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N=2 SUPERSYMMETRIC INTEGRABLE MODELS AND TOPOLOGICAL FIELD THEORY

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N=2 SUPERSYMMETRIC INTEGRABLE MODELS AND TOPOLOGICAL FIELD THEORY.

Since I am the first speaker on the subject of N=2 supersymmetric theories, it means that I have the task of introducing the subject, and laying the ground work for later speakers, Nemeschansky and Vafa. My intention is to discuss the following topics

N=2 SCFT; C.R. & LG

(i) N=2 superconformal field theories, — chiral rings & Landau Ginzburg
(ii) Topologically twisted N=2 superconformal field theories
(iii) Perturbed N=2 superconformal models,
Bogomolny bounds on kink masses
(iv) Effective Landau-Ginzburg potentials via perturbations of topological models
(v) N=2 supersymmetric integrable models.
Soliton structure
(vi) Scattering matrices — Nemeschansky

The techniques that I will describe have many applications: string theory, mirror symmetry, topological field theory, integrable models, exactly solved lattice models, and even polymer physics.

It is therefore my intention to start by giving a hopefully elementary discussion of the technology of $N=2$ superconformal field theory and its perturbations, and then move off in the direction of integrable field theories. This choice reflects a combination of my personal interest, and a desire to mesh with the themes of this conference — the other areas of research cited above are also extremely active.

In an $N=2$ superconformal field theory the energy momentum tensor, $T(z)$, is supplemented by three other generators: G^+, $G^-(z)$ and $J(z)$ of ~~spins~~ conformal weights $+3/2$, $+3/2$ and 1 respectively. These have operator product expansions:

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots$$

$$T(z)\,J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \cdots$$

$$T(z)\,G^\pm(w) = \frac{3/2\, G^\pm(w)}{(z-w)^2} + \frac{\partial_w G^\pm}{(z-w)} + \cdots$$

$$J(z)\,G^\pm(w) = \pm\frac{G^\pm(w)}{(z-w)} + \cdots$$

$$J(z)\,J(w) = \frac{c/3}{(z-w)^2} + \cdots$$

$$G^\pm(z)\,G^\mp(w) = \frac{2/3\, c}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm \partial_w J(w)}{(z-w)} + \cdots$$

$$G^\pm(z)\,G^\pm(w) = 0 + \cdots$$

where $+ \cdots$, means plus terms that are finite as $z \to w$.

Note the presence of the combination $2T(w) \pm \partial_w J(w)$ in $G^\pm(z)\,G^\mp(w)$. This will be important later

One can pass to modes (on the plane | sphere)

$$T(z) = \sum_{n=-\infty}^{\infty} L_n\, z^{-n-2}$$

$$J(z) = \sum_{n=-\infty}^{\infty} J_n\, z^{-n-1}$$

$$G^\pm(z) = \sum_{n=-\infty}^{\infty} G^\pm_{n\pm a}\, z^{-(n\pm a)-3/2}$$

The parameter $a \in \mathbb{R}$, and determines the branch cut properties of $G^{\pm}(z)$. The choice $a=0$ is usually called the Ramond sector, and $a=\frac{1}{2}$ is the Neveu Schwarz sector. I will ~~simplify~~ simplify my life here by working in the Neveu-Schwarz sector – but one should bear in mind that parallel results can be obtained by changing the value of a.

In modes one has:

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12} m(m^2-1)\,\delta_{m+n,0}$$

$$[L_m, J_n] = -n\,J_{m+n}$$

$$[L_m, G_r^{\pm}] = \left(\frac{m}{2} - r\right) G^{\pm}_{m+r}$$

$$[J_m, G_r^{\pm}] = \pm\, G^{\pm}_{m+r}$$

$$\{G_r^{+}, G_s^{-}\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left[r^2 - \tfrac{1}{4}\right]\delta_{r+s,0}$$

where $r, s \in \mathbb{Z} + \frac{1}{2}$

Primary fields $\Psi(z)$ satisfy:

$$T(z)\,\Psi(w) = \frac{h}{(z-w)^2}\Psi(w) + \frac{\partial_w \Psi(w)}{z-w} + \cdots$$

$$J(z)\,\Psi(w) = \frac{q}{(z-w)}\Psi(w) + \cdots$$

$$G^{\pm}(z)\,\Psi(w) = \frac{\Lambda^{\pm}(w)}{(z-w)} + \cdots$$

where $\Lambda^{\pm}(w)$ are two superpartners of $\Psi(w)$

In modes

$$L_n|\Psi\rangle = J_n|\Psi\rangle = 0 \qquad n \geq 1$$
$$G_r^{\pm}|\Psi\rangle = 0 \qquad r \geq \tfrac{1}{2}$$

$$L_0|\Psi\rangle = h|\Psi\rangle, \quad J_0|\Psi\rangle = q|\Psi\rangle$$

$$G^{\pm}_{-1/2}|\Psi\rangle = |\Lambda^{\pm}\rangle$$

Unitary reps : $c \geq 3$ or

$$c = 3 - \frac{6}{k+2} \quad ; \quad k = 1, 2, 3 \cdots$$

minimal models.

Minimal models – finitely many irreducible highest weight reps. All states are obtained by operating on a state $|h_0, q_0\rangle$ with polynomials in $G^+_{-r}, G^-_{-s}, L_{-n}$ or J_{-n}

Allowed values of h_0, q_0 .

$$h_0 = \frac{l(l+2)-m^2}{4(k+2)} \quad , \quad q_0 = \frac{m}{k+2}$$

$$l = 0, 1, 2, \ldots, k \quad ; \quad m = -l, -l+2, \ldots, l-2, l.$$

General models again:

Unitarity bounds :— Let $|\psi\rangle$ be any state:

N.B. $\Rightarrow$ $\boxed{(G^{\pm}_{r})^{+} \equiv G^{\mp}_{-r}}$

$$0 \leq \left| G^{\pm}_{-1/2}|\psi\rangle\right|^2 + \left|G^{\mp}_{+1/2}|\psi\rangle\right|^2$$

$$= \langle\psi|\left(G^{\mp}_{+1/2}G^{\pm}_{-1/2} + G^{\pm}_{-1/2}G^{\mp}_{-1/2}\right)|\psi\rangle$$

$$= (2h_\psi \mp q_\psi)\langle\psi|\psi\rangle \quad \text{—} \quad (*)$$

$$\boxed{h \geq \tfrac{1}{2}|q| \quad \text{for all states}}$$

Special subset of States — the Chiral Ring

$\phi(z)$ will be called a chiral, primary field if $\phi(z)$ is primary, and

$$(G^+_{-1/2}\ \phi)(z) \equiv 0 \leftarrow$$ chiral means killed by $G^+_{-1/2}$

The set of such states will be denoted by $\mathcal{R}$.

Fact: (i) An alternative characterization of $\mathcal{R}$ is that it is precisely the set of states for which

$$\boxed{h = \tfrac{1}{2} q}$$

To see this, consider the equation (*)

Clearly, chiral, primary $\Longrightarrow$ $h = \tfrac{1}{2} q$

Conversely, suppose that $h = \tfrac{1}{2} q$, then from (*) one must have $G^+_{-1/2}|\psi\rangle = G^-_{1/2}|\psi\rangle = 0$

Now consider $X^\pm_r \equiv G^\pm_r |\psi\rangle$

$$q(X^\pm_r) = q \pm 1 \quad , \quad h(X^\pm_r) = h - r = \tfrac{1}{2}(q - 2r)$$

One can see that such states violate

$$h(X^\pm_r) \geqslant \tfrac{1}{2} q(X^\pm_r)$$

unless

$$G_r^+ \, |\psi\rangle = 0 \qquad r \geq \tfrac{1}{2}$$
$$G_r^- \, |\psi\rangle = 0 \qquad r \geq \tfrac{3}{2}$$

So we have $G_r^{\mp} \, |\psi\rangle = 0 \quad r \geq -\tfrac{1}{2}$

$$G_r^- \, |\psi\rangle = 0 \quad r \geq \tfrac{1}{2}$$

To get

$$L_n |\psi\rangle = J_n |\psi\rangle = 0$$

Consider $\{G_r^+, G_s^{\mp}\} |\psi\rangle = 0$ for $r \geq -\tfrac{1}{2}$, $s \geq \tfrac{1}{2}$

(ii) R has a natural ring structure:

$$\overset{h_1}{\phi_1}(z)\, \overset{h_2}{\phi_2}(w) = \cdots + (z-w)^{h_\psi - h_1 - h_2} \psi(w) + \cdots$$

chiral, primary $h_1 = \tfrac{1}{2} q_1$, $h_2 = \tfrac{1}{2} q_2$

$$2h_\psi \geq q_\psi = q_1 + q_2 = 2(h_1 + h_2)$$

$\Rightarrow \quad h_\psi - h_1 - h_2 \geq 0$ with equality <u>if and only if</u> $h_\psi = \tfrac{1}{2} q_\psi$

$\Leftrightarrow \psi$ is chiral primary.

Define the product on R

$$\phi_1 \cdot \phi_2 = \lim_{z \to w} \phi_1(z)\, \phi_2(w)$$

then either $\phi_1 \cdot \phi_2 \in R$ or $\phi_1 \cdot \phi_2 \equiv 0$.

The ring R may thus be thought of as a polynomial ring. There will be generating fields $x_i \in R$ such that

$$R \equiv \frac{\{\text{Polynomials in } x_i\}}{\mathcal{I}} \equiv \mathcal{P}(x_i)$$

$\mathcal{I} \equiv$ vanishing relations — an ideal of $\mathcal{P}(x_i)$ consisting of all polynomials in x_i that vanish in the product defined above.

Example minimal models ← ground states

(No excited states contribute to R)

$\Phi^l_m(z) \Leftrightarrow$ $h = \frac{l(l+2) - m^2}{4(k+2)}$, $q = \frac{m}{(k+2)}$

$c = 3 - \frac{6}{k+2}$ $h - \frac{1}{2}q = \frac{(l+1)^2 - (m+1)^2}{4(k+2)} = 0 \Rightarrow \underline{m = l}$

Let $x = \Phi^1_1$. One can show that $\Phi^l_l \equiv x^l$ in the obvious sense.

Recall however that $\ell = 0, 1, \ldots, k$.

One can ~~also~~ verify that $x^{k+1} = 0$

$$x^{k+p} \equiv 0 \qquad p \geqslant 1$$

$$\Rightarrow \quad \mathcal{R} \equiv \frac{\text{polynomials in } x}{\{x^{k+p} = 0, \; p \geqslant 1\}}.$$

$$\equiv \{1, x, \ldots, x^k\}.$$

Other facts

purely bosonic

will not be proved here — need spectral flow.

(iii) There is a unique element of $\mathcal{R}$ with maximal $U(1)$ charge, $q = c/3$.

This state, $|\rho\rangle$, also satisfies

$$G^+_{-3/2} |\rho\rangle = 0. \qquad \left(\begin{array}{c}\text{as well as being}\\ \text{chiral and primary}\end{array}\right)$$

Proof:
$$0 \leq |G^+_{-3/2} |\psi\rangle|^2 + |G^-_{3/2} |\psi\rangle|^2$$

$$= \langle\psi| \{G^+_{-3/2}, G^-_{3/2}\} |\psi\rangle$$

$$= \left(2h_\psi - 3q_\psi + \tfrac{2c}{3}\right) \langle\psi|\psi\rangle$$

$$= 2\left(\tfrac{c}{3} - q_\psi\right) \langle\psi|\psi\rangle$$

where I have used $q_\psi = 2h_\psi$.

(iv) There are only finitely many chiral primary fields.

(v) Hodge decomposition.

Given any state, $|\psi\rangle$, there is a decomposition of $|\psi\rangle$ according to

$$|\psi\rangle = G^{+}_{-1/2}\,|\psi_1\rangle + G^{-}_{1/2}\,|\psi_2\rangle + |\phi\rangle$$

where $|\phi\rangle$ is chiral primary. Moreover, if $|\psi\rangle$ is chiral (but not necessarily primary) i.e. $G^{+}_{-1/2}\,|\psi\rangle = 0$, then $G^{-}_{1/2}|\psi_2\rangle = 0$.

Proof Define

$$|\chi\rangle \equiv |\psi\rangle - G^{+}_{-1/2}\,|\chi_1\rangle - G^{-}_{1/2}\,|\chi_2\rangle$$

where χ_1, χ_2 are, at present, arbitrary states.

Choose $|\chi_1\rangle$ and $|\chi_2\rangle$ so as to minimize the norm of $|\chi\rangle$. I now show that $|\chi\rangle$ is chiral primary. Let $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ be any two states and consider

$$|\tilde{\chi}\rangle \quad |\chi\rangle + z_1 G^{+}_{-1/2}|\varepsilon_1\rangle + z_2 G^{-}_{1/2}|\varepsilon_2\rangle$$

One must have

$$\left.\begin{aligned} \frac{\partial}{\partial z_i}\langle\tilde{\chi}|\tilde{\chi}\rangle &= 0 \\ \text{and} \quad \frac{\partial}{\partial \bar{z}_i}\langle\tilde{\chi}|\tilde{\chi}\rangle &= 0 \end{aligned}\right\} \quad i=1,2$$

for $|\chi\rangle$ to be of minimal norm.

$$\Rightarrow \quad \langle\varepsilon_1|G^{+}_{-1/2}|\chi\rangle = \langle\varepsilon_2|G^{-}_{1/2}|\chi\rangle = 0$$

for any $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$

$$\Rightarrow \quad G^{+}_{-1/2}|\chi\rangle = G^{-}_{1/2}|\chi\rangle = 0$$

$$\Rightarrow \quad \{G^{-}_{1/2}, G^{-}_{1/2}\}|\chi\rangle = 0$$

$$\Rightarrow \quad h_\chi = \tfrac{1}{2}q_\chi .$$

as required.

If $G^{+}_{-1/2}|\psi\rangle = 0$ then

$$G^{+}_{-1/2}G^{-}_{1/2}|\psi_2\rangle = 0$$

$$\Rightarrow 0 = \langle\psi_2|G^{+}_{-1/2}G^{-}_{+1/2}|\psi_2\rangle = |G^{-}_{+1/2}|\psi_2\rangle|^2$$

$$\Rightarrow \quad G^{-}_{1/2}|\psi_2\rangle = 0$$

Notes: (i) Anti-chiral primary fields are primaries that satisfy $G^{-}_{-\frac{1}{2}}|\psi\rangle = 0$ (anti-chiral)

$\Leftrightarrow$ they are states with $h = -\frac{1}{2}q$.

Everything said above goes through for antichiral fields.

Notation: $\bar{\phi}$ — antichiral, primary field

(ii) Everything I have said was in the holomorphic sector of the theory. The same may be done in the anti-holomorphic sectors.

Notation: $\tilde{G}^{\pm}(\bar{z})$, $\tilde{J}(\bar{z})$, $\tilde{T}(\bar{z})$

(iii) In a $N=2$ superconformal field theory, the chiral ring, R, is made up of scalar fields (i.e. $h = \tilde{h}$). So one pairs $\phi(z)$ with $\tilde{\phi}(\bar{z})$ to make the scalar chiral primary, $\phi(z,\bar{z})$. It is really these fields that define the chiral ring of the theory.

(iv) There a two rings in any given theory. One can pair (chiral, chiral) (chiral, antichiral), (a, c) and (a, a)

The last two are trivial CPT conjugates of the first two. It is this rather simple observation, and the attempt to find geometrical interpretations of these rings, that leads to mirror symmetry.