described by the diagram



The nodes in this diagram correspond, as labeled, to the species in (5.4). The nodes for species 0 and $\bar{0}$ are open to signify that these species have $m_a=0$ in the equations (5.4)(arising from the fact that these species are not physical particles but, rather, account for the additional u and d degree of freedom); the $\otimes$ node has mass $m_1=2|\Delta W|$. The functions ϕ_{ab} in (5.4) are given by $\phi_{ab}(\theta)=(\cosh\theta)^{-1}l_{ab}$, where l_{ab} is the incidence matrix for the figure, i.e. it is one when species a and b are connected by a line and zero otherwise (and $l_{aa}=0$).

The remaining ingredients required in equation (5.7) are the chemical potentials. These are chosen so that $\beta\sum_a\mu_a N_a=i\alpha F$. Thus we need to express F in terms of the above densities. For a state with a total number k

of u solitons and a total number $N-k$ of d solitons, these distributions are defined to satisfy

$$\int d\theta \rho_1(\theta) = N \quad \text{and} \quad \int d\theta (P_0 - \rho_0 + \rho_{\bar{0}}) = k.$$

The fermion number of such a state is $k-(N/2)$ and so using the above equations along with (5.4) we find

$$F = \int d\theta (\rho_{\bar{0}}(\theta) - \rho_0(\theta)). \tag{6.7}$$

We are ready to use (5.6) and (5.7) to obtain exact integral equations for $\ln \mathrm{Tr} e^{i\alpha F} e^{-\beta H}$. The m_a and functions $\phi_{ab}(\theta)$ in these equations are as given above and, using (6.7), the chemical potentials should be taken to be $\beta\mu_1 = 0$ and $\beta\mu_0 = -\beta\mu_{\bar{0}} = i\alpha$. First note that at $\alpha = \pi$ (Witten's index) the equations (5.7) are solved by

$$e^{-\epsilon_1(\theta)} = \epsilon_0(\theta) = \epsilon_{\bar{0}}(\theta) = 0$$

for all θ. From (5.6) is is then seen that $L^{-1} \log \mathrm{Tr}(-1)^F e^{-\beta H} = 0$ when $L \to \infty$, as expected.

To move slightly away from Witten's index, take $\alpha = \pi + h$ with h small and keep terms only up to $O(h)$. The equations (5.7) are then solved by $e^{-\epsilon_1(\theta)} = he^{-A(\theta)}$, and $\epsilon_0(\theta) = \epsilon_{\bar{0}}(\theta) = hB(\theta)$, where

$$\begin{aligned} A(\theta; z) &= z\cosh\theta - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \ln(1 + B^2(\theta'; z)) \\ B(\theta; z) &= -\int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} e^{-A(\theta'; z)}. \end{aligned} \tag{6.8}$$

Using (5.6) and (2.9), the Q-matrix eigenvalues are given by $\pm Q(z = m\beta)$ where, in terms of the solution to the above equations

$$Q(z) = z \int \frac{d\theta}{2\pi} \cosh\theta e^{-A(\theta; z)}. \tag{6.9}$$

We have two exact expressions for the index $Q(z)$, the differential equation (6.3) and (6.4) (or (6.5)) obtained from tt^* considerations, and the integral

equations (6.8) and (6.9) obtained from S-matrix and thermodynamic considerations. These expressions must agree! We know of no way, however, to show directly from the equations that this is the case. Physics has proven a highly non-trivial statement about the above equations. A check is that the ultra-violet limit for the index using the two different equations give exactly the same result $Q(z=0) = 1/3$. Also, using results from [36] concerning the Painleve III differential equation, it can be seen (after some algebra) that the function $Q(z)$ has an expansion

$$Q(z) = -z\frac{d}{dz} \sum_{n=0}^{\infty} \frac{2(2\cos(\frac{\pi}{4}(2-r)))^{2n+1}}{2n+1} \int \prod_{i=1}^{2n+1} \frac{e^{-z\cosh\theta_i}}{\cosh(\frac{\theta_i - \theta_{i+1}}{2})} \frac{d\theta_i}{4\pi} \tag{6.10}$$

$(\theta_{2n+2} \equiv \theta_1)$ and, for later use, we have restored the boundary condition parameter r from (6.6); for the present example $r = 2/3$. [11] This is the type of infra-red (large z) expansion which we would expect for our index; the first term is the usual Bessel function. Also, only odd numbers of solitons contribute because only then are the vacua at spatial infinity different. It is easy to see that the two-particle contribution computed in section 4 vanishes, because $\mu = i\pi/2$ here. By expanding out the integral equations (6.9) and (6.8) it is easily seen that the one-soliton and the three-soliton contribution agree with the n=0 and n=1 terms in the PIII expression (6.10); after that the comparison becomes more difficult to check directly. We have numerically verified (to real precision) that the function $Q(z)$ obtained from (6.3) and (6.4) does, indeed, agree with that obtained from (6.9) and (6.8). It would be interesting to see how difficult it is to find a direct mathematical argument to verify this. In particular, would we have to re-invent the physics argument, in disguise, to prove their equality?!

6.2. N=2 Super sine-Gordon

As our next example, we consider the N=2 super sine-Gordon theory given by the Lagrangian

$$\int d^2 z d^4\theta X\overline{X} + \frac{m}{4}\left(\int d^2 z d^2\theta \cos gX + h.c.\right) \tag{6.11}$$

[11] A similar expression arose in the computation of Ising-model form factors [37].

The coupling g, by a redefinition of X and $\overline{X}$, can be taken to be real. Because our index is independent of the D-term, it must, in fact, be independent of the coupling g since, by rescaling the chiral fields, g can be eliminated from the F-term and put into the D-term. The general quantity $Z(\alpha,\beta) = \mathrm{Tr}e^{i\alpha F}e^{-\beta H}$ can be calculated via the TBA equations since this is an integrable theory. $Z(\alpha,\beta)$ depends on g as a sign of its dependence on the D-term. We will show that our index, the first derivative with respect to α at $\alpha = \pi$, is independent of g as expected by our general arguments in sects. 2 and 3.

The vacua of the N=2 sine-Gordon theory are the points $gX_a = a\pi$ for $a \in \mathbf{Z}$. We thus have an infinite number of possible vacua for the boundary conditions at $\sigma = \pm\infty$. Because of the symmetry $gX \to gX + \pi$, a configuration with vacuum X_a to the left and vacuum X_b to the right is equivalent to one with vacuum X_{a+n} to the left and vacuum X_{b+n} to the right. Consider the contribution Q_{ab} to our index from a fixed boundary condition (ab). Then, this symmetry implies that $Q_{ab} = Q_{a+n,b+n}$. The eigenvalues of a matrix M_{ij} whose entries depend only on $i-j$ are easily found by Fourier transform. The eigenvalues $Q(\Theta)$ are parametrized by an angle Θ and given by

$$Q(z;\Theta) = \sum_{l=-\infty}^{\infty} e^{i\Theta l} Q_{i,i+l}(z). \tag{6.12}$$

In other words, we weight configurations by $e^{i\Theta T}$ where T is the topological charge.

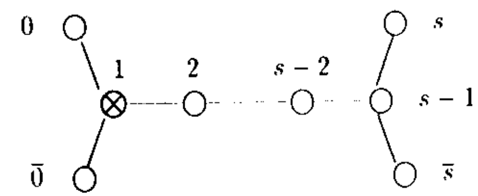
Using the results of [3] and (3.8), the index eigenvalues $Q(z;\Theta)$ are all solutions of the same PIII differential equation (6.3) obtained in the last example, where only the boundary condition (6.6) depends on Θ. Regularity requires [3] the solution $Q(z,\Theta)$ to behave (with $z = m\beta$) as (6.6) with

$$r(\Theta) = 2(1 - \frac{2\Theta}{\pi}) \qquad \text{for } \ 0 \le \Theta \le \pi, \tag{6.13}$$

with $r(\Theta)$ defined outside this interval by $r(\Theta+\pi) = -r(\Theta)$, a consequence of the fact that, in (6.12), only configurations with odd l contribute. By varying Θ, we obtain all regular solutions of Painleve III.

Now we come to the analysis of this theory from the viewpoint of the TBA. The close connection between the index in this example and that in the previous example can also be seen from the exact S-matrix and associated integral equations. The important point is that every sine-Gordon soliton is the same (u,d) supermultiplet, with fermion numbers $(1/2,-1/2)$, seen in the previous example. The (conjectured [38]) S-matrix for the N=2 sine Gordon theory is simply the tensor product of the S-matrix for the theory considered in the previous example with the S-matrix for the N=0 sine-Gordon [9] at coupling $g^{bare}_{N=0} = g$. The TBA integral equations for this theory are obtained by combining those of the previous section with the TBA system for N=0 sine-Gordon[12]. The TBA system of integral equations for N=0 sine-Gordon at generic coupling g is of the usual form (5.7) but with an infinite number of species a and a complicated set of $\phi_{ab}(\theta)$. For the sake of brevity we will thus focus on a nice set of couplings, $g^2 = 8\pi s$ for s a positive integer $s \ge 2$, where the equations simplify[39]. Of course, our index is independent of the coupling g so we can work with any coupling we please. We will verify this fact, though the more general quantity (2.4) does depend on s.

At the coupling $g^2 = 8\pi s$, as discussed in [12], we obtain a TBA system of coupled integral equations of the usual form (5.7) for $s+3$ functions $\epsilon_a(\theta)$. The masses m_a and $\phi_{ab}(\theta)$ entering in the equations (5.7) for this theory are described by the figure



where every node in the diagram corresponds to a species in the equations (5.7). The ⊗ node, labelled by 1, corresponds to the soliton; its mass in the equations (5.7) is that of the soliton. The other species have open nodes to signify that they have m_a=0 in the equations (5.7). Again, the role of these additional species is to account for the additional degrees of freedom (i.e. u or d, and which vacua they connect). As in the previous example, the $\phi_{ab}(\theta) = (\cosh\theta)^{-1} I_{ab}$

where l_{ab} is the incidence matrix for the above figure. The massive node and the nodes labelled 0 and $\bar{0}$ come from the N=2 part of the S-matrix; they correspond precisely to the species in the previous example. The massive node connected to the s open nodes are the TBA species for N=0 sine-Gordon at coupling $(g^{bare}_{N=0})^2 = 8\pi s$. This result is obtained by using a technique known as the algebraic Bethe ansatz to find the eigenvalues of the multi-soliton sine-Gordon transfer matrices (see the appendix of [39]). The TBA system for the tensor-product S-matrix is obtained by joining the two component TBA systems at the massive node as described by the above figure.

As in the last example, the fermion number is given by

$$F = \int d\theta(\rho_{\bar{0}}(\theta) - \rho_0(\theta)),$$

whereas the sine-Gordon topological charge (number of solitons minus anti solitons) is given by [39]

$$T = s\int d\theta(\rho_{\bar{s}}(\theta) - \rho_s(\theta)), \tag{6.14}$$

where the species labels are as given in the above figure. By introducing chemical potentials $\beta\mu_{\bar{0}} = -\beta\mu_0 = i\alpha$ and $\beta\mu_{\bar{s}} = -\beta\mu_s = is\Theta$, with the other chemical potentials zero, the equations (5.7) and (5.6) provide integral equations to compute $\log \mathrm{Tr} e^{i\alpha F} e^{i\Theta T} e^{-\beta H}$ exactly. For generic α, the integral equations, in particular the number of functions $\epsilon_a(\theta)$, clearly depend on s. In the infra-red expansion, one sees that the solutions of these equations are in fact different. Thus $\ln \mathrm{Tr}(e^{i\alpha F} e^{i\Theta T} e^{-\beta H})$ depends on the coupling $g^2 = 8\pi s$, as expected.

At α=π, the solution of the equations (5.7) described by the above figure with the above chemical potentials is given by $\epsilon_a(\theta)$ independent of θ: $\epsilon_0(\theta)$=$\epsilon_{\bar{0}}(\theta)$=$e^{-\epsilon_1(\theta)}$=0, and

$$\begin{aligned} e^{-\epsilon_a(\theta)} &= \left(\frac{\sin a\Theta}{\sin\Theta}\right)^2 - 1 \qquad \text{for } a = 2,\ldots,s-1 \\ e^{-\epsilon_s} &= e^{-\epsilon_{\bar{s}}} = \frac{\sin(s-1)\Theta}{\sin\Theta}. \end{aligned} \tag{6.15}$$

As expected by supersymmetry, (5.6) gives $L^{-1}\log \mathrm{Tr}(-1)^F e^{-\beta H} = 0$ in the $L\to\infty$ limit.

We now move slightly away from Witten's index. At $\alpha = \pi + h$ with h small, the solution of the equations (5.7) is given by $\epsilon_0(\theta)$=$\epsilon_{\bar{0}}(\theta)$=$hB(\theta)$, and $e^{-\epsilon_1(\theta)}$=$hA(\theta)$, where the functions $A(\theta)$ and $B(\theta)$ obey the equations

$$\begin{aligned} A(\theta) &= z\cosh\theta - \ln(2\cos\Theta) - \int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\ln(1 + B^2(\theta')) \\ B(\theta) &= -\int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}e^{-A(\theta')}. \end{aligned} \tag{6.16}$$

The other $\epsilon_a(\theta)$ are (to lowest order in h) still given by the constants (6.15). Using (5.6) and (2.9) we thus see that the Q matrix eigenvalues are given by $Q(z = m\beta;\Theta)$ where

$$Q(z;\Theta) = z\int \frac{d\theta}{2\pi}\cosh\theta e^{-A(\theta;z;\Theta)}, \tag{6.17}$$

with $A(\theta;z;\Theta)$ obtained by solving (6.16).

As expected, our index is independent of the coupling g (i.e. s). One can check by for example studying the IR expansion of the solutions to the full TBA equations that all the other quantities I_l for $l > 1$ *do* depend on g and are thus dependent on D-terms, in accord with the arguments of section 2. In fact, the entire N=0 part of the theory has dropped out of the integral equations, leaving just the constant $\ln 2\cos\Theta$ in (6.16), which resulted from the constants (6.15). This constant piece is the only reflection of the N=0 sine-Gordon structure. The close connections between these integral equations and those computed in the previous example was also to be expected from the tt^* considerations; the $\ln(2\cos\Theta)$ term in (6.16) specifies the boundary conditions in the Painleve III equation. Again, though (6.16) and (6.17) have no obvious connection to the radial sinh-Gordon (or Painleve III) differential equation (6.3) and (6.4), physics proves that the regular solutions are the same.

It is easily seen that (6.16) and (6.17) lead to an expansion of the form

$$Q(z;\Theta) = \sum_{n=0}^{\infty}(2\cos\Theta)^{2n+1}A_{2n+1}(z). \tag{6.18}$$

A_{2n+1} is the contribution from the $(2n+1)$-soliton sector, and is independent of Θ.[12] In the large $m\beta$ limit, A_{2n+1} is $O(e^{-(2n+1)m\beta})$. We can compare this result with (6.10). We immediately arrive at the expression (6.13) relating the boundary condition r on the PIII differential equation to our parameter Θ. This is a further confirmation that the TBA solution is related to PIII solution, as in both cases the dependence on PIII boundary conditions r or equivalently Θ has the same structure. As a further check, the UV limit $z \to 0$ of (6.17) and (6.16) is obtained (using results from [40] for taking the UV limit of TBA systems with imaginary chemical potentials) to be

$$Q(z=0;\Theta) = (1 - \frac{2\Theta}{\pi}), \tag{6.19}$$

for $0 < \Theta < \pi$, in agreement with (6.3) and (6.4) with the boundary condition specified by (6.6) and (6.13).

At $\Theta=0$ mod π we have to be more careful in evaluating the UV limit of the equations (6.17) and (6.16). Exactly as in [41], there is a log piece in the ultra-violet limit. This log piece agrees with the expression (6.6) for the Painleve III solution at $r = \pm 2$.

6.3. $N=2$ minimal models with least-relevant perturbation

An $N=2$ minimal model remains integrable when perturbed by its least-relevant operator [32,42]. The effective Landau-Ginzburg superpotential for the perturbed theory is identified to be a Chebyshev polynomial (as conjectured in [3] and confirmed in [11]); e.g. for the perturbed A_{k+1} theory the superpotential is $W_k(X=2\cos\theta) = (2/k+2)\cos(k+2)\theta$, expressed as a polynomial in X. This is the least relevant perturbation of the conformal field theory $W = X^{k+2}/k+2$ in the flat direction of [23]. The chiral ring of this theory yields the $SU(2)_k$ fusion rules [43].

The tt^* computation of our index here is a simple application of the results from the previous example. The reason is that if we change variables $(k+2)\theta \to Y$, the above Chebyshev polynomial becomes the superpotential for $N=2$ sine-Gordon. The eigenvalues of the index are thus obtained to be

$$Q(z=m\beta;\Theta = \frac{\pi n}{k+2}), \tag{6.20}$$

for $n = 1,\ldots,k+1$, where $Q(z;\Theta)$ is the function discussed in the previous example.

We now discuss the TBA calculation of the index using the exact S-matrix for these theories. The vacua of the $W_k(X)$ Chebyshev theory are at $X^{(n)} = 2\cos(\pi n/k+2)$, for $n=1,\ldots,k+1$. The solitons are $N=2$ Bogomolnyi doublets (u_j,d_j) connecting vacuum $X^{(j)}$ with vacuum $X^{(j+1)}$, for $j=1,\ldots k$, each identical to that of the example in sect 6.1; they all have the same mass and fermion numbers $(\frac{1}{2},-\frac{1}{2})$. The structure of $k+1$ vacua on a line is that of the k-th RSOS theory [44], which describes the $N=0$ minimal models with least-relevant perturbation. The $N=2$ Chebyshev S-matrix is a direct product of this $N=0$ RSOS S-matrix with the $N=2$ S-matrix discussed in sect. 6.1, just as the $N=2$ sine-Gordon S-matrix of the previous subsection was the tensor product of the $N=0$ sine-Gordon S-matrix with the $N=2$ S-matrix of sect. 6.1.

There is a well-known reduction from $N=0$ sine-Gordon at a particular coupling to the k-th RSOS theory. This same reduction can be used to obtain our $N=2$ Chebyshev theories from the $N=2$ sine-Gordon theory; the common $N=2$ structure just goes along for the ride. We start with the $N=2$ sine-Gordon TBA equations appropriate for $N=2$ sine-Gordon coupling $g^2 = 8\pi(k+2)$. The equations are described by the diagram of the previous subsection, with $s = k+2$. The reduction of this to our Chebyshev theory requires taking

[12] This leads to an amusing intuition for our index. Write this result as

$$Q(z;\Theta) = \sum_{n=0}^{\infty} A_{2n+1}(z)(e^{i\Theta} + e^{-i\Theta})^{2n+1},$$

and compare with (6.12). The factor $e^{i\Theta}$ counts the solitons connecting vacuum i to vacuum $i+1$ (topological charge 1) and the factor $e^{-i\Theta}$ counts anti-solitons connecting vacuum i to vacuum $i-1$ (topological charge -1). The fact that they are all weighted by the same factor means that we can obtain the index by weighting each $2n+1$-soliton configuration by A_{2n+1}. This intuition only applies to the computation of our index, and not to the other thermodynamic quantities.

$e^{\beta\mu_s} = e^{\beta\mu_{\bar{s}}} = -1$ [39]. This reduction simply eliminates the nodes labelled s, $\bar{s}$, and $s-1$ from the diagram in the previous subsection, leaving the TBA system discussed in [11].

The solution for this TBA system is the sine-Gordon solution $Q(z;\Theta = \pi/(k+2))$, giving the largest eigenvalue of the Q-matrix for the Chebyshev theory. The remaining eigenvalues of the matrix index are given by $Q(z,\Theta = \pi n/(k+2))$, for $n = 1,\ldots,k+1$, and can be seen as other branches of this solution.[13] It follows from (6.17) that the non-zero eigenvalues come in pairs of opposite sign, as they should. As another check, note that in the ultra-violet limit we obtain (6.19) $Q(z=0,\Theta=\pi n/(k+2)) = 1-(2n/k+2)$; these are the correct expressions for the left plus right chiral $U(1)$ charges for the Ramond ground states of the conformal theory obtained in the ultra-violet limit.

6.4. Supersymmetric CP^1 sigma model

The supersymmetric sigma model on $CP^1 = S^2$ is a massive N=2 theory. In principle we can consider an arbitrary kahler metric on CP^1. Varying the metric without changing its kahler class (i.e., preserving the area) is a D-term perturbation of the theory and therefore the index only depends on the area. Letting X denote the Kahler form of CP^1, the chiral ring, which is to be identified as an instanton-modified cohomology ring, is $X^2 = e^{-A}$ where A is the area of CP^1 (the two-sphere) [45]. The tt^* considerations have been applied to this and CP^{n-1} examples in [46]: using the above ring the index is again given by (6.3) where $z = 8\beta e^{-A/2}$ and, by regularity, the boundary condition (6.6) is determined to be $\Theta = 0$, i.e. r=2.

For a generic metric on CP^1, the supersymmetric sigma model is not an integrable theory. However, since the index only depends on the area, for computation of the index we may as well use a convenient choice of the metric. If we choose the constant-curvature metric on CP^1 then this theory is integrable

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13 The resulting factors of $2\cos n\pi/(k+2) \equiv \lambda_n$ in (6.18) are the eigenvalues of the RSOS soliton incidence matrix. The number of N-soliton configurations can be expressed (for any boundary condition) as $\sum_n c_n \lambda_n^N$ with N independent c_n. This is in accordance with the intuition of the footnote following (6.18).

[47] and so we can compare the tt^* analysis with the TBA analysis. As discussed in [12], the TBA system for this theory is obtained by taking the $k\to\infty$ limit of the Chebyshev TBA system. Our index for the supersymmetric CP^1 sigma model is thus given by (6.16) and (6.17) with $\Theta = 0$, in agreement with the tt^* result, given our equality between the Painleve III differential equations and these integral equations.

6.5. The Basic $\mathbf{Z}_n$-type N=2 Integrable Theories

All of the previous examples displayed a spontaneously broken $\mathbf{Z}_2$ symmetry. The tt^* equations for the index in all these examples were the same, the only difference being in the boundary conditions. Likewise, the TBA integral equations for the index in all of these examples were the same, the only difference being in the value of Θ in (6.16). We will now consider N=2 integrable theories with a spontaneously broken $\mathbf{Z}_n$ symmetry. The basic such theory is the A_n N=2 minimal conformal field theory perturbed by the most-relevant supersymmetry preserving operator. It can be described by the Landau-Ginzburg superpotential [26]

$$W = \lambda\left(\frac{X^{n+1}}{n+1} - X\right). \qquad (6.21)$$

This theory is integrable [31]; in fact, it can be described by an affine Toda theory with an imaginary coupling and a background charge [32].

The tt^* equations for this example were discussed in [3]. Because theory (6.21) has the $\mathbf{Z}_n$ symmetry $X \to e^{2\pi i/n}X$, the Ramond ground state metric g is diagonal in the chiral ring basis spanned by $1, X, \ldots, X^{n-1}$. Denoting these diagonal elements by e^{q_p}, equation (3.3) with the chiral ring relation X^n=1 yields the following relations for the eigenvalues $Q(z;p)$ of our matrix index:

$$Q(z;p) = z\frac{d}{dz}q_p(z) \quad \text{where} \quad \frac{d^2 q_p}{dz^2} + \frac{1}{z}\frac{dq_p}{dz} + e^{q_{p+1}-q_p} - e^{q_p - q_{p-1}} = 0, \qquad (6.22)$$

for $p = 0,\ldots,n-1$, with $q_{p+n} \equiv q_p$ and with the constraint $q_{n-p} = -q_p$. For n=2, (6.22) reduces to the sinh-Gordon (6.3) and (6.4) of our previous examples. The $\hat{A}_{n-1}$ Toda equations with the constraint $q_{n-p} = -q_p$ can be put in the form of $\hat{C}_m$ Toda theory for $n = 2m$ or $\hat{BC}_m$ Toda theory for $n = 2m+1$ [3].

Exact integral equations for the above eigenvalues of our matrix index follow from the exact S-matrix and TBA analysis discussed in [12]. The vacua of theory (6.21) are at $X^{(j)} = e^{2\pi i j/n}$. The soliton content consists of $2(n-1)$ solitons forming doublets (u_r, d_r) under supersymmetry; the soliton species label r runs from $1, \ldots, n-1$ corresponding to solitons connecting initial and final vacua with $X_f = e^{2\pi i r/n} X_i$. The fermion numbers of (u_r, d_r) are given by $(r/n, r/n - 1)$, and their mass is given by the Bogomolnyi bound $m_r = M \sin r\pi/n$ where $M = 4n\lambda/(n+1)$.

The conserved currents require that when a soliton of type r, i.e. u_r or d_r, scatters with a soliton of type s, the labels r and s scatter diagonally along with the rapidities. The number N_r of solitons of type r, i.e. the number of u_r solitons plus the number of d_r solitons, is thus conserved for $r = 1, \ldots, n-1$, as is the total number of u solitons, and the total number of d solitons. A multi- soliton state can thus be characterized by distributions $\rho_r(\theta)$ of rapidities occupied by solitons of type r, i.e. u_r or d_r, along with, as in the previous examples, two additional distributions $\rho_l(\theta)$ and $l = 0, \bar{0}$ [12]. The fermion number of the multi-soliton state is again found to be given in terms of the various distributions by (6.7).

Using the exact S-matrix it was found in [12] that the distributions $P_a(\theta)$ of available levels for the above species a are given in terms of the above occupied distributions $\rho_a(\theta)$ by relations of the usual form (5.4) where $m_r = M\sin(r\pi/n)$ is the mass of species $r = 1, \ldots, n-1$, $m_l = 0$ for $l = 0, \bar{0}$, and the ϕ_{ab} are given by

$$\phi_{l,l'} = 0, \qquad \phi_{r,l}(\theta) = \frac{\sin(r\mu)}{\cosh(\theta) - a_l \cos(r\mu)},$$
$$\phi_{r,s}(\theta) = \int dt e^{it\theta}\left(\delta_{rs} - 2\frac{\cosh \mu t \sinh(\pi - r\mu)t \sinh s\mu t}{\sinh \pi t \sinh \mu t}\right), \tag{6.23}$$

for $r \geq s = 1, \ldots n-1$, with $\phi_{ab} = \phi_{ba}$, where $a_0 \equiv -a_{\bar{0}} \equiv 1$ and $\mu \equiv \pi/n$.

We now calculate the eigenvalues of $\mathrm{Tr}_{ab}(e^{i\alpha F} e^{-\beta H})$. The p-th eigenvalue of this matrix can be obtained by weighting solitons of type r by $e^{2\pi i r p/n}$, for $p = 0, \ldots, n-1$, and summing over al' vacua (ab) at spatial infinity. We will thus calculate $\log \mathrm{Tr} e^{i\alpha F} e^{2\pi i p r N_r/n} e^{-\beta H}$ by using (5.1) with the chemical potentials $\beta\mu_r = -2\pi i r p/n$ and, from (6.7), $\beta\mu_{\bar{0}} = -\beta\mu_0 = i\alpha$. Plugging these chemical potentials and the (6.23) into (5.7) and (5.6), we obtain integral equations which determine exactly $\mathrm{Tr}(e^{i\alpha F} e^{-\beta H})$.

At $\alpha = \pi$ it is seen by inspection that the solution of the coupled integral equations (5.7) is given by $e^{-\epsilon_r(\theta)} = \epsilon_l(\theta) = 0$, and thus $L^{-1} \log \mathrm{Tr}(-1)^F e^{-\beta H} = 0$ in the large L limit. We now consider $\alpha = \pi + h$ with h small. The solution of (5.7) is of the form $e^{-\epsilon_r} = h e^{-2\pi i r p/n} e^{-A_r}$ and $\epsilon_l = h B_l$, where the A_r and B_l satisfy the coupled integral equations

$$A_r(\theta) = m_r\beta\cosh\theta + 2ir p\mu - \sum_{l=0,\bar{0}} \int \frac{d\theta'}{2\pi} \frac{\sin(r\mu)\ln(ia_l + B_l(\theta'))}{\cosh(\theta - \theta') - a_l\cos(r\mu)}$$
$$B_l(\theta') = -\sum_{r=1}^{n-1} \int \frac{d\theta''}{2\pi} \frac{\sin(r\mu) e^{-A_r(\theta'')}}{\cosh(\theta' - \theta'') - a_l \cos(r\mu)}, \tag{6.24}$$

where, again, $a_0 = -a_{\bar{0}} = 1$, $m_r = M\sin(r\pi/n)$, and $\mu = \pi/n$. Using (2.9) and (5.6) we have the index eigenvalues $Q(z = M\beta; p)$, for $p = 0, \ldots, n-1$, given by

$$Q(z;p) = \sum_{r=1}^{n-1} m_r\beta \int \frac{d\theta}{2\pi} \cosh\theta e^{-A_r(\theta;z;p)}. \tag{6.25}$$

It follows from (6.24) that $A_r(\theta; z; p)^* = A_{n-r}(\theta; z; p)$ and $B_0(\theta; z; p)^* = B_{\bar{0}}(\theta; z; p)$ and, thus, the above eigenvalues $Q(z;p)$ are all real, as they should be. Furthermore, $e^{-A_r(\theta;z;p)} = -e^{-A_{n-r}(\theta;z;n-1-p)}$ is a consistent solution of (6.24) for all $r = 1, \ldots, n-1$ and $p = 0, \ldots, n-1$. It follows that $Q(z; n-1-p) = -Q(z;p)$; the non-zero eigenvalues come in opposite pairs as they should.

We can compare these exact results (6.25) and (6.24) with the infrared expansion of sect. 4. Setting $B_l(\theta') = 0$ in (6.24), we obtain the first approximation $e^{-A_r} \approx i e^{-i\pi r(2p+1)/n} e^{-m_r\beta\cosh\theta}$. Plugging this into (6.25) we obtain the one-soliton sector contribution to the index, in agreement with (4.1). Plugging this first approximation back into (6.24) we obtain the next approximation

$$e^{-A_r} \approx i e^{-i\pi r(2p+1)/n} e^{-m_r \cosh\theta}$$
$$(1 - \sum_s e^{-i\pi s(2p+1)/n} \int \frac{d\theta'}{2\pi} \phi_{r+s,0}(\theta - \theta') e^{-m_s\beta\cosh\theta'}).$$

Plugging this into (6.25) gives the contribution to the index coming from the two-soliton sector, in agreement with (4.8) and (4.13).

We have again found two different representations of our index: One in terms of solutions to affine Toda equations (6.22) and the other in terms of solutions to coupled integral equations (6.24) and (6.25). It would be interesting to check this equivalence numerically and verify it analytically.

6.6. Other $\mathbf{Z}_n$ Integrable Theories

There are a variety of other integrable $N{=}2$ theories with spontaneously-broken $\mathbf{Z}_n$ symmetry whose index can be obtained from the equations of the previous subsection. Examples are the affine Toda generalizations of $N{=}2$ sine-Gordon, integrable perturbations of $N{=}2$ Kazama-Suzuki theories described by the $SU(n)_k$ generalized Chebyshev polynomial superpotentials in $n-1$ variables [43] and the CP^{n-1} sigma models [12]. We first consider the $SU(n)$ affine Toda theories described by the action

$$\int d^2z d^4\theta \sum_{j=1}^{n-1} X_j\overline{X}_j + \frac{M}{4n}\left(\int d^2z d^2\theta \sum_{j=1}^{n} e^{ig(X_j-X_{j-1})} + h.c.\right), \tag{6.26}$$

where $X_0 \equiv X_n \equiv 0$. The vacua of this theory form the $n-1$ dimensional weight lattice of $SU(n)$. This theory has $n-1$ topological charges T_r and, as in the sine-Gordon case, the eigenvalues of the Q matrix index can be written as

$$Q(z;\Theta_1,\dots,\Theta_{n-1}) = i\beta L^{-1}\mathrm{Tr}F(-1)^F e^{i\sum_{r=1}^{n-1}\Theta_r T_r} e^{-\beta H}, \tag{6.27}$$

where the trace runs over all boundary conditions.

The tt^* analysis has been applied to this example in [3] where the solutions are expressed in terms of solutions to the affine toda equations (6.22) but now with different boundary conditions, which should now depend on Θ_i.

The S-matrix for the theory (6.26) is (conjectured to be) the tensor product of the $N{=}2$ theory discussed in the previous subsection with an additional $N{=}0$ structure with vacua corresponding to the weight lattice of $SU(n)$: vacua labelled by $SU(n)$ weights μ and ν are connected by a soliton doublet of the

r-th type, with fermion numbers $(r/n, r/n-1)$ and mass $m_r = M\sin(r\pi/n)$, provided the representations satisfy $\mu \otimes \Lambda_r = \nu \oplus \dots$ where Λ_r is the r-th fundamental representation of $SU(n)$ ($r = 1,\dots,n-1$). As far as $N{=}2$ supersymmetry is concerned, every r-type soliton doublet is identical to the r-type doublet in the basic $\mathbf{Z}_n$ theory discussed in the previous section. We will not explicitly carry out the TBA analysis for this theory. Rather, we will use the fact that, as seen in the previous examples, the only effect of the additional $N{=}0$ structure on our index is to modify the $A_r(\theta)$ equations in (6.24) with some θ-independent constants $C_r(\Theta_j)$ (generalizing the term $2\cos\Theta$ in (6.16) for the affine toda case):

$$Q(z;\Theta_1,\dots,\Theta_{n-1}) = \sum_{r=1}^{n-1} m_r\beta\int\frac{d\theta}{2\pi}\cosh\theta e^{-A_r(\theta)}, \tag{6.28}$$

where the $A_r(\theta)$ are solutions to the coupled integral equations

$$\begin{aligned} A_r(\theta) &= m_r\beta\cosh\theta - \ln C_r(\Theta_j) - \sum_{l=0,\bar{0}}\int\frac{d\theta'}{2\pi}\frac{\sin(r\mu)\ln(ia_l + B_l(\theta'))}{\cosh(\theta-\theta') - a_l\cos(r\mu)} \\ B_l(\theta') &= -\sum_{r=1}^{n-1}\int\frac{d\theta''}{2\pi}\frac{\sin(r\mu)e^{-A_r(\theta'')}}{\cosh(\theta'-\theta'') - a_l\cos(r\mu)}. \end{aligned} \tag{6.29}$$

These equations lead to an expansion of the form

$$Q(z;\Theta_j) = \sum_{N_1=1}^{\infty}\cdots\sum_{N_{n-1}=1}^{\infty} C_1^{N_1}\cdots C_{n-1}^{N_{n-1}} A_{N_1,\dots,N_{n-1}}(z), \tag{6.30}$$

where $A_{N_1,\dots N_{n-1}}(z)$ is the contribution from the sector with N_r solitons of type r for $r = 1,\dots,n-1$. Comparing (6.30) with (6.27) it is seen that the $C_r(\Theta_j)$ are given by the character functions

$$C_r(\Theta_j) = \sum_{\lambda\in L(\Lambda_r)} e^{i\lambda\cdot T},$$

where $L(\Lambda_r)$ are the weights in the r-th fundamental representation Λ_r of $SU(n)$ (the $SU(n)$ representation with r vertical boxes for its Young tableau), for

$r = 1, \ldots, n-1$, and $T = \sum_n \Theta_n \alpha_n$ where α_n are the simple roots of $SU(n)$. The above characters can be written (as in [43]) as

$$\sum_{l=0}^{n} C_l(\Theta_1, \ldots, \Theta_{n-1}) t^l = \prod_{j=1}^{n} (1 + t e^{i(\Theta_j - \Theta_{j-1})}), \tag{6.31}$$

where $\Theta_0 \equiv \Theta_n \equiv 0$; expanding the product in t and equating coefficients of t^l on both sides yields the above sums over the fully antisymmetric representations. Plugging these $C_r(\Theta_i)$ into (6.29) and (6.28) yields the index $Q(z;\Theta_j)$. It follows from (6.31) that $C_r(\Theta_i)^* = C_{n-r}(\Theta_i)$ from which it follows from (6.29) that $A_r(\theta;z;\Theta_i)^* = A_{n-r}(\theta;z;\Theta_i)$ and $B_0(\theta;z;\Theta_i)^* = B_{\bar{0}}(\theta;z;\Theta_i)$. It then follows that the index eigenvalues $Q(z;\Theta_i)$ are real.

The above integral equations must provide the regular solutions of the Toda differential equations (6.22). As in the PIII case, physics has proven a statement for which there is, as of yet, no purely mathematical proof. In particular this gives an $n-1$ parameter family of regular solutions to radial affine Toda equations (6.22). The dependence of the solutions on these parameters (Θ_i) is again in line with the intuition discussed in the case of PIII (see footnote 12).

We now consider the theory with superpotential given by the $SU(n)_k$ Chebyshev polynomial $W_{n+k}(X_1, \ldots, X_{n-1})$ in $n-1$ variables discussed in [43]; the generating function for these potentials is

$$\sum_p W_p(X_1, \ldots, X_{n-1}) t^p = -\log(1 + \sum_{r=1}^{n-1} X_r(-t)^r + (-t)^n). \tag{6.32}$$

These theories have been discussed in [43,3,48,49,12,50] ($Sp(N)_k$ theories, which might also be integrable, have been found in [51,50]). From the tt^* analysis it is possible to see that the index is again related to the affine toda equation [3]. From the TBA integral equations the $(n+k-1)!/(n-1)!k!$ eigenvalues of the matrix index for the $SU(n)_k$ theory are expected to be obtained from the equations (6.28) and (6.29) by setting $C_r(\Theta_i) = X_r(\mu)$ where $X_r(\mu)$ are the solutions of $dW_{n+k}(X_r) = 0$. This is equivalent [43] to setting $C_r(\Theta_i) = S_{\Lambda_r\mu}/S_{0\mu}$, where $S_{\mu\nu}$ is the $SU(n)_k$ modular transformation matrix, Λ_r is the r-th fundamental representation $SU(n)$, μ is one of the $(n+k-1)!/(n-1)!k!$ highest weight representations of $SU(n)_k$, and 0 is the identity. These integral equations for the index eigenvalues of the $SU(n)_k$ Chebyshev theory generalize the $SU(2)_k$ results in subsect. 6.3 and the $SU(n)_1$ results in sect. 6.5. Finally, we consider the CP^{n-1} sigma model. Here again, the tt^* equations give the affine toda equations [46] (with logarithmic boundary conditions). The S-matrix for CP^{n-1} sigma model is obtained from the $k \to \infty$ limit of this $SU(n)_k$ Chebyshev theory [12]. Thus, the index for this sigma model is also given by the above integral equations by setting the $\Theta_r = 0$, i.e. $C_r = n!/r!(n-r)!$, in (6.29).

7. Conclusions

We have seen that for two-dimensional N=2 supersymmetric theories $\mathrm{Tr}(-1)^F F e^{-\beta H}$ is a (matrix) index in a generalized sense, i.e., it is independent of $D-$term perturbations. Though the index is not topological, it is determined exactly via non-linear differential equations which are obtained using only topological data, namely the chiral ring. These non-linear differential equations encode the geometry of the vacua of the theory. This allows us to read off, by an IR expansion, the spectrum of Bogomolnyi saturated states of the theory as well as some aspects of their interaction. In case the theory is integrable and the exact S-matrix is known, the index can be computed using TBA methods in terms of solutions to coupled integral equations. It is a very non-trivial statement that in these cases the integral equations thus obtained are equivalent to the differential equations characterizing the geometry of vacua.

It is amusing that one can apply N=2 formalism to study polymer physics [39]. The index in this context is the partition function of a single polymer wrapped around a cylinder of perimeter β.

Given the fact that the integral equations which arose for us in the context of the TBA are equivalent to differential equations, it is very natural to ask if this can be done more generally. In other words, is it *always* possible to relate integral equations arising for integrable theories through the TBA to ordinary differential equations? A first step in this direction may be to try to prove mathematically why in our case the integral equations of the TBA were equivalent to differential equations.

For non-integrable theories the tt^* equations can still be used to compute the index. In the infra-red the leading contribution to the index is universal (4.1). However, even though the normalization of this term can be easily deduced from a Hilbert space interpretation, from the viewpoint of solving the tt^* equations this is only fixed by requiring the regularity of the solution (even in the UV regime). Therefore, it is a very non-trivial test of these ideas that the normalization coming from solving the tt^* equations in the IR agrees with the Hilbert space interpretation. This has recently been confirmed even in a non-integrable case by solving tt^* equations numerically [25].

We have seen that the index basically captures the geometry and interaction of kinks interpolating between supersymmetric vacua. It would be interesting to write a general solution (say for Landau-Ginzburg theories) of the tt^* equations in terms of these kinks. Such a thing is not unexpected, given the fact that tt^* equations depend only on the superpotential (which is equivalent to knowing the kink spectrum and their geometry) and that the equations are integrable as they can be rephrased as flatness conditions even if the underlying quantum field theory is not integrable, a fact which has been recently elaborated upon in [24].

Given the power of the new supersymmetric index in encoding exact results, it would be tempting to look for similar objects in other supersymmetric theories. In particular a very similar setup to what we have discussed in this paper appears naturally in the context of four-dimensional N=2 Yang-Mills theory. Again this theory is related to a topological theory [52] and the analog of the chiral fields are the two-cycle observables. In particular for the $SU(2)$ gauge theory in the Higgs phase where $SU(2)$ is broken to $U(1)$, all the known particles of the theory such as the massive gauge particles and the monopoles are known to saturate the Bogomolnyi bound [15] very much as our kinks in the two-dimensional theory saturate the Bogomolnyi bound. In this case the natural generalization of our index seems to be $\mathrm{Tr}(-1)^F J^2 e^{-\beta H}$ where J is the

generator of $U(2)$ symmetry of N=2 theories[53]. It would be exciting to see what exact information about the S-matrix of these four-dimensional theories are encoded in such an index.

To formulate our index we needed to put the Hilbert space on infinite line to allow for kinks. If we put the Hilbert space on a periodic circle and thus compute the index on the torus we get zero. This can also be seen by CPT invariance. However if we replace F by F_L, the left-moving fermion number, in the definition of the index, CPT no longer requires it to vanish (as even on the torus $(-1)^{F_L}$ is in general not ± 1). This quantity has already appeared in the context of conformal theories where it is related to the moduli dependence of the gauge and gravitational coupling constants [54,55]. This modified index resembles the generalization of Ray-Singer torsion [56] to conformal theories.

We have seen that N-kink configurations each contribute to our index through an 'anomaly' resulting from an inequality in the density of states for a supersymmetric multiplet (2.13). Each of these contributions reminds one of (though it is not the same as) the Callias-Bott-Seeley index [14]. It would be very exciting to uncover the meaning of such a 'topological invariant' for each N-kink contribution. Our new index, which sums up the contribution of all N-kink configurations, would then encode infinitely many topological invariants into a single function!

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Appendix A. An Operatorial Proof of the tt* Equations

In this appendix we show how the ideas of the present paper can be used to give a quick (although less rigorous) proof of the tt^* equations of [3].

We begin by rewriting the tt^* equations in terms of the Q matrix only. We have seen in the main body of the paper that

$$Q_{ab} = \frac{i\beta}{L}\mathrm{Tr}_{ab}(-1)^F F e^{-\beta H}, \tag{A.1}$$

where a, b label boundary conditions at spatial infinity associated to some basis $|a\rangle$ of vacua.

In terms of Q the basic tt^* equations read

$$\begin{aligned} \overline{D}_{\bar{i}}Q &= 2\beta^2[C_\tau, \overline{C}_{\bar{i}}] \\ D_i Q &= 2\beta^2[C_i, \overline{C}_{\bar{\tau}}]. \end{aligned} \tag{A.2}$$

$$\begin{aligned} \tfrac{1}{2}[Q, C_j] &= -C_j + D_j C_\tau \\ \tfrac{1}{2}[Q, \overline{C}_{\bar{j}}] &= \overline{C}_{\bar{j}} - \overline{D}_{\bar{j}}\overline{C}_{\bar{\tau}}. \end{aligned} \tag{A.3}$$

where D_i ($\overline{D}_{\bar{i}}$) denotes the metric covariant derivative and C_τ is the matrix representing multiplication by W (in the case of a Landau-Ginzburg theory, which we assume in this section for simplicity) on the vacua $|a\rangle$. One can think of Q in the special parametrization of the couplings of the theory motivated from the renormalization group as $Q = 2(A_\tau - A_{\tau^*})$ with $\beta = \exp(\tau/2 + \tau^*/2)$, from which one can deduce the above equations from the usual tt^* form (note that a chiral operator has an explicit τ dependence given by $e^\tau \phi_i$). The above equations are written so as to make sense in an arbitrary basis (see also [8]). Eq.(A.3) has the following interpretation. For a quasihomogeneous family of superpotentials $C_\tau = 0$ and (A.3) just states that the marginal chiral (resp. anti-chiral) deformations have $U(1)$ charge $+1$ (resp. -1). The extra term in the RHS measures the 'deviation from marginality'.

On the other hand, (A.3) allows us to write the full metric connection A_i in terms of Q and the ring coefficients. Then we can compute its curvature in terms of Q. Because of this, (A.2) and (A.3) together with

$$\begin{aligned} D_i\overline{C}_{\bar{j}} &= \overline{D}_{\bar{j}}C_i = 0 \\ D_i C_j &= D_j C_i, \end{aligned} \tag{A.4}$$

reproduce all the tt* equations. (In fact, the second line is a consequence of (A.3) together with known properties of the chiral ring). In particular, we get

$$\begin{aligned} [D_i, \overline{D}_{\bar{j}}]Q &= -\beta^2[[C_i, \overline{C}_{\bar{j}}], Q] \\ [D_i, \overline{D}_{\bar{j}}]C_\tau &= -\beta^2[[C_i, \overline{C}_{\bar{j}}], C_\tau] \end{aligned}$$

from which we read the curvature of the metric connection[14].

One has also the identity (Δ is defined in (2.2))

$$[C_\tau, \overline{C}_{\bar{j}}]_{ab} = \tfrac{1}{2}\mathrm{Tr}_{ab}(-1)^F \Delta\overline{\phi}_{\bar{j}} e^{-\beta H}. \tag{A.5}$$

This equation deserves a comment. The simplest way of proving it is to choose the vacuum basis $|a\rangle$ to correspond to the canonical basis, i.e. the holomorphic point basis (normalized so that $\eta = 1$). Then the central charge has a definite value $\Delta_{ab} = 2[W(a) - W(b)]$, and (A.5) follows from the definition of C_τ and the obvious identity

$$(\overline{C}_{\bar{i}})_{ab} = \mathrm{Tr}_{ab}(-1)^F \overline{\phi}_{\bar{i}} e^{-\beta H}. \tag{A.6}$$

However, the canonical basis is not the natural one from a 'thermodynamical' viewpoint. In this framework one decomposes the Hilbert space $\mathcal{H}$ into sectors for which Δ *and* $\overline{\Delta}$ have a definite value. Such sectors should exist on general grounds. Now, whereas it is manifest that the canonical boundary conditions give a definite value for Δ, it seems unlikely that they also have a definite $\overline{\Delta}$. Roughly speaking, the natural boundary conditions should correspond to the 'real' point basis for vacua, defined by prescribing the asymptotical value of the scalar fields to to be a classical vacuum[15]. We have two comments: First the identity above, being covariant under changes of bases, should be valid even in such a 'real' basis. Second, in the simplest situations we can explicitly construct the 'real' point basis and check the consistency of our formal manipulations.

[14] At least for a generic superpotential, these equations fix the curvature unambiguously.

[15] This can be made more precise by defining the 'real' point vacua by starting from a large circle to quantize the theory where the point basis is unambiguous and adiabatically change the radius of the circle.

At any rate it would be worthwhile understanding the real point basis more clearly.

The new proof of tt* consists in showing that Eq.(A.2) and (A.3) follows from the representation (A.1) of Q and the 'AB argument'. At the formal level we have (using the 'AB argument')

$$\begin{aligned}\delta_i \mathrm{Tr}_*(-1)^F F e^{-\beta H} &= \\ &= i\beta L \mathrm{Tr}_*(-1)^F F\{Q^-,[\overline{Q}^-,\phi_i]\}e^{-\beta H} \\ &= -i\beta L \mathrm{Tr}_*(-1)^F\{Q^-,\overline{Q}^-\}\phi_i e^{-\beta H} \\ &= -i\beta L \mathrm{Tr}_*(-1)^F \overline{\Delta}\phi_i e^{-\beta H},\end{aligned}$$

where $*$ means some sector (a,b) of the Hilbert space. Clearly, in view of (A.1) and (A.5), this is the same as (A.2) provided we interpret δ_i as D_i, i.e. as the metric covariant derivative. This is the correct interpretation. In general we get some contribution to the derivative of $\mathrm{Tr}_*(-1)^F F \exp[-\beta H]$ from the variation of the boundary condition $*$. Such terms have a structure which allows to absorb them in the definition of the connection in D_i. This is natural, because in a sense the path integral variation should 'dress' the vacua at infinity as well to make them be the new vacua. In this interpretation, for example, what the 'AB argument' for invariance of the Witten's index discussed in section 2 really shows is that ground state metric g is covariantly constant. Similarly here this suggests that we have *some* covariant derivative such that

$$D_i \mathrm{Tr}_*(-1)^F F e^{-\beta H} = -i\beta L \mathrm{Tr}_*(-1)^F \overline{\Delta}\phi_i e^{-\beta H}.$$

(And analogously for $\overline{D}_{\bar{i}}$). The connection cannot be trivial. Indeed, as shown above, we can use the resulting equation to compute its curvature which turns out to be non-vanishing. It remains to show that the connection predicted by this argument is the metric one. Indeed the AB argument predicts

$$D_i \mathrm{Tr}_*(-1)^F e^{-\beta H} = \overline{D}_{\bar{i}} \mathrm{Tr}_*(-1)^F e^{-\beta H} = 0,$$

for the connection induced by the variation of the boundary condition $*$.

The same reasoning applied to (A.6) gives

$$D_i(\overline{C}_{\bar{j}})_* = D_i \mathrm{Tr}_*(-1)^F \overline{\phi}_{\bar{j}} e^{-\beta H} = -i\beta L \mathrm{Tr}_*(-1)^F \overline{\phi}_{\bar{j}}\{Q^-,[\overline{Q}^-,\phi_i]\}e^{-\beta H} = 0$$

$$D_i(C_j)_* = D_i \mathrm{Tr}_*(-1)^F \phi_j e^{-\beta H} = -i \mathrm{Tr}_*(-1)^F \phi_j\{Q^-,[\overline{Q}^-,\phi_i]\}e^{-\beta H} = D_j(C_i)_*,$$

showing (A.4).

Finally we show (A.3). By definition, one has (as $L \to +\infty$)

$$\begin{aligned}[Q,C_j]_* &= \frac{i\beta}{L}\mathrm{Tr}_*(-1)^F F e^{-\beta H}\Big[\phi_j(x=+L/2)-\phi_j(x=-L/2)\Big] \\ &= \frac{i\beta}{L}\mathrm{Tr}_*(-1)^F F e^{-\beta H}\int_{-L/2}^{+L/2} d_x\phi_j.\end{aligned}$$

For L large, we can replace

$$\int_{-L/2}^{+L/2} d_x\phi_j \to \frac{iL}{2}\{Q^+,[Q^-,\phi_j]\} - \frac{iL}{2}\{\overline{Q}^+,[\overline{Q}^-,\phi_j]\}.$$

In this way we get an expression to which we can apply the 'AB argument'. One gets

$$\begin{aligned}[Q,C_j]_* &= \frac{i\beta}{2}\mathrm{Tr}_*(-1)^F e^{-\beta H}[\{Q^+,Q^-\}+\{\overline{Q}^+,\overline{Q}^-\}]\phi_j \\ &= -\beta \mathrm{Tr}_*(-1)^F e^{-\beta H} H\phi_j = \beta D_\beta\left[\mathrm{Tr}_*(-1)^F \phi_j e^{-\beta H}\right].\end{aligned}$$

In view of (A.4) and dimensional analysis (i.e. dependence of the fields on the scale) this equation is equivalent to (A.3). Indeed, this is the covariant version of the statement (true only in special 'gauges') that Q is the connection in the τ direction. To compare with computations done in such special gauges recall that $\beta = e^{\tau/2+\tau^*/2}$ and so $\beta g\partial_\beta g^{-1} = 2g\partial_\tau g^{-1}$.

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Topological Mirrors and Quantum Rings

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Aspects of duality and mirror symmetry in string theory are discussed. We emphasize, through examples, the importance of loop spaces for a deeper understanding of the geometrical origin of dualities in string theory. Moreover we show that mirror symmetry can be reformulated in very simple terms as the statement of equivalence of two classes of topological theories: Topological sigma models and topological Landau-Ginzburg models. Some suggestions are made for generalization of the notion of mirror symmetry.

11/91

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1. Introduction

One of the most fascinating aspects of string theory is the way it modifies our intuition of classical geometry. It modifies it in ways which in some sense makes the classical geometry *more symmetrical*, and thus, in a sense simpler. This is probably most manifest in the principle of duality in string theory, which states that two classically inequivalent geometries (target spaces for strings) can nevertheless be identical from the string point of view. The aim of this paper is to develop this notion emphasizing the basic physical reasons for believing in its universal existence. My presentation is written with the mathematically oriented reader in mind and even though I will not be fully rigorous I hope that the main ideas are more or less clear to mathematicians.

I will first discuss some general aspects of Hilbert space of strings propagating in a target space in a geometrical way and discuss the notion of duality in this set up (section 2). Then I give some simple examples of this duality for bosonic strings (section 3). In section 4, I will discuss aspects of fermionic (super-) string vacua highlighting aspects which are relevant for mirror symmetries. As we will see an important ingredient in this setup is the notion of *quantum cohomology ring* of Kahler manifolds which is a deformation of the ordinary cohomology ring. In section 5 the relation between singularity theory and solutions of superstrings is discussed. This turns out to be a convenient bridge between target space interpretation and abstract conformal field theory definition of string theory. In section 6 the topological formulation of mirror symmetry is discussed. This turns out to be a very effective language to describe mirror symmetry. In this setup, mirror symmetry is stated as the equivalence of two seemingly inequivalent topological theories. This topological formulation has the advantage of simplifying the conformal theory to a much simpler theory which is the relevant piece needed for the discussion of mirror symmetry. Finally in section 7 I discuss some puzzles for mirror symmetry and their potential resolutions. I also discuss some potential generalizations of mirror symmetries and some possible connections with quantum groups and Donaldson theory.

2. String Hilbert space

In this section we discuss the basic structure of string vacua which involves the Hilbert space and operatorial formulation of the theory (this aspect is discussed much more extensively in the talks of Friedan in this conference; for a mathematical introduction see [1]). Consider a closed string (one dimensional parametrized circle) sitting in a Riemannian manifold M. The space of all such configurations is given by the (parametrized) loop space of M which we denote by $\mathcal{L}M$. The geometrical questions that arise in string theory basically correspond to probing the geometry of $\mathcal{L}M$. The Hilbert space of *bosonic strings* is an 'appropriate' category of function space on $\mathcal{L}M$, which we denote by

$$\mathcal{H}_{bosonic} = \Phi(\mathcal{L}M)$$

with norm inherited from the metric on M. The Hilbert space of *fermionic or superstrings* is the space of semi-infinite forms on $\mathcal{L}M$:

$$\mathcal{H}_{fermionic} = \Lambda^{\infty}(\mathcal{L}M)$$

In addition to this Hilbert space, there is a more or less canonical one to one correspondence between the states $|v\rangle$ in the Hilbert space and some 'special' operators O_v acting on the Hilbert space. Roughly speaking, these operators are characterized by the fact that they are 'invariant' under reparametrizations of the string and that when they act on a special state $|0\rangle$ (the vacuum state) in the Hilbert space, they give the corresponding state $(O_v|0\rangle = |v\rangle)$. These form a complete operator product algebra, in the sense that the product of any two of these operator is another such operator. Choosing a basis, we have

$$O_i O_j = \sum_k C_{ij}^k O_k$$

where the sum over k is generically an infinite sum.

A convenient method of computing C_{ij}^k is as follows: In string theory to find the amplitude of how a number of loops $l_i \in \mathcal{L}M$ ends up changing to the loops $l_j \in \mathcal{L}M$ we have to sum over all interpolating surfaces Σ immersed in M, $f(\Sigma) \subset M$ whose boundary is

$$\partial f((\Sigma)) = \bigoplus l_i - \bigoplus \tilde{l}_j$$

weighed by $exp(-E)$ where E is the energy functional of the surface immersed in M (a natural extension of this applies to fermionic strings). We can choose a 'basis' for our Hilbert space of delta functions corresponding to fixed loops in the manifold. The above prescription then gives a way to compute the amplitude that two of these basis elements ends up with the third one. This can be extended to the full Hilbert space by multi-linearity of the amplitude. The amplitude thus computed for the two string state $|i\rangle$ and $|j\rangle$ to end up with the third one $|k\rangle$ can be obtained by integrating the 'wave function' of these states against the basic amplitude with the delta functions. The resulting answer is in fact the same as C_{ij}^k.

There are consistency conditions that Hilbert space and these coefficients need to satisfy for a consistent theory (following from the associativity of the operator products and modular invariance of string amplitudes). Once we are given such a structure, we can forget about M altogether and talk about the 'string vacuum', meaning this abstract Hilbert space with some canonical set of operators satisfying some 'nice' operator product properties. Let us denote such a structure by S and call it a *string vacuum*. Then two string vacua are equivalent, or isomorphic, if there is an isomorphism between the corresponding Hilbert spaces and the operators. Now it may happen that strings on two different manifolds M_1 and M_2 give rise to isomorphic string vacua

$$M_1 \neq M_2 \quad but \quad S(M_1) = S(M_2).$$

In other words *the map from manifolds to string vacua may be many to one.* In such a case we call the manifolds M_1 and M_2 *dual or mirror pairs*. Actually the choice of the terminology is unfortunate, as it may happen that more than two manifolds may give rise to the same string vacuum. One could also ask the reverse question: Does every string vacuum come from a manifold, i.e., is this map onto? The answer seems to be no (see for example [2]).

The existence of mirror symmetry is thus simply the statement of the existence of different geometrical ways to realize a string vacuum. We can use any representation we please. In such cases, if we try to study some aspects of the string vacuum we can choose any realization and may thus end up equating a 'hard' geometrical computation in one representation to an 'easy' one in another realization. In this lies the power of mirror symmetry transforming a hard problem to an easy one. In the next section we give some examples of mirror pairs in the context of bosonic strings.

3. Examples of Bosonic Mirrors

In this section we consider examples of mirror manifolds which lead to the same string vacuum for the bosonic strings. We will give two classes of examples: In one class the mirror Riemannian manifolds are topologically the same but geometrically distinct, and in the second class the mirror manifolds are even topologically distinct.

Let M_1 be the d dimensional torus identified (as a Riemannian manifold) with

$$M_1 = \frac{E^d}{\Gamma}$$

where Γ is a d dimensional discrete lattice group acting by isometry on flat Euclidean space E^d. Let us consider the Hilbert space of strings on M_1 which is related to the function space on $\mathcal{L}M_1$. First note that $\mathcal{L}M_1$ naturally splits to infinitely many components, corresponding to each element of Γ which can be identified with $H_1(M_1, Z)$. Moreover, the function space on each component splits to the functions of the center of strings which is isomorphic to ordinary function space on M_1, and functions of oscillations of loops (which is universal and independent of Γ). The function space on M_1 is canonically isomorphic to Γ^*, the dual lattice to Γ using Fourier transform. So the dependence of Hilbert space of strings on Γ, appears as a choice of loop component (an element of Γ) and the Fourier component of functions of center of string (an element of Γ^*), i.e., the dependence comes through a choice of element of

$$\Gamma + \Gamma^*$$

This implies that if we consider the second manifold M_2

$$M_2 = \frac{E^d}{\Gamma^*}$$

Then the Hilbert spaces of strings based on M_1 and M_2 are isomorphic, both depending on the *self dual* lattice $\Gamma + \Gamma^*$. This turns out to extend to the full string vacuum structure, i.e., to the operators and their products. So M_1 and M_2 are mirror pairs. In physical terms this implies that there is no physical experiment one can do in string theory to distinguish strings on M_1 from strings on M_2. This means in particular that the notion of 'length' is not a universally invariant way to decide if two manifolds are different as far as strings are concerned. This simple example illustrates the basic structure of duality or mirror symmetry in bosonic strings. This in fact was the first example of mirror symmetry discovered in string theory [3]. The rest of the examples are just extensions of this to more intricate cases.

For our second class of example we consider a simply laced compact Lie group G. Let H denote its Cartan torus. Consider an element $g \in G$ of finite order which belongs to the normalizer of H (i.e., it acts as a Weyl transformation on H). This means that

$$H \to gHg^{-1}$$

Let us denote the cyclic group generated by this transformation Λ_1 (we take g to act non-trivially on H). Choose an element $h \in H$ conjugate to $g \in G$. Consider the action

$$H \to h\ H$$

and denote the group action generated by this cyclic group Λ_2. Consider taking the quotients of H by these two different group actions:

$$M_1 = \frac{H}{\Lambda_1} \qquad M_2 = \frac{H}{\Lambda_2}$$

These two spaces are completely different. In fact M_1 is not even a manifold, but an orbifold, as g acts by fixed points on H, but M_2 is simply another torus, as h simply generates translations on H. It turns out that (bosonic) strings propagating on M_1 and M_2 are equivalent [4]. It is somewhat surprising that M_1 which is not even a manifold behaves very much like the smooth manifold M_2 as far as strings are concerned. This means, in the mathematical sense (as is also seen in examples for superstrings [4][5]) that loop space of an orbifold is a far better behaved object than the orbifold itself and in a sense provides a kind of universal space for resolution of orbifold singularity.

It should also be clear from the above examples that we can construct examples where three (or more) inequivalent Riemannian manifolds lead to the same string vacuum.

4. Superstring Vacua and Quantum Cohomology Rings

Most of our discussion up to now has been on bosonic strings. This is the case in which the Hilbert space is roughly speaking the function space on the loop space of manifold. However fermionic string is the physically (and mathematically) more interesting case. This is the case corresponding to the Hilbert space of semi-infinite forms on the loop space. In most applications one considers target spaces which are Kahler manifolds. In this case the Hilbert space and the operators acting on it naturally admit $Z \oplus Z$ grading, corresponding to the (holomorphic, anti-holomorphic) degree of the differential forms. Let $\mathcal{O}$ denote the space of physical operators. Then we have the decomposition according to the degrees of the forms:

$$\mathcal{O} = \bigoplus_{p,q \in Z} \mathcal{O}_{p,q}$$

Naturally under operator products the degrees add, as expected. Note that since we are dealing with semi-infinite differential forms, the degree of operators runs from $-\infty$ to $+\infty$. This is an important difference with respect to the differential forms on the ordinary manifolds where the degree of differential forms is positive. As we shall see later this is one of the main reasons for the prediction of mirror symmetry in the fermionic strings. There is an anti-unitary involution which implies that $O_{p,q}$ is the conjugate of $O_{-p,-q}$. The existence of this anti-unitary involution is the statement of CPT invariance of the theory

In the language of forms, since we are dealing with semi-infinite forms, it is roughly the statement that operation of 'adding' and 'subtracting' forms are conjugate operations. This turns out to be an important piece of physics in the story of mirror symmetry.

Since the manifold M is naturally embedded in $\mathcal{L}M$, one expects that at least the differential forms on M are related to a subset of those on $\mathcal{L}M$ and in particular the cohomology ring of M should correspond to some closed operator algebra (modulo addition of cohomologically trivial elements) of operators acting on the fermionic Hilbert space. Let d denote the complex dimensions of M. Then we expect that there exist a special set of operators $A_\alpha \in \mathcal{O}_{p,q}$ with $0 \leq p,q \leq d$, such that the operator algebra of A_α correspond to the cohomology ring of M. This expectation turns our to be correct and we denote this subsector of the operators by $H^{*,*}$. In fact more is true [6]: There is a natural way to define the product of these operators which yields a closed truncated operator algebra when restricted to this special finite subspace of operators which becomes finite and related to the cohomology ring[2]. There is one important subtlety however: Unlike the ordinary cohomology ring, the ring we get *depends* on the Kahler class of the metric on M. Only in the limit where we rescale the metric $g \to \lambda g$ and let $\lambda \to \infty$ do the ring of A_α's become exactly the cohomology ring of M. The deviation from the classical result is due to instanton corrections [7] (an explicit exact result for instanton correction on Z orbifold is discussed in [8]). So string theory deforms the cohomology ring. A nice description of this deformation is as follows [9]. In order to describe this it is more convenient to go to the dual basis (i.e., homology). Let A^α denote the dual basis. Each α can be represented by a cycle in M. In order to specify the ring, it is sufficient to give the trilinear pairing between cycles. The ordinary ring is obtained by defining this pairing to be

$$< A^\alpha A^\beta A^\gamma >= \#(C^\alpha \cap C^\beta \cap C^\gamma)$$

[2] This is unlike the ordinary cohomology ring of manifold, in that the actual product of harmonic representatives does not form a closed operator algebra.

i.e., the number of common intersection points of the three cycles (and defining it to be zero if the common intersection has dimension bigger than zero). To define the ring we obtain in string theory we have to consider the space of holomorphic maps from CP^1 to the manifold M (rational curves in M), with the restriction that three fixed points on CP^1 get mapped to points in C^α, C^β and C^γ respectively. Again if the dimension of moduli of such maps is positive they do not contribute to the cohomology ring. The isolated ones contribute weighed by the instanton action. Let us denote an element of the space of such holomorphic maps by $h^{\alpha\beta\gamma}$. Let U denote the image of the sphere under h. Let k denote the Kahler form on M. Then the definition of the deformed ring (which is commutative and associative as shown in [9]) is

$$< A^\alpha A^\beta A^\gamma >= \sum_{h^{\alpha\beta\gamma}} \#(C^\alpha \cap U)\cdot\#(C^\beta \cap U)\cdot\#(C^\gamma \cap U) \exp - \int h^{*\alpha\beta\gamma}(k) \quad (4.1)$$

Note that in the limit $k \to \infty$ only the constant holomorphic maps survive in this sum and that gives back the ordinary definition of intersection between cycles. So in this way we have a ***quantum deformed*** cohomology ring. To actually derive (4.1) in the context of string theory (and define it properly for multiple covers of holomorphic maps)[3] is achieved by showing the topological nature of computation (and showing that on the cylinder it can be rephrased as a computation in a topological sigma model [9] which is discussed briefly in section 6). Without going to much detail let me at least indicate why its form is reasonable from what we have discussed up to this point. As we have discussed before to compute the algebra of operators in string theory we have to consider maps of a sphere with three discs cut out, to the manifold with three fixed boundary circles mapped to specific loops on the manifold. For constant loops or loops which are 'close' to being constant, we can take the limit in which the discs shrink to points, and map a specific point on CP^1 to a particular point on the manifold. Now the string loop amplitude computation tells us that we have to sum over all such loops weighed with e^{-E}, which in this case is

[3] Recent progress from this viewpoint has been made in [10].

nothing but the exponential of the pull back of the Kahler form on the map, as it appears in (4.1). The factors in front of exponential simply counts how many inequivalent ways a fixed rational curve could map to the three cycles (which is accomplished by an $SL(2,C)$ transformation of CP^1 to move the three points on the sphere). The fact that we sum over only holomorphic maps in (4.1) and get an exact answer and its precise definition can be best understood in the topological description of sigma models [9].

It is quite natural to speculate that this deformed ring may be the actual cohomology ring on a properly defined loop space. One way this may be realized is to consider the space of holomorphic maps from the disc to the manifold. The map from the boundary of the disc to the manifold induced from such maps may be viewed as a 'modified' loop space. In this loop space the points of the manifold will be represented more than once in the loop space; in fact if we look for the space of constant loops which was previously isomorphic to the manifold, that would be the same as looking for holomorphic maps which take the boundary of the disc to a point, which is basically a holomorphic map from the sphere to the manifold. So in this case the manifold and all the holomorphic curves in it are representing the original manifold in this loop space. In this set up it is likely to expect that there exists a fixed point formula for the cohomology elements (corresponding to the circle action on the loop) which reduces the computation of cohomology elements to the fixed point subspace which consists of the manifold and the holomorphic curves in it. This would then (presumably) give rise to the cohomology ring defined in (4.1) with $k=0$. We can then expect to get the deformed ring by twisting the cohomology ring, which allows us to weigh the different fixed points (i.e., different holomorphic maps) differently, and thus obtain the formula (4.1) with $k\neq 0$. This line of thought is worth pursuing further and may lead to a better geometrical understanding of the loop space itself.

As an example, if one considers strings on CP^1, if we denote by x the standard $(1,1)$ cohomology element, the classical cohomology ring is generated by x with

$$x^2=0$$

Let $\beta=\exp-\int k$ integrated over the nontrivial 2-cycle. Then the quantum deformed cohomology ring can be computed from its definition given above and is generated by x but the relation is deformed to [9]

$$x^2=\beta$$

This can be generalized to CP^n [11] with the result that the quantum cohomology ring is defined by

$$x^{n+1}=\beta$$

We will discuss the conjectured generalization of this to the Grassmanians in the next section (see also [11]).

Note that in the above examples the deformed or quantum cohomology ring does not respect the grading of differential forms (in physics terminology we say that the instantons have destroyed chiral fermion number conservation), but the amount of violation of grading can be understood. The point is that the (formal) dimension of moduli space $\mathcal{M}$ of holomorphic maps h is given by

$$\dim\mathcal{M}=d+c_1(h)$$

where d is the dimension of manifold and $c_1(h)$ denotes the evaluation of the first chern class on the image of h. By the definition of quantum cohomology ring we see that the sum of dimensions of cohomology elements will have to be $d+c_1(h)$ in order to get a non-vanishing result, which means that we have a violation by $c_1(h)$. This explains the cohomology ring structure for CP^m discussed above (where $c_1=n+1$ for the fundamental cycle). Note the fundamental role played by Kahler manifolds where $c_1=0$, i.e., the Calabi-Yau manifolds. In this case there is no violation of the grading, and we indeed get a quantum cohomology ring which respects the cohomology grading. For Calabi-Yau manifolds of dimensions one and two (torus and $K3$), there are generically no holomorphic maps (this is due to the fact that if there were any there would be a three dimensional family of them by Mobius transformations, and so this would be in contradiction with the above formal dimension). So the first case of interest in terms of the deformation of cohomology rings is the case of Calabi-Yau 3-fold, which has also been the case of most interest for string theory[4].

To obtain a 'static' solution to superstring theory, it turns out that the target Kahler manifold M should admit a Ricci-flat metric, i.e., by Yau's theorem it should be a Calabi-Yau manifold[5]. In such a case the dimensions of $H^{d,0}(M)$ is one, and thus we have in our theory an operator corresponding to this element in $H^{d,0}$. This operator induces an isomorphism on the space of operators by multiplication [6]. This is known as the spectral flow and gives the isomorphism

$$O_{p,q} \sim O_{p+d,q}$$

The fact that this is an isomorphism is related to the existence of the conjugate (or inverse) operator. In other words By conjugation *there must also exist conjugate operators* in $H^{-d,0}$. This operator induces a correspondence between operators:

$$O_{p,q} \sim O_{p-d,q}$$

(similar statements of course hold for conjugate sectors of the Hilbert space and amounts to shifting the anti-holomorphic degree by $-d$). This isomorphism in particular applies to the special operators operators $H^{p,q}$ with $0 \leq p,q \leq d$ which represent cohomology of M and thus suggests that there are also 'special'

4 For manifolds which have $c_1 < 0$, by which I mean there are some two cycles where c_1 evaluates to a negative number, the underlying theory is not very well behaved (i.e., it is not asymptotically free) and it seems that similarly the quantum cohomology ring is somewhat ill defined (in the Landau-Ginzburg description to be mentioned in section 5 it corresponds to perturbing the action by non-renormalizable terms with charge greater than 1). So quantum cohomology rings make better sense for $c_1 \geq 0$. However it would be interesting to see, and there is some indication [12] that maybe the mirror map acts on the space of *all* Kahler manifolds (possibly non-compact) by flipping the sign of c_1, which in particular sends a Calabi-Yau manifold to another Calabi-Yau manifold.

5 Physically we should not ignore other manifolds as is commonly done, since one can use them to construct interesting non-static solutions of string theory, of the type relevant for cosmology (see for example [13]

operators which we denote by $H^{p-d,q}$, by shifting the holomorphic degree by $-d$. These special operators have the following properties which follows by the above isomorphism:

$$\dim \mathrm{H}^{0,0} = \dim \mathrm{H}^{-d,0} = \dim \mathrm{H}^{0,d} = \dim \mathrm{H}^{-d,d} = 1$$

$$\dim \mathrm{H}^{-p,q} = \dim \mathrm{H}^{d-p,q} = \mathrm{h}^{d-p,q}$$

where $h^{*,*}$ denote the hodge numbers of M. It looks as if the operators in $H^{-p,q}$ describe the cohomology of a d-dimensional manifold which has the same hodge diamond as M except that it is flipped. In fact from the structure of string vacuum [6] it follows that there is a closed operator ring among these states which is additive in terms of their $Z \oplus Z$ grading just as was the case for the operators $H^{p,q}$ with $0 \leq p,q \leq d$. Note that the correspondence between cohomology elements of $H^{-p,q}$ and $H^{d-p,q}$ do not respect the ring structure and is thus not an isomorphisms of these rings. So we learn that *for any Calabi-Yau manifold we find not one but two rings–only one of which is related to the deformed cohomology ring of the manifold*. This second ring we call the *complex ring* of the manifold as it will turn out to (generically) characterize the complex structure of the Calabi-Yau manifold[6].

So far we have described the Hilbert space and operators corresponding to strings imbedded in a Calabi-Yau manifold M. But usually we are given not a Calabi-Yau manifold, but the string vacuum itself, i.e., a Hilbert space and a set of operators acting on it. Note that in the isomorphism class of string vacua, if we just relabel the labels of $O_{p,q}$ by $O_{-p,q}$ we have not changed the string vacuum, and we obtain an isomorphic vacuum. This involution of one of the gradings simply exchanges the two rings that we discussed above. In this abstract setting how do we decide which of these two rings are 'preferred' in the sense that it corresponds to the deformation of the cohomology ring

6 The complex ring can be viewed geometrically as the ring generated by wedging $H^q(\Lambda^p\Theta)$ where Θ represents the holomorphic tangent bundle [14]. That their dimension is related to that of $H^{d-p,q}$ can be easily infered from the existence of a holomorphic d form for the Calabi-Yau case.

of a manifold? Since these two rings are absolutely on the same footing as far as the string vacuum is concerned, i.e., that there is an isomorphic string vacuum which relabels the sign of one of the gradings, the only way to restore the impartiality is *to postulate that for every Calabi-Yau manifold M there is another manifold $\tilde{M}$, such that the string vacuum on either M or $\tilde{M}$ gives rise to both cohomology rings.* This in particular means that

$$h^{p,q}(\tilde{M}) = h^{d-p,q}(M) \tag{4.2}$$

This is the basic idea of mirror symmetry [15] [6]. Note that this idea applies to a Calabi-Yau manifold of any dimension (not just three as is mostly applied to). Also note that the dimension of complex deformations of M which is equal to $h^{1,d-1}(M)$ is equal to the dimension of Kahler deformations of $\tilde{M}$ and vice versa. So under this mirror symmetry the shape and size of the manifolds get exchanged. Since the quantum cohomology ring encodes the information about the Kahler class in it, and under mirror symmetry shape and size get exchanged, this explains why the second ring, the complex ring, is characterizing the complex structure of the manifold.

Let us consider the simplest examples of mirror pairs: As we have discussed before for bosonic strings, strings propagating on a torus and the dual torus give identical vacua and form mirror pairs. It turns out that these are in fact also the simplest examples of mirror vacua for fermionic strings. Let us explain this briefly in the context of simplest complex torus, a one dimensional complex torus which is geometrically the product of two circles with radii R_1 and R_2. Then the complex structure τ of the torus and its volume $-i\rho$ are given by

$$\tau = i\frac{R_1}{R_2} \qquad \rho = iR_1R_2$$

Now we apply the duality described in section 2 for bosonic strings in the case of target space being a torus. This duality works equally well for bosonic and fermionic strings. Let us apply that to the second circle of this example sending $R_2 \to 1/R_2$ and we thus end up exchanging $\rho \leftrightarrow \tau$. This is an example of the general phenomena described above namely that the moduli controlling the shape and the size of the Calabi-Yau manifolds are exchanged under such a duality[7]. This is the simplest example of mirror symmetry. It is worth emphasizing that the other beautiful examples that have been found are highly non-trivial to describe geometrically [16] [17] [18] and have far more out reaching consequences. Nevertheless the basic idea remains the same, and fits very naturally into the general framework of duality just as we saw for the bosonic strings.

5. Catastrophes and Superstring Vacua

In this section we describe a link between string vacua and catastrophe theory. The origin of this direction of study of strings was motivated by trying to ignore geometry of target space and classify all string vacua directly (as had been emphasized by Friedan). So far we have mostly described string vacua arising from strings propagating in some target space. However, there are other useful ways to describe string vacua which may or may not be related to such a picture. The main idea is to note that the string amplitude was defined as a sum over all interpolating Riemann surfaces weighed by energy functional $exp(-E)$. Here $E = \int |Dx|^2$ where x denotes the map which defines an immersion of the Riemann surface into the target space (with appropriate addition of fermionic terms in the case of superstrings). The basic idea to generalize this is to think of E as a functional of some fields (functions) defined on the Riemann surface. This defines a quantum field theory in two dimensions. There are many interesting examples of such field theories, but we will mention the one most relevant for superstring vacua which is the case of Landau-Ginzburg theories. Without going to too much detail it turns out that in this case the field theory is characterized by a single holomorphic function $W(x_i)$ where x_i are superfields. It was found [19] that quasi-homogeneous W's which have an

[7] This duality extends to the full moduli of the torus not just to the case that it is geometrically the product of two circles. In the more general case the size also is a complex modulus due to the appearance of the anti-symmetric tensor fields which effectively complexifies the Kahler cone.

isolated critical point at $x_i = 0$ give rise to a nice class of (super conformal) theories. In this way the classification of quasihomogeneous singularities became very relevant for the classification of string vacua. Moreover, it was found [20] that if the index of the singularity[8] is integral and equal to the number of variables x_i minus 2, they are related to string vacua propagating on the Calabi-Yau 'manifold' defined by (the possibly singular variety) $W(x_i) = 0$ in weighted projective space with a very particular Kahler metric. This clarified the geometrical meaning of the important discovery of Gepner [21] in his construction of string vacua. Note that the complex structure of the Calabi Yau is fixed by $W = 0$, but the Kahler structure of Calabi-Yau is only implicitly specified by W (through its quantum symmetries) [22][20]. As an example if we take

$$W = x^4 + y^4 + z^2 + a\, x^2 y^2$$

Setting $W = 0$ in weighted projective two space, we get a one dimensional torus whose moduli is fixed by a. The volume of the torus is implicitly fixed (by the existence of quantum Z_4 symmetry) which teaches us that the volume of this torus is 1 for all a (and the anti-symmetric field vanishes) [22]. So in this way the study of strings propagating on Calabi-Yau manifolds can be very effectively studied using this picture, and this has become an important tool in the recent discovery of interesting class of examples of mirror symmetric pairs of string vacua.

For strings on Calabi-Yau manifolds, as we discussed before we automatically get two rings, only one of which is the cohomology ring of the manifold. What is the other, the complex ring, geometrically? Well, a *subring* of this second ring can be described geometrically, when the Calabi-Yau theory is represented by a variety defined by $W = 0$ in a weighted projective space. In this case if we consider the (integral dimension) ring of the singularity defined by

$$\mathcal{R} = \frac{C[x_i]}{dW} \qquad (5.1)$$

[8] For a quasihomogenous function the index is defined as follows: By assigning degree one to a quasi-homogeneous W we can obtain fractional weights q_i of variables x_i. The index of W is simply $\sum(1 - 2q_i)$.

they generate a subring of $H^{-p,p}$ discussed before (where p corresponds to the degree of the ring element). It would be interesting to see if one can extend this picture to the full ring for all $H^{-p,q}$ (and not just the diagonal elements). Note that this ring certainly does depend on the complex moduli of Calabi-Yau manifold (as that changes as we change W). This is consistent with the mirror picture, namely the mirror ring depends on Kahler moduli (as the quantum deformed cohomology ring does depend on Kahler moduli).

So can we describe the quantum deformed cohomology ring of some manifolds using singularity ring for some W? The answer to this question should be in the affirmative if the mirror picture is valid. After all the mirror map changes $H^{-p,p} \to H^{p,p}$ and so maps (part of) the singularity ring to the diagonal elements of the deformed cohomology ring. The computation of cohomology ring for Calabi-Yau manifolds is in general rather difficult. So in this way we map a difficult problem (computation of deformed cohomology ring) to a simple problem (computation of the ring of a singularity) once we know the right transformation.

The quantum cohomology rings are easy to compute in some cases, as we mentioned before. For example for CP^n we mentioned that the deformed cohomology ring is

$$x^{n+1} = \beta$$

This of course can be written in the 'mirror' picture by the ring of W according to (5.1):

$$W(x) = \frac{x^{n+2}}{n+2} - \beta\, x$$

This can also be generalized to Grassmanians[9]. As discussed before for the case where $c_1 \neq 0$ we expect to violate the grading of the ring, which means the

[9] The cohomology ring of Grassmanian $U(n+k)/U(n) \times U(k)$ can be written as the singularity ring [6] generated by a single potential $W(x_i) = \sum z_i^{n+k+1}/n+k+1$ where x_i are symmetric polynomials of degree i in z_j (with no monomial appearing more than once) and i runs from 1 to n. The x_i correspond to the chern classes of the n-dimensional tautological vector bundle on the Grassmanians. The quantum deformation of this ring is naturally conjectured to be $W \to W - \beta x_1$. The motivation for this comes from the fact that $c_1 = n+k$.

corresponding W would not be quasi-homogeneous. For Calabi-Yau manifolds as we mentioned before the grading of the ring is respected by the deformation, so if the deformed ring is that of a singularity ring the corresponding W will again be quasi-homogeneous.

6. Topological Mirrors

So far we have talked about mirror symmetry in the following sense: We have strings propagating on two manifolds M_1 and M_2, which lead, as described before, to two Hilbert spaces each equipped with an infinite set of operators acting on them. Then if these two structures, or vacua, are isomorphic, we call M_1 and M_2 mirror pairs. Establishing this isomorphism at the level of Hilbert spaces is in general a complicated task. It would have been nice if there were a simple criterion to establish their equivalence. This question is also the same as asking how do we find a simple way to classify string vacua.

Classifying string vacua (and in particular static solutions which correspond to conformal field theories in two dimensions) has been investigated intensively in the past seven years. We are unfortunately still far from a complete classification. However for the fermionic vacua, an interesting class of vacua have, as discussed before, a simple description in terms of quasi-homogeneous singularities. In fact it is believed that for any quasi-homogeneous function W there is a unique string vacuum. In other words it is believed that the information about W is enough to reconstruct the full Hilbert space of strings and operators acting on it. More generally, whether or not the theory comes from a quasi-homogeneous singularity, it is believed that essentially given the chiral rings in the theory one has enough information to reconstruct the full theory. Applied to the special case of strings propagating on manifolds this may sound a little strange: We seem to be saying that given the cohomology ring of a manifold, we can find the manifold, which is certainly false. However it is for the special case of Calabi-Yau manifolds that we are considering this and in such cases just specifying the hodge numbers may go a long way in determining the manifold itself. Moreover we have *two* rings the *quantum cohomology ring* and the *complex ring*, which fix the Kahler class and the complex structure of the manifold respectively. Thus from Yau's proof of Calabi's conjecture which shows that knowing the Kahler class uniquely fixes the Ricci flat metric we can reconstruct the metric on the manifold by the information encoded in these rings.

Having said all these, it becomes clear that the phenomena of mirror symmetry can be formulated more compactly by stating that the two rings we get for one manifold are exchanged in the mirror manifold. In other words we can forget about the rest of the structure of string vacua and Hilbert spaces and the full set of operators acting on them and concentrate simply on this finite dimensional subset of special operators. In fact this concept can be formalized. Consider strings propagating on a Kahler manifold. It turns out there is a *twisted* or *topological* version of this theory [9][23] which can be obtained by a simple modification of the definition of the theory (by shifting the spin of fermions) which has the effect that the only physical operators we obtain are the ones corresponding to the cohomology classes and that they form the quantum cohomology ring of the manifold. If in addition the manifold in question is a Calabi-Yau manifold this twisting can be done in two inequivalent ways (by shifting the spins of fermions chirally, which is allowed for Calabi-Yau manifolds because of absence of sigma model anomalies since $c_1(M) = 0$), one of which gives the quantum cohomology ring and the other gives the complex ring, which (except for the diagonal elements) has a less clear geometrical meaning. In this way we can get both rings depending on which twist we choose. However, it is clear that in this topological description the ordinary cohomology ring has a more 'natural' origin, and it seems to be 'preferred'. However, there is another way to describe (fermionic) string vacua and that is via a Landau-Ginzburg theory. In this case we can also twist the theory and obtain a topological version [24] whose only (physical) operators correspond to the singularity ring of W. Again, if W is quasi-homogeneous, this can be done in two different ways, one of which corresponds to the singularity ring which when W describes a Calabi-Yau manifold correspond to its complex ring, and the other which has a less clear geometrical meaning (as it appears in the twisted sectors) correspond to the deformed cohomology ring. So we see that again we have two rings, but the complex ring is 'preferred'.

The notion of mirror symmetry can be simply translated to the *equivalence of a topological sigma model with a topological Landau-Ginzburg model*, where the 'preferred' ring of the sigma model (the quantum cohomology ring or Kahler ring) gets mapped to the 'preferred' ring of the Landau-Ginzburg model (the singularity ring or complex ring). Stated in this way this mirror symmetry is more general than Calabi-Yau manifolds, as the W may or may not correspond to a Calabi-Yau manifold (even if it is quasi-homogeneous (see next section)). Also W may not be quasi-homogeneous as the example of the Grassmannians mentioned before illustrates (i.e., it goes beyond conformal theories) but nevertheless we have a mirror symmetry in the sense defined above.

7. Some Puzzles and Conclusion

It would be nice to be able to state the mirror symmetry in full generality. In geometrical terms, in the sense that strings on manifold M_1 behave the same way as strings on manifold M_2 this would be rather difficult to do. It is difficult even to fix precisely which category of geometrical objects we are considering. If we fix the category to be that of Calabi-Yau manifolds this would be false because there are examples of Calabi-Yau manifolds which are rigid (i.e., do not admit complex deformations) therefore their mirror would not admit Kahler deformations (i.e., $h^{1,1}=0$), which means that the mirror would not even be a Kahler manifold! So in this sense we have lost the mirror. However in the sense of equivalence of two topological theories, i.e., equivalence of a topological theory based on a sigma model and that on a Landau-Ginzburg model this may still be possible. In fact now we will give an example where this is indeed what happens. Consider a three-fold Calabi-Yau manifold defined by taking the product of three two dimensional tori, with Z_3 symmetry, and modding out by a $Z_3 \times Z_3$ symmetry generated by the elements $(\omega,\omega^{-1},1),(1,\omega,\omega^{-1})$ acting on the three tori, where ω denotes the Z_3 action. It is possible to resolve the fixed point singularities and obtain a smooth Calabi-Yau manifold. This

manifold is rigid, in that it does not admit any complex deformations $h^{1,2}=0$. The dimension of Kahler deformations is $h^{1,1}=84$. What is the mirror for this manifold? The answer turns out to be easy in this case: It is the Landau-Ginzburg theory defined by

$$W=\sum_{i=1,..,9} x_i^3+\sum_{i\neq j\neq k} a_{ijk}\, x_i x_j x_k$$

The way we know this is that at $a_{ijk}=0$ we can explicitly construct the Landau-Ginzburg theory and compare it explicitly with the geometrical description which also turns out to be exactly solvable (before blowing up the singularities) and one finds that (with the metric and the antisymmetric field of tori corresponding to the point of enhanced Z_3 symmetry) they agree. Moreover one can map the fields $x_i x_j x_k$ to the Kahler classes of the manifold. So in this way the 84 Kahler deformations of the manifold (which includes the blow up modes) will get mapped to the deformation of W which are captured through varying a_{ijk} above. This description of mirror symmetry is enough to capture the counting of instantons on the original manifold by studying variations of Hodge structure characterized by W [14]so for the purposes of 'simplifying' the instanton counting it works as well. So in a sense we do not really need a geometrical mirror; or if we insist we can say that the geometrical mirror in this case is a 7 fold defined by $W=0$ in CP^8. But this description is only valid as far as we are relating the variation of its Hodge structure with the deformed cohomology ring of the original Calabi-Yau manifold[10]. This example reinforces another view of mirror symmetry, namely, the abstract property of the rings that may arise in conformal theory is the same whether or not they come from the cohomology ring or the complex ring. So somehow the lesson is to *forget*

[10] It would be interesting to see if turning on (possibly singular) dilaton fields and torsion on this 7-fold, gives a sigma model which is equivalent to the three fold Calabi-Yau we started with. As is well known turning on dilaton field shifts the effective dimension (central charge) of the theory. In such a picture the freezing of Kahler degrees of freedom would be related to solving dilaton equations of motion.

about the underlying manifold altogether and concentrate on abstract properties of the rings, and the classification of the kinds of rings that can appear. This is very much the question of classification of variation of Hodge structures[25]. This is in fact the point of view advocated by Cecotti [14]. In this setup the existence of mirror symmetry is probably related to the 'scarcity' of inequivalent types of variations of hodge structure (with some given topological invariants).

The same idea of mirror picture applies even to the general case of manifolds with $c_1 \neq 0$, for example the Grassmanians, where the 'mirror symmetry' allows us to compute exactly instanton corrections to the analog of 'Weil-Petersson' metric for such manifolds [26]. This follows from the structure of special geometry which exists even off criticality. This reinforces the picture that *we should not restrict our attention to Calabi-Yau manifolds if we are to have a deeper understanding of mirror symmetry.*

The notion of 'quantum' cohomology ring might remind one of seemingly unrelated subject of 'quantum' groups. As is well known these groups have representation ring which is a 'quantum deformation' of the classical representation ring of the group. The deformations being parametrized by a parameter k which is the level of quantum group, and as $k \to \infty$ we recover the classical representation ring. Indeed this k seems to play a very similar role to the role kahler class k plays in quantum cohomology ring in the infinite limit of which one recovers the classical cohomology ring. It turns out these two different 'quantum rings' are not as unrelated as might seem at first sight! In particular it has been shown [27] that for special class of such theories the fusion ring (representation ring) of quantum groups get mapped to the chiral ring of a Landau-Ginzburg theory (see also [28] [11]). For example, if one considers $SU(n)$ quantum group with level $k = 1$, its representation ring is isomorphic to the quantum cohomology ring of CP^{n-1} (discussed before). It would be interesting to see whether or not all the rings of quantum groups can be interpreted as the quantum cohomology ring of some manifold (for example which manifold has the quantum cohomolgy ring related to Chebychev polynomial?). This connection has become even more intriguing with the discovery [26] that precisely these Landau-Ginzburg theories seem integrable field theories in the sense that they have an integrable classical equation describing the generalized special geometry (some further evidence for their integrability has been found in [29]). It was further conjectured in [26] that whenever the ring of a supersymmetric theory corresponds to that of a RCFT (rational conformal field theories), i.e. a solution to quantum group representation ring, the corresponding field theory is integrable. These connections we believe are very important to understand better for a more abstract understanding of 'mirror symmetry' and 'quantum rings'.

We have learned that mirror symmetry is the statement of equivalence of two topological theories, one which is difficult to compute and the other which is easy. It is natural to continue this line of thinking and suggest that the same thing happens for other topological theories. In particular Donaldson theory which captures some invariants for differentiable manifolds in four dimensions, has a topological field theory description [30]. It is in general very difficult to compute Donaldson invariants, just as it is in general difficult to compute the number of rational (holomorphic) curves in a manifold. But we have seen in the latter case that there is a simpler topological theory which is the Landau-Ginzburg description. It is tempting to conjecture that there is a similar thing going to happen in four dimensions [31], namely that there must be a topological mirror theory, far simpler than Donaldson theory, which via an appropriate mirror map allows us to effectively compute Donaldson invariants. It remains to be seen if this conjecture is valid.

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Topological–anti-topological fusion

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We study some non-perturbative aspects of $N = 2$ supersymmetric quantum field theories (both superconformal and massive deformations thereof). We show that the metric for the supersymmetric ground states, which in the conformal limit is essentially the same as Zamolodchikov's metric, is pseudo-topological and can be viewed as a result of fusion of the topological version of $N = 2$ theory with its conjugate. For special marginal/relevant deformations (corresponding to theories with factorizable S-matrix), the ground state metric satisfies classical Toda/Affine Toda equations as a function of perturbation parameters. The unique consistent boundary conditions for these differential equations seem to *predict* the normalized OPE of chiral fields at the conformal point. Also the subset of $N = 2$ theories whose chiral ring is isomorphic to $SU(N)_k$ Verlinde ring turns out to lead to affine Toda equations of $SU(N)$ type satisfied by the ground state metric.

1. Introduction

$N = 2$ supersymmetric quantum field theories have recently undergone an intensive investigation from many different points of view: From the string point of view $N = 2$ superconformal models in 2 dimensions constitute the building blocks of $N = 1$ space-time supersymmetric string vacua [1]. From the point of view of classification of conformal theories, they are in a sense the simplest type to classify, and a nice subset of them, supersymmetric Landau–Ginzburg theories, is related to catastrophe theory [2–5]. From the point of view of topological characterization of the theory, they have a finite ring of operators (chiral primary fields) [4] which is believed to basically characterize them. There is a "twisted" version of these theories [6], the topological version, which has as its physical degree of freedom *only* these operators. These topological theories have been studied from the view point of 2d superconformal [7] and topological Landau–Ginzburg theories [8], and from the viewpoint of their properties under coupling to topological gravity in [9–11].

From a slightly different point of view, $N = 2$ supergravity theories has also been studied in four dimensions, and it was found that for the construction of the theory a very special type of Kähler geometry is needed [12]. This in turn is related to the fact that in the type II superstring compactification, leading to 2-dimensional $N = 2$ superconformal theories, the metric on moduli space of a three-fold Calabi–Yau has special properties, and is basically characterized by a holomorphic, topological object (pre-potential) [13,14]. This geometry is called "special geometry". The metric on moduli space of Calabi–Yau is the same as the Zamolodchikov metric of the underlying $N = 2$ SCFT, thus relating geometry with SCFT correlation functions.

In the topological description of $N = 2$ theories, one of the two supersymmetry charges plays the role of a BRST operator and the physical operators of the theory get truncated to the chiral ring. In this way the computations can be performed in a more or less closed form. The topological correlation functions are basically combinatorial objects, holomorphic functions of moduli. In the case of special geometry these topological correlation functions serve as *coefficients* of differential equations characterizing Zamolodchikov's metric on moduli space, which thus makes the Zamolodchikov's metric pseudo-topological. The Zamolodchikov metric which appears for example in the low-energy *dynamics* of the effective field theory description of strings is thus characterized by purely *kinematical / combinatorial* topological data. In these cases one finds that the Zamolodchikov (Weil–Petersson) metric is Kähler and the Kähler potential is written as a finite sum of holomorphic and anti-holomorphic "blocks" (periods) in the moduli of target space.

In the context of supersymmetric quantum mechanics related to LG theories it was found in ref. [14] that the same system of differential equations that characterizes the ground state metric (basically the Zamolodchikov metric) at the conformal point and gave rise to special geometry are also valid off the conformal point. That naturally raises a question of whether there is a generalization of special geometry off the conformal point as well.

One of the aims of this paper is to uncover the special geometry for massive (i.e. non-conformal) theories as well, and explain the rationale for finding a pseudo-topological metric from the topological viewpoint for both massless and massive theories. Basically what we find is, that if one fuses a topological theory with itself, one ends up with topological objects such as the holomorphic pre-potential which arise in special geometry. If on the other hand we fuse a topological theory with its conjugate, which we call *anti-topological*, we end up with pseudo-topological objects such as Zamolodchikov's metric. The generalized notion of special geometry simply encodes this relation between topological–topological fusion versus topological–anti-topological fusion and their variation with respect to moduli. We

show from this viewpoint in precisely what sense they are topological and derive the equations that characterize them, thus generalizing the results derived in ref. [14] for Landau–Ginzburg theories to arbitrary $N = 2$ QFTs. In this way we find a *generalized special geometry* to be equally valid on- or off-criticality. Even though the equations are the same in the two cases, we find a sharp difference between the *solutions* to these equations on- and off-criticality. In both cases we find the metric to be a sum over a finite block of objects, but in the critical theory these objects (periods) are *holomorphic* while in the off-critical theory these objects (the generalized periods) are *not holomorphic* functions of moduli and are generically far too complicated to give in closed form. From the viewpoint of chiral rings the main reason for complication of solutions to special geometry in the massive case is that in this case the ring is *not* nilpotent.

These ideas are made more concrete using many explicit examples of massive deformations of $N = 2$ LG theories, which is the main reason for the unusual length of the present work. The special examples that we obtain, which are of the form of generalized affine Toda equations, bring a completely orthogonal direction of interest to the present work. Namely, many of our examples provide interesting non-singular solutions to some affine Toda equations in terms of correlations (the metrics) of $N = 2$ theories. In this way we can use the methods available to us from the $N = 2$ theories, to gain insight into the solutions of (self-similar) affine Toda equations, which one generally does not have a good handle on. Along the way we are able to reproduce some deep mathematical results for solutions to Painlevé III [15] and Bullough–Dodd [16], which had been obtained using isomonodromic deformation technique and generalize them to other affine Toda theories. We basically find that the OPEs of SCFT *solve* the boundary conditions needed for a non-singular solution to (self-similar) affine Toda equations.

As is the case with many works on integrable systems there are many mysteries which need explanation. We find a number of intriguing results which beg for a deeper understanding. In particular many of our $N = 2$ massive supersymmetric theories are themselves described by *quantum* affine Toda theories (some non-supersymmetric and some $N = 2$ supersymmetric affine Toda lagrangians). In these cases we find that the ground state metric, which could be viewed as some particular correlation functions in these theories, as a function of the overall coupling (temperature or scale parameter) satisfy ordinary *classical* affine Toda equations *of the same type* (or reductions thereof). This is somewhat reminiscent of the space-time–target duality obtained for critical $N = 2$ strings [17]. The magic is even more mysterious: some of the cases corresponding to $N = 2$ supersymmetric affine Toda lagrangians (the SU(N) case) turn out to be related to Verlinde's ring for SU$(N)_k$ RCFT [18].

The structure of this paper is as follows: In sect. 2 we review some topological aspects of $N = 2$ theories, and introduce the idea of topological–anti-topological fusion. In sect. 3 we derive some equations satisfied by the ground state metric by considering a family of $N=2$ theories. We also discuss some general properties of the metric. In sect. 4 we discuss the relation to renormalization group flows, the c-function and Zamolodchikov metric. In sect. 5 we discuss the reductions to SQM and in particular derive a rule which allows us to relate different models by non-invertible change of variables. Moreover we find a "period" decomposition of the metric which generalizes the known result at the conformal point to the massive theories. In sect. 6 we discuss some Lie-algebraic aspects of our equations, which are very helpful in a classification of their solutions. In sect. 7 we consider some examples related to minimal models and some special massive perturbations of them. In sect. 8 we consider a few of the examples discussed in sect. 7 in more detail, using properties of the solutions to Painlevé III and Bullough–Dodd [15,16]. In sects. 9 and 10 we study more tricky models related to Verlinde rings. In sect. 11 we present our conclusions. In appendices A and B the properties of the metric in the UV and IR are discussed respectively. Finally in appendix C the relationship with the "special coordinates" of special geometry is uncovered.

2. Topological aspects of $N=2$ theories

In this section we review some of the background work which is needed for this paper. Our main interest for most of the paper is $N=2$ Landau–Ginzburg theory, but many of our constructions are more general, and so in this section we will not commit ourselves to the Landau–Ginzburg theory, and consider the more general class of $N=2$ quantum field theories. Moreover we do not make the assumption that the quantum field theory is conformal, and our treatment applies to both massive and massless (conformal) cases. We will be mostly interested in the 1- and 2-dimensional descriptions, but some of what we say generalizes in a simple way to higher dimensions (and in particular to Donaldson theory [19]).

In an $N=2$ theory, there are two supersymmetry charges, which we label by Q^+ and Q^-. The main property of these supersymmetry charges is that

$$(Q^+)^2=(Q^-)^2=0, \qquad \{Q^+,Q^-\}=H, \qquad (Q^+)^\dagger=Q^-, \tag{2.1}$$

where H is the hamiltonian. Topological theory is obtained by declaring Q^+ to be a BRST operator [6] and by identifying the BRST cohomology of Q^+ with the physical Hilbert space (note that in the context of two-dimensional theories, this means that we put periodic boundary conditions on the circle in order to have a supersymmetry operator, i.e. we are in the Ramond sector)

$$Q^+|\psi\rangle=0, \qquad |\psi\rangle\sim|\psi\rangle+Q^+|\rho\rangle .$$

We can fix the ambiguity of the topological theory in identifying the state, by using the Q^- operator and demanding that the physical states be also annihilated by

Q^-. This is the analog of picking a harmonic representative in the standard cohomology. As is clear from the standard arguments, this fixes the ambiguity of adding Q^+-exact states to the ground state. In fact using (2.1) we can identify the topological states with the ground states of the supersymmetric theory.

The topological operators ϕ_i are defined to be operators which commute with Q^+, i.e.

$$[Q^+,\phi_i]=0. \tag{2.2}$$

These fields are called chiral fields. A field which itself is a Q^+-commutator acts trivially on the Hilbert space. It is obvious that chiral fields form a ring, because of OPE of two of them is Q^+-closed and so can be expanded in terms of chiral fields. But most of the elements that appear in the product are themselves Q^+-commutators, and thus are trivial operators in the topological theory. Since the translation operator is itself a Q^+-commutator (following from supersymmetry) the chiral fields and their translations differ by Q^+-trivial operators. Thus we see that in order to obtain the topological product of two chiral fields at different points it is sufficient to take their product *at the same point*. This will differ from the fields at different points by fields which are Q^+-commutators. So to specify this ring we do not have to specify the points at which we put the fields. If we choose a basis ϕ_i for the physical chiral fields, we get a ring

$$\phi_i\phi_j=C_{ij}^k\phi_k+Q^+\text{-commutator terms.}$$

This ring is in generic cases a finite ring. In the context of critical theory this ring, the chiral primary ring, was defined and studied in ref. [4].

The question arises as to whether there is a natural identification of the states with the operators in the topological theory. This would be obvious if we can identify a unique vacuum state in the topological theory which we denote by $|0\rangle$. Once we have such a state then we simply identify the states by the operation of ϕ_i on the vacuum

$$\phi_i|0\rangle=|i\rangle .$$

The property (2.2) guarantees that the resulting state is Q^+-closed and is thus itself a topological state. So the main question is how we identify the vacuum state. In general there are a number of ground states which all have zero energy (in the LG case the number is equal to Witten's index) and it might at first sound impossible to pick a "preferred" one. If we were dealing with the SCFT there is a canonical choice. Namely in that case we have two U(1) charges (the left and right charges) which labels the vacua and we look for the unique state with minimum (left and right) charge and identify that as $|0\rangle$. All the other states are obtained from it by applying the physical fields (chiral primary fields which all have positive U(1) charge) on it. Here we have crucially used the properties of the conformal theory, and in particular the existence of an additional U(1) charge, which is the property of the critical theory. In the general massive case there is only one U(1) charge and that counts the fermion number (the difference between the left and right charges at the conformal point). In particular this is not enough to pick a unique state (for example in the LG theories all the ground states have equal left and right charges and thus are neutral under this charge).

One might be led to believe that a canonical choice for the ground state of the Ramond sector does not exist off criticality, but that turns out not to be so. To see this we can use the spectral flow to give an alternative definition of the vacuum state [4]. Consider the Hilbert space based on the NS sector, i.e. circle with antiperiodic boundary condition for fermions. The spectral flow is obtained by changing the boundary condition for fermions *continuously* from antiperiodic to periodic. This can be done because we do have a conserved fermion number in the theory even off criticality. In this way we can identify each state in the NS sector with a unique state in the Ramond sector. In particular the unique vacuum of the NS sector will flow to a unique ground state of the Ramond sector which we identify as $|0\rangle$. Note that this description of spectral flow is equally valid *whether or not the theory is conformal*. So in this way we see that there is a canonical isomorphism between the operators in the NS sector and the topological states (in the Ramond sector).

There is a nice way to implement spectral flow in the path-integral language which will be very useful for us: Consider a hemisphere with the standard metric and with some operators inserted on it. The boundary of the hemisphere is a circle on which we base our Hilbert space. The path integral will give us a state in the Hilbert space. Now if we were doing the standard $N=2$ quantum field theory on the hemisphere, the fermion spin structure is trivial on it, but that induces an *antiperiodic* boundary condition for the fermions on the boundary. So the standard path-integral, if we do not put spin operators on the hemisphere, will give us a state in the NS sector as is familiar from the study of SCFTs. The trick is to consider the topological version of this path-integral. This is equivalent [7] to putting a background gauge field which couples to fermions number and is set to be half of the spin connection. In this background, over the sphere the fermion number is violated by one unit, and over half of the sphere the fermion number is violated by one half, which is precisely the flow from the NS to R sector. Put differently, the boundary condition for the fermions at the circle boundary of the hemisphere is still antiperiodic, but there is a U(1) Wilson line which couples to fermion number. We can get rid of the Wilson line by changing the boundary condition of fermions by the holonomy

$$\exp\left(i\int_{S^1} A\right) = \exp\left(i\int_{\text{hemisphere}} F\right) = \exp(i\pi) = -1,$$

i.e. it is equivalent to changing the boundary conditions from the NS to the R. This is the magic of topological theory: it automatically "knows" about spectral flow.

The topological description guarantees that as long as we put fields which commute with Q^+ (i.e. chiral fields) on the hemisphere we get a state at the boundary which is in the topological Hilbert space, i.e. it is Q^+-closed. In fact the topological nature of the theory guarantees that the topological state that we get will not depend on the precise metric we put on the hemisphere. Changing the metric has the effect of shifting the resulting state by the addition of a Q^+-closed state. If we wish to obtain the actual ground state representative we will have to choose the metric on the hemisphere which makes it look like the standard hemisphere with an infinite cylinder glued at the boundary to it. In this way the propagation by $\exp(-TH)$ for large T along the infinite cylinder will project the topological state onto the actual ground state of the hamiltonian.

In this way we see that for each chiral field ϕ_i we get a state $|i\rangle$ in the Ramond sector by doing the path integral with that chiral field on the hemisphere. In particular $|0\rangle$ is the state associated to the identity operator. The topological nature of the theory will guarantee independence of where we put that field precisely within the hemisphere. In particular we can move it to the boundary of the hemisphere, in which case by operator formulation we see that the state is the same as multiplication of the state by the field ϕ_i

$$|i\rangle = \phi_i|0\rangle,$$

thus agreeing with the previous definition. Note that in this equation by $|i\rangle$ we mean the topological class of the state, i.e. $|i\rangle$ may differ from an actual ground state of the theory by Q^+-closed states. Again if we wish to obtain an actual ground state we should propagate the state along a cylinder for a long time. From the above we also learn that

$$\phi_i|j\rangle = \phi_i\phi_j|0\rangle = C_{ij}^k\phi_k|0\rangle = C_{ij}^k|k\rangle, \tag{2.3}$$

where again the equalities are modulo Q^+-trivial states. We can thus represent the action of the chiral fields in the subsector of vacuum states by the matrix C_i

$$(C_i)_j^k = C_{ij}^k$$

Everything we have said in the above can be repeated replacing everything by its adjoint. In particular this means replacing Q^+ by its adjoint Q^-, the chiral fields ϕ_i by their adjoint antichiral fields $\bar\phi_{\bar i}$, and the chiral ring coefficients C_{ij}^k with their complex conjugate $C_{\bar i\bar j}^{\bar k} = (C_{ij}^k)^*$ for the antichiral ring. In the path-integral definition of the states, we introduce a background gauge field which is now *minus* half the spin connection. In this way we get another topological theory which is simply the conjugate one and we call it the *anti-topological* theory.

It turns out to be crucial for us to have a deeper understanding of the relation between these two topological theories. The crucial link between the two theories turns out to be the Ramond sector. Namely, the physical states in both theories are in one to one correspondence with the Ramond vacua, as we have discussed above. So now let $|i\rangle$ and $|\bar{j}\rangle$ denote the *actual ground states* corresponding to the fields ϕ_i and $\bar{\phi}_j$ respectively. In this way we have found two "preferred" bases for the Ramond ground states. Of course we can write one in terms of the other, so we must have

$$\langle\bar{i}| = \langle j|M_{\bar{i}}^{j}. \tag{2.4}$$

The matrix M defined above is referred to as the *real structure*. It is crucial for us because it is precisely an intertwiner between the topological theory and its conjugate. In a sense it allows us to compare a topological theory with its conjugate. Since the Hilbert space is coming from a quantum field theory we have a CPT operator which is an anti-unitary operator of order 2. Acting on (2.4) with this operator and using its anti-unitarity one easily deduces that the real-structure matrix satisfies

$$MM^{*} = \mathbb{1}, \tag{2.5}$$

where M^{*} denotes the complex (not hermitian) conjugate matrix to M. In order to completely understand the structure of our Hilbert space, in addition to the operator content of the Hilbert space we also need to know its inner product. Since we have a natural $N=2$ field theory underlying our constructions we automatically have an inner product. That is simply the inner product in the Ramond ground state. To write it down, we need to choose bases. In particular we can use the basis where the left and right states are taken to be the chiral basis

$$\langle j|i\rangle = \eta_{ij}, \tag{2.6}$$

or the chiral and antichiral basis

$$\langle\bar{j}|i\rangle = g_{i\bar{j}}, \tag{2.7}$$

and the complex conjugate of the above inner products. Of course the two metrics η_{ij} and $g_{i\bar{j}}$ are related using the real-structure matrix M

$$g_{i\bar{k}} = \eta_{ij}M_{\bar{k}}^{j}. \tag{2.8}$$

Note that we can deduce from eqs. (2.5) and (2.8) the very useful identity which relates g and η

$$\eta^{-1}g(\eta^{-1}g)^{*} = \mathbb{1}. \tag{2.9}$$

The inner product of immediate interest in $N=2$ theories is the g metric, because when we take the inner product of states we take the *adjoint* of a state on the left and the adjoint of the state $|i\rangle$ is $\langle\bar{i}|$ and not $\langle i|$. In particular the metric η is not hermitian whereas g is obviously a hermitian metric. However, as we shall see η is much simpler to compute, and is in fact a purely topological object (it will be clear as we proceed that η is a symmetric matrix).

To understand the structure of these two metrics better we represent them by path-integrals. The path-integrals are represented by two hemispheres, one on the left and the other on the right, joined by an infinitely long cylinder. We need an infinitely long cylinder to project onto the ground states. In addition we have a background gauge field, which for the computation of η_{ij} is set equal to half the spin connection throughout the sphere, and we insert the operator ϕ_i on the right hemisphere and the operator ϕ_j on the left hemisphere. For the computation of $g_{i\bar{j}}$, on the right hemisphere and the right half of the infinite cylinder we have a background gauge field which is half the spin connection while on the left hemisphere and the left half of the infinite cylinder we have a background gauge field which is minus half the spin connection. The fact the the region were the left and right meet is flat, means that the gauge fields glue smoothly from one to the other, and we have a well defined gauge field. Also we insert the field ϕ_i on the right hemisphere and the field $\bar{\phi}_j$ on the left hemisphere.

From the above path-integral definition it follows that essentially both metrics are topological, where by topological we mean if we perturb the corresponding positions of inserted fields or the metric on the hemisphere, as long as there is an infinitely long intermediate cylinder, with a fixed perimeter β, the result of the path-integral does not change. This is due to the fact that local perturbations of this kind, as noted above, are equivalent to operations by Q^{+} or Q^{-} on some state, and propagation along the cylinder of length T results in $\exp(-TH)Q^{\pm}$ which goes to zero as $T\to\infty$. However η is *more* topological in the sense that even if we change the length of the intermediate cylinder or even completely change its metric, or even move the positions of fields from one hemisphere to the other, the result will still not change. This follows from the usual definitions of the topological theory, as all such variations are Q^{+}-commutators and since the fields commute with Q^{+} we immediately see that the variations do not change the result of path-integral. Note that since we can exchange the position of the operators between the two hemispheres η is symmetric. The fact that η is purely topological allows us to give a simple closed form for it in many cases. The result for the LG theory will be mentioned below. This general notion of topological invariance, i.e. without necessitating an infinitely long intermediate cylinder, would not work for $g_{i\bar{j}}$ because we get both Q^{+} and Q^{-} variations on the right and left hemispheres respectively which does not allow us to complete the argument. In this sense $g_{i\bar{j}}$ which is obtained by "fusing" a topological theory with its complex conjugate is only "partially" topological in the sense discussed above *. In particular it depends on the perimeter of the cylinder β. It turns out that changing β is equivalent to changing the scale of the theory, and the one-parameter family of metrics $g_{i\bar{j}}$ that we obtain can be viewed as the trajectory of the metric under RG flow. In the following we set $\beta = 1$, and implement the change of scale by flow in the coupling-constant space of the theory. Even though g looks only pseudo-topological, as we shall see later in this paper the purely topological correlations allow us to completely determine it.

It is also easy to see these constructions in the operator language. In particular in this way we can show that even though in the above definitions of η and g we have used the ground states themselves the metric η_{ij} is independent of which representative we choose. This follows by noting that changing for example the representative $\langle i|$ is equivalent to shifting it by $\langle \alpha | Q^+$, but this does not affect the inner product $\langle i | j\rangle$, because $Q^+ | j\rangle = 0$ (for any representative of $|j\rangle$). Note that the same argument to show independence of $g_{i\bar{j}}$ of the choice of the representative would fail and so this quantity *does* depend on the fact that we have to actually choose the precise ground states representing the cohomology classes.

So far we have been general. We will now illustrate these ideas in the context of $N = 2$ LG theories. These models are defined by taking a number of superfields X_i in two-dimensional space with two left and two right moving anti-commuting coordinates denoted by θ^+ and $\bar{\theta}^+$. The superfields are taken to be chiral which means that

$$D^+ X_i = \left(\frac{\partial}{\partial \theta^+} + \theta^- \frac{\partial}{\partial z} \right) X_i = 0 = \bar{D}^+ X_i ,$$

and similarly $\bar{X}_i$ is anti-chiral (and satisfies the above equations with θ^+ and θ^- exchanged). Then one writes down a lagrangian

$$\mathcal{L} = \tfrac{1}{2} \int d^4\theta \; K\left(X_i, \bar{X}_i\right) + \int d^2\theta \; W(X_i) + \text{h.c.},$$

which has $N = 2$ supersymmetry. This consists of two terms, the term involving K, the D-term, and the F-term W, the superpotential, which is a holomorphic function of X_i. If we represent the operators corresponding to D^+ and $\bar{D}^+$ acting on the Hilbert space as

$$D^+ \to Q_L^+ , \qquad \bar{D}^+ \to Q_R^\pm .$$

* The construction of topological–anti-topological fusion can be extended to arbitrary genus. We take a surface consisting of alternating regions, supporting the topological theory and its conjugate respectively, which are separated by infinitely long tubes. However, once we know η, g and C on the sphere for arbitrary β we can write down the corresponding answer at higher genus using simple ideas of sewing.

we can write the two supersymmetry operators $Q^\pm$ discussed above as

$$Q^\pm = Q_L^\pm + Q_R^\pm .$$

When W is quasi-homogeneous the IR fixed-point of the LG theory is believed to describe an $N = 2$ superconformal theory [2,3]. Here we will not make these restrictions, and our discussion is equally valid for the critical as well as the non-critical (massive) theory. It is easy to find the chiral ring $\mathcal{R}$ for LG models. In fact the chiral ring is generated by the X_i themselves. All we have to do then is find the relations in this ring, or put differently, which product of the X_i is Q^+-closed. These relations will come from the variations of the lagrangian, which are the equation of motion for this theory. Varying the action with respect to X_i and doing the θ^+, $\bar{\theta}^+$ integrals in the D-term gives us

$$\partial_i W(X_j) = -D^+ \bar{D}^+ \partial_i K(X_j, X_j).$$

This means that the chiral fields containing $\partial_i W$ are Q^+-commutators and thus are trivial in the ring. Therefore we learn that the chiral ring of the theory is simply

$$\mathcal{R} = \mathbf{C}[X_i]/\partial_i W.$$

An important thing to note here is that W completely determines the ring (known as the singularity ring of W) and the D-term K does not affect the ring. In particular the D-term is trivial in the sense of both supercharges $Q^\pm$. This in particular implies that the variations of K is trivial in the sense of both the topological theory and its conjugate. Thus, it will not affect the metrics we defined above, and so the two metrics just depend on W. The metric η turns out to be particularly simple to compute and it is simply computable using the techniques of topological theories. A topological description of LG theories and the computation of its correlation functions is given in ref. [8]. Alternatively, one can apply dimensional reduction to supersymmetric quantum mechanics, and compute the metric η using properties of solutions to the supersymmetric Schrödinger equation [14]. The answer is

$$\eta_{ij} = \langle \phi_i \phi_j \rangle_{\text{top.}} = \langle i | j \rangle = \text{Res}_W\left[\phi_i \phi_j\right]$$

in terms of the Groothendieck residue symbol $\text{Res}_W[\cdot]$ defined by

$$\text{Res}_W[\phi] = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\phi(X)\, dX^1 \wedge \ldots \wedge dX^n}{\partial_1 W \partial_2 W \ldots \partial_n W} = \sum_{dW=0} \phi(X) \mathfrak{H}^{-1}, \quad (2.10)$$

where $\mathfrak{H}$ denotes the hessian of W: $\mathfrak{H} = \det \partial_i \partial_j W$ and we are assuming that the critical points are non-degenerate in writing the last equality. Note that with the above result, under field redefinition, the fields do not transform as scalars. This "anomalous" behaviour, is connected with the fermion zero-modes in the background gauge field which we discussed for the topological theory. This anomalous behaviour will also be explained geometrically below in the context of SQM.

The computation of $g_{i\bar{j}}$, or equivalently the real-structure matrix M, turns out to be far more complicated and the study of its properties is the main focus of this paper. In order to study these we need to review some techniques developed for this purpose. This will be done in sect. 3.

3. General properties of the metric and its variation

The basic method to compute the metric g is to study its behaviour under perturbations which preserve the $N=2$ supersymmetry. In this setup, using standard perturbation theory techniques, one can derive differential equations which are satisfied by g. The coefficients of these differential equations turn out to be completely fixed by the chiral ring $\mathscr{R}$ and thus, in the case of LG theories, they only depend on W as they should.

The idea that there should be a differential equation on the coupling-constant space is not surprising. In fact in the context of non-degenerate perturbation theory in quantum mechanics it is well known that there is a canonical curvature on the perturbation space, and the integral of this curvature leads to the Berry phase [20]. In the case of degenerate perturbation theory, this leads to non-abelian gauge fields on the coupling space [21]. Our case is generically of this type, with the added structure that we have a *holomorphic* parameter space and that gives us some additional structure.

We will discuss the idea in the general setting. Again, our considerations apply to conformal and non-conformal cases with equal validity. We consider changing the action by giving expectation value to chiral and anti-chiral operators. This means that we vary the action by

$$\Delta L = \int d^2\theta \, \Delta t_i \phi_i + \text{c.c.},$$

where t_i correspond to the (complex) couplings in the theory. As we change t_i the Ramond vacua change. In perturbation theory one usually defines the variation of the state to be orthogonal to itself (and to the other states with the same energy). It is however more convenient to first allow an arbitrary basis for the perturbation and introduce a connection in the space of vacua which projects out the components of the perturbed vacua which are not orthogonal to the vacuum states. Let us denote this covariant derivative by D_i. Its basic property is that

$$\langle \bar{b} | D_i | a \rangle = \langle \bar{b} | \partial_i - A_i | a \rangle = 0,$$

where a, b label the Ramond vacua in some unspecified basis and $\langle \bar{b} |$ denotes the state adjoint to $| b \rangle$. Put differently, we can define a gauge field A_i on the coupling constant space given by

$$A_{ia\bar{b}} = \langle \bar{b} | \partial_i | a \rangle. \tag{3.1}$$

It is easy to see that under a coupling-constant-dependent change of basis for the vacua, the quantity A transforms as a gauge field. Similarly we can define $\bar{D}_i$ and $A_{\bar{i}}$. Let $\mathscr{V}$ be the space (the vector bundle) of Ramond ground states over coupling-constant space on which g defines a hermitian metric. Then it is easy to see that g is covariantly constant with respect to the gauge field we just introduced. In fact, this is how we defined A. One obtains

$$D_i g_{a\bar{b}} = \partial_i g_{a\bar{b}} - A_{ia}^{c} g_{c\bar{b}} - A_{i\bar{b}}^{\bar{c}} g_{a\bar{c}} = 0 = \bar{D}_i g,$$

where

$$A_{ia}^{b} = A_{ia\bar{c}} g^{b\bar{c}}, \qquad A_{\bar{i}\bar{b}}^{\bar{c}} = (A_{ib}^{c})^*.$$

It is natural to compute the curvature of these connections. We find *

$$[D_i, D_j] = [\bar{D}_i, \bar{D}_j] = 0,$$
$$[D_i, \bar{D}_j] = -[C_i, \bar{C}_j]. \tag{3.2}$$

Moreover one has

$$D_i C_j = D_j C_i, \qquad \bar{D}_i \bar{C}_j = \bar{D}_j \bar{C}_i, \qquad D_i \bar{C}_j = \bar{D}_j C_i = 0. \tag{3.3}$$

where C_i and $\bar{C}_i$ are the matrices which represent the action of ϕ_i and $\bar{\phi}_i$ on the vacuum states. Since in the topological phase we can moves fields around, it is clear in addition that

$$[C_i, C_j] = 0 = [\bar{C}_i, \bar{C}_j].$$

In the conformal limit this system of equations was derived and studied in the physics literature from many different view points [13,14], and gives rise to what is called "special geometry". In fact it has been shown in the context of Landau-Ginzburg models [14] that these very same equations remain valid even off the conformal point. The technique used there involves a careful study of the zero-en-

* These equations are a natural generalization of equations studied by Hitchin corresponding to a reduction of self-dual Yang-Mills equations to two dimensions.

ergy solutions to the Schrödinger equation in the context of SQM. As we will see these turn out to be quite general and apply to arbitrary $N=2$ theories, regardless of whether they come from LG theories and they can be easily derived from the path-integral viewpoint of fusing the topological theory with the anti-topological theory.

In the usual non-topological setup, one can derive incorrect "theorems" by a naive treatment of supersymmetric Ward identities which would lead one to believe that the metric is constant, independent of the coupling constants. The "argument" goes as follows:

$$\frac{\partial}{\partial t^a}\langle \bar{k} \mid h\rangle \doteq \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, \langle \bar{k} \mid \phi_a \mid h\rangle = 0,$$

i.e. using the fact that the ground states are annihilated by both Q^+, the Grassmann integral seems to kill the above term. However, this is incorrect. The difficulty lies in ignoring contact terms. It fact it is shown in ref. [22] that in the conformal case such terms are crucial in obtaining the correct Zamolodchikov metric. In the critical $N=2$ SCFT theories (corresponding to strings on Calabi–Yau manifolds), the contact terms were found to be crucial in getting the correct answer [23]. However, amazingly the topological theory allows us to be "naive" about contact terms and ignore them and get the correct answer! This is precisely because contact terms which are UV singularities have no invariant definitions in the topological phase, as we can move fields around with no consequence for correlation functions.

Before turning to the derivation of these equations let us describe their interpretation. The first line in eq. (3.2) is telling us that the gauge connection is unitary, and the second is telling us that its curvature is computable using the commutators of the ring of the topological and anti-topological theory. Combined with eqs. (3.3) one sees that we can introduce "improved" connections which are actually flat, namely consider *

$$\nabla = \mathrm{d}t^i(D_i + C_i),$$
$$\bar{\nabla} = \mathrm{d}\bar{t}^i(\bar{D}_i + \bar{C}_i). \tag{3.4}$$

Then the new connection $\nabla + \bar{\nabla}$ is flat,

$$\nabla^2 = \bar{\nabla}^2 = \nabla\bar{\nabla} + \bar{\nabla}\nabla = 0. \tag{3.5}$$

$\nabla + \bar{\nabla}$ is the analog of the Gauss–Manin (GM) connection well known to mathematicians [24] which in the physics language plays a role when we are dealing with

* One could as well consider the dual connection $\nabla' = D - C$, $\bar{\nabla}' = \bar{D} - \bar{C}$ which is also flat.

marginal instead of massive perturbations of conformal theory. Indeed, when the $N=2$ theory is a LG theory which has a σ-model interpretation, it is the usual GM connection (see ref. [14] for details).

In order to prove eqs. (3.2) and (3.3) our strategy will be as follows: We will first show that it is possible to choose a holomorphic basis in which a, b run over chiral indices and with

$$A_{ij}^{k} = g^{k\bar{r}} A_{ij\bar{r}} = 0.$$

Once we show this (and similarly the conjugate version of it) the first line in eq. (3.2) will follow. Similarly, the fact that in this basis C_i is holomorphic implies that

$$\bar{D}_j C_i = \bar{\partial}_j C_i = 0,$$

which with its conjugate version will prove the second line of (3.3). For the other equations we will have to work harder.

Let us start by showing that in the chiral basis we can choose a holomorphic gauge, i.e. a gauge in which the antiholomorphic components of the gauge field are zero. As we shall see, the topological path-integral automatically picks this gauge. By definition of the gauge field we have to compute

$$A_{\bar{i}j}^{k} = \eta^{kl}\langle l \mid \bar{\partial}_i \mid j\rangle. \tag{3.6}$$

The matrix element in the above equation can be conveniently represented by a path-integral: We represent the state $|j\rangle$ by a topological path-integral on a hemisphere with a long tube attached to it with the field ϕ_j inserted in it. This space (with the long tube attached) we call the right-hemisphere S_R. In order to find $\bar{\partial}_i |j\rangle$ all we have to do is to insert the operator

$$\int_{S_R} \mathrm{d}^2 z\, \mathrm{d}^2\theta^+\, \bar{\phi}_i = \int_{S_R} \mathrm{d}^2 z\, D^+ \bar{D}^+ \bar{\phi}_i$$

in the path integral. To compute the matrix element in (3.6) we can create the state $\langle l|$ by a topological path integral on a left-hemisphere S_L, with ϕ_l inserted, again with a long tube attached, and glue it to the path-integral on the right sphere S_R. This we will represent symbolically by

$$A_{\bar{i}jl} = \left\langle \phi_l \middle| \left(\int_{S_R} D^+ \bar{D}^+ \bar{\phi}_i \right) \phi_j \right\rangle.$$

Since Q^+ is a symmetry of the topological theory and ϕ_j is closed under it, we can write this as

$$\left\langle \phi_l \middle| Q^+ \middle| \left(\int_{S_R} \bar{D}^+ \bar{\phi}_i \right) \phi_j \right\rangle.$$

This vanishes because the topological theory on S_L produces a state which is annihilated by Q^+. Thus we have seen that the path integral in the chiral basis provides a holomorphic basis for the connection in which the anti-chiral components of the connection vanish. This concludes the first thing we wished to show.

Now we turn to harder parts of the derivation and show how the second line in eq. (3.2) can be derived. To do that we have to show that

$$\partial_j A_{ik}^l - \partial_i A_{jk}^l = \left[C_i, \bar{C}_j\right]_k^l. \tag{3.7}$$

where we have used that fact that in our basis the anti-holomorphic component of the connection vanishes, and thus there are no commutator terms on the left-hand side. In fact we know that even the second term on the left-hand side is identically zero, but we will keep this as it cancels some of the terms from the first term on the left-hand side and slightly simplifies our analysis.

Using the path-integral representation of the left-hand side of eq. (3.7) it is easy to see that, after some obvious cancellation between terms, we get a path integral on the sphere which symbolically can be represented by

$$\begin{aligned}\partial_j A_{ikl} - \partial_i A_{jkl} = &\left\langle \phi_k \left(\int_{S_L} D^+ \bar{D}^+ \bar{\phi}_j \right) \middle| \left(\int_{S_R} D^- \bar{D}^- \phi_i \right) \phi_l \right\rangle \\ &- \left\langle \phi_k \left(\int_{S_L} D^- \bar{D}^- \phi_i \right) \middle| \left(\int_{S_R} D^+ \bar{D}^+ \bar{\phi}_j \right) \phi_l \right\rangle \end{aligned} \tag{3.8}$$

Now we will show that these two terms give $-\bar{C}_j C_i$ and $C_i \bar{C}_j$ respectively (up to terms which cancel between them). Let us concentrate on the first term

$$\left\langle \phi_k \left(\int_{S_L} D^+ \bar{D}^+ \bar{\phi}_j \right) \middle| \left(\int_{S_R} D^- \bar{D}^- \phi_i \right) \phi_l \right\rangle$$

Just as discussed before we can move D^+ to the right where it kills everything except for D^- acting on ϕ_i which converts that into ∂ (by using the fact that D^+ kills ϕ_i and using the (anti-)commutators of $N = 2$ algebra). Similarly we can move $\bar{D}^+$ to the right and again the net effect on the path-integral on S_R is to replace $\bar{D}^-$ with $\bar{\partial}$. So we are left with *

$$\left\langle \phi_k \left(\int_{S_L} \bar{\phi}_j \right) \middle| \left(\int_{S_R} \partial \bar{\partial} \phi_i \right) \phi_l \right\rangle .$$

* In order to move D^+ and $\bar{D}^+$ we have used the fact that there are two topological charges: Q^+ and $\bar{Q}^+$.

Now we can do the integral of the field on the right hemisphere and get a contribution on the boundary circle C on the cylinder which separates the two regions $S_{L,R}$. We get

$$-\left\langle \phi_k \left(\int_{S_L} \bar{\phi}_j \right) \middle| \left(\oint_C \partial_n \phi_i \right) \phi_l \right\rangle ,$$

where ∂_n denotes derivative in the normal direction to the circle C, i.e. in the infinitely-stretched direction of cylinder. We can replace

$$\partial_n \phi_i = [H, \phi_i].$$

Since $|\phi_l\rangle$ is the same as the vacuum state $|l\rangle$, it is killed by H and so we can write the above matrix element as

$$-\left\langle \phi_k \left(\int_{S_L} \bar{\phi}_j \right) \middle| \left(H \oint_C \phi_i \right) l \right\rangle$$

We will divide the integral on the left-hand side to two roughly equal parts each of which is infinitely stretched, the first part includes the field ϕ_k and contains the curved piece of S_L with roughly half the infinitely stretched cylinder, while the second part includes only the other half of the infinite cylinder of S_L. The integral on the part further on the left will not contribute to the above matrix element, because the state one gets propagates infinitely long on the second part of the space, and so the net effect is projection on ground state which is accomplished by the $\exp(-TH)$ for large T, and the final state we get on the circle C is thus killed by H in the above matrix element. We are thus left with the second part of the integral on the left which is on a very long cylinder. Let τ denote the long direction on this cylinder and let us take it to run from 0 to $T \gg 1$. Meanwhile the empty first part of the path integral will convert the path integral with the insertion of ϕ_k to an actual ground state given by $\langle k|$. So we are left with

$$-\langle k | \int d\tau \oint \bar{\phi}_j(\tau) H \oint \phi_i | l \rangle ,$$

where we have written the integral on the cylinder as first running around the perimeter on the cylinder at a fixed time τ and then integrating over all τ. Since the H kills the ground state on the left, we can replace H with its commutator with $\oint \bar{\phi}_j(\tau)$ which gives us a $-\partial_\tau \oint \bar{\phi}_j$. Thus doing the integral over τ becomes easy and we get the contributions from the boundaries at $\tau = 0, T$. The contribution at $\tau = T$ is on the same circle as the one the operator $\oint \phi_i$ is inserted and is canceled by the same term from the second term of eq. (3.8). We are thus left with

$$-\langle k | \oint \bar{\phi}_j \exp(-TH) \oint \phi_i | l \rangle ,$$

where we have to send $T \to \infty$. This has the effect of projecting the intermediate states to the ground states of the theory, and we recover the definition of the chiral ring matrices * and so we get

$$-(\bar{C}_j C_i)_{kl}$$

And similarly for the second term in eq. (3.8) we get the same as above with C_i and $\bar{C}_j$ exchanged places and with the opposite sign. We thus get the commutator on the right-hand side, thus completing the proof of the second line eq. (3.2). Using very similar techniques, which we hope the reader would be able to reproduce, one can verify the validity of the first line of eq. (3.3).

On closing this section let us note that in this holomorphic basis, we can write everything in terms of the metric g and the holomorphic chiral ring elements C_{ij}^k. Namely from the fact that g is covariantly constant and that the antiholomorphic component of the gauge field vanishes we have

$$A_{ik}^l = -g_{k\bar{j}}(\partial_i g^{-1})^{\bar{j}l}.$$

Moreover, just from the definition of the basis we have

$$(\bar{C}_i)_k^l = (g C_i^\dagger g^{-1})_k^l.$$

Putting everything together, the zero-curvature conditions (3.5) become differential equations for the metric g. We get

$$\bar{\partial}_j(g \partial_i g^{-1}) - [C_i, g(C_j)^\dagger g^{-1}] = 0, \qquad (3.9)$$

$$\partial_i C_j - \partial_j C_i + [g(\partial_i g^{-1}), C_j] - [g(\partial_j g^{-1}), C_i] = 0, \qquad (3.10)$$

all other conditions being either trivially satisfied, or consequences of these two together with known properties of the topological functions C_{ik}^h and η_{ij}.

As we shall see in more detail in subsequent sections these equations have "magical" properties, making them a natural generalization of the so-called *Special Geometry* which plays a key role in understanding the geometry of the moduli space of $N=2$ conformal field theories (related to CY manifolds). One important miracle is already evident from this discussion: our non-linear differential equations are always in the form of a consistency requirement for a set of linear equations, i.e. they always admit a zero-curvature (Lax) representation. Instead of eq. (3.9) and (3.10), we can study the associated linear problem

$$\Gamma \psi = \bar{\Gamma} \psi = 0. \qquad (3.11)$$

* We have taken the perimeter of the cylinder β to have unit length, otherwise the commutators will be accompanied with a factor of β^2.

In this abstract sense our equations are always solvable. As we shall see below, because of this Lax representation, for simple models the equations have a tendency to reproduce celebrated equations of mathematical physics. More surprisingly, generally speaking, to solvable models in the world-sheet sense, models which lead to infinitely many conserved currents and connected with factorizable S matrices, we find solvable (classical) systems for the dependence of ground state metric as a function of coupling-constant space (the "target" space). Moreover these equations tend to be of the sample type! (Quantum affine Toda theory as the world-sheet theory, and classical affine Toda theory of the same type (or its reductions) as the equations satisfied by the ground state metric!) This bizarre duality between world-sheet and target phenomena is reminiscent of what one finds in the case of critical $N=2$ string theories [17].

Not all the solution to the above equations can be accepted as ground state metrics. There are other conditions to be satisfied. First of all, g should be a positive-definite hermitian matrix. Furthermore the metric should have all the symmetries of the problem and in particular in the LG case, it inherits all the (pseudo) symmetries of W. Moreover, as mentioned before we have the "reality constraint"

$$\eta^{-1} g (\eta^{-1} g)^* = \mathbb{1}.$$

There are some general properties of the metric which follows from the above equations. Take the trace of eq. (3.9) which gives us

$$\bar{\partial}_i \partial_j \log \det g = 0,$$

i.e.

$$\det g = |f(t)|^2 \quad \text{with} \quad f(t) \quad \text{holomorphic}.$$

In particular, we can find a holomorphic basis such that $\det g = 1$.

Another general property of g which should be consistent with our equations is that the metric should not depend on t^0, the coupling associated to the operator $\mathbb{1}$. Indeed, adding a constant to the lagrangian in chiral superspace does not change the model because the Grassman integration over superspace kills it. This is consistent with our equations. In fact, $C_0 = \mathbb{1}$ and hence it commutes with everything. This simple remark has a very useful generalization. Sometimes the $N=2$ theory has a (pseudo)-symmetry such that the space of vacua viewed as a representation of a subring $\mathscr{R}'$ of $\mathscr{R}$ generated by some ϕ_i decomposes into orthogonal representations. Then if in a given irreducible representation some non-trivial operator reduces to a multiple of unity, $\langle \bar{i} | j \rangle$ ($|i\rangle$ in the given representation) is (essentially) independent of the corresponding coupling.

At this point a natural question arises. Are these conditions sufficient to uniquely determine the metric or not? A priori, one would think that the above differential equation should be supplemented with boundary conditions in order to predict g. However, the analogy with the geometrical case (the variational Schottky problem) which is the geometrical interpretation of these in the context of marginal operators of conformal theories suggests that *generically* the above conditions already lead to a very overdetermined problem. Then just *one* boundary condition would give a solution satisfying all the requirements simultaneously. In this sense, the equations *predict* their own boundary conditions. Although we do not have a general proof of this statement *, below we shall show in many explicit models how the equations are strong enough to predict their highly non-trivial boundary conditions. In particular the OPE of conformal theories are predicted by consistency alone and they agree with the results previously obtained. As a by-product we shall also reproduce some deep mathematical results in the context of isomonodromic deformation theory (together with some generalizations).

4. RG flow, Zamolodchikov metric and c-function

In the context of perturbing quantum field theories one usually defines a one parameter family of quantum field theories related to each other by a change in scale. This defines a "flow" on the space of quantum field theories which is known as the renormalization group flow. Conformal theories are precisely the fixed points of this flow. For a given theory characterized by a point on the coupling constant space, one defines an UV (ultra-violet) and an IR (infra-red) fixed point defined as the short-distance, and long-distance limits of that given theory. Generically one starts with a theory which is obtained by relevant perturbations of conformal theory so that the UV fixed point is the theory we started with. The infrared fixed points are generically infinitely massive theories; however, if one chooses the perturbation of the quantum field theory judiciously, one can end up with another conformal field theory as an IR fixed point. The study of this kind of situation in 2-dimensional quantum field theories was given a big boost by the work of Zamolodchikov [25]. In that work a function was defined on the parameter space, the "c-function", which has the beautiful property of decreasing along the renormalization group flows, and whose critical points correspond to fixed points of RG flow, i.e. CFTs. Moreover Zamolodchikov defined a metric on the parameter space, using the two-point function of perturbing operators on the plane at a fixed distance.

As we have been studying perturbed $N = 2$ SCFTs in this paper, it is natural to ask how the RG flows look in this context. Some aspects of this has been studied

* Even in the geometrical (conformal) case there is no general proof.

[26]. We will focus on the case of Landau–Ginzburg theories. The non-renormalization theorems of $N = 2$ theory come to our aid in the study of RG flows. These state that the superpotential W of $N = 2$ theories does not get corrected perturbatively. We will take this to be true non-perturbatively. In fact it appears that the non-perturbative non-renormalization theorem can be proven along the following line of argument. In flat space, where the spin-connection vanishes, the functional measure for the LG model is identical to that of the topological theory with a certain gauge-fixing term. The topological theory is not renormalized just because there are no local degrees of freedom *. Then its quantum effective action Γ should have the form

$$\Gamma = \Gamma_{\text{cl}} + s\Delta\Gamma,$$

where s is the topological Slavnov operator. In the LG context this equation is interpreted as the $N = 2$ non-renormalization theorem. Indeed, the usual superdiagrammatic proof of this result [27] consists of a loop expansion of this equation. For other viewpoints, see sect. 4 of ref. [5]. Anyhow, some evidence for the validity of this kind of conjecture is the correctness of some of its consequences [2,3]. To be more precise, even though we take W not to change, the action will pick up the supervolume factor. If we take $z \to \lambda z$, $\theta \to \lambda^{1/2}\theta$ we get

$$\int d^2z\, d^2\theta\, W(X_i) \to \lambda \int d^2z\, d^2\theta\, W(X_i).$$

This overall factor of λ can be gotten rid of in the leading terms of W (the highest degrees of fields) by a field redefinition with the effect of rescaling the rest of the couplings. In this way the rescaling of W by λ *generates* a flow. The IR limit is when $\lambda \to \infty$ and UV is obtained when $\lambda \to 0$. This we take as our working hypothesis as to what the RG flow is for us. Needless to say the D-terms are expected to get corrected in a much more severe way, but as we have seen in previous sections, luckily our computations for the ground state metrics are independent of that **.

Now it is natural to see whether we can compute the form of the metric g in the UV and IR limits. These will also be a kind of "boundary condition" for the differential equations we have discussed, eqs. (3.9), (3.10). In the UV, as $\lambda \to 0$, we start from a conformal theory. In other words, in this limit we can take W to be quasi-homogeneous by rescaling of the fields. For $N = 2$ LG theories, this problem has been solved in ref. [14] which shows how the differential equations (3.9) and (3.10) and other basic properties of the metric discussed above lead to the answer.

* The topological Green functions are computable. From their explicit form the non-renormalization is obvious.

** In the formulation of topological–anti-topological fusion of sect. 2, the perimeter β of the intermediate cylinder can be identified with λ.

It turns out that the answer can be written in a simple closed form that we will now discuss. Let $\phi_i(X_k)$ be a basis for the chiral primary fields of the LG theory. Then the metric can be given by finite-dimensional integrals over the variables X_i. For instance, if $\phi_i(X_k)$ is *relevant* (i.e. $q(\phi_i) < 1$) one has the very compact formula

$$g_{i\bar{j}} = \langle \phi_i \bar{\phi}_j \rangle = \int \prod \mathrm{d}X_l \, \mathrm{d}\bar{X}_l \, \phi_i(X_k) \bar{\phi}_j(\bar{X}_k) \exp(W - \bar{W}). \qquad (4.1)$$

We have to be a little careful with this integral. For one thing for large values of fields it is typically a highly oscillatory integral. Of course our intuition says that these highly oscillatory parts should not contribute appreciably to the integral. This intuition can be made more precise by defining the above integral using surfaces of constant W. Alternatively, we can define the above integrals by demanding Riemann bilinear identities to hold: Let $B^{\pm} \subset \mathbb{C}^n$ denote the asymptotic regions in $\mathbb{C}^n$ where $\mathrm{Re}\, W \to \pm\infty$. Here n denotes the number of variables. Let $\gamma_i^{\pm}$ label a basis of equivalence class of the n-chains in $\mathbb{C}^n$, whose boundary $\partial\gamma_i^{\pm} \subset B^{\pm}$, in other words they define a basis of the relative homology classes

$$\gamma_i^{\pm} \in H_n(\mathbb{C}^n, B^{\pm}).$$

Moreover, let C_{ij} denote the intersection matrix between these cycles

$$C_{ij} = \gamma_i^{+} \cap \gamma_j^{-}.$$

Then applying the idea of the Riemann bilinear identity to the above integral we come up with the following result *:

$$g_{i\bar{j}}\big|_{\mathrm{relev.}} = \int_{\gamma_l^-} \phi_i(X_k) \exp(W) C^{lm} \int_{\gamma_m^+} \bar{\phi}_j(\bar{X}_k) \exp(-\bar{W}). \qquad (4.2)$$

The residue can also be described in this way. One has

$$\eta_{ij} = \int_{\gamma_l^-} \phi_i(X_k) \exp(W) C^{lm} \int_{\gamma_m^+} \phi_j(X_k) \exp(-W)$$

$$\text{for} \quad q(\phi_i) + q(\phi_j) \leqslant c/3$$

$$= 0 \quad \text{otherwise}. \qquad (4.3)$$

Note that the above integrals are well defined by the choice of the cycles on which we integrate them. In appendix A we derive these formulas, by showing why they provide solutions to eqs. (3.9) and (3.10). It is important to notice that eq. (4.2) is

* Technically speaking, the symbols $\gamma_k^{\pm}$ represent locally-constant families of homology cycles rather than given cycles. This remark applies throughout the paper.

valid *only* at the conformal point where W is quasi-homogeneous. For more general W the story is far more complicated and cannot be described by such a simple integral. However, using SQM, even in those cases one can write similar expressions but one has to replace the fields in the above by the exact solutions to Schrodinger equation. This will be discussed in sect. 5. Instead eq. (4.3) is valid for *arbitrary* W's. More precisely, the general formula is

$$\eta_{ij} = \int_{\gamma_l^-} \phi_i(X_k) \exp(W) C^{lm} \int_{\gamma_m^+} \phi_j(X_k) \exp(-W). \qquad (4.4)$$

However here there is a subtlety. Whereas both sides of these equations transform the same way under a change of basis in $\mathscr{R}$, they transform differently under a change of the representative of BRST-classes

$$\phi_i(X_k) \to \phi_i(X_k) + \sum_l h_i^l(X_k) \, \partial_l W.$$

Then eq. (4.4) holds only for *special* representatives. The special operators ϕ_i are those associated to the special coordinates of TFT [10,28]. These coordinates are discussed in appendix C. There eq. (4.4) is proven. With generic representatives, the r.h.s. of eq. (4.4) would differ from η because of spurious mixings of the operators of charge q with those of charge $q-k$ (k a positive integer). Modifying the definition as in eq. (4.3) we disentangle this mixing. Then eq. (4.3) holds for all choices of the operators ϕ_i. See also ref. [29].

In the case that $W = 0$ defines a Calabi–Yau manifold in weighted projective space these results are all consistent with what is known as special geometry. In fact the integral representation of the metric (4.1) is very reminiscent of the period integrals of special geometry, but now in the context of general LG theory. We will see more connections below.

We can vary W by marginal operators, and remain in the class of conformal theories. Then it is natural to ask what is the relation between the g we have computed, and Zamolodchikov's definition, which gives a natural metric on moduli space of conformal theory. As we have discussed spectral flow relates chiral operators to the ground states, and so the metrics that we have computed must be related to the metric that Zamolodchikov defines. This relation is quite precise in the case that the perturbations are marginal and preserve the conformal properties of the theory. In particular using conformal Ward identities it is easy to show that what we have computed in this case is

$$g_{i\bar{j}} = \langle \phi_i(0) \bar{\phi}_j(1) \rangle$$

evaluated on the sphere. This is not precisely the metric that Zamolodchikov defines for two reason: The important reason is that ϕ_i and $\bar{\phi}_j$ are *not* themselves the perturbing operators, but rather $\int d^2\theta\phi_i$ and the complex conjugate of it are the perturbing operators. That is easy to implement, as again the superconformal Ward identities relate these to the above computation by multiplication by a factor of q_i^2 where q_i is the U(1) charge of the field ϕ_i (which we assume to have equal left and right charge – otherwise we would get $q_{i,L}q_{i,R}$). Note in particular that the identity operator gets projected out once integration over Grassmann coordinates is performed. For marginal operators the charges are all 1 and so this does not affect the above metric at all. The other point to bear in mind is that Zamolodchikov's definition is the expectation value of two operators, and we need to choose a correct normalization for the vacuum by dividing out by $\langle 0|0\rangle$. So for conformal deformations we see that the Zamolodchikov metric G is related to our g simply by (the index 0 labels the identity operator)

$$G_{i\bar{j}} = g_{i\bar{j}}/g_{0\bar{0}}, \tag{4.5}$$

where i, j run over the marginal directions.

It turns out that quite generally, one can show that the metric G for the metric on moduli space of $N = 2$ SCFTs is Kähler. This is in fact true for arbitrary $N = 2$ SCFTs and not just LG theories. In the conformal limit we have an extra U(1) symmetry, with respect to which all chiral primary states, except for the identity operator which is neutral, have positive charge. Then by charge conservation we have

$$g_{0\bar{k}} = g_{k\bar{0}} = 0 \quad \text{for} \quad k \neq 0,$$

$$(gC_i^{\dagger}g^{-1})_k{}^0 = 0 \quad \text{for} \quad k \neq 0.$$

Let the indices i, j correspond to *marginal* perturbations, i.e. chiral primary fields of charge $q = 1$. Then from (3.9) we find

$$-\partial_{\bar{j}}\partial_i \log\langle 0|0\rangle = \left[\partial_{\bar{j}}(g\partial_i g^{-1})\right]_0{}^0 = (C_i)_0{}^k g_{k\bar{l}} C^{*\bar{l}}_{\bar{j}\bar{0}} g^{\bar{0}0} = g_{i\bar{j}}/g_{0\bar{0}} = G_{i\bar{j}},$$

where we used that $C_{i0}^k = C_{0i}^k = \delta_i^k$. Let $|\rho\rangle$ be the Ramond state of maximal charge dual to $|0\rangle$ with respect to the pairing η_{jk}. Using eq. (2.9) we see that

$$\partial_j\partial_i \log\langle\rho|\rho\rangle = -\partial_j\partial_i \log\langle 0|0\rangle.$$

So we get

$$G_{ij} = \partial_i\partial_j \log\langle\rho|\rho\rangle.$$

Thus we find that in the $N = 2$ case the Zamolodchikov metric (along the marginal directions) is Kähler with potential $K = \log\langle\rho|\rho\rangle$. This is a result due to Periwal and Strominger [30].

In the case of LG theories the integral representation of the metric (4.1) implies that we can write the Kähler potential as an integral

$$g_{0\bar{0}} = e^{-K} = \int \prod dX_i\, d\bar{X}_i \exp(W - \bar{W}). \tag{4.6}$$

In the case that W is of a form to be directly related to Calabi–Yau manifolds [31], i.e. with integer $\hat{c}$ and the number of variables $n = \hat{c} + 2$, then doing the integral above with respect to one of the variables (with a suitable change of variables) results in $\delta(W)$ in the integrand. It was observed by Greene that if one continues this formal integration one more step one ends up with $\int\omega \wedge \bar{\omega}$, where ω is the representative of the $(\hat{c}, 0)$-form on the manifold $W = 0$ defined in weighted projective space. So in this case we have

$$e^{-K} = \int \omega \wedge \bar{\omega},$$

which is a well-known result due to Tian [32]. One should emphasize that (4.6) is valid *regardless* of a Calabi–Yau interpretation of the LG theory. From the other equations in (3.9) we get additional constraints on this Kähler potential. It is easy to show that they reproduce the conditions valid for a variation of Hodge structure on the algebraic hypersurface $W = 0$ in weighted projective space, which may or may not be a CY manifolds. This was discussed at length in ref. [14].

All we have said so far is only valid at the conformal point, i.e. the limit where $\lambda \to 0$. Now we wish to discuss what is the form of the metric in the IR, i.e. when $\lambda \to \infty$. In such a case the critical points of W, i.e. $dW = 0$ which are the minima of energy, become infinitely separated from each other, and to leading order do not see each other. In other words to leading order the metric becomes diagonal in basis of chiral fields corresponding to excitations near the vacua. So we can base our physical vacuum by shifting fields to correspond to each one of the vacua we wish to study. If the critical point is not a simple zero of dW, then the field configurations near that critical point will still describe a (massless) conformal theory and what we said above about the computation of g remains valid for this part of the metric. However at the critical points of W for which dW has a simple zero, we end up with a massive theory. In the limit that $\lambda \to \infty$ the mass goes to infinity proportional to λ. Again in this case the metric is trivial to compute using free massive field theory.

These vacua will not completely decouple from each other, in the sense that there are instanton corrections which tunnel from one vacuum to another and provide off-diagonal elements for the metric which are exponentially small as $\lambda \to \infty$. In order to describe this situation, let us take the case where all the critical points of W are simple, i.e. that they all give rise to massive theories. It is convenient to use the "point" basis for $\mathscr{R}$. Two holomorphic functions $f_1(X)$ and $f_2(X)$ represent the same element in $\mathscr{R}$ iff

$$f_1(X_k) = f_2(X_k) \quad \forall X_k,$$

where X_k are the critical points; this follows from the residue formula (2.10). So we can label each equivalence class by its values at the critical points. We denote by ϕ_k the class such that we get 1 at X_k and 0 at X_h $(h \neq k)$. In this basis, as $\lambda \to \infty$ we get

$$g_{k\bar{h}} \simeq \frac{\delta_{kh}}{|\mathfrak{H}(X_k)|},$$

where $\mathfrak{H}$ denotes the hessian of λW evaluated at the critical point. In the case of one field, one can also give a general form for the first correction to this classical limit. One finds that if there is a primitive soliton connecting the two vacua, the condition for the existence of which has been studied in ref. [33], one obtains a correction of the form

$$\frac{g_{k\bar{h}}}{(g_{k\bar{k}}g_{h\bar{h}})^{1/2}} \simeq \alpha_{k\bar{h}}(4\pi z_{kh})^{-1/2} \exp[-2z_{kh}] \quad k \neq h, \tag{4.7}$$

where

$$z_{kh} = \lambda |W(X_k) - W(X_h)|,$$

and $\alpha_{k\bar{h}}$ is a phase factor. Here $2z_{kh}$ is equal to the mass of the soliton connecting the two vacua. This result is discussed in appendix B.

Having discussed the two limiting cases of UV and IR, it is natural to ask what can be said in general about the properties of the flow in between. In particular, does there exist a natural "c-function" for us? What is the relation of Zamolodchikov's metric to our ground state metric g away from the conformal point? We will now address these questions in turn.

The central charge of the SCFT is proportional to the maximum charge in the ring of chiral primary fields [2,4]. Indeed

$$c/3 = \hat{c} = q_{\max}.$$

In the Ramond sector the charges are shifted by $-\hat{c}/2$, and they are symmetrically distributed between $-\hat{c}/2$ to $\hat{c}/2$. It is natural to try to define this charge, even off criticality, and view it as a "c-function". We should in fact be able to do more: The charges q_k of the chiral primary fields are all on the same footing from an abstract point of view. So we must be able as well to define q-functions corresponding to the charges of all these operators. In fact there is a theorem in *Singularity Theory*

[34] stating that all these functions would satisfy a "c-theorem". More precisely, suppose we perturb a singularity (which corresponds to a given $N=2$ critical theory) in order to get a simpler singularity (which is interpreted as the corresponding IR fixed point). Let Δ denote the number of chiral primary fields. Order the charges of the chiral primary operators in a non-decreasing sequence

$$0 = q_1 \leq q_2 \leq \ldots \leq q_\Delta = c/3,$$

then one has

$$q_k \leq q_k' + \tfrac{1}{6}(c - c') \leq q_{k+\delta}, \tag{4.8}$$

where the primed quantities refer to the IR fixed point and $\delta = \Delta - \Delta'$ is the difference of Witten indices between the UV and the IR theories.

Motivated by these observations one naturally looks for a definition of a "charge" matrix. Note that by a change of phase of the Grassmann variables, we see that the phase of λ is not a physical degree of freedom and all quantities depend on $|\lambda|$. Let

$$\lambda = e^{\tau}.$$

In other words, the metric and all the other physical quantities depend on τ through its real part $\tau + \bar{\tau}$. Now we are to define a notion of a charge matrix, using the only quantity available to us, namely the ground state metric g. Near a conformal point g becomes diagonal in a basis of ground state vacua with definite charge. One can easily see using the Ward identities that, in the basis defined by our path integral, as $\lambda \to 0$ g behaves as *

$$g_{i\bar{i}} \sim (\lambda\bar{\lambda})^{-q_i - n/2},$$

where here q_i denotes the charge of the ith Ramond vacuum. We thus see that near the critical point the matrix

$$g\partial_\tau g^{-1} - n/2$$

is a diagonal matrix with eigenvalue equal to the charges of the Ramond vacua. In particular the maximum eigenvalue of this matrix reproduces $c/6 = \hat{c}/2$ near criticality. So let us define the Ramond charge matrix q as

$$q = g\partial_\tau g^{-1} - n/2, \tag{4.9}$$

i.e. the "gauge connection" in the direction of flow minus the "anomalous" part.

* The shift of q_i by $n/2$ is related to the behaviour of η under a rescaling of W (which is a kind of "anomaly" arising from the Fermi zero modes). Indeed, from eqs. (2.9) and (2.10) we have $\det[g] = |\det[\eta]| \sim |\lambda|^{-n\Delta}$.

This q has a simple field-theoretical interpretation. Since nothing depends on the D-terms, we fix them to be the "standard" ones

$$K = \sum_i \overline{X}_i X_i.$$

If W is quasi-homogeneous we have a conserved U(1) current J_μ, and we must have

$$q_{h\bar{k}} = \langle \bar{k} | \oint J_0(\sigma)\, \mathrm{d}\sigma | h \rangle.$$

Noether's theorem gives the following expression * for J_μ:

$$J_\mu = J_\mu^5 + U|_{A_\mu},$$

where

$$J_\mu^5 = -\tfrac{1}{2} \sum_i \overline{\psi}_i \gamma_\mu \gamma_5 \psi_i,$$

$$U = \sum_i q_i \overline{X}_i X_i.$$

Since $U|_{A_\mu}$ is a Q-commutator, we have

$$q_{h\bar{k}} = \langle \bar{k} | \oint J_0 | h \rangle = \langle \bar{k} | \oint J_0^5 | h \rangle.$$

Consider now a generic superpotential W. The current J_μ^5 is still partially conserved. Indeed, it is only softly broken by the superpotential W

$$-\partial_\mu J_\mu^5 = i \int \mathrm{d}^2\theta^+ W - i \int \mathrm{d}^2\theta^- \overline{W}. \qquad (4.10)$$

Hence it makes sense to consider its matrix elements. Then the natural definition of the off-critical charge is

$$q_{h\bar{k}} = \langle \bar{k} | \oint J_0^5 | h \rangle.$$

This definition agrees with the previous one, eq. (4.9). To see this we compare (4.10) with the path-integral definition of the connection. In our context, eq. (4.10) should be modified. Indeed, in order to produce the correct vacuum state we have

* $U|_{A_\mu}$ means the vector component of the superfield U.

introduced a background gauge field in the right hemisphere. Then the axial current develops an anomaly

$$-\partial_\mu J_\mu^5 = i \int \mathrm{d}^2\theta^+ W - i \int \mathrm{d}^2\theta^- \overline{W} + (n/2\pi) F.$$

Consider the connection along the flow

$$A_{\tau k h} \equiv \langle h | \partial_\tau | k \rangle = \langle h | (\partial_\tau - \partial_{\bar{\tau}}) | k \rangle.$$

It has the following functional representation:

$$i \langle \phi_h | \left(\int_{S_R} D^+ \overline{D}^+ W - \int_{S_R} D^- \overline{D}^- \overline{W} \right) \phi_k \rangle$$

$$= -\langle \phi_h | \left(\int_{S_R} \left[\partial_\mu J_\mu^5 + (n/2\pi) F \right] \right) \phi_k \rangle$$

$$= -\langle \phi_h | \oint J_0^5 | \phi_k \rangle - \tfrac{1}{2} n \langle \phi_h | \phi_k \rangle,$$

which shows that the two definitions agree. This also guarantees the "gauge independence" of the eigenvalues of q, which is not manifest from eq. (4.9). Under a "gauge transformation" the variation of the anomalous term compensates the change in the connection. From the QFT viewpoint it is manifest that the spectrum of q is real and symmetric about zero. This follows most clearly in a basis where $\eta = \eta^* = \eta^{-1}$. Then from eq. (2.9) we see that

$$q\eta = -\eta q.$$

Now we can show that the criticality of q as a function of couplings occurs only at the conformal points. This is an easy consequence of eq. (3.9), namely we have *

$$\bar{\partial}_i q = \left[C_\tau, g C_i^\dagger g^{-1} \right],$$

and at the conformal point the matrix C_τ is represented by multiplication by W, and since at the conformal point W is quasi-homogeneous, it follows that W itself is in the ideal generated by dW and thus is trivial in $\mathscr{R}$. Therefore $C_\tau = 0$ precisely at the conformal point and thus from the above equation we see that q is critical precisely at these points. This is also true the other way around, namely, $C_\tau = 0$ implies W is quasi-homogeneous [35]. This is the algebraic characterization of a fixed point, in the sense that when this happens the chiral ring has the properties

* Because of reality of the eigenvalues it is enough to check stationarity with respect to the couplings $\bar{t}_i$.

prescribed for a critical point. Whether it is actually a fixed point is a more tricky question depending, of course, on the D-term too.

At criticality eq. (3.10) reduces to

$$[C_i, q] = C_i,$$

which merely states that only perturbations by operators of charge 1 are compatible with conformal invariance.

From this definition of the q-function it is not obvious that this quantity satisfies a "c-theorem". This should be globally true, in the sense that the inequalities (4.8) between the eigenvalues at the UV and IR points hold true. What is not manifest, is that pointwise along the "RG-trajectory" the derivative of these quantities has a definite sign. However, experience with concrete models suggests this is also true. Moreover, using the connection with Special Geometry it is easy to show that $\dot{c}$ is non-positive near a critical point. So, at least our version of the "c-theorem" holds in perturbation theory.

There is another way of getting the q-function which is more convenient since it holds in an arbitrary basis (provided the operators ϕ_k do not depend explicitly on the t's) without need of a compensating "anomalous" term. Consider the matrix

$$Q_k{}^h = G_{kl}\partial_\tau (G^{-1})^{lh},$$

where G is the above normalized metric. It is easy to see that near the critical point this definition of charge Q gives the list of the charge of chiral primary fields and in particular the range of the eigenvalues goes from 0 to $c/3 = \hat{c}$. Three times the maximal eigenvalue is then a candidate c-function. Obviously, the two definitions agree. We will refer to this function as *algebraic c-function*. It would be interesting to see what is the precise relationship of this c-function with that of Zamolodchikov.

Now we turn to the question of the relation between the Zamolodchikov metric off criticality with the ground state metric g. If we wished to write the Zamolodchikov metric for both marginal and relevant perturbations, *at the conformal point* all we have to do is to multiple G by factors of charge mentioned above. It is now clear that we cannot expect a simple relation between our metric g and Zamolodchikov's metric G off-criticality, because we already see that even near criticality we have to know the charges of fields in order to relate the two, and the notion of U(1) charges of fields is well defined only at criticality. It is natural to suspect that given the off-critical definition of charge discussed above there might be a way to define a natural metric which is related to Zamolodchikov's definition. Even though there are some obvious guesses, we leave a carefully study of this for the future.

5. Reduction to SQM

There are other useful points of view about the ground-state metric. In ref. [5] it was shown that g can be computed by dimensional reduction to one dimension (i.e. in Supersymmetric Quantum Mechanics). Roughly, this follows from the fact that one can find a susy (but not Lorentz) invariant D-term which suppresses all the non-zero modes in the Fourier expansion of the fields. Thus, independence from the dimensions is a special instance of independence from the Kähler potential. Although the computations can be done directly in 2 dimensions, the reduction to SQM is useful for two reasons: first of all, here one has an explicit construction for the isomorphism of primary fields and states in the Ramond sector in terms of the wave functions of the SQM vacua. This also naturally encodes in a geometric way the "anomalous" transformation under field redefinitions, which as we mentioned is related to the violation of fermion numbers in the topological description of the theory. The second reason is that we can give a general solution to the linear problem (3.11) in terms of the vacuum wave functions. This also turns out to be very closely related to the generalization of special geometry in the context of massive theories. As customarily, we identify SQM wave functions with differential forms via

$$\Phi_{i_1 i_2 \dots i_r \bar{k}_1 \bar{k}_2 \dots \bar{k}_s}(X_j)\psi^{i_1}\dots\psi^{i_r}\bar{\psi}^{\bar{k}_1}\dots\bar{\psi}^{\bar{k}_s}|0\rangle$$

$$\to \Phi_{i_1 i_2 \dots i_r \bar{k}_1 \bar{k}_2 \dots \bar{k}_s}(X_j)\, \mathrm{d}X^{i_1}\wedge\dots\wedge \mathrm{d}X^{i_r}\wedge \mathrm{d}\bar{X}^{\bar{k}_1}\wedge\dots \mathrm{d}\bar{X}^{\bar{k}_s}.$$

Then in the Schrödinger representation, Q_R^+ is represented as

$$Q_R^+ = \bar{\partial} + \mathrm{d}W\wedge$$

and Q_L^- is represented as

$$Q_L^- = \partial + \mathrm{d}\overline{W}\wedge .$$

The isomorphism between the realizations of $\overline{W}$-cohomology on fields and states becomes

$$\phi_k \to \frac{1}{(-2\pi)^{n/2}}\phi_k\, \mathrm{d}X^1\wedge\dots\wedge \mathrm{d}X^n + Q_R^+\xi_k .$$

Note that this isomorphism takes into account the topological violation of fermions number mentioned before. In fact from the path-integral description of sect. 2 it should be clear that once we see why the identity operator can be represented cohomologically by $\mathrm{d}X^1\wedge\dots\wedge\mathrm{d}X^n$ the above follows, and that representation of the identity operator can be shown by taking a very tiny hemisphere, represented by a little disc and perform the topological path integral. In the language of SQM it is manifest that under a field redefinition the Ramond state representing ϕ_k should transform as a $(n, 0)$-form rather than as scalar. This is the origin of the "anomalous" jacobian. Clearly,

$$Q = Q_L^+ + Q_R^- \equiv \exp\left[-W(X) - \overline{W}(\overline{X})\right] \mathrm{d} \exp\left[W(X) + \overline{W}(\overline{X})\right],$$

where d is the exterior differential. Since the vacuum wave-forms ω_k are annihilated by Q and its adjoint $Q^\dagger$, the modified forms

$$\hat{\omega}_k \equiv \exp\left[W(X) + \overline{W}(\overline{X})\right]\omega_k,$$

$$\tilde{\omega}_k \equiv \exp\left[-W(X) - \overline{W}(\overline{X})\right] * \omega_k$$

are d-closed. They represent some kind of cohomology of the d-operator. Obviously this cannot be the usual deRham one, since for $\mathbb{C}^n$ it is trivial. In fact, these forms are representative of *relative* deRham classes. For $\tilde{\omega}_k$ the relevant cohomology is $H^n(\mathbb{C}^n, B)$, where $B \subset \mathbb{C}^n$ is the region where Re W is greater than a certain (large) value. The $\hat{\omega}_k$ correspond to the dual cohomology space. This dual space * can be identified with (equivalence classes of) n-chains γ_i^+ such that on $\partial\gamma_i^+$ we have Re $W \to +\infty$. We put

$$\Pi_k^j = \int_{\gamma_j^+} \tilde{\omega}_k. \tag{5.1}$$

One checks that Π_k^j is finite and $\det[\Pi] \neq 0$. From ref. [14] one sees that there exists $\lambda_{i,k}$ such that

$$D_i\omega_k = (\partial + \mathrm{d}\overline{W}\wedge)\lambda_{i,k},$$

where

$$(\bar{\partial} + \mathrm{d}W\wedge)\lambda_{i,k} = \partial_i W\omega_k - (C_i)_k^h\omega_h.$$

Then one gets

$$D_i\Pi_k^j = -(C_i)_k^h\Pi_h^j, \qquad \overline{D}_a\Pi_k^j = -(\overline{C}_i)_k^h\Pi_h^j,$$

that is

$$\nabla\Pi = \overline{\nabla}\Pi = 0. \tag{5.2}$$

* This dual space can be viewed as providing an *integral* basis for the vacua

The matrix Π gives the general solution to the linear problem (3.11). These remarks give a simple description of the geometry of the bundle over the parameter space discussed in sect. 3. Indeed, we see that the vacuum wave-forms, after projection into the relevant relative homology, represent sections of the bundle discussed there.

The real-structure matrix M_k^j has a simple meaning in SQM. The Schrödinger equation is real, and hence the complex conjugate of a vacuum wave-form ω_k is again a vacuum wave-form and should be a linear combination of the ω_h. If the ω_k corresponds to the basis ϕ_k, we have

$$(\omega_k)^* = M_k^h\omega_h,$$

from which the reality constraint is obvious. In particular, we have

$$\Pi^* = M\Pi \quad\Rightarrow\quad M = \Pi^*\Pi^{-1}, \tag{5.3}$$

which gives an alternative way of computing the metric from the solution of the linear problem.

In SQM, eq. (2.8) follows from the definition of the ground-state metric

$$\langle\bar{k}\,|\,h\rangle = \int * \omega_k^* \wedge \omega_h, \tag{5.4}$$

and the cohomological identity

$$\int * \omega_k \wedge \omega_h = \mathrm{Res}_W[\phi_k\phi_h] \equiv \eta_{kh}, \tag{5.5}$$

which is a consequence of the Bochner–Martinelli theorem (see the appendix of ref. [14] for details). In analogy with (5.1) we write

$$\hat{\Pi}_k^j = \int_{\gamma_j^-} \hat{\omega}_k,$$

where γ_j^- are cycles with Re $W = -\infty$ at the boundary. Using the fact that

$$D_i\hat{\Pi} = C_i\hat{\Pi}, \qquad \overline{D}_i\hat{\Pi} = \overline{C}_i\hat{\Pi},$$

one can easily show, using the uniqueness of solutions to linear differential equations, that

$$\hat{\Pi}_k^j = \eta_{kh}\rho^{ji}(\Pi^{-1})_i^h,$$

where ρ'' is some pairing * of the above cycles which is independent of the couplings t'. Then (5.3) gives

$$\eta = \hat{\Pi}\rho\Pi^{\mathrm{T}}, \qquad g = \hat{\Pi}\rho\Pi^{\dagger},$$

which are a kind of Riemann bilinear identities for the integrals (5.4) and (5.5).

This SQM viewpoint is quite suggestive of the geometry of a variation of Hodge structure (*special geometry* in the physics languages). Indeed, the matrix Π is just the period map for the relative classes $\bar{\omega}_k$. Note though, the similarities are somewhat misleading in that the period matrix which is holomorphic in the case of special geometry (or variation of Hodge structure) has the distinctive property of *not* being holomorphic in terms of couplings t_i. And even though we have an integral representation for the metrics in terms of solutions of Schrödinger equation, it is *not* possible to give a closed form answer for them as integrals of simple objects, as it was the case in the quasi-homogeneous (conformal) case discussed in the previous section. In this sense the problem is much more difficult to solve in the massive case. We have already mentioned that ∂ can be identified with the Gauss–Manin connection. In fact eq. (5.2) can be seen as the defining property of the GM connection in terms of periods. So, the structure arising out of $N = 2$ susy is a generalization of *special geometry*.

The SQM viewpoint is very useful from another view point, and that arises when one considers changes of variables. Indeed it turns out that one can do non-invertible field redefinitions and still be able to relate the metrics between the two models. That this is possible is essentially why the formal arguments in ref. [31] relating LG theories to geometry of CY can be justified – at least as far as the metrics on the moduli space is concerned [14]. Moreover this will also justify, to the extent of getting the same moduli metric, the more recent work on relating different LG theories with each other by non-invertible changes of variables [36]. It turns out that for many of the applications that we will consider this is a very important technique.

The simplest way to understand how it works is in the language of SQM. We will use a mathematical language as it is most convenient to describe it in that setting, where we sometimes refer to the nice properties of non-invertible changes of variables as "functoriality with respect to branched coverings". Let ω_k ($k = 1,\dots,\Delta$) be the vacuum wave-forms for some superpotential $W(X)$. In this superpotential we make a substitution

$$X = f_i(Y_j),$$

* Just as in the conformal case, ρ'' is simply the inverse of the intersection matrix $\gamma_i^* \cap \gamma_j$. It is possible to show this by multiplying the integrand in eq. (5.4) by one represented by $\exp(W + \overline{W})\exp(-W - \overline{W})$, and using the Riemann bilinear identity.

where the map f is holomorphic but not globally invertible (otherwise we would get just an irrelevant field redefinition). Then consider the new superpotential

$$W_f(Y_j) = W(f_i(Y_j)) \equiv f^*W.$$

For the supercharges one has

$$Q^+_{\mathrm{R},f} = \bar{\partial} + \mathrm{d}W_f \wedge \; \equiv f^*Q^+_{\mathrm{R}},$$

$$Q^-_{\mathrm{L},f} = \partial + \mathrm{d}\overline{W}_f \wedge \; \equiv f^*Q^-_{\mathrm{L}},$$

so that the forms $f^*\omega_k$ satisfy

$$Q^+_{\mathrm{R},f} f^*\omega_k = Q^-_{\mathrm{L},f} f^*\omega_k = 0.$$

IN the case of just one field, these equations imply that $f^*\omega_k$ ($k = 1,\dots,\Delta$) are vacuum wave-forms for the superpotential W_f. (Recall that if $n = 1$ the wave forms are independent of K as form, not just as cohomology classes). In the general case, the new wave functions are

$$\Omega_k = f^*\omega_k + Q^+_{\mathrm{R},f} Q^-_{\mathrm{L},f} \mathfrak{H}^{-1}\Lambda f^*\omega_k,$$

where the dependence on the Kähler metric is hidden in Λ and $\mathfrak{H}$. The Ω_k's are manifestly cohomologous to the pullbacks of the forms ω_k. Indeed, if W is not degenerate, $\mathfrak{H}^{-1}$ is a continuous operator in the $(n-2)$-form sector. Of course, these functions are just a subset of all vacuum wave-forms for W_f since $\Delta_f > \Delta$ for a branched cover. Now for $n = 1$, one has simply

$$\langle \bar{k} | h \rangle|_{W_f} = \int * \overline{f^*\omega_k} \wedge f^*\omega_h$$

$$= (\deg f) \int * \overline{\omega_k} \wedge \omega_k = (\deg f) \langle \bar{k} | h \rangle|_W . \tag{5.6}$$

(for $n = 1$, the Hodge dual $*$ on 1-forms depends on the complex structure only). The equality is true for the general case as well, the only difference being that in order to prove it one has to use the full machinery of the cohomological computation for overlap integrals, see ref. [14]. Alternatively, functoriality follows from the (conjectural) uniqueness of the solution to our equations. Indeed, the topological functions η_{ij} and C_{ij}^k are trivially functorial, and hence the equations themselves behave as expected under non-invertible change of variables. Therefore, if we know the ground-state metric for W_f we can get the metric for W just by restricting ourselves to the cohomology classes $\phi_k\, dY^1 \wedge \ldots \wedge dY^n$ (with $\phi_k \in \mathscr{R}_f$) which can be written as

$$\phi_k\, dY^1 \wedge \ldots \wedge dY^n = f^*\left(\psi_k\, dX^1 \wedge \ldots \wedge dX^n\right), \quad \psi_k \in \mathscr{R}. \tag{5.7}$$

Note that in this way we automatically reproduce the "anomalous" jacobian.

The presence of a jacobian in the transformation has another implication. Suppose that both W and f are quasi-homogeneous. Then so is W_f. Both models are critical and we can speak of their central charge. Then using the fact that hessian is the maximum charged element in the ring with charge $c/3$, eq. (5.7) implies

$$c = c_f - 6q_f(J), \tag{5.8}$$

where $q_f(J)$ is the U(1) charge of the jacobian

$$\det\left[\partial f_i / \partial Y_j\right] \in \mathscr{R}_f.$$

The insertion of the jacobian just soaks up the excess of vacuum charge of the branched model with respect to the original one. Note that we can use this technique to relate different conformal theories even with *different* central charges, as far as the metric on chiral primary fields are concerned. It would be interesting to investigate the precise relation between the full conformal theories in such cases.

6. Lie-algebraic aspects

Our equations have an interesting group-theoretical meaning. This is well known in the conformal case where W is quasi-homogeneous, where it is related to the Lie-algebraic aspects of the period map of the corresponding hypersurface (or the Lie-algebraic structure of the Variation of Hodge structure). It turns out that the Lie-algebraic point of view is very useful even for *massive* perturbations of our theories as well and they help us understand the geometrical content of the equations as well as to actually solve them. Our discussion here is modelled on the classical one for the topology of algebraic hypersurfaces (which arises in the conformal limit). This case we will refer to as the "geometrical case" below.

We begin by discussing the reality condition on the metric (2.9). One can find a "special" holomorphic basis such that the residue pairing is independent of the couplings t^i and

$$\eta^* = \eta^{-1} = \eta.$$

Such special bases have been considered before in the context of topological field theories [10]. Their existence is a deep property of TFT and they are also technically convenient. See appendix C for details. In such a basis, the reality constraint on the metric becomes

$$g\eta g^{\mathrm{T}} = \eta,$$

i.e. g is orthogonal with respect to the real metric η. Then, $g\,\partial g^{-1}$ belongs to the corresponding Lie algebra, namely $g(\partial g^{-1})\eta$ is antisymmetric. Thus the first term in eq. (3.9) is skew-symmetric with respect to η. This is consistent with our equations. Indeed, the topological 3-point functions C_{ij}^k are η-symmetric (that is, $C_{ijk} \equiv C_{ij}^l \eta_{lk}$ is symmetric. So is $gC_j^\dagger g^{-1}$ (since $\eta M = (M^{-1})^{\mathrm{T}}\eta^*$). Then $[C_i, gC_j^\dagger g^{-1}]$ is also η-antisymmetric. Note that, without loss of generality, we can choose $\eta = \mathbb{1}$, so g is orthogonal in the standard sense. Of course, g belongs to the complexified orthogonal group, not to the usual compact form.

To go on with the discussion, it is better to rewrite the linear problem (3.11), in a more convenient way. Let $g = \exp[\mathscr{G}]$ and put $e = \exp[\mathscr{G}/2]$. We perform the gauge transformation

$$\psi \to eT.$$

Then the linear problem becomes

$$\left[\partial - (\partial e)e^{-1} + e^{-1}Ce\right]T = 0,$$
$$\left[\bar{\partial} + e^{-1}(\bar{\partial} e) + eC^\dagger e^{-1}\right]T = 0. \tag{6.1}$$

From now on, by T we mean the fundamental solution, i.e. T is the matrix solution such that $T(0) = \mathbb{1}$. By adding an irrelevant constant to W, we can assume that tr $C = 0$. Then from (6.1) it is manifest that T belongs to SL(Δ, $\mathbb{R}$). This is similar to what one finds in the geometrical case, where however there are additional algebraic restrictions coming from the topology. They reflect the so-called Riemann bilinear relations. Under certain circumstances, similar restrictions apply to the massive case as well. They are quite important, since restricting the Lie group in which T takes values is a crucial step in solving the equations for particular models. Let G be this group and H be the subgroup gauges by the connection for D, $\bar{D}$. One had H ⊂ K, where K is the maximal compact subgroup of G (this follows from the fact that the connection is metric – or put differently, from the eq. (6.1) and recalling that e is hermitian). Of course, g (and e) belong to $H_{\mathbb{C}}$ (i.e. they are complex gauge transformations). The importance of identifying G and H is best understood by realizing what the equations become for special G and H.

Suppose we have a family of superpotentials depending on just a single coupling t. This will be the case of most interest for us in the rest of this paper. The simplest case is when H is the maximal torus of G. In this case eqs. (3.5) are just the usual Lax-representation of a Toda system (indeed, consistency alone implies that the matrix C_i is the sum of an *admissible* set of roots for G). Then eqs. (3.5) reduce to the standard equations of Toda field theory.

The Gauss–Manin equations for the variation of Hodge structure for an algebraic manifold X (of dimension m) are also of the form (6.1) with

$$G = SO(b_m^+, b_m^-), \qquad H = SO(h^{k,k}) \bigotimes_{p=0}^{k-1} U(h^{m-p,p}) \quad m = 2k,$$

$$G = Sp(b_m, \mathbb{R}), \qquad H = \bigotimes_{p=0}^{k} U(h^{m-p,p}) \quad m = 2k+1,$$

where $h^{p,q}$ (resp. b_m) are the Hodge (resp. Betti) primitive numbers and

$$b_m^+ + b_m^- = b_m, \qquad b_m^+ - b_m^- = \tau \quad \text{(Hirzebruch signature)}.$$

In this case C_i is the class in $H^1(\Theta)$ of the complex deformation corresponding to an infinitesimal variation of the parameter t^i, seen as the matrix of the endomorphism in $H^m(X)$ induced by wedge product (where Θ represents the tangent bundle).

In particular, if we have a Hodge (sub-)structure such that for some integer a

$$h^{m-p,p} = 1 \quad \text{for} \quad |m-2p| \leq a$$
$$= 0 \quad \text{otherwise},$$

and $(\partial_i W')^a \neq 0$ in $\mathcal{R}$, then the GM equations reduce to those of the G-Toda molecule (i.e. the non-affine version). The simplest example of this state of affairs is the torus. The σ-model on a torus is equivalent to an orbifold of the LG theory [31] with superpotential

$$W = X_1^3 + X_2^3 + X_3^3 + tX_1X_2X_3.$$

in this case $h^{1,0} = h^{0,1} = 1$ and hence G = Sp(2, $\mathbb{R}$) and H = U(1). In other words, in this case the monomial $X_1X_2X_3$ generates a *nilpotent* subring of order 2, and that is how we end up with Sp(2, $\mathbb{R}$). Solving the linear problem one gets (for details, see ref. [37])

$$\frac{\langle q=1/2 | q=1/2\rangle}{\langle q=-1/2 | q=-1/2\rangle} = \frac{1}{4|\text{Im}\,\tau(t)|^2}\left|\frac{d\tau}{dt}\right|^2,$$

for some holomorphic function $\tau(t)$. This is precisely the general (real) solution to the SL(2)-Toda equation, i.e. the Liouville equation

$$\frac{\langle q=1/2 | q=1/2\rangle}{\langle q=-1/2 | q=-1/2\rangle} = \exp[\phi_{\text{Liouville}}].$$

However, in the LG language the function $\tau(t)$ is further restricted by the boundary conditions. It turns out that this function is just equal to the period for the torus $W=0$ as it should from the general correspondence between LG theories and geometry [2,31]. Indeed, one can use the degeneration structure of the algebraic surface to find out what $\tau(t)$ exactly is.

This example can be generalized. Take the CY manifolds $\mathscr{X}_n$ associated to the superpotentials

$$W = X_1^n + X_2^n + \ldots + X_n^n + tX_1X_2\ldots X_n$$

and consider the Hodge substructure (i.e. the subset of $\mathcal{R}$) corresponding to the subspace of $H^{n-2}(\mathscr{X}_n)$ invariant under the automorphisms

$$X_j \to \exp[2\pi i a_j/n]X_j \quad \sum_j a_j = 0 \bmod n.$$

(It is precisely modding out the LG theory by this symmetry that has been shown to be a beautiful example of mirror symmetry [38].) The ring invariant under the above transformation is generated by $X_1 \ldots X_n$. In this case the equations one gets for the metric g is the same as Toda molecule with G = Sp($n-1$, $\mathbb{R}$) (resp. SO($n/2$, $n/2-1$)) for n odd (resp. even). These follow very easily from eq. (3.9). In particular these Toda theories emerge as a $\mathbb{Z}_2$ reduction of sl($n-1$) Toda, with

$$\left\langle (\bar{X}_1 \ldots \bar{X}_n)^r \middle| (X_1 \ldots X_n)^r \right\rangle = \exp(q_r)$$

with $0 \leq r \leq n-2$, and one identifies the vector v in $q_r - q_{r-1} = q_i v^i$ with a simple root of sl($n-1$). The $\mathbb{Z}_2$ reduction follows from (2.9) implying $q_r + q_{n-2-r} = 0$. It is the nilpotent structure of the ring generated by the symmetric monomial $X_1 \ldots X_n$ which directly reflects the sl($n-1$) Toda molecule structure in these equations.

The general case of arbitrary deformations of algebraic hypersurfaces is a very natural generalization of the Toda molecule. In ref. [14] the ground state metric for quasihomogeneous superpotentials was written in terms of holomorphic countour integrals. This explicit representation is just the extension to the more general case of the standard Leznov–Saveliev algorithm to solve Toda equations [39]. (This is common knowledge in Algebraic Geometry). Indeed, this algorithm reduces the solution of the Toda molecular to finding a triangular holomorphic matrix Π satisfying

$$\partial_t \pi_i{}^k = C_{ti}^h(t)\Pi_h{}^k .$$

The period integrals of ref. [14] (after filtration á la Griffiths) give the special Π matrix satisfying the correct boundary condition. Of course, this method works for all variations of Hodge structure, even if H is not abelian and we have a multiparameter family.

Now we come back to the more general case of massive perturbations, and wish to determine G and H. There is a simple method to determine H. Decompose the vacuum subspace $\mathscr{V}$ of the Hilbert space into orthogonal subspaces corresponding to different irreducible representations of the (pseudo)-symmetries of W. A priori from the above discussion it is clear that H is a subgroup of product of $\mathrm{U}(N_R)$ where N_R denotes the dimension of the representations in question. However η, which is of order 2, acts on the representations, and because of the eq. (2.9) relates the $\mathrm{U}(N_R)$ for each pair and so cuts the number of $\mathrm{U}(N_R)$ by half. Also, if η maps a representation to itself, eq. (2.9) implies that the corresponding H is in $\mathrm{SO}(N_R)$. Put differently, an irreducible representation we call *real* if it is real with respect to the real-structure M. Then, a real subspace of dimension N_r contributes a factor $\mathrm{SO}(N_r)$ to H, and a conjugate pair of complex subspaces of dimension N_c contribute a factor $\mathrm{U}(N_c)$. I.e.

$$\mathrm{H} \subset \bigotimes_{\text{pairs}} \mathrm{U}(N_c) \bigotimes_{\text{real}} \mathrm{SO}(N_r).$$

In particular, H is abelian if all complex subspaces have dimension 1 and the real ones at most dimension 2. In the geometrical case H is given by this recipe with $\mathscr{V} = \mathrm{H}^m(\mathrm{X})$, the relevant subspaces being $\mathrm{H}^{p,q}(\mathrm{X})$ and under complex conjugation $p \leftrightarrow q$.

The problem of determining G is more deep. A typical case when we have special restrictions on this group is in the presence of a special $\mathbb{Z}_2$ symmetry P; this occurs in a theory which has the property that for all values of the coupling t,

$$PW = -WP. \tag{6.2}$$

Such a symmetry operator P appears in the geometrical case as well and is called the "Weil operator" [40]. This operator is order 2 as far as the NS is concerned, but since the vacua are in the Ramond sector and two Ramond states produce an NS state P^2 acting on Ramond states can end up being ± 1. Since the spectral flow from NS to NS is equivalent to product of two Ramond vacuum states, and this is accomplished by the hessian of W, we learn that the phase of P^2 is simply the same as the phase of $\mathfrak{H}$ under P. Let us write

$$P^2 = (-1)^m .$$

Working in the holomorphic basis we represent P by

$$P|k\rangle = P_k{}^h|h\rangle .$$

Note that we have

$$\eta P = (-1)^m P^{\mathrm{T}}\eta .$$

This follows from the fact that a state and its dual with respect to η transform under P the same way up to the phase $(-1)^m$ which is the way the spectral flow (given by the hessian) relates them. We thus see that

$$\Omega^{ij} = \eta^{ik}P_k{}^j$$

is symmetric for m even and antisymmetric (a symplectic form) for m odd. Now if we consider

$$\Phi = \psi^{\mathrm{T}}\Omega\psi ,$$

where ψ is the solution to the linear problem in the holomorphic basis (3.11), and note that eq. (6.2) implies that $PC_i = -C_iP$ we see that *

$$\partial\Phi = \bar{\partial}\Phi = 0 \quad \Rightarrow \quad \Phi = \Omega .$$

Then for m even (resp. odd) ψ is orthogonal (resp. symplectic) with respect to the constant pairing Ω. If the signature of Ω is (r, s), $\mathrm{G} \subset \mathrm{SO}(r, s)$. The geometrical case is just of this type, with $\Delta = b_m$, $r = b_m^+$ and $s = b_m^-$ (of course we can rewrite all these in the other gauge for T).

7. Minimal models perturbed by the most relevant operator and related models

In the remaining sections of the paper we shall discuss particular classes of Landau–Ginzburg models for which the computation of the ground state metric can be done explicitly. We do this both for the intrinsic interest of the "solvable" models in various physical applications and also in order to illustrate the general phenomena of the previous sections (in particular, the overdeterminate nature of the problem).

Among the perturbations of conformal theories by relevant operators Zamolodchikov [41] found a technique to find which directions give rise to integrable models. The integrability is in the sense of having factorizable S-matrices for the massive excitations of the resulting theory. The idea is to look for an infinite

* We are mimicking the geometrical case. In that case the bilinear form Ω is the intersection in $\mathrm{H}^m(X, \mathbb{R})$.

number of conserved currents which survive the perturbations away from the conformal point. These ideas were applied to $N=2$ minimal superconformal theories in ref. [33] where it was found that these models perturbed by the (last component of the) chiral primary field of lowest (non-trivial) dimension, i.e. most relevant operator, leads to an integrable theory. Moreover it was found that there is a beautiful interplay between the structure of the superpotential W and the solitons and their masses. Then essentially self-consistency alone fixes the S-matrix in these models. It was shown in ref. [42] how these models (and their generalizations) can be realized in terms of quantum affine Toda field theories with very specific couplings. Also, the geometry of solitons and their conservation laws for specific perturbations of certain Kazama–Suzuki models (and in particular the grassmannians) has been uncovered in an interesting recent paper [43].

As we will see it turns out that precisely these perturbations (and some natural generalizations to be mentioned below) which can be described by $N=0$ *quantum* (affine) Toda field theories [42] lead to equations for the ground state metric which as a function of the perturbing parameter t (which can be identified with RG flow parameter) satisfy *classical* (affine) Toda equations of the *same type* (and their natural reductions). This is an intriguing connection between the quantum theory and the correlation functions of that quantum theory, which begs for a deeper understanding. That we should get Toda equations is already clear from the discussion of sect. 6. In fact that discussion will help us organize what we should expect for our equations. The general arguments of sect. 6 can be explicitly verified in the concrete examples we study in this section. The models of the present section are basically the ones for which the equations can be recast in a Toda form by elementary tricks. In sect. 9 and 10 we shall consider other model which are related to Verlinde rings whose equations are reduced to Today by more sophisticated techniques.

Here we limit ourselves to a discussion of the relevant equations. However, the real magic of the subject stems from the unique properties of the *solutions* corresponding to the actual metric rather than from the fact that the *equations* themselves are among the nicer ones in mathematical physics. Part of the magic will be discussed in some detail in sect. 8.

7.1 THE A_n SERIES

In the LG approach, the A_n minimal model corresponds to the superpotential $W=X^{n+1}/(n+1)$. The (non-trivial) chiral field of lowest dimension is X. Then we consider the superpotential

$$W(X,t)=\frac{X^{n+1}}{n+1}-tX, \tag{7.1}$$

and look for the dependence of the ground state metric g on t. As a basis on

$$\mathscr{R}\equiv \mathbb{C}[X]/(X^n-t)$$

we choose

$$1,\ X,\ X^2,\dots,X^{n-1}.$$

The vacuum state associated to X^k will be denoted by $|k\rangle$.

The model described by (7.1) has the discrete symmetry

$$X\to\exp[2\pi i/n]X, \tag{7.2}$$

under which the state $|k\rangle$ picks up a phase $\exp[\pi i(2k+1-n)/n]$. Then $\langle k|h\rangle=0$ for $k\neq h$, i.e. g is diagonal in this basis (from here till the end of the paper we have changed our notation and take $\langle k|$ to be the adjoint of $|k\rangle$). Therefore the group H defined in sect. 6 is abelian. From the discussion there it follows that our equations are of the Toda type. This system is rather peculiar in that the metric belongs to an abelian group just on symmetry grounds, i.e. before using the reality constraint to further reduce the number of independent elements of g. Imposing the reality constraint will lead to a consistent truncation of the Toda system to one with less degrees of freedom. Such consistent truncations are well known in the Toda theory [44] and are understood algebraically as foldings of the corresponding Dynkin diagrams.

To start with, ψ takes values in SL(n) and hence the equation for the t-dependence is that of some A_{n-1} Toda system. Which one depends on the admissible root system to which C_t corresponds. Multiplication by operator X is denoted by the matrix C_t given in the above basis of vacuum as

$$C_t=\begin{pmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&0&1\\t&0&0&\cdots&0&0\end{pmatrix}$$

i.e. (up to conjugacy) C_t is the sum of primitive roots of sl(n) minus the longest root. Then we get the affine A_{n-1} equation.

To see what the truncated "real". Today system is, it is better to distinguish between even and odd n. If n is even ($=2m$) we have a "Weil operator" P. This is just the generator of the symmetry $X\to -X$. This is an element of the group in (7.2). From the phase a state picks up under such a transformation, we see that $P^2=-\mathbb{1}$. Then, according to the discussion in sect. 6 we have

$$\mathrm{G}=\mathrm{Sp}(2m,\mathbb{R}),$$

i.e. we get the $\hat{C}_m$ Toda equations (for $m=1$ this is $\hat{A}_1$ and for $m=2$ this is $\hat{B}_2$). This can be checked explicitly using eq. (3.9) as we will show below.

The situation for $n=2m+1$ odd is less simple. The truncated Toda equations are associated to a root system (denoted by $\widehat{BC}_m$) which do not correspond to any Lie algebra. The corresponding equations are called the generalized Bullough–Dodd equations, since the first equation in the series is precisely the usual BD equation.

Let us see how they arise. In our basis, the residue pairing is independent of t. The only non-vanishing entries are

$$\eta_{k,n-1-k}=1.$$

Then the reality constraint reads

$$\langle k|k\rangle\langle n-1-k|n-1-k\rangle=1.$$

In particular, if n is odd ($n=2m=1$) one has

$$\langle m|m\rangle=1 \quad \text{for all } t.$$

In this way we reduce to $[n/2]$ unknown functions, namely $\langle k|k\rangle$ for $k=0,1,\dots,[n/2]-1$. In particular, for $n=2$ or 3 we have a *single* unknown function.

Writing

$$\varphi_i=\log\langle i|i\rangle, \quad i=0,\dots,n-1,$$

and using the explicit form of C_t, eq. (3.9) becomes

$$\begin{aligned}\partial_t\partial_{\bar t}\varphi_0+e^{(\varphi_1-\varphi_0)}-|t|^2\,e^{(\varphi_0-\varphi_{n-1})}&=0,\\ \partial_t\partial_{\bar t}\varphi_i+e^{(\varphi_{i+1}-\varphi_i)}-e^{(\varphi_i-\varphi_{i-1})}&=0, \quad i=1,\dots,n-2,\\ \partial_t\partial_{\bar t}\varphi_{n-1}+|t|^2\,e^{(\varphi_0-\varphi_{n-1})}-e^{(\varphi_{n-1}-\varphi_{n-2})}&=0.\end{aligned} \tag{7.3}$$

To put these equations in standard form, we put ($i=0,\dots,n-1$)

$$\varphi_i=q_i+\frac{2i-n+1}{2n}\log|t|^2,$$

$$z=\frac{n}{n+1}t^{(n+1)/n}.$$

We extend the definition of q_i to all i's by setting

$$q_{i+n}\equiv q_i.$$

Then eqs. (7.3) take the standard form for $\hat{A}_{n-1}$ Toda equations

$$\partial_z\partial_{\bar z}q_i+e^{(q_{i+1}-q_i)}-e^{(q_i-q_{i-1})}=0. \tag{7.4}$$

However, we have still to use the reality constraint which in the new variables reads

$$q_i+q_{n-i}=0.$$

If n is even ($n=2m$), using this constraint we reduce to the $\hat{C}_m$ Toda theory. To write it in the canonical form, just write (notations as in ref. [45])

$$q_i=-2\phi_{i+1}+\frac{2i+m}{2(m-1)}\log 2,$$

$$z\to 2^{1/2(m-1)}z.$$

Then eqs. (7.4) become

$$\begin{aligned}2\partial\bar\partial\phi_1&=e^{2(\phi_1-\phi_2)}-2\,e^{-4\phi_1},\\ 2\partial\bar\partial\phi_j&=e^{2(\phi_j-\phi_{j+1})}-e^{2(\phi_{j-1}-\phi_j)} \quad j=2,\dots,m-1,\\ 2\partial\bar\partial\phi_m&=2\,e^{4\phi_m}-e^{2(\phi_{m-1}-\phi_m)}.\end{aligned}$$

For n odd ($n=2m+1$), the redefinition

$$q_i=-2\phi_{i+1}-\frac{1}{2}\left(\frac{i+1}{2m+1}-1\right)\log 2$$

$$z\to 2^{-1/2(2m+1)}z$$

puts the reduced equations into the canonical $\widehat{BC}_m$ form

$$\begin{aligned}2\partial\bar\partial\phi_1&=e^{2(\phi_1-\phi_2)}-2\,e^{-4\phi_1},\\ 2\partial\bar\partial\phi_j&=e^{2(\phi_j-\phi_{j+1})}-e^{2(\phi_{j-1}-\phi_j)}, \quad j=2,\dots,m-1,\\ 2\partial\bar\partial\phi_m&=e^{2\phi_m}-e^{2(\phi_{m-1}-\phi_m)}.\end{aligned}$$

Of course, not all solutions to the above equations are acceptable as ground state metrics. At least two additional conditions are needed: first of all, $\langle k|k\rangle$ should be real, positive, and regular for all values of the couplings, and second the solution should not depend on the phase of the coupling t since this phase can be re-absorbed by the field redefinition

$$t\to e^{i\varphi}t, \quad X\to e^{i\varphi/n}X, \quad \theta\to e^{-i(n+1)\varphi/2}\theta.$$

Then only solutions invariant under rotations of z are acceptable. This property applies to all models we consider in the present section.

There is strong evidence that these two conditions uniquely fix the solutions. This will be discussed in sect. 8.

7.2 THE D_n SERIES

In the D_n case the most relevant perturbation of superpotential reads

$$W = \frac{X^{n-1}}{n-1} + XY^2 - tX.$$

As basis for $\mathscr{R}$ we choose

$$1, Y, Y^2, X, X^2, \dots, X^{n-3}.$$

This model has two symmetries, namely

$$X \to \exp[2\pi i/(n-2)]X \quad Y \to Y,$$
$$X \to X \qquad Y \to -Y.$$

It follows, that in this basis the only non-vanishing off-diagonal element of g is $\langle Y^2 | 1 \rangle$. One has

$$\mathrm{Res}[X^a] = \tfrac{1}{2}\delta_{a,n-2}, \qquad \mathrm{Res}[Y^b X^c] = 0 \quad \text{for } b, c \neq 0,$$
$$\mathrm{Res}[Y^{2k+1}] = 0, \qquad \mathrm{Res}[Y^2] = -\tfrac{1}{2}, \qquad \mathrm{Res}[Y^4] = -\tfrac{1}{2}t. \tag{7.5}$$

Then, decomposing $\mathscr{R}$ according the representations of these symmetries, for n even (resp. odd) we have $n/2 - 1$ (resp. $(n-1)/2 - 1$) one-dimensional complex orthogonal subspaces, 1 (resp. 2) one-dimensional real subspace, and 1 two-dimensional real subspace spanned by $(1, Y^2)$. Then (cf. sect. 6)

$$\mathrm{H} = \mathrm{SO}(2) \otimes \mathrm{U}(1)^{[(n-2)/2]},$$

is abelian and we get again a Toda system.

If $n = 2m + 2$ is even, the general arguments of sect. 6 uniquely fix the Toda system our equations correspond to. Indeed, we have a "Weil symmetry" P,

$$P : X \to -X.$$

This time $P^2 = \mathbb{1}$. Indeed, the hessian of W is even with respect to P, not odd as in the A-case. On $\mathscr{R}$ (neglecting the "decoupled" state $|Y\rangle$) the $+1$ eigenvalue of P has multiplicity $m+1$. Then,

$$\mathrm{G} = \mathrm{SO}(m+1, m),$$

and we have the $\hat{\mathrm{B}}_m$ Toda system. Instead, the Toda for n odd does not correspond to a root system and cannot be deduced by symmetry arguments alone.

Explicitly the reality constraint reads

$$\langle X^a | X^a \rangle \langle X^{n-2-a} | ^{n-2-a} \rangle = \tfrac{1}{4}, \quad a = 1, \dots, n-3,$$
$$\langle Y | Y \rangle = \tfrac{1}{2}, \qquad \langle 1 | Y^2 \rangle = \tfrac{1}{2}t\langle 1|1 \rangle, \qquad \langle Y^2 | 1 \rangle = \tfrac{1}{2}\bar{t}\langle 1|1 \rangle,$$
$$2\langle Y^2 | Y^2 \rangle = \frac{1}{2\langle 1|1 \rangle} + \frac{|t|^2}{2}\langle 1|1 \rangle.$$

The coefficients C_t are

$$X | X^a \rangle = | X^{a+1} \rangle, \quad a = 0, \dots, n-4,$$
$$X | Y \rangle = X | Y^2 \rangle = 0,$$
$$X | X^{n-3} \rangle = t | 1 \rangle + | Y^2 \rangle.$$

Let $n = 2m + 2 - s$ with $s = 0, 1$. The independent entries of g are $\langle X^a | X^a \rangle$ for $a = 0, 1, \dots, m-1$. In terms of these variables, our equations become

$$-\partial_t \partial_{\bar{t}} \log\langle 1|1 \rangle = \frac{\langle X | X \rangle}{\langle 1|1 \rangle} - |t|^2 \langle 1|1 \rangle \langle X | X \rangle,$$
$$-\partial_t \partial_{\bar{t}} \log\langle X | X \rangle = \frac{\langle X^2 | X^2 \rangle}{\langle X | X \rangle} - \frac{\langle X | X \rangle}{\langle 1|1 \rangle} - |t|^2 \langle X | X \rangle \langle 1|1 \rangle,$$
$$-\partial_t \partial_{\bar{t}} \log\langle X^a | X^a \rangle = \frac{\langle X^{a+1} | X^{a+1} \rangle}{\langle X^a | X^a \rangle} - \frac{\langle X^a | X^a \rangle}{\langle X^{a-1} | X^{a-1} \rangle},$$
$$(a = 2, \dots, m-2),$$
$$-\partial_t \partial_{\bar{t}} \log\langle X^{m-1} | X^{m-1} \rangle = \left(\frac{1}{2\langle X^{m-1} | X^{m-1} \rangle} \right)^{1+s} - \frac{\langle X^{m-1} | X^{m-1} \rangle}{\langle X^{m-2} | X^{m-2} \rangle}. \tag{7.6}$$

To put these equations in canonical form, we define

$$\phi_{\pm} = \log\langle X | X \rangle \pm \log\langle 1|1 \rangle \mp F(|t|) + (1 \pm 1) \log |t|,$$
$$\phi_j = \log\langle X^{j+1} | X^{j+1} \rangle - \log\langle X^j | X^j \rangle + F(|t|), \quad (j = 2, \dots, m-2),$$
$$\phi_{m-1} = -(1+s) \log\langle X^{m-1} | X^{m-1} \rangle + (1+s)(m-1)F(|t|) - (1+s) \exp |t|,$$
$$z = \left(\frac{1+s}{2^{2+s}} \right)^B \frac{t^{1+(1+s)B}}{1 + (1+s)B},$$

where

$$B = \frac{1}{2[1 + (m-1)(1+s)]},$$

and

$$F(|t|) = 2B[(1+s)\log(|t|/2) + \log(1+s) - \log 2].$$

Then eqs. (7.6) become

$$\partial\bar\partial\phi_{+} = 2\,e^{\phi_+} - e^{\phi_1},$$

$$\partial\bar\partial\phi_1 = 2\,e^{\phi_1} - e^{\phi_+} - e^{\phi_-} - e^{\phi_2},$$

$$\partial\bar\partial\phi_a = 2\,e^{\phi_a} - e^{\phi_{a-1}} - e^{\phi_{a+1}} \quad (a = 2,\dots,m-3),$$

$$\partial\bar\partial\phi_{m-2} = 2\,e^{\phi_{m-2}} - e^{\phi_{m-3}} - (2/(1+s))\,e^{\phi_{m-1}},$$

$$\partial\bar\partial\phi_{m-1} = 2\,e^{\phi_{m-1}} - (1+s)\,e^{\phi_{m-2}}.$$

In general, the Toda equations can be written in the form [44]

$$\partial\bar\partial\phi_a = C_{ab}\,e^{\phi_b},$$

where C_{ab} is the Cartan matrix of some root system. From the above explicit formula, we see that for $s=0$ (n even) we get the Cartan matrix of $\hat{B}_m$, as expected from the general argument. Instead for $s=1$ (n odd) we get the transpose Cartan matrix. This is the Toda system denoted by $D^T(SO(2m+1))$ in ref. [44].

7.3. THE E-SERIES

The only new model is E_7, since E_6 and E_8 can be obtained as tensor products of A minimal models. In the E_7 case the most relevant perturbation of the superpotential reads

$$W = \tfrac{1}{3}X^3 + \tfrac{1}{3}XY^3 - tY.$$

As basis in $\mathscr{R}$ we take ($i = 1,\dots,7$)

$$\phi_i = \{1, Y, X, Y^2, XY, X^2, X^2Y\}. \qquad (7.7)$$

This model has a $\mathbb{Z}_7$ symmetry

$$X \to eX, \quad Y \to e^3 Y, \quad e^7 = 1.$$

Under this symmetry no two fields in (7.7) transform the same way, and hence the metric g is diagonal. So H is abelian and we have again a Toda theory.

In the above basis the residue pairing reads

$$\eta_{ij} = (1 - 4\delta_{i,4})\delta_{i+j,8},$$

and the reality constraint reads

$$\langle i|i\rangle\langle 8-i|8-i\rangle = 1, \quad i \neq 4,$$

$$\langle 4|4\rangle = 3.$$

Then $H = U(1)^3$. The non-vanishing elements of C_t are

$$C_1{}^2 = C_2{}^4 = C_3{}^5 = C_6{}^7 = 1, \qquad C_5{}^1 = C_7{}^3 = t, \qquad C_4{}^6 = -3.$$

Putting

$$2\varphi_1 = \log\langle 3|3\rangle + \tfrac{1}{7}\log|t|^2 + \tfrac{1}{7}\log 24,$$

$$2\varphi_2 = -\log\langle 1|1\rangle - \tfrac{4}{7}\log|t|^2 - \log 2 + \tfrac{3}{7}\log 24,$$

$$2\varphi_3 = -\log\langle 2|2\rangle - \tfrac{2}{7}\log|t|^2 + \log 6 - \tfrac{2}{7}\log 24,$$

$$z \equiv \frac{7}{8\sqrt{2}}(24)^{1/7}t^{8/7},$$

one gets the equations in the form

$$2\partial\bar\partial\varphi_1 = e^{2(\varphi_1-\varphi_2)} - 2\,e^{-4\varphi_1},$$

$$2\partial\bar\partial\varphi_2 = e^{(2\varphi_2-\varphi_3)} - e^{2(\varphi_1-\varphi_2)},$$

$$2\partial\bar\partial\varphi_3 = e^{2\phi_3} - e^{2(\varphi_2-\varphi_3)},$$

which is the $\widehat{BC}_3$ Toda in the notations of ref. [45] (i.e. $GD(H_3)$ in the language of [44]).

7.4. THE A_n MODELS PERTURBED BY NEXT RELEVANT OPERATOR

Next we consider the models

$$W = \frac{X^{n+1}}{n+1} - t\frac{X^2}{2}.$$

For $\mathscr{R}$ we use the basis $1, X, \ldots, X^{n-1}$. These models have the discrete symmetry

$$X \to \exp[2\pi i/(n-1)]X, \qquad \theta \to \exp[-\pi i(n+1)/(n-1)]\theta.$$

This implies

$$\langle k|h\rangle = 0 \quad \text{for} \quad k \neq h \quad \text{except for} \quad \langle n-1|0\rangle \quad \text{and} \quad \langle 0|n-1\rangle.$$

Since the two-dimensional subspace spanned by 1 and X^{n-1} is *real*, H is still abelian and therefore we get again a Toda system. In fact one has

$$\mathrm{H} = \mathrm{SO}(2) \otimes \mathrm{U}(1)^{[(n-2)/2]}.$$

In the present case the residue pairing is

$$\eta_{kh} = \delta_{k+h,n-1} + t\delta_{k,n-1}\delta_{h,n-1},$$

so the reality constraint becomes

$$\langle k|h\rangle\langle n-1-k|n-1-k\rangle = 1 \quad \text{for} \quad k \neq 0,\ n-1,$$

$$\langle 0|n-1\rangle = \tfrac{1}{2}t\langle 0|0\rangle, \qquad \langle n-1|0\rangle = \tfrac{1}{2}\bar{t}\langle 0|0\rangle,$$

$$\langle n-1|n-1\rangle = \frac{1}{\langle 0|0\rangle} + \frac{|t|^2}{4}\langle 0|0\rangle.$$

If $n+1$ is even $(=2m)$ the model can be reduced to already solved ones. Indeed,

$$W(X) = W_0(X^2)$$

with

$$W_0(Y) = \frac{Y^m}{n+1} - \frac{t}{2}Y,$$

so the "odd" states

$$|2k+1\rangle \quad (k = 0, 1, \ldots, m-2),$$

are just the pullbacks of the vacua for the A_{m-1} minimal model perturbed by the most relevant operator. For our purposes, these states decouple from the others and, by functoriality, the corresponding ground state metric

$$\langle 2k+1|2h+1\rangle$$

is the solution to a $\mathrm{Sp}(m-1)$ or a $\widehat{\mathrm{BC}}_{m-1}$ Toda system according to whether m is odd or even.

Instead, the metric for the "even" states $|2k\rangle$ is equal to that of the D_{m+1} model. This follows from the fact that the D models are the orbifolds of the A_{2k} ones with respect to the symmetry

$$X \to -X.$$

Then for the even states we get $\hat{\mathrm{B}}_{(m-1)/2}$ or $\mathrm{D}^{\mathrm{T}}(\mathrm{SC}(m+1))$ Toda according whether m is odd or even.

On the contrary, when n is even $(=2m)$ we have no "Weil operator" and hence we expect a Toda theory associated to a generalized Cartan matrix. Indeed, let

$$q_i = -\log\langle 2(i-1)|2(i-1)\rangle \quad \text{for} \quad i = 1, 2, \ldots, \left[\frac{m+1}{2}\right]$$

$$= \log\langle 2(m-1)+1|2(m-i)+1\rangle \quad \text{for} \quad i = \left[\frac{m+1}{2}\right] + 1, \ldots, m.$$

Then the equations become

$$\partial\bar{\partial}q_1 = \tfrac{1}{4}\,\mathrm{e}^{(q_1-q_2)} - \tfrac{1}{16}|t|^2\,\mathrm{e}^{-(q_1+q_2)},$$

$$\partial\bar{\partial}q_2 = \tfrac{1}{4}\,\mathrm{e}^{(q_2-q_3)} - \tfrac{1}{4}\,\mathrm{e}^{(q_1-q_2)} - \tfrac{1}{16}|t|^2\,\mathrm{e}^{-(q_1+q_2)},$$

$$\partial\bar{\partial}q_i = \tfrac{1}{4}[\mathrm{e}^{(q_i-q_{i+1})} - \mathrm{e}^{(q_{i-1}-q_i)}] \quad (i = 3, \ldots, m-1),$$

$$\partial\bar{\partial}q_m = \tfrac{1}{4}|t|^2\,\mathrm{e}^{2q_m} - \tfrac{1}{4}\,\mathrm{e}^{(q_{m-1}-q_m)},$$

which, after an obvious re-interpretation of the symbols, is the same as eqs. (7.6). Then by a redefinition of the variables it can be recast in the standard $\mathrm{D}^{\mathrm{T}}(\mathrm{SO}(2m+1))$ Toda form.

7.5. PERTURBED GRASSMANNIAN COSET MODELS

The Landau–Ginzburg description of some of the superconformal models proposed by Kazama and Suzuki [46] has been found in ref. [4] *. As another application of our techniques, we will focus on an interesting subclass of such models given by the level-1 superconformal grassmannian coset models

$$\mathscr{G}/\mathscr{H} = \mathrm{SU}(n+m)/\mathrm{SU}(m) \otimes \mathrm{U}(n), \qquad c = \frac{3nm}{n+m+1},$$

perturbed by the most relevant operator. Again, these models are solvable as quantum field theories and related to $N=0$ quantum Toda systems [42].

* Actually this has been conjectured for many cases but not proven in full generality yet.

Let us summarize it in a way convenient for our purposes. We assume, with no loss of generality, that $m \geqslant n$. We start with n fields Y_k $(k = 1, \cdots, n)$ with charge $q = 1/(n+m+1)$ and consider the elementary symmetric functions

$$X_i = \sigma_i(Y_k) \equiv \sum_{1 \leqslant l_1 < l_2 < \ldots l_i \leqslant n} Y_{l_1} Y_{l_2} \ldots Y_{l_i} \quad (i + 1, \ldots, n). \tag{7.8}$$

Then take the function

$$W_f(Y_k) = \frac{1}{n+m+1} \sum_k Y_k^{n+m+1}.$$

By the fundamental theorem on symmetric functions, it can be rewritten (in a unique way) as a quasi-homogeneous polynomial in the $\sigma_i(Y)$, i.e. in terms of t he X_i one finds

$$W_f(Y_k) = f^* W(X),$$

where the map f is given by eq. (7.8). The function $W(x)$ so obtained is the superpotential for the grassmannian model. Thus the canonical branched covering of the grassmannian model is just n copies of the A_{n+m} minimal model. To check this picture of coset models, let us compute their central charge, using the formula for the change of c under covering maps, eq. (5.8). One has

$$J = \det\left(\frac{\partial X_i}{\partial Y_j}\right) = \Delta(Y_1, \ldots, Y_n),$$

where $\Delta(Y_i)$ is the Vandermonde determinant. Then

$$q_f(J) = \frac{n(n-1)}{2(n+m+1)} \quad \Rightarrow \quad c = \frac{3nm}{n+m+1},$$

as it should.

As perturbed superpotential we take

$$W(X_i, t) = W(X_i) - tX_1. \tag{7.9}$$

By going to the canonical covering, we get

$$W_f(Y_k, t) = f^* W(X, t) = \sum_{k=1}^{n} \left(\frac{Y_k^{n+m+1}}{n+m+1} - tY_k\right).$$

Thus the perturbed model goes over to n copies of the already solved perturbed A_{n+m} minimal model. The ground state metric for W_f is just the product of the known one for each factor.

Now the metric for the grassmannian models can be obtained using change of variables. Let $P_r(X_i)$ $(r = 1, \ldots, (n+M)!/n!m!;\ i = 1, \ldots, m)$ be a set of polynomials making up a basis for the chiral ring $\mathscr{R}$ of the models in (7.9). Then eq. (5.6) gives,

$$\langle P_s | P_r \rangle = (1/n!) \langle \Delta(Y) P_r(\sigma_i(Y)) | \Delta(Y) P_s(\sigma_i(Y)) \rangle_f$$

(here $\langle \cdot | \cdot \rangle_f$ denotes the known metric for W_f).

By the same token, we can also solve the grassmannian models perturbed by the operator $(X_1^2 - 2X_2)$. Indeed,

$$f^*\left[W(X_i) - t(X_1^2 - 2X_2)\right] = \sum_{k=1}^{N} \left[\frac{Y_k^{n+m+1}}{n+m+1} - tY_k^2\right]$$

and we are reduced to n copies of the model we solved in subsect. 7.4.

7.6. PARTIALLY ABELIAN MODELS

In addition to the models that can be reduced to Today systems there are those for which the ground state metric decomposes in two "non-interacting" sectors one of which can be recast in a Toda form. Many of these models can be related to theories leading to Toda equations by a simple change of variables. Then the sector arising as the pull-back of the simpler theory "decouples" and has the Toda form.

There are however, other more interesting examples. We make no attempt to completeness, but we merely mention an example to show how it works.

Consider the model

$$W = X^4/4 + Y^4/4 + Z^4/4 - tXYZ.$$

It has a $\mathbb{Z}(4) \otimes \mathbb{Z}(4)$ discrete symmetry. Using the rules of sect. 6, one finds

$$\mathrm{H} = \mathrm{SO}(3) \otimes \mathrm{U}(2)^3 \otimes \mathrm{SO}(2)^3 \otimes \mathrm{U}(1)^3.$$

The part of the metric corresponding to the "abelian" part of H, $\mathrm{SO}(2)^3 \otimes \mathrm{U}(1)^3$, (corresponding to 12 chiral primary operators out of 27) decouple from the rest, and hence it is Toda. What is remarkable, is that the ground state metric for these 12 operators is a rational function of the metric for the theory with $W = X^4 - tX$.

8. The magic of the solutions

Up to now we have just discussed how equations take, for special models, the form of interesting differential systems of mathematical physics, typically Toda equations. However, the real magic of the ground state geometry appears only when we consider the corresponding *solutions*.

In particular, we want to illustrate how the conditions we have already stated uniquely fix the metric. Basically, the requirement that g is a non-singular positive-definite metric will fix it uniquely. Thus, in particular, the boundary conditions for the differential equations are *predicted*. These boundary conditions correspond to the values of the ground state metric for the unperturbed conformal theory which is well understood. For the models of sect. 7, this implies that the absolute normalization of the OPE coefficients for, say, the minimal models can be deduced from our equations as the unique boundary condition allowed by regularity. This will be shown here and, in a more general class of examples, in sect. 9. On the other hand, the behaviour as $|t| \to \infty$ should be the semiclassical one, as described at the end of sect. 4. Thus the equations also encode in a beautiful way the geometry of solitons in the theory. Finally, the unique solution should also lead to the correct behaviour for the algebraic c-function.

8.1. THE MODEL $W = X^3/3 - tX$

Consider the first model in (7.1). The equation in this case is $\hat{A}_1$ Toda, i.e. the sinh-Gordon equation. We know that the metric is a function of $|t|$ only. Let $|t|^2 = x$ and $y(x) = \langle 1|1 \rangle$. Then the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \frac{\mathrm{d}}{\mathrm{d}x} \log y \right) = y^2 - \frac{x}{y^2}.$$

Consistency requires that, as $t \to 0$, we get back the result for the A_2 minimal model, i.e.

$$y^2(t=0) = \frac{\langle 1|1 \rangle}{\langle 0|0 \rangle} = 3^{2/3} \left[\frac{\Gamma(2/3)}{\Gamma(1/3)} \right]^2. \tag{8.1}$$

On the other hand, as $t \to \infty$, the two classical vacua at $X = \pm\sqrt{t}$ decouple. Denoting by $l_\pm$ the corresponding chiral primary operators (the "point" basis) we must have

$$\langle l_\pm | l_\pm \rangle = \frac{1}{2|t|^{1/2}} + \dots,$$

$$\langle l_\pm | l_\mp \rangle = \frac{\beta}{2|t|^{1/2}} z^{-1/2} \exp[-2z] + \dots$$

where

$$z = |W(\sqrt{t}) - W(-\sqrt{t})| = \tfrac{4}{3}|t|^{3/2} \equiv \tfrac{4}{3}x^{3/4},$$

and β is some numerical coefficient. β is real by "Weil symmetry". Its sign would be predicted by the "c-theorem". Since

$$1 = l_+ + l_-, \qquad X = \sqrt{t}\,(l_+ - l_-),$$

we get

$$y^2(x \sim \infty) \simeq \sqrt{x}\left[1 - 2\beta\sqrt{\tfrac{3}{4}}\, x^{-3/8} \exp\left(-\tfrac{8}{3}x^{3/4}\right) + \dots\right].$$

We write

$$y^2(x) = \sqrt{x}\, Y^2\left(\tfrac{4}{3}x^{3/4}\right), \tag{8.2}$$

where $Y(z)$ satisfies *

$$Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{z} + Y^3 - \frac{1}{Y}. \tag{8.3}$$

This is just the special third Painlevé transcendent equation (PIII). The general form of this equation is

$$Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{z} + \frac{1}{z}(\alpha Y^2 + \beta) + \gamma Y^3 + \frac{\delta}{Y},$$

the special case corresponds to $\alpha = \beta = 0$, $\gamma = -\delta = 1$.

Our metric $y(x)$ should be regular, real and strictly positive on the positive real axis. The solutions to this equation without poles on the positive real axis are well known. Following ref. [15] we introduce the function

$$u(z) = 2 \log Y(z).$$

u is a solution to the self-similar sinh-Gordon equation

$$u_{zz} + \frac{u_z}{z} = 4 \sinh(u).$$

In ref. [15] it is shown that this equation arises from an isomonodromy problem. In fact, it turns out that the associated isomonodromy (= zero-curvature) problem is nothing else than our linear problem (3.11), for the model at hand. Indeed, let

$$z = \tfrac{4}{3}x^{3/4}, \qquad \lambda = -\tfrac{3}{2}i \frac{1}{(\bar{t})^{3/2}},$$

* It is assuming that this very same equation is satisfied by the spin-spin correlation functions of the 2d Ising model off criticality [47].

and make the "gauge transformation"

$$\psi \to (1/\sqrt{2})\sigma_3(1+i\sigma_2)\, e^{\frac{1}{4}\sigma_3 \log t}\psi.$$

In the new variables, the linea: problem becomes

$$\partial_z\psi = \partial_\lambda\psi = 0,$$

with

$$\partial_\lambda = \frac{\partial}{\partial\lambda} + \frac{zu'(z)}{4}\sigma_1 + i\frac{z^2}{\lambda^2}\sigma_3 \cosh u(z) - \frac{1}{\lambda^2}\sinh u(z),$$

$$\partial_z = \frac{\partial}{\partial z} + \tfrac{1}{2}u'(x)\sigma_1 + \tfrac{1}{2}iz\lambda\sigma_3,$$

which is the isomonodromy problem discussed in ref. [15]. The relevant monodromy which remains constant is precisely the monodromy of the period-map Π for the SQM vacuum wave-forms introduced in sect. 5. In fact, this is true for the general case. The linear problem (the generalized Gauss–Manin connection) is always an isomonodromy problem for the SQM period map Π. Exploiting this interpretation of the equation, one finds the properties of its solutions [15].

The real solutions (for which the origin is not an accumulation point of poles *) are classified by their asymptotic behaviour as $z \to 0$

$$u(z) \simeq r \log z + s + O(z^{2-|r|}) \quad \text{for } |r| < 2,$$

$$u(z) \simeq \pm 2 \log z \pm 2 \log\left[-(\log \tfrac{1}{2}z + C)\right] + O(z^4 \log^2 z) \quad (r = \pm 2). \tag{8.4}$$

(C is the Euler constant). For each pair (r, s) with $|r| \leqslant 2$ there is a solution. A real solution is regular (no poles on the positive real axis) if and only if the two boundary data r and s are related by the equation

$$e^{s/2} = \frac{1}{2^r}\frac{\Gamma(\frac{1}{2} - \frac{1}{4}r)}{\Gamma(\frac{1}{2} + \frac{1}{4}r)}. \tag{8.5}$$

So, requiring regularity fixes s as a function of r. Note that a regular solution $Y(s)$ has no zero on the positive real axis. Indeed, Y^{-1} is also a solution of eq. (8.3), with just the opposite signs for r and s. Since (8.5) is invariant under this change of signs, Y^{-1} has no poles and hence Y no zeros.

* By "pole" we mean a pole of the associated Painlevé transcendent of the third kind $Y(z)$.

The connection formula for PIII states that the asymptotic behaviour of these real solutions as $z \to \infty$ is

$$u(z) \sim \frac{\alpha(r)}{z^{1/2}}\exp[-2z], \quad z \to \infty, \tag{8.6}$$

where

$$\alpha(r) = -\frac{2}{\sqrt{\pi}}\sin\left(\frac{\pi r}{4}\right).$$

From eq. (8.2) one gets

$$u(z) = 2 \log\langle 1|1\rangle(z) - \tfrac{2}{3}\log(\tfrac{3}{4}z).$$

Since the ground state metric is regular and non-zero as $z \to 0$, we have

$$r = -\tfrac{2}{3},$$

$$s = 2 \log\langle 1|1\rangle|_{t=0} - \tfrac{2}{3}\log\tfrac{3}{4}.$$

Using the regularity condition (8.5) one gets

$$\frac{\langle 1|1\rangle}{\langle 0|0\rangle} = 3^{2/3}\left[\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right]^2(1 + O(|t|^2)),$$

in agreement with eq. (8.1).

More generally, all the elements of g for the A_n minimal models can be obtained (in fact in many ways) from regularity constraints on the solutions of our equations.

On the other hand, the asymptotic behaviour predicted by eq. (8.6) precisely matches with that predicted by semiclassical arguments (cf. appendix B). The sign of the asymptotic behaviour of u may be surprising at first, since a naive classical picture might suggest the opposite one. In fact, the intuitive picture would apply to the leading semiclassical correction, which in this case just vanishes by supersymmetry. The sub-leading one has a sign which cannot be inferred by classical ideas. However, the sign is fixed from the point of view of the c-theorem. Let us work in the point basis, normalizing $l_\pm$ so that det $g = 1$. Then the metric reads

$$g = \exp[-u(z)\sigma_3/2].$$

By the redefinition $X \to t^{3/2}X$, we put W in the standard form with an overall coupling $\lambda = t^{3/2}$. Then the charge matrix q introduced in sect. 4 becomes

$$q = \tfrac{1}{4}\sigma_3 z\frac{\partial u(z)}{\partial z}.$$

So the algebraic c-function is

$$c = -\tfrac{3}{2} z \frac{\partial u(z)}{\partial z},$$

as $z \to 0$, we get $c \to 1$, and as $z \to \infty$, $c \to 0$, as expected. The derivative of c with respect to the scale is

$$\frac{\partial c}{\partial z} = -6z \sinh(u).$$

c is stationary only if $z = 0$ or $u = 0$. $u = 0$ implies

$$\langle I_{\pm} | I_{\mp} \rangle = 0,$$

i.e. the "classical" theory. In between, c is obviously monotonic with the scale. Since for $z \to \infty$ we have $c = 0$, for large, but finite z, c should be a small *positive* number. Using the asymptotic expansion (8.6) we get

$$c \simeq (3/\sqrt{\pi}) z^{1/2} \exp[-2z] > 0.$$

If the leading behaviour of u had the opposite sign, c would be negative in this regime. Thus the c-theorem explains physically the peculiar sign of the "instanton" correction.

8.2. OTHER MODELS LEADING TO SPECIAL PIII

In the list of models discussed in sect. 7 there are other whose equations can be reduced to special PIII.

The first one is

$$W(X) = \frac{X^4}{4} - \frac{t}{2} X^2.$$

Again we put $x = |t|^2$ and $y(x) = (\langle 0|0 \rangle)^{-1}$. Then this equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \frac{\mathrm{d}}{\mathrm{d}x} \log y \right) = \frac{1}{4}\left(y^2 - \frac{x^2}{16} \frac{1}{y^2} \right). \qquad (8.7)$$

By the redefinition

$$y = \sqrt{z}\, Y(z), \qquad z = \tfrac{1}{4} x$$

we reduce eq. (8.7) to the standard form of special PIII, (8.3).

For $t = 0$ we have

$$y^2(0) = \frac{\langle 2|2 \rangle}{\langle 0|0 \rangle}\bigg|_{t=0} = \left[2 \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right]^2.$$

The soliton mass is

$$2\,|W(\sqrt{t}) - W(0)| = \tfrac{1}{2} |t|^2 \equiv 2z.$$

As in the above model we put

$$u(z) = 2 \log Y(z) = 2 \log y(z) - \log z.$$

So $u(z)$ is the solution to PIII with

$$r = -1,$$
$$s = 2 \log y(0) \equiv 2 \log 2 + 2 \log \Gamma(\tfrac{3}{4}) - 2 \log \Gamma(\tfrac{1}{4}).$$

These numbers satisfy the regularity condition (8.5) (i.e. $y^2(0)$ is predicted by regularity alone). The large-t expansion is

$$y(|t|) \simeq \frac{|t|}{2}\left(1 + \sqrt{\frac{2}{\pi}} \frac{2}{|t|} \exp\left[-|t|^2/2 \right] + \cdots \right)$$

in agreement with the semiclassical analysis.

By the same token as in the previous model, the c-function reads

$$c = -\tfrac{3}{2} z \frac{\partial}{\partial z} u(z). \qquad (8.8)$$

In this case, as $z \to 0$ we get $c = 3/2$, as we should. The comments above on the sign of the "instanton" corrections apply to the present model as well.

Note that the boundary data r is (essentially) the central charge at the UV fixed point. That is, the UV central charge is a monodromy data (basically, the Stokes multiplier). The condition $|r| < 2$ is just

$$c < 3,$$

i.e. restricts to the minimal models! Then the PIII regularity condition (8.5) can be seen as saying that in order to have a regular solution $\exp[s]$ should be the OPE coefficient appropriate for the given central charge. These remarks will become clear in full generality in sect. 9.

Another model that can be reduced to special PIII is

$$W(X) = \frac{X^6}{6} - \frac{t}{2} X^2.$$

The matrix elements

$$\langle 1|1\rangle \quad \text{and} \quad \langle 3|3\rangle$$

can be obtained from the $X^3/3 - tX$ model by a change of variable

$$f: \quad X \to X^2. \tag{8.9}$$

Then there remains a single unknown function

$$y(x) = (\langle 0|0\rangle)^{-1},$$

which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \frac{\mathrm{d}}{\mathrm{d}x} \log y\right) = \tfrac{1}{4}y - \frac{x}{16}\frac{1}{y}.$$

At $t = 0$ we must have

$$y^2(0) = \left.\frac{\langle 4|4\rangle}{\langle 0|0\rangle}\right|_{t=0} = \left[6^{2/3}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})}\right]^2.$$

Putting

$$y = \sqrt{\tfrac{1}{4}x}\, Y^2(z), \qquad z = \tfrac{1}{3}x^{3/4},$$

we get again special PIII for $Y(z)$. Then

$$u(z) = 2 \log Y(z) = \log y - \tfrac{2}{3} \log z - \tfrac{2}{3} \log 3 + \log 2.$$

which gives

$$r = -\tfrac{2}{3},$$

$$s = \log y(0) - \tfrac{2}{3} \log 3 + \log 2.$$

Since r is as in the cubic model, $Y(s)$ – if regular – should be the same. Thus regularity implies an algebraic relation between the two independent elements of the ground state metric.

One has

$$|W(t^{1/4}) - W(0)| = \tfrac{1}{3}|t|^{3/2} \equiv z,$$

so the large-t behaviour is again the correct one.

By the same argument as above, we have

$$c = -3z\frac{\partial}{\partial z}u(z).$$

(The factor 2 with respect to eq. (8.8) is due to the fact that now $(\langle 0|0\rangle)^{-1}$ is proportional to $Y^2(z)$ rather than $Y(z)$). So, as a function of z the central charge is just twice that of the perturbed A_2 model which, pulled back by the map (8.9), gives the present model. In particular, for $t = 0$ we get $c = 2$, as we should.

There are other models whose equations can be reduced to special PIII. A very important class will be discussed in sect. 9. There are a few other models that we omit for brevity. We have explicitly checked that all these models satisfy the regularity and consistency criteria.

8.3. THE MODEL $W = X^4/4 - tX$

Next we consider the model leading to $\widehat{BC}_1$ Toda. Putting $y = \langle 2|2\rangle$ and $x = |t|^2$, we get

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{y^2}{x} - \frac{1}{y},$$

which is again a special case of the third Painlevé equation, with $\alpha = -\delta = 1$, $\beta = \gamma = 0$. This is the so-called "degenerate" PIII. Putting

$$\tau = \tfrac{9}{16}x^{4/3}, \qquad \log y = u(\tau) + \tfrac{1}{4} \log(\tfrac{16}{9}\tau), \tag{8.10}$$

we recast this equation in the form of the self-similar Bullough–Dodd equation

$$(\tau u_\tau)_\tau = e^u - e^{-2u}.$$

The properties of the asymptotically regular solutions were studied in ref. [16], again by the isomonodromic deformation method. It turns out that these solutions are parametrized by four complex numbers g_1, g_2, g_3, and s satisfying

$$g_1 + g_2(1-s) + g_3 = 1, \qquad g_2^2 - g_1 g_3 = g_2,$$

so we have a two-dimensional manifold of solutions. From eq. (8.10) we see that regularity implies that, as $\tau \to 0$,

$$\exp[u] \sim \frac{\text{const.}}{\tau^{-1/4}}.$$

This selects $s = 1$. In this case, one has

$$\exp[u] \sim \tfrac{9}{8} r_1 \frac{C_2}{C_0} \tau^{-1/4} \qquad \tau \to 0,$$

where (for $s = 1$)

$$\frac{C_2}{C_0} = \frac{8}{3^{3/2}} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})},$$

$$r_1 = g_3 - g_1 + (1 - i)(g_1 - g_2).$$

To fix the residual ambiguity of the solution, we require that, as $\tau \to \infty$, there are no exponentially growing terms (i.e. no negative-mass solitons). Then one gets

$$g_1 = g_2 = 0, \quad g_3 = 1 \quad \Rightarrow \quad r_1 = 1,$$

and the solution is uniquely fixed.

At this point, both the value of the metric at $t = 0$ and the strength of the "instanton" correction are *predicted*. One gets

$$\langle 2 | 2 \rangle |_{t=0} = 2 \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})},$$

the expected value. The asymptotical expansion for $\tau \to \infty$ is

$$\exp[u(\tau)] \simeq 1 + \frac{1}{2} \sqrt{\frac{3}{\pi}} (3\tau)^{-1/4} e^{-2\sqrt{3\tau}} + \dots.$$

This is the correct strong-coupling behaviour, because

$$z \equiv | W(t^{1/3}) - W(e^{2\pi i/3} t^{1/3}) | = \frac{3\sqrt{3}}{4} | t |^{4/3} = \sqrt{3\tau}$$

and the coefficient in front of the exponential agrees with the soliton picture discussed in appendix B.

Again one has

$$c(\tau) = -3z \frac{\partial u}{\partial z} \equiv -6\tau \frac{\partial u(\tau)}{\partial \tau}.$$

As $\tau \to 0$, we get $c = 3/2$, the correct value. To the best of our knowledge, no mathematician has ever studied in detail the properties of the higher equations in sect. 7. However, we can easily work the other way around, namely, start from the

known physical properties of the metric and deduce the corresponding mathematical theorems, analogous to the above ones for the $\hat{\mathrm{A}}_1$ and the $\widehat{\mathrm{BC}}_1$ cases. In some sense, this is just what a mathematician would do. In fact, the known results are obtained by exploiting the isomonodromic method, which is somehow built-in the physical approach.

9. Models associated to Verlinde rings: the $SU(2)_k$ case

Recently Gepner [18] has shown that the Verlinde rings of some rational CFTs have the same algebraic structure as the chiral rings of the $N = 2$ LG models, namely they are polynomial rings modulo the ideal generated by the derivatives of a certain superpotential $W(X_i)$. This has been considered further recently [48,49]. The main case considered in ref. [18] is that of $SU(N)_k$ theories. From the $N = 2$ viewpoint, the corresponding superpotentials correspond to particular (relevant) perturbations of $N = 2$ coset models. Then it is natural to ask whether, for these special perturbations, the equations for g (as we vary the RG scale) are "solvable" in the sense that they can be reduced to Toda. The answer to this question is yes! Moreover, the trick to solve them is based on the interpretation of the corresponding $\mathcal{R}$'s as fusion rings. In particular, for the model associated to the $SU(N)_k$ Verlinde ring the ground state metric is written in terms of k linearly independent * solutions to the (self-similar) affine $SU(N)$ Toda equations.

In this section we discuss in detail the $SU(2)_k$ situation, the generalization to arbitrary N being discussed in sect. 10. In this case, the superpotentials are the Chebyshev polynomials [18]

$$W_K(X) = \lambda T_{k+1}(X), \quad \text{where} \quad T_m(\cos Y) = \cos(mY).$$

Rescaling the field X, we see that as the coupling $\lambda \to 0$ one gets back the minimal model A_k, which is equivalent to the grassmannian model at level 1

$$SU(k+1)_1 / U(k).$$

The fact that one gets Chebyshev polynomials is remarkable, since for these polynomials the SQM Schrödinger equation is separable, and hence the ground state metric is computable *by brute force*. In fact, separability for the SQM Schrödinger equation (with one field) is equivalent to separability for the 2d Helmholtz equation (related in turn to SU(2) Toda). However, the corresponding wave functions are not very managable, so it is more convenient to use the information coming from separability to simplify our equations, rather than to compute g directly. It has not yet been shown, in the sense of having infinitely

* However, the reality constraint gives non-linear algebraic relations between these solutions.

many conserved currents, that the Chebyshev perturbation of minimal models is in that class, but the fact that we find an affine Toda equation even for this case suggests that this must be true. In fact for the A_n model $W = X^{n+1}$ it has only been shown that X and X^2 perturbations are integrable [33,42], and it was suspected that perturbation by X^{n-1} is also integrable. Chebyshev perturbation to leading order (as $\lambda \to 0$) is of this type. So what we are finding is that this is, to leading order, integrable but to get it to be fully integrable it must be "dressed" by lower-dimension operators which make it become precisely the Chebyshev polynomial. It would be very interesting to verify this by studying perturbation theory near the conformal point.

The method we use for solving the Chebyshev models is again using the change of variables trick discussed in sect. 5. This will in fact allow to solve them all at once. We take

$$W = \lambda T_n(X),$$

$$f = \cos(Y/n) \equiv X,$$

$$W_f(Y) = \lambda \cos(Y). \tag{9.1}$$

Then, if we are able to compute the ground-state metric for the $N=2$ sine-Gordon model, $W_f(Y)$, we get all Chebyshev superpotentials at once by truncation to the operators $\phi_k \in \mathscr{R}_f$ of the form

$$\phi_k(Y) = P_k(\cos(Y/n)) \sin(Y/n),$$

where $P_k(X)$ are polynomials of degree $k \leqslant n-2$.

9.1. N = 2 SINE-GORDON

For the sine-Gordon model we identify an element of $\mathscr{R}$ with the set of its values at the (non-singular) critical points (the "point" basis). For $W_f(X)$ the critical points are

$$X_r = \pi r, \quad r \in \mathbb{Z},$$

and we identify an element $\phi \in \mathscr{R}_f$ with the sequence

$$\{(\phi)_r \equiv \phi(\pi r),\ r \in \mathbb{Z}\}.$$

The ring operations act componentwise on ϕ. One has (using definition (2.10))

$$\mathrm{Res}[\phi] = \frac{1}{\lambda} \sum_{r \in \mathbb{Z}} (-1)^{r+1} (\phi)_r.$$

We choose as basis in $\mathscr{R}_f$ the elements a_k $(k \in \mathbb{Z})$ such that

$$(a_k)_r = \delta_{kr}. \tag{9.2}$$

In this basis we have

$$\eta_{kh} = (-1)^{(k+1)} \frac{1}{\lambda} \delta_{kh},$$

$$(C_\lambda)_k^h = (-1)^k \delta_k^h.$$

The superpotential (9.1) is invariant (up to phase) for

$$T:\quad Y \to Y + \pi,$$

$$P:\quad Y \to -Y.$$

Then, in our basis one has

$$g_{i+1,\overline{j+1}} = g_{i,j},$$

$$g_{-i,-j} = g_{i,j}. \tag{9.3}$$

Given an integer i, there is a unique decomposition

$$i = \langle i \rangle + 2\{i\}, \quad \text{with} \quad \langle i \rangle = 0, 1.$$

Using (9.3) we write

$$g_{ij} = g_{\langle i \rangle \langle j \rangle}(\{i\} - \{j\}),$$

and introduce its Fourier series

$$g_{\langle i \rangle \langle j \rangle}(\theta) = \sum_r e^{i\theta r} g_{\langle i \rangle \langle j \rangle}(r).$$

Next, we consider the 2×2 matrix $(0 \leqslant \theta \leqslant 2\pi)$

$$g(\theta) = \begin{pmatrix} g_{0\bar{0}}(\theta) & g_{0\bar{1}}(\theta) \\ g_{1\bar{0}}(\theta) & g_{1\bar{1}}(\theta) \end{pmatrix}.$$

Eq. (9.3) implies

$$g_{0\bar{0}}(\theta) = g_{1\bar{1}}(\theta),$$

$$g_{0\bar{1}}(\theta) = e^{i\theta} g_{1\bar{0}}(\theta),$$

and

$$g_{0\bar{0}}(\theta) = g_{0\bar{0}}(-\theta),$$

$$g_{1\bar{0}}(\theta) = e^{-i\theta} g_{1\bar{0}}(-\theta).$$

Then we can parametrize the metric as

$$g(\theta) = \begin{pmatrix} A(\theta) & e^{i\theta/2}B(\theta) \\ e^{-i\theta/2}B(\theta) & A(\theta) \end{pmatrix},$$

where

$$A(\theta) = A(-\theta), \qquad B(\theta) = B(-\theta).$$

The transpose and the conjugate of the ground state metric in terms of the 2×2 matrix $g(\theta)$ read

$$g^{\mathrm{T}}(\theta) = [g(-\theta)]^{\mathrm{T}}, \qquad g^*(\theta) = [g(-\theta)]^*.$$

Then

$$g^{\dagger}(\theta) = [g(\theta)]^{\dagger},$$

and $g(\theta)$ is hermitian in the 2×2 sense. Therefore

$$A(\theta) = A(\theta)^*, \qquad B(\theta) = B(\theta)^*.$$

Moreover, $A(\theta) > 0$, since the metric is positive.

Finally, we must impose the "real structure" constraint on $g(\theta)$, namely

$$\eta^{-1}(\theta) g(\theta) (\eta^*)^{-1}(\theta) g^*(\theta) = \mathbb{1}. \qquad (9.4)$$

In the 2×2 notation, one has

$$\eta(\theta) = \frac{1}{\lambda}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C_\lambda(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So eq. (9.4) reduces to

$$|\lambda|^2 \left(A(\theta)^2 - B(\theta)^2\right) = 1.$$

Therefore, we can parametrize $g(\theta)$ in terms of a single function of $x (= |\lambda|^2)$

$$A(x, \theta) = (1/\sqrt{x}) \cosh[L(x, \theta)],$$

$$B(x, \theta) = (1/\sqrt{x}) \sinh[L(x, \theta)].$$

Putting everything together, we get

$$g(x, \theta) = (1/\sqrt{x}) U(\theta) \exp[\sigma_1 L(x, \theta)] U(\theta)^{-1},$$

where

$$U(\theta) = \exp(\tfrac{1}{4} i\theta\sigma_3).$$

Now,

$$\partial_{\bar{\lambda}}[g\,\partial_\lambda g^{-1}](\theta) = -U\sigma_1 U^{-1} \partial_\lambda \partial_{\bar{\lambda}} L(x, \theta),$$

$$[G_\lambda, \overline{G}_\lambda](\theta) = -2U\sigma_1 U^{-1} \sinh[2L(x, \theta)],$$

and the final equation reads

$$\partial_\lambda \partial_{\bar{\lambda}} L(x, \theta) = 2\sinh[2L(x, \theta)],$$

i.e. for each θ, $2L(x, \theta)$ is a self-similar solution to the sinh-Gordon equation and we are back with our old friend the special PIII. To put the equation in canonical form, let

$$2L(x, \theta) = u(z, \theta) \quad \text{where} \quad z = 2x^{1/2}.$$

For $z \to 0$ we have the asymptotics (cf. sect. 8)

$$u(z, \theta) \simeq r(\theta) \log z + s(\theta) + \dots, \quad \text{with} \quad |r(\theta)| \leqslant 2,$$

that is

$$L(x, \theta) \simeq \tfrac{1}{4} r(\theta) \log x + \tfrac{1}{2}[s(\theta) + r(\theta) \log 2] + \dots,$$

whereas for $x \to \infty$ we get (cf. (8.6))

$$L(x, \theta) \simeq \frac{\alpha(\theta)}{2\sqrt{2}} \frac{\exp(-4x^{1/2})}{x^{1/4}} \equiv \frac{\alpha(\theta)}{2\sqrt{2}\,|\lambda|^{1/2}} e^{-4|\lambda|}.$$

Notice that the exponent is precisely the soliton mass

$$2|\Delta W| = 2|\lambda|\,|\cos(k\pi) - \cos((k+1)\pi)| = 4|\lambda|,$$

in agreement with the semiclassical picture

To specify completely the metric for $N=2$ sine-Gordon, it remains only to fix the boundary conditions, i.e. the function $r(\theta)$. This will be done below.

In terms of $L(x,\theta)$, the point-basis metric reads

$$g_{kj}(x)=\frac{1}{4\pi\sqrt{x}}\int_0^{2\pi}\mathrm{d}\theta\ \mathrm{e}^{i\theta(j-k)/2}\Big\{\exp[L(x,\theta)]+(-1)^{(j-k)}\exp[-L(x,\theta)]\Big\}. \tag{9.5}$$

Since $g(\theta)$ is periodic with period 2π, one has

$$L(x,\theta+2\pi)=-L(x,\theta),$$

$$L(x,-\theta)=L(x,\theta).$$

In particular,

$$L(x,\pi)=0.$$

9.2. BACK TO CHEBYSHEV

Now we return to the original Chebyshev superpotentials,

$$W=\lambda T_n(X).$$

The critical points are

$$X_r=\cos\left(\frac{r\pi}{n}\right)\quad r=1,\dots,n-1.$$

Again we work in the point basis. We denote by I_r the chiral field with value 1 at the rth critical point and zero elsewhere. From each I_r, by pull-back, we get a chiral primary operator in the sine-Gordon theory. Taking into account the jacobian, we get ($j=1,\dots,n-1$)

$$f^*I_j=-\frac{1}{n}\sin\left(\frac{\pi}{n}j\right)\sum_{r\in Z}[a_{2nr+j}-a_{2nr-j}],$$

where a_k is as in (9.2).

Then eq. (9.5) gives

$$2\delta(0)\langle I_j^*I_k\rangle=\frac{1}{n^2}\sin\left(\frac{\pi}{n}j\right)\sin\left(\frac{\pi}{n}k\right)$$
$$\times\sum_{r,s\in Z}[g_{2nr+j,\overline{2ns+k}}-g_{2nr-j,\overline{2ns+k}}-g_{2nr+j,\overline{2ns-k}}+g_{2nr-j,\overline{2ns-k}}],$$

where $2\delta(0)$ is the degree of the cover. The sums in the r.h.s. can be computed via the Poisson formula

$$\sum_{r,s\in Z}g_{2nr+j,\overline{2ns+k}}=\frac{1}{2n\sqrt{x}}\delta(0)\sum_{r=0}^{n-1}\mathrm{e}^{i\pi r(k-j)/n}\left\{\exp\left[L\left(x,\frac{2\pi r}{n}\right)\right]\right.$$
$$\left.+(-1)^{(k-j)}\exp\left[-L\left(x,\frac{2\pi r}{n}\right)\right]\right\}.$$

Putting everything together, we get the ground state metric for the model $W=\lambda T_n(X)$,

$$\langle I_j^*I_k\rangle=\frac{1}{n^3\sqrt{x}}\sin\left[\frac{\pi}{n}j\right]\sin\left[\frac{\pi}{n}k\right]$$
$$\times\sum_{r=1}^{n-1}\sin\left(\frac{\pi}{n}rk\right)\sin\left(\frac{\pi}{n}rj\right)\left[\mathrm{e}^{L(x,2\pi r/n)}+(-1)^{(k-j)}\mathrm{e}^{-L(x,2\pi r/n)}\right], \tag{9.6}$$

which expresses the metric as a combination of a finite number of solutions to special PIII. All these solutions are bounded for $x\to\infty$ and regular on the positive real axis. Taking into account that

$$L(x,2\pi-\alpha)=-L(x,\alpha),$$

we see that the metric for the T_n-model involves $[(n-1)/2]$ independent solutions to PIII. In particular, for $n=2$ we have just elementary functions, and for $n=3,4$ we have a single Painlevé transcendent. This is in full agreement with previous work, since T_2 is equivalent to the free theory, $W=X^2/2$, T_3 is equivalent to $W=X^3/3-tX$, and T_4 to $W=X^4/4-tX^2/2$. These last two models have already been solved in sect. 8 in terms of a single Painlevé transcendent. In fact, by going through the field redefinitions needed to put these superpotentials in the standard form (paying attention to the "anomalous" jacobian) one checks that for $n=2,3,4$ the above results reproduce the results of sects. 7 and 8. For brevity, we omit the details of this check.

9.3. REGULARITY VERSUS BOUNDARY CONDITIONS

As in sect. 8, the boundary condition $r(\theta)$ is fixed by requiring that the metric is finite and non-zero as $\lambda\to 0$. Then the value of $s(\theta)$ is predicted by the condition of no pole on the positive real axis. We recall that for $W=Y^n$ the ground state metric reads

$$\langle Y^k|Y^k\rangle=\Gamma\left(\frac{k+1}{n}\right)\Big/n\Gamma\left(1-\frac{k+1}{n}\right)\quad(k=0,\dots,n-2). \tag{9.7}$$

One has

$$T_n(X) = 2^{n-1}X^n + k_{n-2}X^{n-2} + \dots .$$

The field redefinition

$$Y \equiv 2(\lambda/2)^{1/n} X,$$

puts the superpotential in the form

$$W = \lambda T_n(X) = Y^n + O(\lambda^{2/n}).$$

Consistency requires that, as $\lambda \to 0$, the Chebyshev metric reproduces (9.7).

The critical points for $T_n(X)$ are $X_k = \cos(k\pi/n)$. Then in the point basis the monomials $X^k \in \mathscr{R}$ read as

$$X^k = \sum_{r=1}^{n-1} \left[\cos\left(\frac{r\pi}{n}\right)\right]^k l_r \qquad (k = 0, 1, \dots, n-2).$$

Taking into account the jacobian, one has

$$\langle Y^k | Y^h \rangle = \left[2(\bar{\lambda}/2)^{1/n}\right]^{k+1} \left[2(\lambda/2)^{1/n}\right]^{h+1} \langle X^k | X^h \rangle.$$

Let us define the sums

$$A_{k,t} = \sum_{r=1}^{n-1} \cos^k\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}rt\right),$$

$$B_{k,t} = \sum_{r=1}^{n-1} (-1)^r \cos^k\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}rt\right).$$

Explicitly, one has

$$A_{0,t} = \tfrac{1}{2}n\left[\delta_{(t)_n,1} - \delta_{(t)_n,2n-1}\right],$$

and, for $k \neq 0$

$$A_{k,t} = \frac{n}{2^{k+2}}\left[1 - (-1)^{(k+t)}\right]\left\{\begin{pmatrix} k \\ \frac{1}{2}(k+t-1)_n \end{pmatrix} - \begin{pmatrix} k \\ \frac{1}{2}(k+t+1)_n \end{pmatrix}\right\},$$

where $(a)_n$ is a short-hand notation for the unique number $0 \leqslant (a)_n < 2n$, which is congruent to a modulo $2n$. Moreover,

$$B_{k,t} = A_{k,t+n}.$$

Putting everything together, the metric in the monomial basis reads

$$\langle Y^k | Y^h \rangle = \frac{1}{n^3|\lambda|}\left[2(\bar{\lambda}/2)^{1/n}\right]^{k+1}\left[2(\lambda/2)^{1/n}\right]^{(h+1)}$$
$$\times \sum_{t=1}^{n-1}\left[A_{k,t}A_{h,t}\,e^{[L(x,2\pi t/n)]} + B_{k,t}B_{h,t}\,e^{-[L(x,2\pi t/n)]}\right].$$

The coefficients $A_{k,t}$, $B_{k,t}$ satisfy the "selection rules" (for $0 < t < n$)

$$\begin{aligned} A_{k,t} &= 0 \quad \text{for} \quad t > k+1, \\ B_{k,t} &= 0 \quad \text{for} \quad t < n-1-k, \end{aligned} \tag{9.8}$$

The first non-vanishing coefficients are

$$A_{k,k+1} = -B_{k,n-1-k} = n/2^{k+1}. \tag{9.9}$$

A consequence of the selection rules is that $\langle 1|1 \rangle$ is equal to (up to trivial factors) $\exp[L(x, 2\pi/n)]$, i.e. it is expressed in terms of a single Painlevé transcendent. More generally, the matrix element $\langle Y^k | Y^h \rangle$ involves, at most, $\min(k+1, h+1)$ transcendents.

The asymptotic behaviour of the diagonal elements of the metric as $\lambda \to 0$ is

$$\langle Y^k | Y^k \rangle \simeq (1/n^3)(2^{1-1/2})^{2k+2}|\lambda|^{((2k+2)/n)-1}$$
$$\times\left[\sum_{t=1}^{n-1} A_{k,t}^2 (2|\lambda|)^{r(2\pi t/n)/2}\,e^{s(2\pi t/n)/2} + \sum_{t=1}^{n-1} B_{k,t}^2 (2|\lambda|)^{-r(2\pi t/n)/2}\,e^{-s(2\pi t/n)/2}\right].$$

Using the selection rules, the requirement that the r.h.s. has a finite non-zero limit, gives

$$r\left(\frac{2\pi}{n}t\right) = 2\left(1 - \frac{2t}{n}\right) \quad (t = 1, \dots, n-1). \tag{9.10}$$

Note that in particular $|r| < 2$, as required by regularity. Assuming that the solutions are regular, we get (8.5)

$$\exp\left[\tfrac{1}{2}s\left(\frac{2\pi}{n}t\right)\right] = 2^{(4t-2n)/n}\,\Gamma\left(\frac{t}{n}\right)\Big/\Gamma\left(1 - \frac{t}{n}\right).$$

This, using (9.9), implies

$$\langle Y^k \mid Y^k\rangle|_{\lambda\to 0} = (1/n^3)2^{(n-1)(2k+2)/n}\left[A^2_{k,k+1} + B^2_{k,n-k-1}\right]2^{(n-2k-2)/n}\, e^{[s(2\pi(k+1)/n)/2]}$$

$$= (1/n)\left(\Gamma\left(\frac{k+1}{n}\right)\Big/\Gamma\left(1-\frac{k+1}{n}\right)\right),$$

in full agreement with eq. (9.7). Moreover, the off-diagonal elements

$$\langle Y^k \mid Y^h\rangle \quad k \neq h,$$

go to zero in this limit, as they should. Therefore regularity implies the correct boundary conditions for Chebyshev superpotentials. It is amusing that *all the normalization coefficients of the A_m minimal models can be deduced from regularity theorems on Painlevé transcendents of third kind and vice versa.*

It remains to specify the boundary conditions for the solution of the $N=2$ sine-Gordon model. We assume that $r(\theta)$ is a continuous (albeit not smooth) function of θ. From eq. (9.10) we know it at all rational values of θ/π. Then it should be

$$r(\theta) = 2\left(1-\frac{\theta}{\pi}\right) \quad \text{for} \quad 0 \leq \theta \leq 2\pi.$$

Outside this interval, the function is obtained by using

$$r(\theta) = -r(\theta+2\pi), \quad r(\theta) = r(-\theta).$$

Then the bound-state metric for the $N+2$ sine-Gordon is completely determined. Note that $|r(\theta)| \leq 2$, and that *all* the regular solutions to special PIII appear in the metric for the $N+2$ sine–Gordon model. The points $\theta = 2\pi k$ where $|r(\theta)| = 2$ coincide with the points where $r(\theta)$ is not smooth. These are also the points where $L(x, \theta)$ even if continuous in θ changes its asymptotic behaviour for $\lambda \to 0$ (cf. sect. 8). At the se points one has "logarithmic violations of scaling". This is precisely the boundary condition satisfied by the Ising model correlation functions [47].

9.4. STRONG-COUPLING LIMIT

Let us take the limit $\lambda \to \infty$. in this limit the various vacua at different critical points, $X_k = \cos(\pi k/n)$, decouple (up to exponentially small corrections corresponding to soliton corrections). Then we must have

$$\langle l_j^* l_k\rangle \simeq \frac{\delta_{jk}}{|W''(X_k)|} + \frac{\alpha_{jk}}{\sqrt{|W''(X_j)W''(X_k)|}}\,\frac{1}{\sqrt{2|\lambda|}}\,e^{-4|\lambda|},$$

for certain constants α_{jk}. Since

$$W''(X_k) = \lambda n^2 \frac{(-1)^{k+1}}{\left[\sin\left(\frac{\pi}{n}k\right)\right]^2},$$

we must get

$$\langle l_j^* l_k\rangle = \frac{\sin(\pi k/n)\sin(\pi j/n)}{n^2|\lambda|}\left(\delta_{jk} + \alpha_{jk}\frac{1}{\sqrt{2|\lambda|}}e^{-4|\lambda|} + \dots\right). \quad (9.11)$$

Using the asymptotics of $u(z, \theta)$, eq. (8.6), and the identity (valid for $j, k = 1, \dots, n-1$)

$$\frac{1}{n}\left[1+(-1)^{(k-j)}\right]\sum_{r=1}^{n-1}\sin\left(\frac{\pi}{n}rk\right)\sin\left(\frac{\pi}{n}rj\right) = \delta_{k,j},$$

the r.h.s. of eq. (9.6) for large λ has the behaviour of eq. (9.11) with

$$\alpha_{j,k} = \frac{1}{2n}\left[1-(-1)^{(k-j)}\right]\sum_{s=1}^{n-1}\sin\left(\frac{\pi}{n}sk\right)\sin\left(\frac{\pi}{n}sj\right)\alpha\left(\theta=\frac{2\pi s}{n}\right)$$

$$= -\frac{1}{2n\sqrt{\pi}}\left[1-(-1)^{(k-j)}\right]\sum_{s=0}^{2n-1}\sin\left(\frac{\pi}{n}sk\right)\sin\left(\frac{\pi}{n}sj\right)\cos\left(\frac{\pi}{n}s\right)$$

$$= -\frac{1}{2\sqrt{\pi}}(\delta_{j,k+1} + \delta_{k,j+1}),$$

in agreement with the results of sect. 8 and appendix B.

9.5. THE c-FUNCTION

Next we consider the c-function. By the same agreement as in sect. 8, for the T_n model we have ($z \equiv 2|\lambda|$)

$$c(z) = \tfrac{3}{2}z\frac{\partial}{\partial z}u(z, 2\pi/n),$$

(in particular for $n=2$, c is identically zero, and for $n=3, 4$ it is just what we got in sect. 8). This follows from the fact that the Ramond operator associated to 1 is the one with lowest charge. As z goes to 0, we get for the UV central charge

$$c_{uv} = \tfrac{3}{2}r\left(\frac{2\pi}{n}\right) = 3\left(1-\frac{2}{n}\right),$$

which is the well known result for the A_{n-1} minimal model. The leading correction to this result is of order $|\lambda|^{4/n}$, i.e. the modulus square of the perturbation.

The "running" U(1) charges of the Ramond ground state are

$$q_k(z) = \tfrac{1}{4}z\frac{\partial}{\partial z}u(z, 2\pi k/n) \quad (k = 1, 2, \ldots, n-1).$$

As $z \to 0$, we get back the result of the A_{n-1} minimal model, whereas as $z \to \infty$ they all go to zero, as they should since the IR fixed point is trivial.

For the $N = 2$ sine-Gordon theory itself, we have

$$c(z) = \tfrac{3}{2}z\frac{\partial}{\partial z}u(z, 0),$$

which in the UV limit gives $c = 3$. However, now the corrections are logarithmic,

$$c(z) \simeq 3 + \frac{3}{2(\log z + C)}, \quad z \to 0.$$

It is tempting to speculate about the relation of this logarithmic scaling violation with the ones appearing in 2d gravity at $c = 1$. This is in particular tempting in view of the conjecture of Li [9] about the relation of topological $N = 2$ minimal models with 2d quantum gravity.

All the discussion in sect. 8 on the properties of these c-functions applies word-for-word to the present general case.

9.6. VARIATIONS ON THE THEME

One interesting aspect of the equations for g is that they have a tendency to reproduce nice field equations. For example, above we got the equations of 2d sinh-Gordon. There are other models leading to even more suggestive equations. As a divertissement we present a class of model which lead to 3d chiral models.

We consider the multicritical sine-Gordon models. By this we mean a model which has the same critical points as the sine-Gordon one, but with a multiplicity $\mu > 1$. All the critical points are assumed to have the same multiplicity μ. For simplicity, we assume μ to even ($= 2m$). Then the superpotential is

$$W(X) = \lambda\int \sin^{2m}X \,\mathrm{d}X$$

$$= \frac{\lambda}{2^{2m}}\frac{(2m)!}{(m!)^2}X + \frac{\lambda}{2^{2m}}\sum_{k=0}^{m-1}\frac{(-1)^{m-k}}{(m-k)}\binom{2m}{k}\sin[2(m-k)X],$$

which has the pseudosymmetries

$$X \to X + k\pi, \qquad X \to -X. \tag{9.12}$$

An element $\phi \in F$ is uniquely specified by its $(2m-1)$-jets at the critical points, i.e. by the set of data

$$\left\{\phi(k\pi), \partial\phi(k\pi), \frac{1}{2!}\partial^2\phi(k\pi), \ldots, \frac{1}{(2m-1)!}\partial^{(2m-1)}\phi(k\pi) \,|\, k \in \mathbb{Z}\right\},$$

(the "point" basis). Then F is identified with this set of numbers written as a two-index object

$$\phi \equiv (\phi)_{k,r} \quad k \in \mathbb{Z},\ r = 1, \ldots, 2m.$$

In this notation the ring product reads

$$(\phi\psi)_{k,r} = \sum_{s=1}^{2m}(\phi)_{k,s}(\psi)_{k,r-s}.$$

Consider the ground state metric in such a basis $g_{i,r;\bar{j},\bar{s}}$. From (9.12) we have

$$g_{i+1,r;\overline{j+1},\bar{s}} = g_{i,r;\bar{j},\bar{s}},$$

$$g_{-i,r;\overline{-j},\bar{s}} = (-1)^{(r+s)}g_{i,r;\bar{j},\bar{s}}.$$

As above we introduce the Fourier transform

$$g_{i,r;\bar{j},\bar{s}} \equiv g_{r,s}(i-j), \qquad g_{r,s}(\theta) = \sum_k e^{ik\theta}g_{r,s}(k).$$

The $2m \times 2m$ matrix $g(\theta)$ satisfies

$$g(-\theta) = \Sigma_3 g(\theta)\Sigma_3, \tag{9.13}$$

where

$$\Sigma_3 = \mathrm{diag}(1, -1, 1, -1, \ldots, 1, -1).$$

In this notation, the residue pairing is

$$\eta(\theta) = (1/\lambda)\Sigma_1,$$

where

$$(\Sigma_1)_{ij} = \delta_{i+j,2m+1}.$$

As in the sine-Gordon case, we have

$$[g(\theta)]^\dagger = g(\theta).$$

The reality structure constraint reads

$$\Sigma_1 g(\theta) \Sigma_1 g(-\theta)^* = \frac{1}{|\lambda|^2} \mathbb{1}$$

Let $\mathscr{G}(\theta) = |\lambda| g(\theta)$. Then the above equation becomes

$$\mathscr{G}(\theta) \Sigma_1 \mathscr{G}(-\theta)^{\mathrm{T}} = \Sigma_1,$$

or, using eq. (9.13),

$$\mathscr{G}(\theta) \Omega \mathscr{G}(\theta)^{\mathrm{T}} = \Omega,$$

where

$$\Omega = -\Sigma_1 \Sigma_3$$

is a symplectic matrix. Hence

$$\mathscr{G}(\theta) \in \mathrm{Sp}(2m).$$

The matrix C_λ reads

$$(C_\lambda)_{k,r}^{h,s} = \delta_k^h \frac{(2n)!}{2^{2m}(m!)^2} \pi k,$$

or, in the θ basis,

$$C_\lambda = -i\pi \frac{(2m)!}{2^{2m}(m!)^2} \frac{\mathrm{d}}{\mathrm{d}\theta}.$$

To save print we put

$$z = \lambda \frac{(2m)!}{2^{2m}(m!)^2}.$$

Then the equations become

$$\partial_{\bar z}\left[\mathscr{G}(\theta)\, \partial_z \mathscr{G}(\theta)^{-1}\right] = -\pi^2 \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\mathscr{G}(\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} \mathscr{G}(\theta)^{-1}\right].$$

Putting

$$x_1 = \mathrm{Re}\, z, \qquad x_2 = \mathrm{Im}\, z, \qquad x_3 = \theta/2\pi,$$

and using the fact that $\mathscr{G}$ is invariant under rotations in the (1, 2) plane, this equation is rewritten as ($\mu = 1, 2, 3$)

$$\partial_\mu\left[\mathscr{G}\, \partial_\mu \mathscr{G}^{-1}\right] = 0, \quad \mathscr{G} \in \mathrm{Sp}(2m),$$

which are the field equations of the (complexified) Sp($2m$) principal chiral model in three dimensions. This is the model corresponding to the lagrangian

$$\mathscr{L} = \mathrm{Tr}\left[\partial_\mu \mathscr{G}\, \partial_\mu \mathscr{G}^{-1}\right].$$

Of course, the metric is a very special solution to these field equations. $\mathscr{G}$ should be a positive hermitian matrix, invariant under rotations in the (1, 2) plane, periodic with respect to translations in the orthogonal direction, and such that

$$\mathscr{G}(x_1, x_2, -x_3) = \Sigma_3 \mathscr{G}(x_1, x_2, x_3) \Sigma_3.$$

Nevertheless, it is amusing that we get a formal "unification" of the coupling constant λ with θ which labels the different critical points!

10. Generalization to SU$(N)_k$

In this section we generalize the results of sect. 9 to arbitrary SU$(N)_k$. The ground state metric of the associated models will be expressed as a finite combination of (self-similar) solutions to $\hat{\mathrm{A}}_{N-1}$ Toda theory.

10.1. N CHEBYSHEV POLYNOMIALS

We start by describing the superpotentials corresponding to SU$(N)_k$ Verlinde rings, i.e. the generalization of Chebyshev polynomials to arbitrary N. These superpotentials are closely related to those for the grassmanian coset models of sect. 7, and indeed reduce to them in the UV limit.

Following Gepner [18], we introduce the variables q_i ($i = 1, \ldots, N$). These variables are subject to the constraint

$$\prod_{i=1}^{N} q_i = 1. \tag{10.1}$$

As in sect. 7, we denote by $\sigma_r(q_i)$ the rth elementary symmetric function of the q_i. Obviously, $\sigma_N(q_i) = 1$.

The superpotential corresponding to the $SU(N)_k$ Verlinde ring

$$W_{N,k}(X_1, X_2, \dots, X_{N-1})$$

is the unique polynomial such that

$$W_{N+k}(\sigma_1(q), \sigma_2(q), \dots, \sigma_{N-1}(q)) = \frac{\lambda}{N+k} \sum_{i=1}^{N} q_i^{N+k},$$

the only difference with respect to the grassmannian case being the constraint (10.1). Of course, this is a major difference since it spoils quasi-homogeneity. These polynomials are mutually orthogonal with respect to the L^2-measure defines * by the weight $\sqrt{\Delta(q_i)}$ and obey the recursion relation

$$(m+N)W_{m+N}(X_j) + \sum_{i=1}^{N-1} (-1)^i X_i (m+N-i) W_{m+N-i}(X_j) + (-1)^N m W_m(X_j) = 0.$$

Let us parametrize q_i as $(m = N+k)$

$$q_i = \exp\left[\frac{1}{m}(\phi_i - \phi_{i-1})\right] \quad i = 1, 2, \dots, N,$$

with the understanding that

$$\phi_0 = \phi_N = 0.$$

Let $f_{(m)}$ be the map

$$X_r = \left(f_{(m)}(\phi_j)\right)_r \equiv \sigma_r\left(\exp\left[(\phi_j - \phi_{j-1})/m\right]\right).$$

Then,

$$f_{(m)}^* W_m = \frac{\lambda}{m}\left[e^{\phi_1} + \sum_{i=1}^{N-2} e^{(\phi_{i+1}-\phi_i)} + e^{-\phi_{N-1}}\right],$$

which, up to an obvious field redefinition, is just the $N=2$ $SU(N)$ Toda superpotential. Then, by a change of variables, to solve the problem for $W_{N,k}(X_i)$ it is enough to compute the ground state for the supersymmetric Toda models. The jacobian is again $\Delta(q_i)$, the Vandermonde determinant.

* As in sect. 7, $\Delta(q_i)$ is the Vandermonde determinant.

10.2. $N=2$ TODA THEORIES

We are reduced to compute the ground state metric for the $N=2$ $SU(N)$ Toda theories,

$$W(\phi_1, \phi_2, \dots, \phi_{n-1}) = \frac{\lambda}{N}\left[e^{\phi_1} + \sum_{i=1}^{N-2} e^{(\phi_{i+1}-\phi_i)} + e^{-\phi_{N-1}}\right].$$

This model has two symmetries:

$$\phi_r \to \phi_r + i\frac{2\pi}{N} rk + 2\pi i l_r, \quad \text{with} \quad k = 0, 1, \dots, N-1, \quad l_r \in \mathbb{Z},$$

and

$$\phi_j \to \phi_{N-j}.$$

The critical points correspond to the orbit of the origin with respect to the first symmetry. Then a critical point is labelled by the numbers

$$(k, l_1, l_2, \dots, l_{N-1}), \quad k = 0, 1, \dots, N-1, \quad l_r \in \mathbb{Z}.$$

As usual, we denote by $a_{(k,l_r)}$ the chiral operator with value 1 at the given critical point and zero elsewhere. The value of W at the critical point (k, l_r) is

$$W_{(k,l_r)} = e^{2\pi i k/N}.$$

So,

$$(C_\lambda)_{(k,l_r)}^{(h,m_s)} = e^{2\pi i k/N} \delta_{(k,l_r)}^{(h,m_s)},$$

and the residue pairing is

$$\eta_{(k,l_r);(h,m_s)} = C_N\, e^{2\pi i k/N} \delta_{(k,l_r);(h,m_s)}.$$

Here C_N is a numerical constant depending on N only

$$(C_N)^{-1} = (1/N)^{N-1} \det[C_{ab}],$$

where C_{ab} is the $SU(N)$ Cartan matrix.

The above symmetries imply the following conditions on the metric

$$\langle (k, l_r) | (h, m_s) \rangle = \langle (k, l_r + a_r) | (h, m_s + a_s) \rangle, \quad a_r \in \mathbb{Z},$$

$$\langle (k, l_r) | (h, m_s) \rangle = \left\langle \left(\{k+p\}, l_r + \left[\frac{k+p}{N}\right]_r \right) \middle| \left(\{h+p\}, m_s + \left[\frac{k+p}{N}\right]_s \right) \right\rangle$$

($p = 0, \ldots, N-1$), where $\{a\}$ is the unique number between 0 and $N-1$ which is congruent to a modulo N. Moreover,

$$\langle (k, l_r) | (h, m_s) \rangle = \langle (k, -k - l_{N-r}) | (h, -k - m_{N-s}) \rangle .$$

The first property allows us to introduce the Fourier transform

$$g_{k\bar{h}}(\theta_1, \ldots, \theta_{N-1}) = \sum_{l_r \in \mathbb{Z}} \exp\left(i \sum_r l_r \theta_r \right) \langle (h, 0) | (k, l_r) \rangle .$$

Then the other two properties read

$$g_{k+1, \overline{h+1}}(\boldsymbol{\theta}) = g_{k\bar{h}}(\boldsymbol{\theta}) \quad (\text{for } 0 \leq k, h \leq N-2),$$

$$g_{k+1, \bar{0}}(\boldsymbol{\theta}) = \exp\left[-i \sum_r r\theta_r \right] g_{k, \overline{N-1}}(\boldsymbol{\theta}),$$

$$g_{0, \overline{h+1}}(\boldsymbol{\theta}) = \exp\left[i \sum_r r\theta_r \right] g_{N-1, \bar{h}}(\boldsymbol{\theta}),$$

$$g_{k\bar{h}}(\theta_1, \theta_2, \ldots, \theta_{N-1}) = \exp\left[-i(h-k) \sum_r \theta_r \right] g_{k\bar{h}}(-\theta_{N-1}, -\theta_{N-2}, \ldots, -\theta_1).$$

To put the equations in the Toda form, we have to diagonalize the $N \times N$ matrix $g(\boldsymbol{\theta})$. It has the structure

$$g_{k\bar{h}}(\boldsymbol{\theta}) = A_{(h-k)}(\boldsymbol{\theta}) + \exp\left[-i \sum_r r\theta_r \right] A_{(N+h-k)}(\boldsymbol{\theta}),$$

where

$$A_h(\boldsymbol{\theta}) = \begin{cases} g_{0\bar{h}}(\boldsymbol{\theta}) & \text{for } h = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

Given the peculiar structure of $g(\boldsymbol{\theta})$, its diagonalization is elementary. We introduce a new basis in $\mathcal{H}$ ($k = 0, \ldots, N-1$)

$$\psi_k(\boldsymbol{\theta}) = \sum_{r=0}^{N-1} \exp\left[\frac{ir}{N} \left(2\pi k + \sum_s s\theta_s \right) \right] a_r(\boldsymbol{\theta}),$$

$$a_r(\boldsymbol{\theta}) = \sum_{l_s \in \mathbb{Z}} \exp\left[i \sum_s l_s \theta_s \right] a_{(r, l_s)}.$$

In this basis, the ground state metric is diagonal, indeed

$$\langle \psi_k(\boldsymbol{\theta}) | \psi_h(\boldsymbol{\theta}') \rangle = \delta(\boldsymbol{\theta} - \boldsymbol{\theta}') \delta_{h,k} N \sum_{l=0}^{N-1} \exp\left[\frac{-il}{N} \left(2\pi k + \sum_s s\theta_s \right) \right] A_l(\boldsymbol{\theta}).$$

In the new basis,

$$W\psi_k(\boldsymbol{\theta}) = \psi_{\{k+1\}}(\boldsymbol{\theta}),$$

i.e.

$$(C_\psi)_k^h(\boldsymbol{\theta}) = \delta^h_{\{k+1\}}.$$

Therefore, for each value of $\theta_1, \ldots, \theta_{N-1}$ the ground state metric $\mathcal{G}_k(\boldsymbol{\theta})$,

$$\langle \psi_h(\boldsymbol{\theta}) | \psi_k(\boldsymbol{\theta}') \rangle = \delta(\boldsymbol{\theta}' - \boldsymbol{\theta}) \delta_{k,h} \mathcal{G}_k(\boldsymbol{\theta}),$$

satisfies the $\hat{\mathrm{A}}_{N-1}$ Toda equation,

$$-\partial_{\bar{\lambda}} \partial_\lambda \log \mathcal{G}_k(\boldsymbol{\theta}) = \frac{\mathcal{G}_{\{k+1\}}(\boldsymbol{\theta})}{\mathcal{G}_k(\boldsymbol{\theta})} - \frac{\mathcal{G}_k(\boldsymbol{\theta})}{\mathcal{G}_{\{k-1\}}(\boldsymbol{\theta})}.$$

However, in this basis the residue pairing is rather involved,

$$\mathrm{Res}\left[\psi_k(\boldsymbol{\theta}) \psi_h(\boldsymbol{\theta}') \right] = \delta(\boldsymbol{\theta}' - \boldsymbol{\theta}) C_N \sum_{r=0}^{N-1} \exp\left\{ \frac{ir}{N} \left[2\pi(k + h + 1) + 2 \sum_s s\theta_s \right] \right\}.$$

so the reality constraint is not as simple as in sect. 9. Notice that – contrary to the SU(2) case – the reality constraint gives $\mathcal{G}(-\boldsymbol{\theta})$ in terms of $\mathcal{G}(\boldsymbol{\theta})$ instead of putting a condition on the metric for fixed $\boldsymbol{\theta}$.

This completes the argument showing that for $N = 2$ *quantum* SU(N) affine Toda, associated to SU$(N)_k$ Verlinde rings, the ground state metric can be written as a finite combination of solutions to the *classical* $\hat{\mathrm{A}}_{N-1}$ (self-similar) affine Toda equation. Here we see the group SU(N) in operation in three seemingly unrelated ways!

11. Conclusions

We have seen that the metric on the space of ground state vacua of $N = 2$ QFTs can in principle be determined by solving certain interesting differential equations which express the flatness of certain holomorphic and antiholomorphic connections for the vacuum bundle over the parameter space. Not surprisingly, this flatness condition reduces in special cases to well known systems of equations of mathematical physics (of the Toda type) which are expressible in the Lax form. In examples which lead to equations which had been studied by mathematicians we were able to reproduce some of their results, derived from isomonodromic deformation techniques, from a purely $N=2$ QFT point of view. The generalizations that this $N=2$ point of view would naturally lead to, are yet to be verified using the isomonodromic deformation techniques.

The system of equations that we have used does not distinguish a "preferred" direction of perturbation, and in a sense treats all the directions on the same footing. This is partly a surprise, because only very special directions are integrable QFT's in the sense of having infinitely many conserved current *. It is precisely in these cases that our equations reduce to equations of the Toda type. Nevertheless it is natural to study the full space of perturbations. In particular it should be possible to flow from one conformal theory to another conformal theory and see how the OPE of the two theories are *predicted* by self-consistency, and in particular by the absence of singularity in the solution to the differential equations. The examples leading to affine Toda are always massive at the IR, and unfortunately do not provide any examples of this type.

We have seen that some examples of $N=2$ theories whose rings are the same as the rings of RCFT ($SU(N)_k$) lead to affine Toda equations. Is this a general property? Is it true that each case where Verlinde ring of a RCFT can be represented by the chiral ring of an $N=2$ theory the equations we get are integrable and lead to Toda equations? Is it true that each time our equations are of the toda type we can interpret the ring as that of a RCFT? These are mysterious links between a *conformal* theory (RCFT) and a *massive* $N=2$ theory, which deserve a serious study. Could it be that $N=2$ theories lead to knot invariants in three dimensions through this link? (if this were true singularity theory might be connected to knot invariants). Do the $N=2$ theories admit a direct three-dimensional interpretation?

We have seen that the affine Toda equations that characterize the metric encode a lot of the information about the solitons in the theory. Can one *derive* the soliton scattering amplitudes from this viewpoint using the techniques of thermodynamic Bethe ansatz [51]? The discussion in appendix B points in this direction.

Many of our constructions work for Donaldson theory and is worth investigating. This might lead to a simpler derivations of Ward identities in the context of $N=2$ supersymmetric Yang–Mills theories [52]. This would be interesting to study.

* It would be interesting to see if one can imbed this in an integrable setup by infinitely extending the number of couplings, similar to what one has in matrix models [50]. We would like to thank authors of the first reference in [50] for discussions on this point.

It is our distinct feeling that we have only found the tip of an iceberg. There are too many different things being related in too many seemingly accidental ways for there not to be a bigger story. We hope that this will motivate further study to find this bigger story.

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Appendix A. The ground state metric in the critical regime

At a conformal point W is quasi-homogeneous. In this case one can give explicit representations of the metric in terms of integrals of holomorphic forms. Basically, this is the generalization of Gepner's correspondence for minimal models: at criticality an $N=2$ model is related to a σ-model and thus can be studied by complex geometry techniques. There are three (equivalent) formulations of these integral representations:

(i) In terms of the integrals ($\{\phi_k\}$ a holomorphic basis of $\mathscr{R}$)

$$\varpi_k{}^j \equiv \int_{\gamma_j^+} e^{-W}\phi_k \, dX_1 \wedge \ldots \wedge dX_n. \tag{A.1}$$

(ii) In terms of the period integrals for the pure (p, q) components of the groups

$$H^{n-2}(E_1) \otimes H^{n-1}(E_2),$$

where E_i are the (weighted) projective manifolds

$$\begin{aligned} E_1: &\quad W(X_i) = 0, \\ E_2: &\quad W(X_i) + X_{n+1}^d = 0. \end{aligned} \tag{A.2}$$

(iii) For marginal operators the ground state metric is Kähler. The Kähler potential has the representation

$$e^{-K} = \int d^nX \, d^n\bar{X} \exp\left[W(X) - \bar{W}(\bar{X})\right], \tag{A.3}$$

which can be rewritten as a bilinear form in the integrals of point i) as explained in sect. 4.

To simplify the arguments notice that (without loss of generality) we can assume W to be homogeneous. Indeed let the fields X_i have U(1) charge $q_i = r_i/d$. Then make the change of variables

$$X_i = Y_i^{r_i}.$$

In terms of the new fields W is homogeneous, and the original ground state metric is related to the new one as in sect. 5.

A part of the above statements is elementary. Indeed, we known that (for marginal deformations) the metric is Kähler. Then it is elementary to show that

$$e^{-K(t,\bar{t})} = \sum_{k,h=1}^{\Delta} I_{kh} \chi_k(t_a)[\bar{\chi}_h(t_b)]^*,$$

where I_{kh} is the intersection matrix and $\chi_k(t_a)$, $\bar{\chi}_k(t_a)$ are holomorphic. In fact (cf. sect. 4) $\exp[-K] = \langle 0|0\rangle$, and (sect. 5)

$$\langle 0|0\rangle = \sum_{h,k} \rho_{hk}\left[\int_{\gamma_k^-} e^{W+\overline{W}}\omega_0\right]\left[\int_{\gamma_h^+} e^{-W-\overline{W}} * \omega_0\right]^*. \tag{A.4}$$

Then it remains to show that

$$\int_{\gamma_k} e^{W+\overline{W}}\omega_0, \qquad \int_{\gamma_k^+} e^{-W-\overline{W}} * \omega_0, \tag{A.5}$$

are holomorphic. Indeed

$$0 = \bar{\partial}_a \int_{\gamma_k^+} e^{-W-\bar{W}} * \omega_0 = \bar{\partial}_a \int_{\gamma_k^+} e^{-W-\overline{W}} * \omega_0,$$

since $(\bar{C}_a)_0^k = 0$ by charge conservation. The same argument (using the dual connection $\bar{\partial}'$) works for the other integral in (A.5).

According to the discussion in sect. 4, to prove eq. (A.3) it remains to show that in (A.4) one can replace the integrals of the vacuum wave-forms with those of the corresponding holomorphic forms. The proofs are hidden in ref. [14]. Here we try to present them in a more "physical" form. We have already mentioned that the basic flatness equations

$$\partial \Pi = \bar{\partial}\Pi = 0, \tag{A.6}$$

have the same general structure as Toda's. In the case of (quasi) homogeneous W they are analogous to the non-affine Toda, and hence can be solved by the usual

Leznov–Saveliev method [39]. One starts from the Gauss decomposition of Π,

$$\Pi = e^D AB$$

(here B is an upper-triangular * matrix, A is a lower-triangular one and D is block-diagonal). In terms of B one gets simpler equations

$$\bar{\partial}B = \bar{\partial}D = 0,$$
$$(\partial + e^{-D}C\,e^{D})B = 0. \tag{A.7}$$

The crucial point of the method is that, once we are given an upper-triangular matrix B satisfying (A.7) (for some D), we can reconstruct the full solution by Lie-algebraic techniques.

A direct computation gives

$$(\partial_a + C_a)\varpi = L_a\varpi, \tag{A.8}$$

with L_a zero above the diagonal. Now, consider the Gauss decomposition of ϖ,

$$\varpi = e^{\tilde{D}}\tilde{A}\tilde{B}.$$

Eq. (A.8) implies that $\tilde{B}$ is a solution to eq. (A.7) (with $D = \tilde{D}$). Thus, out of the periods ϖ we can reconstruct a solution to our equations. The hard part of the argument is to show that this solution coincides with the one given by the SQM "period map" Π. We postpone the discussion of this point to the end.

Then one has

$$\Pi = e^F \mathcal{N}\varpi, \tag{A.9}$$

with F block-diagonal and holomorphic and $\mathcal{N}$ *strictly* lower-triangular, i.e. $\mathcal{N} = 1 + Z$, with Z decreasing the charge by one or more units. The first component of (A.9) gives

$$\int_{\gamma_k^+} e^{-W-\bar{W}} * \omega_0 = \exp(F_0^0(t))\int_{\gamma_k^+} e^{-W}\,dX_1 \wedge \ldots \wedge dX_n.$$

Analogously,

$$\int_{\gamma_k^-} e^{W+\overline{W}}\omega_0 = \exp(\hat{F}_0^0(t))\int_{\gamma_k} e^{W}\,dX_1 \wedge \ldots \wedge dX_n.$$

* By upper-(lower)triangular matrix we mean the identity plus the matrix of an operator which increases (decreases) the U(1) charge. It is actually block-triangular.

Then

$$e^{-K} = \exp\left(\left(F_0^0\right)^* + \hat{F}_0^0\right) \sum_{h,k} \rho_{kh} \int_{\gamma_h} e^{W} \, dX_1 \wedge \ldots$$

$$\wedge \, dX_n \left[\int_{\gamma_k^*} e^{-W} \, dX_1 \wedge \ldots \wedge dX_n \right]^* . \quad \text{(A.10)}$$

This, together with the discussion in sect. 4 shows property (iii). (The factor in front of the sum can be re-absorbed by a Kähler gauge transformation). That ρ can be identified with the inverse intersection matrix C^{ij} can be seen by the same argument used in appendix C to show eq. (4.4).

A slight generalization of this argument leads to eq. (4.1). Let $\phi_j(X)$ be the *relevant* chiral operators with U(1) charges r_j/d $(0 < r_j < d)$. Consider the auxiliary superpotential

$$W_{\text{aux}}(X_k, Y; t_a, s_i) = W(X_k, t_a) + Y^d + \sum_j s_j \phi_j(X) Y^{d-r_j}.$$

W_{aux} is quasi-homogeneous and the couplings s_j are moduli. So the above analysis applies. As $s_j \to 0$ the field Y decouples and then

$$\left\langle \overline{\phi_j Y^{d-r_j}} \phi_i Y^{d-r_i} \right\rangle_{\text{aux}} \Big|_{s_i=0} = \left\langle \overline{\phi}_j \phi_i \right\rangle \langle \overline{Y}^{d-r_j} Y^{d-r_i} \rangle_d , \quad \text{(A.11)}$$

where $\langle \ldots \rangle_d$ denotes the metric for the A_{d-1} minimal model. On the other hand, the l.h.s. of eq. (A.11) is equal to

$$-\langle 0|0\rangle_{\text{aux}} \, \partial_{\bar{s}_j} \partial_{s_i} \log\langle 0|0\rangle_{\text{aux}} \big|_{s_i=0} .$$

Replacing the integral representation (A.3) for $\langle 0|0\rangle_{\text{aux}}$ and neglecting terms which vanish by symmetry reasons, we get

$$g_{i\bar{j}} = \left\langle \overline{\phi}_j \phi_i \right\rangle = \int \prod dX_l \, d\overline{X}_l \, \phi_i(X_k) \overline{\phi}_j(\overline{X}_k) \exp\left[W(X) - \overline{W}(\overline{X})\right] . \quad \text{(A.12)}$$

In this form the equality holds only for *relevant* operators. Let us explain why the irrelevant ones are different. First of all, it would be contradictory to assume eq. (A.12) to be true for all fields. In fact, $\langle \overline{\phi}_j \phi_i \rangle = 0$ if $q_i \neq q_j$, whereas the r.h.s. of eq. (A.12) does not vanish for $q_i - q_j$ *integral*. In other words, the bilinear form in the r.h.s. mixes operators with charges differing by an integral amount. More precisely, an operator ϕ_i of charge q_i gets mixed with operators of *lower* charge $q_i - 1$, $q_i - 2, \ldots$. Only the relevant operators are well defined, whereas the marginal ones can mix only with the identity. In this last case, the problem is solved by taking the "connected" part of the integral in eq. (A.12), i.e. one takes the *logarithm* of the integral as Kähler potential. The fundamental reason behind this mixing is the dependence on the choice of a particular representative for the classes in $\mathscr{R}$. Under a change of representatives (preserving their U(1) charges)

$$\phi_j \, dX_1 \wedge \ldots \wedge dX_n \to \phi_j \, dX_1 \wedge \ldots \wedge dX_n + Dw \wedge \alpha_j ,$$

the periods ϖ change as

$$\varpi \to \varpi + \mathscr{Z} \varpi ,$$

where the matrix $\mathscr{Z}$ decreases the charge by an integral amount. Then mixing in unavoidable unless we have a preferred representative to start with. Instead the SQM period Π is unambiguous since it is defined in terms of given forms. A change of representatives is compensated in eq. (A.9) by a change in the matrix $\mathscr{F}$. Restricting to operators with $0 \leqslant q \leqslant 1$, in eq. (A.9) we can replace $\mathscr{F}$ by 1 and hence effectively identify the period ϖ with the SQM periods Π (F is absorbed in the conventions). This explain why for relevant/marginal operators we get nice formulae and why they do not hold for $q > 1$. In fact in the general case the metric can still be written in terms of ϖ though not so explicitly *. The mixing above has deep mathematical meaning. Some aspects are discussed in ref. [14]. To do better than this one has to leave the elementary methods. Luckily the mixing – which at the elementary level is a nuisance – at a more sophisticated level turns into a welcome simplification.

We just sketch the idea of how one can compute the metric for irrelevant operators out of the periods ϖ. More details can be found in ref. [14]. Basically, one has to reconstruct the complete solution of the linear problem (A.6) from its triangular part $e^D B$. In the Toda case this is done by Lie-theoretical methods [39]. The same applies here, but since in our case H is not abelian (in general) the reconstruction is a bit less elementary. It is convenient to present the tricks in a slightly more abstract language than in the abelian case. From sct. 6 we know that $\psi(t, \bar{t})$ is an element of the group G. So it can be seen as a map from coupling-constant space to the group G. However, it is more convenient to project it to a map ρ into the coset space ** G/H. G/H is an open domain in G_c/B where B is the group of lower triangular matrices (in our sense). This space is obviously a homogeneous complex manifold. In fact, it is the classifying space for complex flags of given type. Over GC/B we have universal tautological bundles corresponding to these flags. They are homogeneous with respect to the action of G_c and holomorphic. They have a unique hermitian metric $\langle \cdot | \cdot \rangle$ which is homogeneous

* However, for operators with $\hat{c} - 1 \leqslant q \leqslant \hat{c}$ one also has nice expressions. Indeed they can be connected to the relevant ones by the reality constraint. Then for $\hat{c} \leqslant 2$ elementary methods suffice to get all g.

** H is assumed to act on the left.

and such that G acts by isometries. Correspondingly there is a unique universal connection which can be constructed by Lie-group techniques. Embedding $G_{\mathbb{C}}/H$ into $G_{\mathbb{C}}/B$ enlarges the "gauge group" from H to B. Then ψ and its triangular part are related by a gauge transformation, i.e. define the same map *

$$\varrho: \text{couplings} \rightarrow G_{\mathbb{C}}/B.$$

In the triangular gauge ψ is holomorphic. Hence the map ϱ is holomorphic. Now, the crucial point is that the ground state metric is precisely the pullback of the universal one *via* the map ϱ. This is a consequence of the fact that the group G acts homogeneously on the ground state metric and hence g must correspond to the unique homogeneous one **. Since the universal one is known, we can reconstruct the full g out of the map ϱ. But the triangular part of ψ is sufficient to specify the map.

In fact ϱ is not just a holomorphic map, it is also horizontal. By this we mean that it satisfies eq. (A.7). Horizontal maps are very rigid. Then in various situations we have uniqueness theorems for the metric g. Using these results one can show, e.g. that the map ϱ is the direct sum of the periods maps for the projective manifolds E_1 and E_2 defined in (A.2) [14]. Here we want to exploit them to prove that the map ϱ defined by the SQM period map coincides (at criticality) with the one defined by the periods ϖ. A typical rigidity theorem for horizontal maps [24,40] states that two such maps are equal if: (i) they transform the same way under modular transformations and (ii) they agree at a single point in moduli space.

Then everything is proven if we can show that: (1) under a modular transformation the chiral primary fields transform as the periods ϖ (equivalently, as the periods for the projective manifolds E_i) and (2) that at a particular point in moduli space we have equality between the ground state metric and the metric computed out of the above integrals. Point (1) has been discussed in detail for $\hat{c} = 1$ in ref. [53]. The general proof is very easy. It is enough to check the equality of the monodromy action in the topological theory. In the topological case one can indeed identify the chiral operators with the integrals ϖ (see appendix C). Hence the equality is manifest. To show (2), we assume W to be homogeneous of degree d. Then we consider the family

$$\mathscr{W}(X_i, t; s) = sW(X_i, t) + (1-s)\sum_i X_i^d.$$

* ϱ is the period map in the Griffiths sense [40].

** The reader may wonder about the overall normalization of the metric. It is also fixed. Indeed, we know already that, restricting to marginal deformations, the metric is the curvature of a certain line bundle. Then its overall scale is fixed topologically.

It is enough to check equality at $s = 1$. In this case we end up with a bunch of decoupled A_{d-1} minimal models. For the A-series the equality was explicitly checked in ref. [14].

Appendix B. Semiclassical considerations

In this appendix we discuss the leading semiclassical corrections and show the result quoted in eq. (4.7). So we are interested in the limit where the superpotential λW has simple critical points which are very far from each other (in the limit of large λ) and to leading order decouple from one another.

On general grounds one can argue that the leading off-diagonal semiclassical correction to the metric, which to leading order is diagonal in the basis of critical points is a "universal" function of the mass of the soliton interpolating between critical points (in units of inverse length of the cylinder) if there is a soliton connecting the two points. The mass of the soliton has simple dependence on the superpotential and is given by

$$m = 2|\lambda||\Delta W|.$$

In the case of just one field, which we will mainly concentrate on, a precise statement of this universality is as follows *. Assume there is a convex domain $\tilde{\Omega} \subset \mathbb{C}$ containing only two (distinct) critical values $W(X_j)$ and $W(X_k)$. Suppose that there is a simply connected domain $\Omega \subset \mathbb{C}$ containing only two critical points ($\equiv$ classical vacua), X_j and X_k, such the $W(\Omega) = \tilde{\Omega}$. Finally, assume that the two Milnor vanishing classes associated with these critical points have an intersection number ± 1 (i.e. in the Dynkin diagram of the polynomial $W(X)$ the two points corresponding to X_j and X_k are connected by a single link). These conditions imply in particular that there exists a soliton connecting the critical points. As before le $|l_j\rangle$ label the critical point basis of chiral fields, i.e. up to topologically trivial terms they are eigenstates of X with eigenvalue X_j. Then, as $\lambda \rightarrow \infty$

$$|\lambda|\left[-W''(X_j)\overline{W}''(X_k)\right]^{1/2}\langle l_k|l_j\rangle = U(2|\lambda||W(X_k)-W(X_j)|) + O(\exp[-\mu|\lambda|]), \tag{B.1}$$

where

$$\mu = \min\left\{4\inf_{W\in\partial\tilde{\Omega}}|W-W(X_k)|,\ 4\inf_{W\in\partial\tilde{\Omega}}|W-W(X_j)|\right\}.$$

* More general arguments are available but, unfortunately, they do not give more detailed results.

and $U(m)$ is an universal function. Comparing with the known $W = X^3 - X$ case, we get

$$U(m) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\sqrt{p^2+m^2}} \exp\left(-\sqrt{p^2+m^2}\right) = -\frac{1}{\pi} K_0(m). \tag{B.2}$$

Then as $m \to \infty$ we have the asymptotical expansion

$$U(m) \sim -\frac{e^{-m}}{\sqrt{2\pi m}} \left\{ 1 + \sum_{k=1} \frac{(-1)^k}{k!} [(2k-1)!!]^2 \frac{1}{(8m)^k} \right\}.$$

Since $\hbar \sim \lambda^{-1}$, the various terms in this expansion can be seen as loop corrections to the one-instanton (soliton) process. It is remarkable that all the perturbative corrections are universal.

So stated, universality can be proven in many ways. We will concentrate on three different ways: The first, and the most direct way, is to use our equation (3.9) in the asymptotic region. The second, is to use WKB approximation to write down the overlap of wave functions based at different critical points – this can be done both in the path-integral language as an instanton sum or in the Schrödinger equation. The third one is not as rigorous, but has the advantage of giving the overall normalization in a simple way and suggesting a physical picture of how the corrections to the metric might be related to a kind of partition function in the soliton subsector *. This is very much in the spirit of the thermodynamic Bethe ansatz [54]. We will discuss these three different view points in turn. At the end of this appendix, as an example, we discuss the leading correction of the metric for $W = x^{n+1}/(n+1) - x$ in the asymptotic region.

We first show how this universality property can be shown starting from our basic equations (3.9). We present the details of the argument since it can be easily extended to prove more general "universality theorems" for multi-instanton processes. Assume that all the zeros of W' are simple. In this point basis, we rewrite the metric as

$$g = n \exp[\gamma] n^{\dagger},$$

where

$$n_k^h = \frac{\delta_k^h}{\sqrt{\lambda W''(X_k)}}.$$

At the classical level $\gamma = 0$. As $\lambda \to \infty$, γ is dominated by the (leading) 1-instanton contribution. Neglecting terms exponentially suppressed with respect to the leading instanton, we can work to first order in γ. In this approximation (3.9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}|\lambda|^2}\left(|\lambda|^2 \frac{\mathrm{d}}{\mathrm{d}|\lambda|^2} \gamma_{jk}\right) = |W(X_j) - W(X_i)|^2 \gamma_{jk}.$$

Putting

$$\gamma_{jk} = \gamma_{jk}(z_{jk}),$$

$$z_{jk} = 2|\lambda|\,|W(X_j) - W(X_k)|,$$

one gets

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z}\gamma(z)\right) = z\gamma(z). \tag{B.3}$$

The general solution to this equation (vanishing as $z \to \infty$) is

$$\gamma_{jk} = \beta_{jk} K_0(z_{jk}). \tag{B.4}$$

and universality is proven up to an overall constant β_{jk}. That this argument does not fix the overall constant was to be expected. In particular, in this argument we did not use the fact that there is a soliton connecting the critical points. If there were no solitons connecting the two critical points, the corresponding β_{jk} would have to vanish. However, in case there exists a soliton connecting the two critical points we would still like to determine the overall constant and show its universality. We accomplish this by showing that in such a case the constant β_{jk} is the same we got for the $X^3 - X$ model (which does have a soliton connecting the critical points).

Consider the auxiliary superpotential

$$W(X; s) = \mu_{kj} W_{kj}(X) + s\left[W(X) - \mu_{kj} W_{kj}(X)\right],$$

where

$$W_{kj}(X) = \tfrac{1}{3}X^3 - \tfrac{1}{2}(X_j + X_k)X^2 + (X_j X_k)X,$$

$$\mu_{kj} = \left[W(X_k) - W(X_j)\right] / \left[W_{kj}(X_k) - W_{kj}(X_j)\right].$$

As $s \to 1$ we get back the original superpotential $W(X)$, whereas for $s \to 0$ we get a cubic one *. Note that for this superpotential the mass of the soliton $2|\Delta W(X; s)|$ is independent of s.

* We wish to thank A.B. Zamolodchikov for encouraging us to take this interpretation seriously.

* The limit $s \to 0$ is not smooth in general (the Witten index jumps). However, the limit is smooth for the quantities of interest here.

Assume that $W(X)$ is such that, for λ large enough, we can consistently use the linearized approximation in the whole range $0 \leqslant s \leqslant 1$ (this in particular means that there is a soliton in the original theory at $s=1$). Then the linearized equations read

$$\partial_{\bar{s}}\, \partial_\lambda \gamma_{jk} = \partial_{\bar{\lambda}}\, \partial_s \gamma_{jk} = 0,$$

or, using eq. (B.4)

$$\partial_{\bar{s}} \beta_{jk} = \partial_s \beta_{jk} = 0.$$

Since β_{jk} is independent of s, it takes the same value as in the cubic case, namely $\beta_{jk} = -1/\pi$. It is easy to check this universality result in the models explicitly solved in the main body of the paper.

The second method uses WKB approximation. We first sketch the proof using SQM, omitting technicalities. One writes the restrictions to Ω of the wave functions associated to the states $|l_j\rangle$ as

$$\psi_j = \frac{1}{\sqrt{\lambda W''(X_j)}} f_j^* \psi_0 + \delta\psi_j,$$

where ψ_0 is a certain universal function and f_j is a model-dependent field-redefinition. $\delta\psi_j$ is the deviation with respect to exact universality. Then one uses residue-like techniques to rewrite

$$\|\delta\psi_j\|_{\mathrm{B}}^2 \equiv \int_{\mathrm{B}} |\delta\psi_j|^2 \quad (\mathrm{B}\text{ any domain in } \Omega),$$

in terms of the value of the wave function on the *boundary* of B. To evaluate the error one makes by replacing the true wave function ψ_j by its universal counterpart, we can use domains B such that their boundaries remain at a finite distance from the critical points. Then go to the semiclassical limit, $\lambda \to \infty$. We know that the WKB approximation to the wave functions is reliable in this limit only as long as we are away from the critical points. one cannot compute $\langle l_j | l_k\rangle$ directly by WKB methods, since there is a non-negligible contribution to this quantity from regions of radius $O(\sqrt{\hbar})$ around the critical points where WKB is totally unreliable. However, the tricks above guarantee that we can evaluate the error with respect to the universal answer using only the values of ψ_j away from the critical points. Therefore in the formula for the error we can use the WKB wave functions. In this way we get the result stated above. We will now investigate WKB approximation in more detail from a slightly different viewpoint and show why the leading semiclassical correction is of order

$$O\left(\frac{1}{\sqrt{2|\lambda|}\,|\Delta W|} \exp[-2|\lambda||\Delta W|]\right)$$

(unfortunately, we are not able to get the numerical coefficient in front by this method). This is a tricky point. Indeed at a first glance one would rather expect a vanishing result for $\langle l_j | l_k\rangle$ $(j \neq k)$. In fact, from the topological–anti-topological fusion point of view, ignoring the two hemispheres at the two ends and concentrating on the infinitely long intermediate cylinder with circumference β, one would (naively) identify $\langle l_j | l_k\rangle$ with

$$\mathrm{Tr}_{(j,k)}(-1)^F \exp[-\beta H],$$

the trace being over the soliton sector corresponding to the path integral with boundary conditions

$$X(+\infty) = X_j, \qquad X(-\infty) = X_k.$$

In the soliton subsector all state appear in supersymmetry multiplets (see e.g. ref. [33]) and due to the $(-1)^F$ in the above expression we seem to be getting zero. So it seems with this naive interpretation of the topological–anti-topological fusion we are getting a paradox.

The point is that the identification of $|l_j\rangle$ with the vacuum $|X_j\rangle$, corresponding to the boundary condition $X(\tau = -\infty) = X_j$, is correct only at $\hbar = 0$. Indeed, the "point" basis, which the topological theory gives, is defined as the one which diagonalizes $\mathscr{R}$, i.e. for any holomorphic function f

$$F(X)|l_j\rangle = f(X_j)|l_j\rangle + Q^+|\text{something}\rangle.$$

There is also an anti-point basis, obtained from the anti-topological theory, which diagonalizes the Q^--cohomology ring

$$F(\bar{X})|\bar{l}_j\rangle = f(\bar{X}_j)|\bar{l}_j\rangle + Q^-|\text{something}\rangle.$$

For $\hbar \neq 0$ $|\bar{l}_j\rangle \neq |l_j\rangle$ because the chiral and anti-chiral rings cannot be diagonalized simultaneously. Instead, the definition of the vacua $|X_j\rangle$ is symmetric between Q^+- and Q^--cohomology and hence it is real with respect the real structure M. In other words, the state $|X_j\rangle$ is a "real" admixture of topological and anti-topological states. The correct identification has the general form (using results of sect. 5)

$$2|X_j\rangle = \sqrt{\lambda W''(X_j)}\,|l_j\rangle + \left(\sqrt{\lambda W''(X_j)}\right)^* m_j^{\;k}\,|l_k\rangle$$
$$+ \text{sub-leading instanton corrections.} \qquad \text{(B.5)}$$

Susy predicts $\langle X_j | X_k\rangle = 0$ for $j \neq k$. This is consistent with eq. (B.5). Indeed

$$\langle X_j | X_k\rangle = \tfrac{1}{4}\left[(e^\gamma)_{kj} + (e^\gamma)_{jk} + 2\delta_{jk}\right] + \ldots = \delta_{jk} + O(\gamma^2).$$

and hence (at least at the one-instanton level) there is no tunnelling between distinct classical vacua $|X_j\rangle$. Therefore $\langle l_j | l_k \rangle$ is non-vanishing not because there is a "physical" tunnelling process but because the topological states $|l_j\rangle$ are combinations of different classical vacua.

Despite the fact that $\langle l_j | l_k \rangle$ is not an instanton tunnelling amplitude in an obvious sense its evaluation is quite reminiscent of an instanton computation. We will now make this connection a little more clear. Our finding supports the idea that *loop* corrections in an instanton background is responsible for the leading semiclassical correction to the metric. For the sake of comparison, we recall what we would have found in an actual instanton computation. We would get a factor $\exp[-2|\lambda||\Delta W|]$ from the classical action, a factor $\sqrt{4\pi|\lambda||\Delta W|}$ from the integration over the position of the center of the instanton, no determinant factor (by susy) and, unless we soak them up, a factor 0 from the Fermi zero-modes.

For definiteness we consider the model $W=(X^3/3-X)$, and compute $\langle l_1 | l_2 \rangle$ as $\lambda \to \infty$. There are two (equivalent) techniques available, one can use WKB either in the path integral or in the Schrödinger equation. We choose the second one since using explicit wave functions the identification of the various vacuum states is simpler. In this framework, $\langle l_1 | l_2 \rangle$ is just the overlap integral for the two vacua. However as mentioned above there is a difficulty. In SQM we compute such overlaps by residue techniques. This requires only the knowledge of the leading behaviour of the wave functions at the critical points of W. But these are precisely the points where the WKB approximation breaks down! In other words, for the vacuum wave functions the limits $X \to X_j$ and $\hbar \to 0$ do *not* commute. This is why making reliable semiclassical computations is very hard. Of course, we can try to compute the overlap by integrating the WKB wave functions in the region where they can be trusted but, as we shall see, this will give us only a rough estimate of the amplitude.

We parametrize the wave form corresponding to l_1 as

$$\omega_1 = \frac{1}{\sqrt{2\pi}} \frac{e^{-2|\lambda||W(X)-W(1)|}}{\sqrt{2|\lambda||W(X)-W(1)|}} \left[\lambda\phi_1(X)\, dW + \bar\lambda\phi_2(X)\, d\overline{W}\right].$$

From the Schrödinger equation we know that the functions $\phi_i(X)$ have the properties

$$\phi_1(1+\epsilon\, e^{i\theta}) \simeq e^{-i\arg[W''(1)]/2}\, e^{-i\theta} + \dots,$$

$$\phi_2(1+\epsilon\, e^{i\theta}) = -\left[\phi_1(1+\epsilon\, e^{i\theta})\right]^* + \dots.$$

Moreover, WKB methods give

$$|\phi_i| = 1 + O(1/|\lambda|) \tag{B.6}$$

both near the critical point $X=1$ and in the region where

$$|\lambda||W(X)-W(1)| \gg 1,$$

provided we are away from the other critical point by at least $O(1/|\lambda|)$. It is crucial that the $1/|\lambda|$ corrections in eq. (B.6) cannot vanish identically.

The wave form for $|l_2\rangle$ is

$$\omega_2 = \frac{1}{\sqrt{2\pi}} \frac{e^{-2|\lambda||W(X)+W(1)|}}{\sqrt{2|\lambda||W(X)+W(1)|}} \left[\lambda\tilde\phi_1(X)\, dW + \bar\lambda\tilde\phi_2(X)\, d\overline{W}\right].$$

by "functoriality"

$$\tilde\phi_1(X) = i\phi_1(-X), \qquad \tilde\phi_2(X) = -i\phi_2(-X).$$

The idea is to evaluate the overlap by integrating only over the intermediate region between the two critical points where (apart for points very near the critical ones) the WKB functions are reliable enough. This region dominates the integral. We must compute

$$\int *\,\bar\omega_1 \wedge \omega_2 = \text{const.}|\lambda| \times \int\left[\phi_1^*\tilde\phi_1 + \phi_2^*\tilde\phi_2\right] \times \frac{\exp[-2|\lambda|(|W(X)-W(1)|+|W(X)+W(1)|)]}{\sqrt{|W(X)^2-W(1)^2|}}\, d^2W.$$

The argument of the exponential is of order λ. Since we are interested in $\lambda \to \infty$, we can evaluate this integral by saddle-point methods. In other words, the integral is dominated by the minima of the "action". It is convenient to work in the W-plane. In this plane the "action" at a given point is the sum of the distances from the points $W(1)$ and $-W(1)$, and hence it is minimal along the segment connecting these two critical values. Then, in doing the d^2W integral, we integrate in d(Re W) between $-W(1)$ and $W(1)$, whereas we use the gaussian approximation for the integral in d(Im W). To quadratic order in Im W the exponential is

$$\exp\left[-4|\lambda||W(1)| - 2|\lambda||W(1)|\frac{(\text{Im}\, W)^2}{W(1)^2 - (\text{Re}\, W)^2}\right].$$

Integrating over d(Im W) we get

$$\text{const.}\sqrt{|\lambda|}\, e^{-4|\lambda||W(1)|}\int_{-W(1)}^{W(1)}\left[\phi_1^*\tilde\phi + \phi_2^*\tilde\phi_2\right] d(\text{Re}\, W).$$

This formula is consistent with instanton physics. Apart for the factor involving the ϕ's (related to the fermionic part of the wave function and the sub-leading WKB corrections) this is what we expect: a factor $\exp[-S]$ from the classical action and a factor $\sim\sqrt{|\lambda|}$ from the integration over the collective coordinate. Moreover, the computation realizes manifestly the idea [33] that the soliton is the segment in the W-plane connecting the two critical values. The phases of the ϕ's are such that on this segment one has

$$\phi_1^*\bar{\phi}_1+\phi_2^*\bar{\phi}_2=O(1/|\lambda|).$$

The fact that to leading order this vanishes just reflects the presence of Fermi zero-modes. However, the sub-leading terms need not vanish (in fact, the Schrödinger equation suggests they are not zero). Then we get

$$\langle I_1|I_2\rangle=O\left(\frac{1}{\sqrt{|\lambda|}}e^{-4|\lambda||W(1)|}\right), \tag{B.7}$$

as claimed. The constant in front cannot be computed by these methods both because the sub-leading corrections are poorly understood and because regions where WKB fails may also give contributions of this magnitude. Anyhow, this constant is predicted by our differential equations.

The third idea in getting this universal result is suggested by the form (B.2) that we wrote the universal correction to the metric in. Indeed $U(m)$ is related to the contribution of a single particle of mass m in two space-time dimensions to $\mathrm{Tr}\exp(-\beta H)$ (where we fix a point in space in taking the trace) *, where m is the mass of the soliton connecting the two critical points and we have set $\beta=1$. Note that in particular the normalization (up to the phase) is easily predicted in this way. So this means that the naive picture of soliton partition function, which led to the paradox mentioned above, is essentially right, but with taking the contribution of one soliton from each supersymmetry multiplet to $\mathrm{Tr}[(-1)^F\exp(-\beta H)]$ to avoid vanishing. Somehow the loop corrections to the instantons are responsible for giving this "effective" soliton description. It would be worthwhile understanding this connection more clearly. In particular this may allow one to compute the scattering matrices of solitons from solutions to our equations using the thermodynamic Bethe ansatz. In fact the asymptotic solution to PIII equation, given in the second reference in [47] can presumably be interpreted as giving an *exact* multi-soliton contribution to the $\mathrm{Tr}\exp(-\beta H)$ for the $\lambda(X^3/3-X)$ model (and similarly for the Chebyshev case). In particular the quantity defined in eq. (1.4a) of that

* We would like to thank P. Fendley and K. Intriligator for a discussion on this point.

reference which is simply related to our functions can be viewed as computing the contribution of soliton in the form

$$G=\sum_{n=0}^{\infty}g_{2n+1}, \tag{B.8}$$

where g_{2n+1} (after specializing to our case and a suggestive redefinition of variables) takes the form

$$g_{2n+1}=\int\prod_{i=1}^{2n+1}\frac{\mathrm{d}p_i\exp\left(-\sqrt{p_i^2+m^2}\right)}{2\pi\sqrt{p_i^2+m^2}}\left[\prod_{i=1}^{2n}\left(\sqrt{p_i^2+m^2}+\sqrt{p_{i+1}^2+m^2}\right)^{-1}\prod_{i=1}^{n}(p_{2i}^2)\right].$$

which should clearly have the interpretation of the contribution of $2n+1$ solitons whose contribution to the partition function has been modified from the free case by the presence of "interaction" encoded in the above equation by the term inside [...]. It would be interesting to connect this to the S matrix of the $N=2$ theories computed in ref. [33], using ideas similar to thermodynamic Bethe ansatz.

As another example let us consider

$$W=\frac{x^{n+1}}{n+1}-x$$

considered in this paper. Let $|\bar{I}_r\rangle$ denote the critical points of W as r runs from 0 to $n-1$ with an appropriate phase factor to cancel the hessian term appearing in eq. (B.1). Let $|x^r\rangle$ denote the usual chiral basis for the vacua. Let $\omega=\exp(2\pi i/n)$. We have

$$|x^s\rangle=\frac{1}{\sqrt{n}}\sum_{r=0}^{n-1}\omega^{r(s+1/2)}|\bar{I}_r\rangle.$$

Using eq. (B.1) we see that the phase of the leading correction to $\langle\bar{I}_r|\bar{I}_{r+1}\rangle$ is i, and its absolute value is $\exp(-m)/\sqrt{2\pi m}$, where m is the mass of the soliton connecting the nearest critical points

$$m=2|\lambda(W(r+1)-W(r))|=4|\lambda|\sin\pi/n.$$

Computing q_i defined in sect. 7, as logarithm of $\langle x^i|x^i\rangle$, we see from the above that (for $n>2$)

$$q_i\sim\frac{-2\sin\left[\frac{2\pi}{n}(i+\frac{1}{2})\right]\exp\left(-4|\lambda|\sin\frac{\pi}{n}\right)}{\sqrt{8\pi|\lambda|\sin\frac{\pi}{n}}}.$$

It is easy to check that to leading order this satisfies eq. (7.4), where z defined there is the same as λ here.

Appendix C. Special coordinates and all that

In this paper we used a coordinate-independent formulation of generalized special geometry. However, in the physics literature it is more usual to formulate this geometry using some special coordinates in which the formulae look quite simpler. The only drawback of these coordinates is that one has to work hard just to *define* them. In this appendix we describe the construction of such coordinates in our framework and use them to simplify the proof of some technical results we claimed in the main body of the paper. To avoid all misunderstandings, we use Greek letters to label the various chiral fields in the model.

The basic formula, arising from SQM perturbation theory, is (cf. subsect. 9.1 of ref. [5])

$$D_a\phi_k = \partial_\alpha\sigma^\alpha_{ak} + T^h_{ak}\phi_h, \tag{C.1}$$

where

$$\sigma^\alpha_{ak}\,\partial_\alpha W = \partial_a W\phi_k - C^h_{ak}\phi_h,$$

and T_a is the "torsion". The two terms in the r.h.s. of eq. (C.1) have very different origins. The first is the true variation of the topological operator whereas the torsion arises because of the special representatives of BRST-classes one needs to use in order to get the actual vacuum states *.

T_a has the form

$$T_a = [Z, C_a],$$

with

$$\partial_{\bar a} Z = -\bar C_a, \qquad Z\eta = \eta Z^{\mathrm T}.$$

Hence,

$$T_a\eta = -\eta T^{\mathrm T}_a, \qquad \partial_{\bar a}T_b = -\left[\bar C_a, C_b\right]. \tag{C.2}$$

The first of eqs. (C.2) justifies the name torsion for T_a: It is the antisymmetric part (with respect to η) of the connection. The second one shows that our curvature originates from the torsion. In fact

$$[\partial_{\bar a}, D_b]\phi_k = \partial_{\bar a}(D_b\phi_k) = (\partial_{\bar a}T_b)_k{}^h\phi_h.$$

Now, consider the connection *

$$\mathscr{D}_a = D_a - T_a \quad (= \partial_a - \mathscr{A}_a).$$

With respect to g, $\mathscr{D}_a$ is not metric any longer. But it is still metric for η. This was to be expected since from a purely topological point of view the two connections differ only by a gauge transformation. Next we consider a "curved" basis for $\mathscr{R}$, i.e. of the form

$$\phi_a = \partial_a W.$$

Then one has

$$\sigma^\alpha_{ab}\partial_\alpha W = \partial_a W\partial_b W - C^c_{ab}\partial_c W, \tag{C.3}$$

thus $\sigma^\alpha_{ab} = \sigma^\alpha_{ba}$, or

$$\mathscr{D}_a\phi_b = \mathscr{D}_b\phi_a.$$

Moreover,

$$\mathscr{D}_a\phi_b = \partial_a\partial_b W - \mathscr{A}^c_{ab}\partial_c W,$$

which gives

$$\mathscr{A}^c_{ab} = \mathscr{A}^c_{ba}.$$

Thus $\mathscr{A}$ is torsionless. Then it is the Christoffel connection of η. Let us compute its Riemann curvature. One has

$$[\mathscr{D}_a, \mathscr{D}_b]\phi_c = \partial_\alpha(\mathscr{D}_a\sigma^\alpha_{bc} - \mathscr{D}_b\sigma^\alpha_{ac}). \tag{C.4}$$

From eq. (C.3) one has

$$(\mathscr{D}_a\sigma^\alpha_{bc} - \mathscr{D}_b\sigma^\alpha_{ac})\partial_\alpha W = \partial_\alpha\left[(\phi_b\sigma^\alpha_{ac} + C_{ac}{}^d\sigma^\alpha_{bd} - (b \leftrightarrow a)\right],$$

$$(\phi_b\sigma^\alpha_{ac} + C_{ac}{}^d\sigma^\alpha_{bd})\partial_\alpha W = \phi_a\phi_b\phi_c - (C_aC_b)_c{}^e\phi_e.$$

* Here the tricky point is that, since Q^+ depends on t_a, the derivative of a Q^+-exact state is not Q^+-exact in general. Then computing the derivatives the actual representatives matter. In the definition of D_a they are uniquely fixed by the vacua. This is why a torsion appears.

* In ref. [5] it was shown that $\mathscr{D}_a$ is the Gauss–Manin connection in the sense of versal deformations of a given singularity.

Then the r.h.s. of eq. (C.4) is in the jacobian ideal, and hence the curvature vanishes. Then we can find (local) coordinates t_a such that

$$\eta = \text{const.} \qquad \mathscr{A} = 0.$$

This result is a standard mathematical fact [55]. These are the so-called special coordinates. They are characterized by

$$\partial_a \partial_b W = \partial_a \sigma^{\sigma}_{ab} \tag{C.5}$$

with σ^{σ}_{ab} as in eq. (C.3). Before going to more useful characterizations, let us show that for $n = 1$ this formula reproduces the results obtained in ref. [10] by KdV flows considerations.

In the one-field case

$$\sigma_{ab} W' = \partial_a W \partial_b W - C^c_{ab} \partial_c W,$$

or

$$\sigma_{ab} = \left(\frac{\partial_a W \partial_b W}{W'} \right)_+,$$

where $(\dots)_+$ means the non-negative part. Then eq. (C.5) becomes

$$\partial_a \phi_b = \partial_X \left(\frac{\phi_a \phi_b}{W'} \right)_+$$

which is equivalent to eq. (4.45) in ref. [10].

Put

$$\varpi^{\pm}_{aj} = \int_{\gamma_j^{\pm}} e^{\pm W} \partial_a W \, dX_1 \wedge \dots \wedge dX_n. \tag{C.6}$$

Using eq. (C.5) we find

$$\partial_a \varpi^{\pm}_{bj} = \pm C^c_{ab} \varpi^{\pm}_{cj}. \tag{C.7}$$

This is a characterization of special coordinates which is more convenient for computations. Since $\det[\varpi^+] \neq 0$, we can define the matrix C^{jk} by

$$C = (\varpi^+)^{-1} \eta \left[(\varpi^-)^{\mathrm{T}} \right]^{-1}.$$

Then from eq. (C.7)

$$\partial_a C = (\varpi^+)^{-1} \left[\eta C_a^{\mathrm{T}} - C_a \eta \right] \left[(\varpi^-)^{\mathrm{T}} \right]^{-1} = 0.$$

Then we have the general formula for the residue pairing

$$\eta_{ab} = \varpi^+_{aj} C^{jk} \varpi^-_{bk} \tag{C.8}$$

with C^{jk} a constant matrix. Now we can show that this matrix is precisely the intersection discussed in sect. 4. In fact, we show it for the "good" cases, where in the UV limit we get a non-degenerate quasi-homogeneous W, although it is plausibly true in general. Since C^{jk} does not depend on λ, we can limit ourselves to quasi-homogeneous W, and hence to homogeneous ones. Then we consider the homogeneous superpotential

$$\mathscr{W}(X, t; s) = sW(X, t) + (1-s) \sum_i X_i^d.$$

C^{jk} is independent of s. So we can compute it for $s = 0$, i.e. it is enough to show our statement for Fermat W's. In this case our periods factorize into the product of A_{d-1} minimal model periods. That in this last case C^{jk} is the inverse intersection matrix can be seen by a direct computation.

We end this appendix by showing that our "perturbative" characterization of the special coordinates agrees with the mathematical one [28,55]. Indeed, define

$$u_{kj}(\lambda) = \frac{1}{2\pi i} \int_{\infty}^{0} dg \, g^{\lambda} \varpi_{kj}(g),$$

where

$$\varpi_{kj}(g) = \int_{\gamma_j(g)} e^{gW} \phi_k \, dX_1 \wedge \dots \wedge dX_n.$$

($\varpi^{\pm} \equiv \varpi(\pm)$). Eq. (C.7) generalizes to

$$\partial_a \varpi_b(g) = g C^c_{ab} \varpi_c(g).$$

Taking the Mellin transform, in terms of $u_{kj}(\lambda)$ this becomes eq. (55) of ref. [28].

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