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SMR.626 - 31

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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LECTURES ON COSMOLOGY

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Robert Schaefer

Cosmology

2. Standard Background Cosmology

The fundamental relations for the standard big bang model of the universe come from the Einstein equations

$$R^{\mu}{}_{\nu} - \delta^{\mu}_{\nu} R = 8\pi G T^{\mu}{}_{\nu}$$

where $R^{\mu}{}_{\nu}$ is the Ricci tensor which contains only the metric and metric derivatives, $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature, and $T^{\mu}{}_{\nu}$ is the energy momentum tensor for the matter, and G is the Newtonian gravitational constant. $T^{\mu}{}_{\nu}$ is also conserved:

$$T^{\mu}{}_{\nu;\mu} = 0.$$

~~On the~~ In an average sense, the universe is assumed to be homogeneous and isotropic. Strong observational support for this assumption can be found in the uniformity of the cosmic microwave background radiation, which is ~~uniform~~ constant to 1 part in $\sim 10^5$ over the entire sky. The notion of homogeneity and isotropy has been elevated to a principle "The Cosmological Principle." Using this as our guide constrains the metric to the following form called the

"Robertson-Walker" metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(\sin^2\theta\, d\theta^2 + d\phi^2)\right]$$

where K is the ~~mod~~ scalar curvature of space, and $a(t)$ is a function of time only which describes the expansion of space with time. We can then ~~have~~ make the distinction between physical coordinates $\vec{R}$ and comoving coordinates $\vec{r}$:

$$\vec{R} = a(t)\,\vec{r}$$

comoving coordinates are useful for separating out the general expansion of the universe

To illustrate the effect of expansion let us note the velocity of a test particle e.g. a galaxy at a distance R from us

$$\vec{V} = \frac{d}{dt}\vec{R} = \frac{d}{dt}(a\vec{r})$$

$$= \frac{\dot{a}}{a} a\vec{r} + a\dot{\vec{r}}$$

the first term is the velocity of coordinate expansion and the other is called the peculiar velocity. If the comoving distance does not change $\dot{r}=0$ then

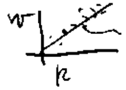
$$\vec{V} = \frac{\dot{a}}{a}\vec{R}$$

and we find at any given time velocity is proportional to ~~the~~ distance

$v \propto \vec{R}$! Hubble law

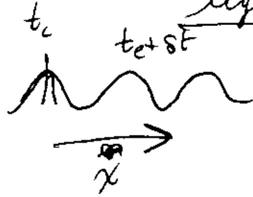
$$v_0 = \frac{\dot{a}}{a}(t_0)\,\vec{R} = H_0\vec{R} \qquad \boxed{H = \frac{\dot{a}}{a}}$$

Hubble constant $H_0 = \frac{\dot{a}}{a}(t_0)$



slope H_0 but $a\dot{r} \ll HR$ if $\dot{r}=0$

other consequence — light



$$ds^2 = 0 \Rightarrow dt^2 = a^2 dr^2$$

$$\frac{dt^2}{a^2} = dr^2 \qquad \frac{dt}{a} = dr$$

two galaxies at rest w.r.t. each other in co moving coords (distance r apart)

$$\int_{t_e}^{t_0} \frac{dt}{a} = r$$

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a} = r \quad \text{if } \delta t\,\dot{a} \ll a$$

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a} - \int_{t_e}^{t_0}\frac{dt}{a} = 0$$

$$\Rightarrow \frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)} \qquad \lambda = c\,\delta t$$

$$\frac{\lambda_0}{a(t_0)} = \frac{\lambda_e}{a(t_e)} \qquad \lambda_0 = \frac{a(t_0)}{a(t_e)}\lambda_e$$

Wavelength of light gets stretched

light gets redshifted (if $a(t)$ increases)

redshift parameter $z = \dfrac{\lambda_0 - \lambda_e}{\lambda_e}$

$$= \frac{a(t_0)}{a(t_e)} - 1$$

$$(1+z) = \frac{a(t_0)}{a(t_e)}$$

two ways to verify expansion of the universe

Hubble Law $v \propto R$

& redshift of light from faraway objects

for convenience we choose $a(t_0) = 1$

Specific form of $a(t)$ depends on the type of matter. For a homogeneous & isotropic universe, $T_\mu{}^\nu$ must take the perfect fluid form $(g_{00} = -1 \quad g_{ij} = \frac{1}{a^2}\delta_{ij})$

$$T_\mu{}^\nu = \begin{bmatrix} -\rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix}$$

where the matter of the universe is "at rest". That is we can write $T_\mu{}^\nu = (\rho + p) u_\mu u^\nu + p\,\delta_\mu{}^\nu$

where $u^\mu = (1, 0, 0, 0)$

Note $u_\mu = g_{\mu 0} u^0 = (-1, 0, 0, 0)$

$$T^0_0 = -\rho - p + p = -\rho \checkmark$$

Conservation equation

$$T_\mu{}^\nu{}_{;\nu} = 0$$

$$\Rightarrow \frac{d\rho}{dt} = -3\frac{\dot a}{a}(\rho + p)$$

Friedmann eqn $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

can be reduced to:

$$\Rightarrow \quad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$

and

$$\left(\frac{\ddot a}{a}\right) = -\frac{4\pi G}{3}(\rho + 3p)$$

The behavior of $a(t)$ will be derived for the 3 types of "matter" which occur

1) Cold particle matter

2) radiation

3) vacuum energy density

1) Non relativistic cold matter $\quad p \ll \rho$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\rho = \frac{\rho_0 a_0^3}{a^3}$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a^3}$$

$$\frac{\dot a}{a} = \sqrt{\frac{8\pi G}{3}\rho_0 a_0^3}\; a^{-3/2}$$

$$\dot a \propto a^{-1/2}$$

$$a \propto t^\beta \Rightarrow \quad \beta - 1 = -\beta/2 \Rightarrow \beta = \frac{2}{3}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

conserv.

$$\frac{d\rho}{dt} = -3\frac{\dot a}{a}(\rho)$$

$$\rho \propto \frac{1}{a^3}$$

you know that:

$$\rho = \frac{N(r)}{\frac{4\pi}{3} a^3(t)\, r^3}$$

2) Radiation $\quad p_R = \frac{1}{3}\rho_R$

$$\frac{d\rho}{dt} = -4\frac{\dot a}{a}(\rho)$$

$$\rho \propto \frac{1}{a^4}$$

2) Radiation $p_r = \frac{1}{3}\rho_r$ $\qquad \frac{d\rho}{dt} = -3\frac{\dot a}{a}(\rho+p) = -4\frac{\dot a}{a}\rho$

$\rho \propto \frac{1}{a^4}$

Note if a light wave leaves at t_e and is received at t_0

$\rho = \frac{\langle E\rangle N}{\frac{4\pi}{3} a^3 r^3}$ ~~the time between crests~~ the length between successive crests observed

$$c\,\delta t_0 = \frac{a(t_0)}{a(t_e)}\,c\,\delta t_e$$

$$\langle E\rangle = \frac{ch}{\langle \text{wavelength}\rangle} = \frac{ch}{\langle\lambda\rangle a(t)}$$

$$E \propto \frac{1}{a}$$

so $\rho_r \propto \frac{N}{r a^3}\frac{\langle E_0\rangle}{a}$

$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_r \propto \frac{1}{a^4}$

$\left(\frac{\dot a}{a}\right)^2 \propto \frac{1}{a^2}$

$\dot a \propto a^{-1} \Rightarrow a \propto t^{1/2}$

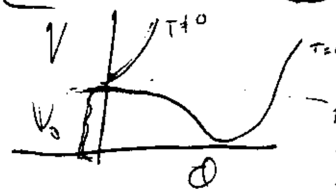
3) Vacuum energy density:

e.g. Spontaneous symmetry breaking

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi) - V(\phi)\right]$$

$$T_\mu{}^\nu = -\left[\partial_\mu\phi\,\partial^\nu\phi - \delta_\mu^\nu\left[\frac{1}{2}\partial^\alpha\phi\,\partial_\alpha\phi - V(\phi)\right]\right]$$

suppose



– flat near top so $(\partial_\mu\phi) << V_0^{1/2}$

$$T_\mu{}^\nu \approx -V_0\,\delta_\mu^\nu$$

equivalent to a cosmological constant.

note if we force

$$T_{\mu\nu} = (\rho_v+p_v)u_\mu u_\nu + p_v g_{\mu\nu}$$

$$p_v = -V_0 = -\rho_v$$

$$\Rightarrow \frac{d\rho_v}{dt} = -3\frac{\dot a}{a}(\rho_v+p_v) = 0$$

$$\rho_v = \text{constant}$$

Friedman eq.

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m+\rho_v) - \frac{k}{a^2}$$

cosmological constant

Λ

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_m(t_0)\left(\frac{a(t_0)}{a(t)}\right)^{3\sim 4}\right) + \frac{8\pi G}{3}\rho_v - \frac{k}{a^2}$$

Regardless of what matter is if $\rho_v \neq 0$ or $k \neq 0$ eventually they will dominate the density of the universe

~~Radiation dominated~~ ~~$a \propto t^{1/2}$~~

Vacuum dominated

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_v = \text{const} = H^2$$

$$\Rightarrow \quad a \propto e^{Ht}$$

$\Rightarrow$ exponential expansion

~~q_0 - decelerat~~

note without

$$H_0^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} \qquad \Lambda = 8\pi G \rho_v$$

$$\rho_{crit} = \frac{3H_0^2}{8\pi G}$$

$$\Omega = \frac{\rho_{matter}}{\rho_{crit}} \qquad \text{note if } \rho_{matt} = \rho_{crit} \Rightarrow k = \Lambda = 0$$

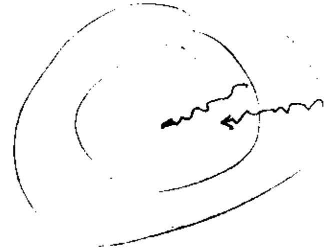
For $\Lambda = 0$

note if $\rho < \rho_{crit} \Rightarrow k < 0$ hyperbolic space

$\rho > \rho_{crit} \Rightarrow k > 0$ closed spatial curved S^4

one more classical def'n. "q_0 - "deceleration parameter"

$$q \equiv -\left(\frac{\ddot a}{\dot a}\right)\Big/\left(\frac{\dot a}{a}\right) = \frac{1}{2}\,\frac{\rho + 3p - \rho_v}{\rho_{crit}} \qquad \text{if } \rho_v = 0,\ p = 0 \quad = \frac{1}{2}\Omega$$



"Horizon" size

how far can a photon have travelled since the big bang

$$\int_0^{r_H} dr = \int_0^{t_0} \frac{dt}{a}$$

universe today is matter dominated

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

$$r_H = t_0^{2/3}\int_0^{t_0} t^{-2/3}\,dt$$

$$= 3t_0^{2/3}\left[t^{1/3}\right]_0^{t_0}$$

$$r_H = 3t_0^{2/3}\,t_0^{1/3} = 3t_0$$

since $a(t_0) = 1$

physical coord $R_H = 3t_0$

$$R_H = \frac{2}{H_0} \qquad H_0 = \frac{\frac{2}{3}\frac{1}{t}\left(\frac{t}{t_0}\right)^{2/3}}{\left(t/t_0\right)^{2/3}} = \frac{2}{3t} \qquad H_0 = \frac{2}{3t_0}$$

Horizon size = $2\left(\frac{1}{H_0}\right) = 2$ (Hubble radius)

often $\frac{1}{H_0}$ is loosely termed horizon size

all that we can see is within our horizon. The observable part of the universe is within our horizon, sometimes called "The universe".

→ Problem Horizon problem

"Cosmo Principle" — how did the universe get so homogeneous & isotropic?

e.g. the cosmic background radiation

$$\rho_r = a_R T^4 \qquad a_R = 7.56\times10^{-15}\ \frac{\text{erg}}{\text{cm}^3}\frac{1}{\text{deg}^4}$$

$$\rho_r(t) = \frac{a_R T_0^4}{a^4(t)} = a_R\left[\frac{T_0}{a(t)}\right]^4$$

$t \to 0$ universe gets hotter → matter is ionized in early un. CMBR + matter were in thermal equil.

$e^- + p \to H$

as universe expanded and cooled eventually the matter combined to form neutral H & became transparent. This is called the decoupling time. it happened when the photon temp ~3000°K

$$\frac{T_0}{a(t)} = 3000$$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} = \frac{2.74}{3000}$$

$$\frac{t}{t_0} = \left(\frac{2.74}{3000}\right)^{3/2} = 2.76\times10^{-5}$$

~ 100,000 yrs

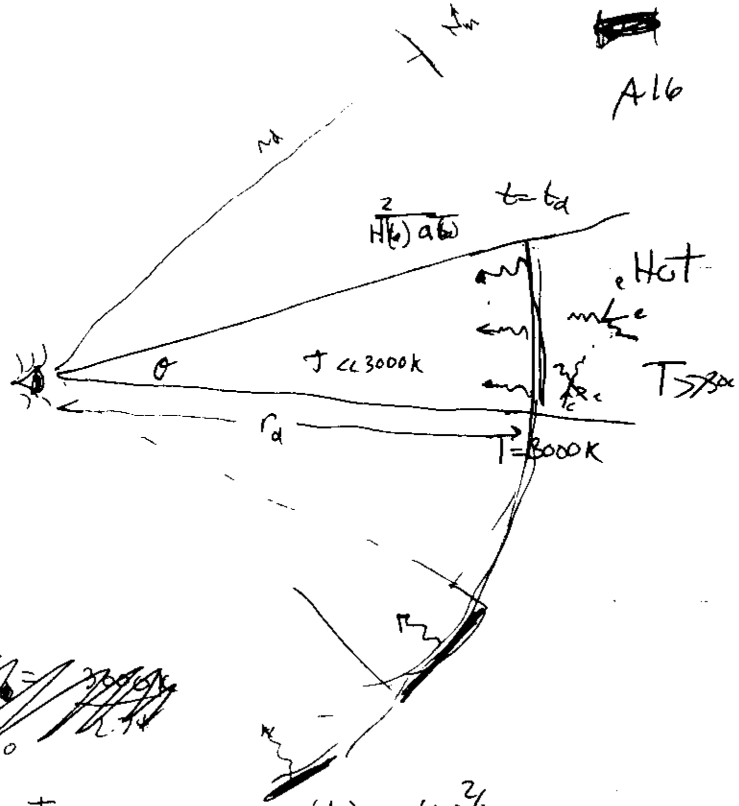
~~Notice~~ $\cancel{R_c(t)} \quad \cancel{r_H(t)} = 3t = 3\left(\frac{t}{t_0}\right)t_0$

~~in today's units~~

$$R_c = \cancel{a(t)}\ R_0\ a(t_0) \quad 3\left(\frac{t}{t_0}\right)t_0$$

$$= \cancel{\left(\frac{t_0}{t_c}\right)^{2/3}\left(\frac{t}{t_0}\right)} 3t_0$$

$$\frac{R_c}{R_0} = \left(\frac{t}{t_0}\right)^{1/3} \approx 0.03$$



$$\frac{T_{\gamma 0}}{T_{\gamma d}} = \frac{2.74}{3000} = \frac{a(t_d)}{a(t_0)} = \left(\frac{t_d}{t_0}\right)^{2/3}$$

$$\frac{t_d}{t_0} = 2.8 \times 10^{-5}$$

$$r_d = \int_{t_d}^{t_0} \frac{dt}{a} = t_0^{2/3} \int_{t_d}^{t_0} t^{-2/3}\, dt$$

$$= 3 t_0^{2/3} \left[t_0^{1/3} - t_d^{1/3} \right]$$

$$= 3 t_0 \left[1 - \left(\frac{t_d}{t_0}\right)^{1/3} \right] \approx 3 t_0 = \frac{2}{H_0}$$

in co-moving coords:

$$r_d \sin\theta \simeq \frac{2}{H_0}\theta = \frac{2}{H_d\, a(t_d)}$$

$$\theta = \frac{H_0}{H_d\, a(t_d)} = \frac{t_d}{t_0 \left(\frac{t_d}{t_0}\right)^{2/3}}$$

$$\theta = \left(\frac{t_d}{t_0}\right)^{1/3} = 0.03$$

$$\theta^\circ = 0.03 \, \frac{180^\circ}{\pi} = 1.7^\circ$$

$$\frac{4\pi}{(0.03)^2} = 14{,}000$$ ~~Horizon size~~

individual patches on the sky corresponding to horizon sizes

these regions could not have had time to communicate

after t_d γ & matter no longer in thermal contact

how did they all get to have the same precise Temp T_0 today?

Horizon proble[m]

Puzzle 2. Flatness Problem

Note Friedman eq.

$$H^2 = \frac{8\pi G}{3}\rho + \frac{8\pi G}{3}\rho_v - \frac{k}{a^2}$$

for the moment say $\rho_v = 0$
some dynamical process

if $k=0$ $\quad H^2 = \frac{8\pi G}{3}\rho.$

$$\rho_{crit} \equiv \frac{3}{8\pi G} H^2$$

if $\rho < \rho_{crit} \Rightarrow k < 0$ negative curvature

$\rho > \rho_{crit}$ $\quad k > 0$ positive curv

today $0.1\rho_{crit} \lesssim \rho \lesssim 2\rho_{crit}$

~~at Match time~~ if $\rho \neq \rho_{crit}$

then $K \lesssim \frac{8\pi G}{3}\rho_{crit}$

Since $k \neq 0$ term quickly dominates

$\Rightarrow$ In the past ρ must be finely tuned

$$(\rho_{crit} - \rho_0) = \frac{3K}{8\pi G}$$

$$\frac{\rho_c - \rho}{\rho_c} = \frac{\frac{3}{8\pi G} K \frac{1}{a^2}}{\rho_c}$$

$$\rho_c \sim \frac{\rho_c^0}{a^3}$$

$$\simeq \quad k \approx \frac{8\pi G}{3}(\rho_c^0)$$

$$\frac{\rho_c - \rho}{\rho_c} \approx \frac{a^3}{a^2} = a$$

since $T_\gamma \sim \frac{T_\gamma^0}{a}$

$$a = \frac{T_\gamma^0}{T_\gamma}$$

$$\frac{\rho_c - \rho}{\rho_c} \simeq \frac{T_\gamma^0}{T_\gamma}$$

In GUT epoc $\quad T_\gamma \simeq 10^{16}\,\text{GeV} = 10^{25}\,\text{eV}$
$\quad T_\gamma^0 = 10^{-4}\,\text{eV}$

$$\frac{\rho_c - \rho}{\rho_c} \simeq 10^{-29}$$

Incredible fine tuning initially

In other words if $\rho \neq \rho_c$
how did K get tuned to
be so small so that it is
only important now?

$\frac{\delta\rho}{\rho}$ problem

cluster of galaxies ~ 10 Mpc

$1\,\mathrm{Mpc} = 3.08 \times 10^{24}$ cm

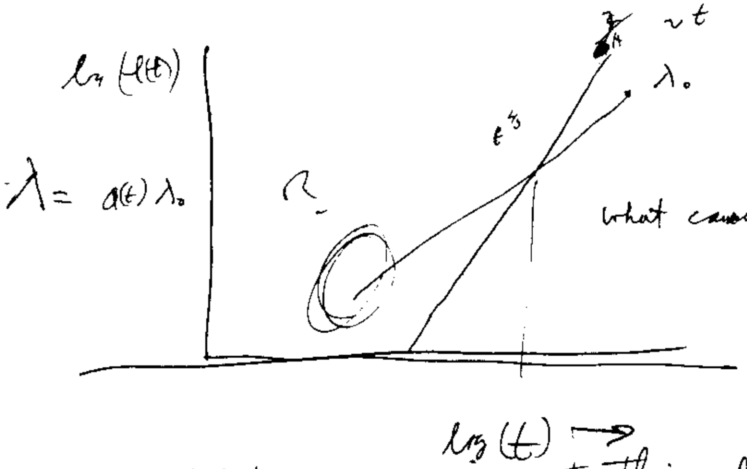
Density fluc. ~~entered the Horizon~~
could not ~~entered the Horizon until~~
have arisen from a causal process
until relatively late in the universe hist

$\cancel{\frac{2}{H(t)}} \approx 10$ Mpc

$$\frac{3t}{\left(\frac{t}{t_0}\right)^{2/3}} = \frac{3t_0\left(\frac{t}{t_0}\right)}{\left(\frac{t}{t_0}\right)^{2/3}} = 3t_0\left(\frac{t}{t_0}\right)^{1/3}$$

$$= \frac{2}{H_0}\left(\frac{t}{t_0}\right)^{1/3} = 12000\ \mathrm{Mpc}\ \frac{t}{t_0}$$

$$\left(\frac{t}{t_0}\right)^{1/3} = \frac{10}{12000} \qquad t = 10^{-9}\, t_0$$



what caused density fluctuation

What gave rise to this density fluctuations
in the early universe?

density fluctuation origin?

A22

Inflation

1) Horizon problem

2) Flatness problem

3) Origin of density fluctuation

B 0.1

1st from yesterday

$$\frac{\pi \times 10^7 \text{sec}}{1 \text{yr}} \times \frac{10^5 \text{cm}}{1 \text{km}} \times \frac{1 \text{Mpc}}{3 \times 10^{24} \text{cm}}$$

$H_0 = 100\, h$ km/(sec − Mpc) $= \frac{\text{Mpc}}{3 \times 10^{24} \text{cm}} = \frac{1}{10^{10}\, \text{yr}}$

$1 \geq h \geq 0.4$

$H_0 = \frac{\dot a}{a}$ if $K = 0$, $\frac{\dot a}{a} = \frac{2}{3t}$

$t_0 = \frac{2}{3}\frac{1}{H_0} = \frac{2}{3} h^{-1} \times 10^{10}$ yrs

$\rightarrow 7 \times 10^9 \leq t_0 \leq 16 \times 10^9$ yr

age of glob. cluster 15–20 yr

ages estimates $11 \times 10^9 \leq t_0 \leq 20 \times 10^9$

$\Rightarrow h \leq 0.6$

if $\Omega_0 = 1$!

other $\rho_{crit} = \frac{3 H_0^2}{8 \pi G} = 1.9 \times 10^{-29} h^2 \frac{\text{gm}}{\text{cm}^3}$

$\frac{2c}{H_0} = 6000\, h^{-1}$ Mpc

Big Bang does not mean everything came from a point!

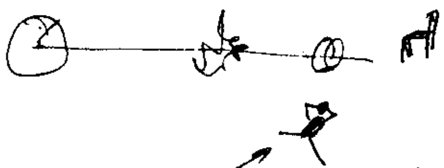
not Hubble Law

take $t = 10^{-46}$ sec $= \frac{1}{M_{pl}}$.

$$ds^2 = -dt^2 + a^2(t)[d\vec{x}\cdot d\vec{x}]$$

at a fixed time $dr^2 = a^2(t)\, d\vec{x}\cdot d\vec{x}$

Space is infinite; it does not stop at the horizon

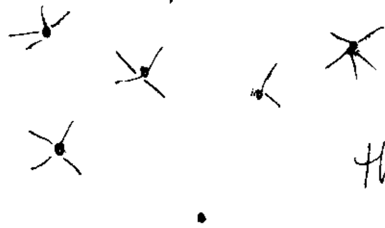


you cannot yet see this stuff

Correct picture

Raisin

at $t = \frac{1}{M_{pl}}$



Raisin bread

the universe fills all space

& creates more space as it expands.

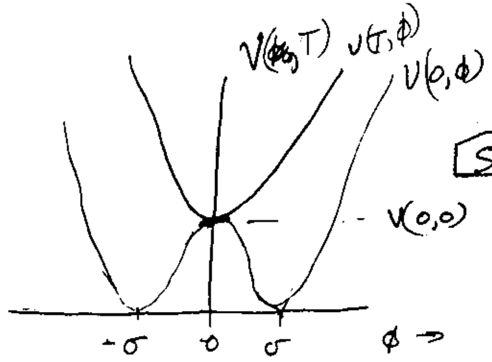
That is why you look anywhere & see 2.74 K CMBR

the big bang happened everywhere → you can see it everywhere you look.

<u>Inflation</u> - The cosmological advantages of having an inflationary type era had been kicking around in the community before the 1980's, but a detailed picture of an inflationary epoch from start to finish did not emerge until the 1980s.

The particle physics connection became more plausible with the success of the Spontaneous Symmetry breaking paradigm for Electroweak theory.

Recall the basic scenario for SSB.

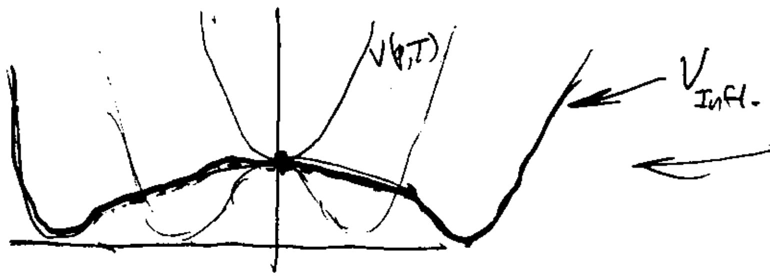


$V(\phi,T) \approx V(\phi,0) + T^2 \ldots$

$\boxed{\text{SSB}}$

At high Temp $T \gg V(0,0)^{1/4}$ minimum is at $\phi = 0$. As T falls below this temp, the minimum is non zero, breaking the symmetry of the potential.

The trick of inflation is to have a potential with a broad flat shoulder to $V(0,0)$.



keep somewhat safe from [illegible]

so it takes some time for the field

to reach its minimum.

$$\mathcal{L} = \partial^\mu \phi \, \partial_\mu \phi / 2 - U(\phi) \simeq \dot\phi^2/2 - V(\phi) \quad \text{+ coupling to matter}$$

notice $\nabla\phi$ term quickly become negligible

classical equation of motion of the field ϕ

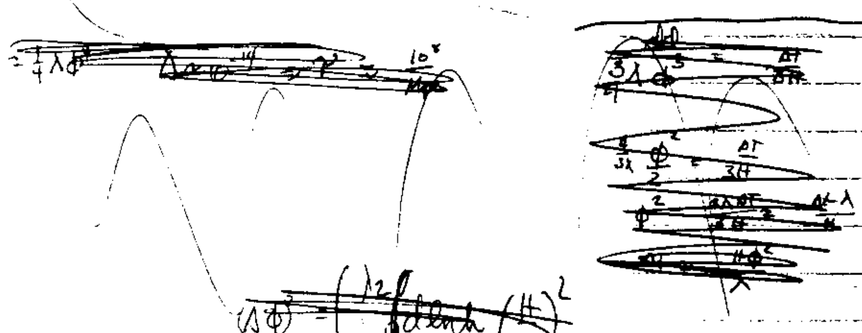
$$\ddot\phi + 3H\dot\phi + \Gamma_\phi \dot\phi + V'(\phi) = 0$$

provided that ϕ particles do not decay immediately and the potential is very flat ($\ddot\phi$ is small by design)

$$\Rightarrow \quad 3H\dot\phi \simeq -V'(\phi)$$

$$\frac{d\phi}{dt} = -\frac{V'(\phi)}{3H} \qquad -\frac{d\phi}{V'(\phi)} = \frac{dt}{3H}$$

to find length of time of inflationary period



$$(\Delta\phi)^2 = \left(\ln \frac{\Lambda}{\Lambda}\right)\left(\frac{H}{2\pi}\right)^2$$

$$H\tau = H\Delta t = \int_{t_i}^{t_f} dt = \frac{-3H^2}{\phi}\int_{\phi_i}^{\phi_f} \frac{d\phi}{V(\phi)} \quad \leftarrow \text{end of inf.}$$

<u>Inflation requires</u> $H\tau = O(10^2)$

to solve horizon & flatness problems

Why?

current horizon size $= \frac{2}{H_0}$ $c=1$

before inflation horizon size was $\frac{2}{H}$

$$G = \frac{1}{M_{Pl}^2} \qquad = \frac{2\phi}{\sqrt{\frac{8\pi G}{3}\rho_V}} \simeq \sqrt{\frac{3}{2\pi}}\left(\frac{M_{Pl}}{M_{GUT}}\right)^{-1}$$

$$\simeq (10^{-17})\,\text{GeV}^{-1} \qquad \text{if } M_{GUT} = 10^{14}\,\text{GeV}$$

in infl expands this scale by $e^{Ht} = e^{N}$

$(10^{-17})\frac{e^N}{\text{GeV}}$ $\Rightarrow$ scale corresponding to $\frac{c}{H_0}$

that scale has expanded to the present physical scale $\quad a \propto T_\gamma^{-1}$

Reheating process

$$10^{-17}\frac{e^{+N}}{\text{GeV}}(10^5) \qquad \frac{10^{10}\,\text{GeV}}{(10^{-4})\times 10^{-9}\,\text{eV}} = \frac{(10^{17})10^{28}e^N}{\text{GeV}}$$

during reheating Γ_ϕ $\quad T_{RH} \leftarrow \Gamma_\phi \quad$ 2.7 K $\quad T_1$

$$= \frac{10^{19}e^N}{\text{GeV}}$$

$$\gtrsim \frac{2}{H_0} = 2(3000\ Mpc)\left(\frac{3\times10^{24}\,cm}{Mpc}\right)\left(\frac{1}{2\times10^{-14}\,GeV\,cm}\right)$$

$$\boxed{\gtrsim 10^{42}\frac{1}{GeV}}$$ save for later

$$\Rightarrow\ e^N\, 10^{19}\frac{1}{GeV} \gtrsim 10^{42}\frac{1}{GeV}$$

$$e^N \gtrsim 10^{23}$$

$$N \gtrsim 50$$

depending on Mass scale at SSB $(\rho_V) = V(0)$
and amount of expansion during reheating

— solves horizon problem
i.e. every part of the currently observable universe had been in causal contact before inflation.

flatness problem

If $\frac{k}{a^2} \approx \frac{8\pi G}{3}\rho_V$ before inflation

today — Ignore Λ for now

$$\frac{k}{a^2} = H_0^2 - \frac{8\pi G}{3}\rho - \frac{\Lambda}{3}$$

$$\frac{k/a^2}{H_0^2} = 1 - \frac{\rho}{[3H_0^2/8\pi G]} = 1 - \rho/\rho_c = 1 - \Omega$$

$$\frac{k}{a_0^2 H_0^2} = \left.\frac{k}{a^2}\right|_{\text{before infl}} \times \underset{\text{infl}}{e^{-2N}} \times \underset{\substack{\text{reheating}\\(10^{-5})}}{10^{-10}} \underset{\text{after reheat}}{\left(\frac{10^{-4}\,10^{-9}}{10^{10}\,GeV}\right)^2} \left(\frac{1}{4\times10^{41}/GeV}\right)^2$$

$\frac{1}{H_0}$

let us say that $\frac{k}{a^2}$ before infl $\approx \frac{8\pi G}{3}\rho|_{vac}$

$$= \frac{8\pi G}{3}\rho_V$$

$$\approx \frac{8\pi}{3}\frac{(10^{14})^4}{(10^{19})^2}(GeV)^{-2}$$

$$\approx 8\times10^{18}(GeV)^2$$

$$\frac{k}{a_0^2 H_0^2} = 8\times10^{18}(GeV)^2\, \underset{\text{infl}}{e^{-2N}}\, \underset{\text{after reheat}}{10^{-56}}\, \underset{\text{reheat}}{10^{-10}}\, \underset{1/H_0^2}{1.6\times10^{-83}(GeV)^{-2}}$$

$$= 1.3\times10^{35}\, e^{-2N}$$

$$\frac{k}{a_0^2 H_0^2} = 1-\Omega \lesssim 10^{35}(10^{-45})$$

$$\lesssim 10^{-10}$$

Curvature is negligible today

without Λ $\quad \rho = \rho_{cr}$ to $\geq O(10^{-10})$

Of course inflation will not set Λ to zero so recall $\left|1-\left(\Omega_0+\frac{\Lambda}{3H_0^2}\right)\right| \lesssim 10^{-10}$. Notice, however

if $\frac{\Lambda}{3H_0^2} \lesssim 1$ $\quad \Lambda = 8\pi G \rho_V$

$$\frac{8\pi G \rho_V}{3H_0^2} \lesssim 1$$

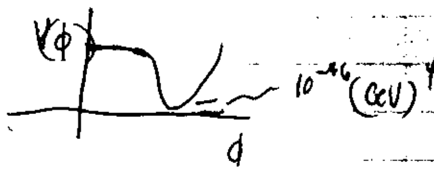
$$\rho_V \lesssim \frac{3}{8\pi}\frac{H_0^2}{G} = \rho_{cr}$$

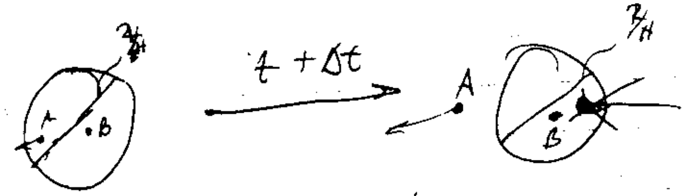
$$\lesssim \frac{1}{8}\frac{(GeV)^2 (10^{-42} GeV)^2}{10^{-38}}$$

$$\lesssim \frac{1}{8} 10^{-45} (GeV)^4 = 10^{-46} (GeV)^4$$

$$\rho_V^{1/4} \lesssim 10^{-12} GeV = 10^{-3} eV \ !!$$

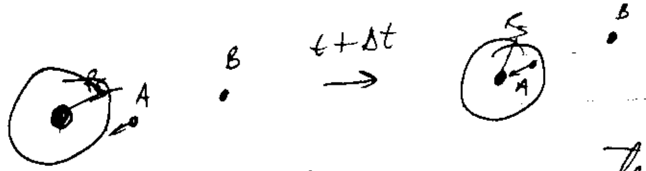
What does this have to do with particle physics of SSB GUT?





A can no longer communicate with B

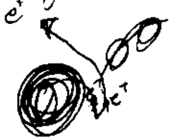
quite like a black hole



A can no longer communicate with B

Similar in another respect.

near event horizon when quantum fluctuations occur on length scales comparable to the event horizon scale get Hawking radiation



Similar effect occurs in de Sitter universe

There are quantum fluc. on scales comparable to the horizon scale of length $\sim \frac{1}{H}$ ant temp

$T_{GH} = \frac{H}{2\pi}$ Gibbons - Hawking temp.

To find the amplitude of density fluctuations $\left(\frac{\delta\rho}{\rho}\right)$ is not trivial. This is because for lengths $\gtrsim$ Horizon scale, such as those which arise during inflation, $\frac{\delta\rho}{\rho}$ is a gauge dependent quantity.

The gauge dependence comes from the fact that we are always comparing the coordinates of a perturbed universe with a background RWF ~~s~~ perfectly homogeneous

and isotropic universe. One must be careful to specify the amplitude of density fluctuations in a gauge invariant way.

It is popular to use the quantity $\zeta = \frac{2}{3}\rho \frac{\left[\frac{1}{H}\dot{\Phi} + \Phi\right]}{(\rho+p)} + \Phi$; $\Phi =$ grav. potential $\quad \zeta' = 0$

when evaluated on the scale of the horizon is $\zeta\big|_{\ell \simeq \frac{1}{H}} \simeq \frac{\delta\rho}{\rho+p}$ $\quad \boxed{\Phi|_H = \frac{3}{2}\ldots}$

Notice during inflation $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

but $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

so although $\rho_\phi \approx V(\phi) \approx -p_\phi$

$\rho_\phi + p_\phi = \dot{\phi}^2$

$\Rightarrow \zeta \approx \frac{2}{3}\rho \frac{\left(\frac{1}{H}\dot{\Phi} + \Phi\right)}{\rho+p} \simeq \frac{2}{3}\frac{\rho\Phi}{(\rho+p)} \approx \frac{\delta\rho}{(\rho+p)}$ later

$T_\mu^\nu = \partial_\mu\phi\,\partial^\nu\phi - \delta_\mu^\nu\left[\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V\ldots\right]$

M.D. $\rightarrow \frac{2}{3}\Phi + \Phi = \frac{5}{3}\Phi = \ldots$

Now $\delta\rho_\phi = \delta\phi\, V'(\phi)$

$V'(\phi) = 3H\dot\phi$ during infl

$$\left.\oint\right|_{H} \approx \frac{\delta\rho_\phi}{(\rho_\phi + p_\phi)} = \frac{\delta\phi\ 3H\dot\phi}{(\dot\phi)^2}$$

$$= 3\,\delta\phi\left.\left(\frac{H}{\dot\phi}\right)\right|_{hor}$$

$\delta\phi \simeq \left(\frac{H}{2\pi}\right) = T_{GH}$

$$\left.\frac{\delta\rho}{(\rho+p)}\right|_H \simeq \frac{3}{2\pi}\left.\left(\frac{H^2}{\dot\phi}\right)\right|_H$$

during infl. $H \approx$ constant $\dot\phi$ very small

"as time goes by"

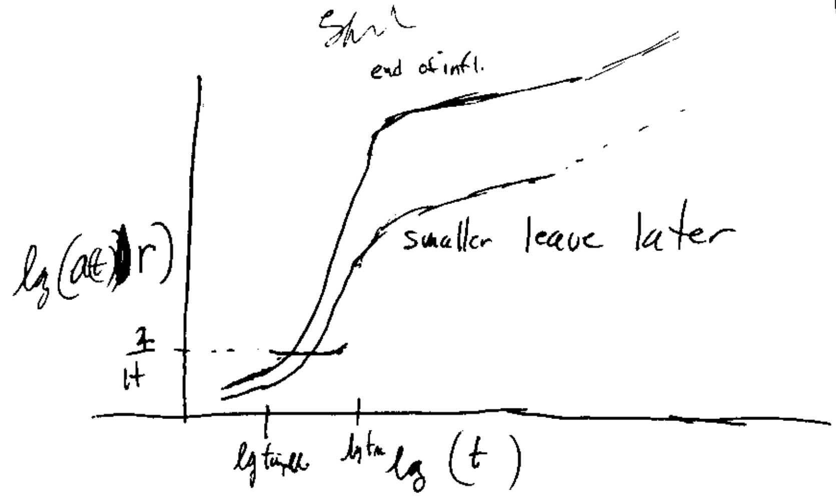
so $\frac{\delta\rho}{\rho} \sim$ constant H-Z spectrum

scale - invariant

$H^2 \sim \frac{8\pi}{3}\frac{V}{M_{pl}^2}$ decrease slightly with t

$\dot\phi^2$ increase slightly with t

as t ~~increase~~ $\frac{\delta\rho}{\rho}$ decrease slightly with t



small length amplitude fluc slightly smaller then large length amplitude

$$\left.\frac{\delta\rho}{(\rho+p)}\right|_H = \left(\frac{3}{2\pi}\right)\left.\left(\frac{H^2}{\dot\phi}\right)\right|_H$$

depends only on the potential

Note $H = \frac{8\pi}{3}\frac{V}{M_{pl}^2}$

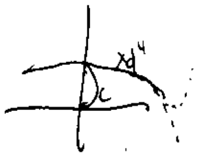
$\dot\phi = -V'(\phi)/3H$

so $$\left.\frac{\delta\rho}{(\rho+p)}\right|_H = -24\sqrt{\frac{2\pi}{3}}\left.\frac{V^{3/2}}{M_{pl}^3\ \frac{dV}{d\phi}}\right|_H$$

B13

Specific Model quartic pot

$$V = \text{const} - \frac{\lambda\phi^4}{4}$$



$$V' = -\lambda\phi^3$$

$$H\int_t^{t_{end}} dt \quad \text{(end of inf)} \quad = H\int_{\phi(t)}^{\phi_e}\frac{dt}{d\phi}d\phi \quad \text{(end of inf)}$$

$$H(t_{end}-t) = \#\text{ of efolds between } t \text{ \& end of inf} = -H\int_{\phi_t}^{\phi_e}\frac{3H}{V'(\phi)}d\phi$$

$$= -3H^2\int_{\phi_t}^{\phi_e}\frac{d\phi}{-\lambda\phi^3}$$

$$\boxed{\frac{a(t_f)}{a(t_i)} = e^{H(t_{end}-t)}}$$

$$= \frac{3H^2}{\lambda}\int_{\phi_t}^{\phi_e} d\phi\,\phi^{-3}$$

$$= -\frac{3H^2}{2\lambda}\left(\frac{1}{\phi_e^2}-\frac{1}{\phi_t^2}\right)$$

$$\phi_e >> \phi_t$$

$$H(t_{end}-t_\phi) \approx \frac{3}{2\lambda}\frac{H^2}{\phi^2}$$

$$N = H(t_{end}-t) = \frac{3}{2\lambda}\frac{H^2}{\phi^2}$$

$$\left(\frac{H}{\phi}\right) = \left[\frac{2\lambda}{3}N\right]^{1/2}$$

Save

B14

Now

$$\left.\left(\frac{\delta\rho}{\rho+p}\right)\right|_H = \frac{3}{2\pi}\left.\frac{H^2}{\left(-\frac{V'}{3H}\right)}\right|_H$$

$$= \frac{9}{2\pi}\left.\frac{H^3}{V'}\right|_H$$

$$= -\frac{9}{2\pi}\left.\frac{H^3}{\lambda\phi^3}\right|_H$$

$$= -\frac{9}{2\pi}\frac{1}{\lambda}\left[\frac{2\lambda}{3}N\right]^{3/2}$$

$$= -\frac{3}{\pi}\cdot\sqrt{\frac{2}{3}}\sqrt{\lambda}\,N^{3/2}$$

this factor should be $\frac{2}{5\pi}\sqrt{\frac{2}{3}}$ with more precise calc.

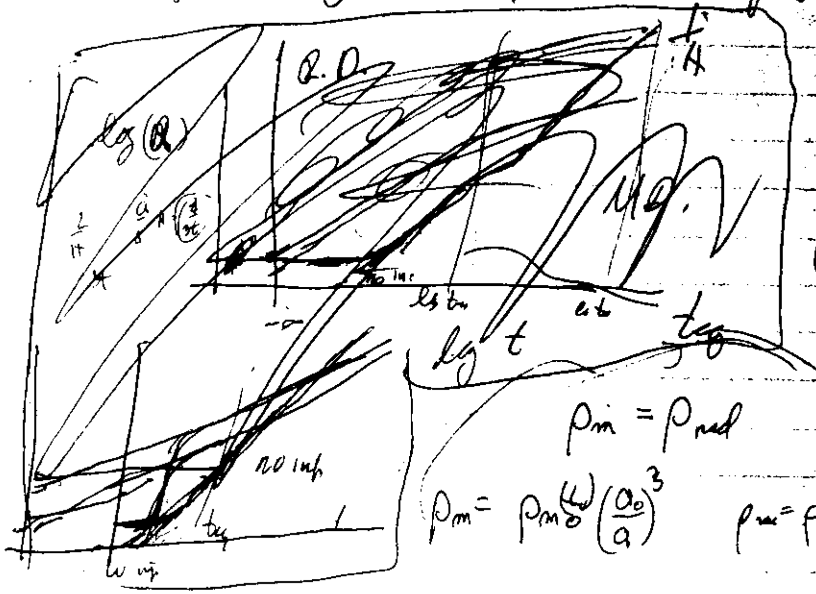
for $\frac{\delta\rho}{\rho} \sim 10^{-5} \Rightarrow \sqrt{\lambda}\,N^{3/2} \lesssim 10^{-4}$

$N \gtrsim 50 \qquad N^{3/2} \approx 350$

$\sqrt{\lambda} \lesssim 3\times10^{-7}$

tiny coupling $\qquad \lambda \lesssim 10^{-13}$ or so.

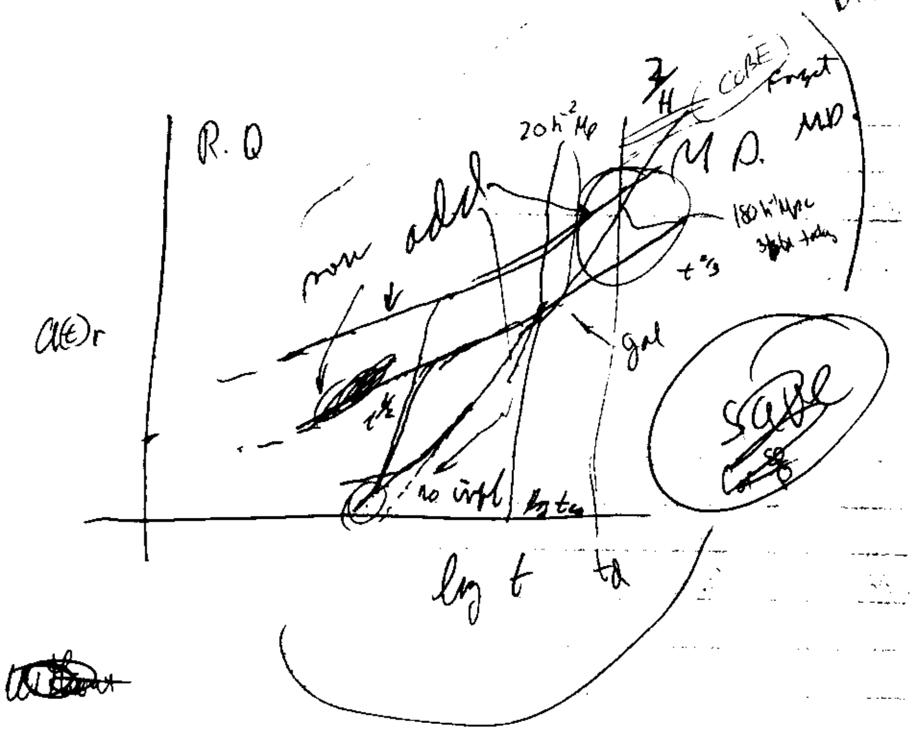
these quantum fluctuations ~~are~~ are then thought to give rise to the (through grav. amplification) ~~the~~ observed structure in the universe. This also gives a physical mechanism for the initial conditions ~~required~~ for these fluctuations, something sorely lacking without infl.



see
cut
inf
MD.
RD phase
stuff

$$\rho_m = \rho_{rad}$$

$$\rho_m = \rho_{m0}(t_0)\left(\frac{a_0}{a}\right)^3 \quad \rho_{rad} = \rho_{rad0}(t_0)\left(\frac{a_0}{a}\right)^4$$



Valery Rubakov "t_a" = t_{eq}

$$\rho_m(t_{eq}) = \rho_{rad}(t_{eq})$$

$$\rho_m(t_{eq}) \quad \rho_m(t_0)\left[\frac{a(t_0)}{a(t_{eq})}\right]^3 = \rho_{rad}\left[\frac{a(t_0)}{a(t_{eq})}\right]^4$$

$$\frac{a(t_{eq})}{a(t_0)} = \frac{\rho_{rad}(t_0)}{\rho_m(t_0)} \qquad aT^4$$

$$= \left(\frac{t_{eq}}{t_0}\right)^{2/3} = \frac{4\times10^{-34}\ \mathrm{erg/cm^3}}{\Omega\, 2\times10^{-29} h^2\ \mathrm{g/cm^3}}$$

$$\left(\frac{t_{eq}}{t_0}\right)^{2/3} = 2\times10^{-5}\,\Omega^{-1}h^{-2} \qquad \text{if } h=1 \quad t_{eq} \approx t_{dec}$$

$$\left(\frac{t_{eq}}{t_0}\right) = 1\times10^{-7}\, h^{-3}\, (\Omega h^2)^{-3/2}$$

$$t_{eq} = 1\times10^{-7}\, h^{-3}\left(\tfrac{2}{3}\times10^{10} h^{-1}\,\mathrm{yrs}\right)$$

$\frac{1}{H} \sim$

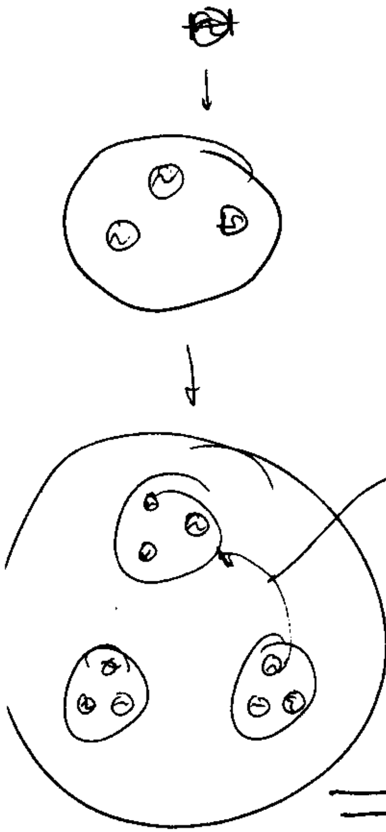
$$3ct_{eq} = \tfrac{4}{3}\times10^{3}\, h^{-4}\ \mathrm{lyr} \times \frac{\mathrm{pc}}{3\,\mathrm{ly}}$$

$$= \tfrac{4}{3}\times10^{3}\, h^{-4}\ \mathrm{pc}$$

(over)

point Gaussian

$3/H$



t_1

$t_1 + \Delta t$

fluctuations are independent & have random phases

$t_1 + 2\Delta t$

$\Longrightarrow$ fluctuations are gaussian.

B18

$\Rightarrow$ ~~structure~~ $\frac{\Delta T}{T}$?

Infl

- homogeneous
- $\Omega_0 \left(1 + \frac{\Lambda}{3H_0^2}\right) = 1$
- roughly scale invariant $\frac{\delta\rho}{\rho}$ set up. over all length scales.
- fluc are gaussian

$\rightarrow \frac{\Delta T}{T}$? Structure?

B19

Aside

$3H\dot\phi = -V'(\phi)$

$V = -\frac{\lambda\phi^4}{4}$

$V' = -\lambda\phi^3$

$$-\frac{d\phi}{(-\lambda\phi^3)} = \frac{\tau}{3H}$$

$$+\frac{1}{\lambda}\int \phi^{-3}\,d\phi = \frac{\tau}{3H}$$

$$-\frac{1}{2\lambda}\phi^{-2}\Big|_{\phi_i}^{\phi_f} = \frac{\tau}{3H}$$

$$-\frac{1}{2\lambda}\left(\frac{1}{\phi_f^2} - \frac{1}{\phi_i^2}\right) = \frac{\tau}{3H}$$

$\phi_i \ll \phi_f$, so

$$+\frac{1}{2\lambda\phi_i^2} \simeq \frac{\tau}{3H}$$

$$H = \sqrt{\frac{8\pi G}{3}\frac{\lambda}{4}\phi_i^4}$$

$$= \sqrt{\frac{2\pi G}{3}\lambda}\;\phi_i^2$$

$a(t) \to$

$\Rightarrow \tau_{GUT} = \frac{M_{pl}}{M_{GUT}} \sim 10^3$!

$$\frac{1}{2\lambda\phi_i^2} = \frac{\tau}{\sqrt{6\pi G\lambda}\;\phi_i^2}$$

$$\tau \simeq \frac{\sqrt{6\pi G\lambda}}{2\lambda} = \frac{1}{M_{pl}}\sqrt{\frac{3\pi}{2}}\frac{1}{\sqrt{\lambda}}$$

$T_{pl} \sim E^{-1}$

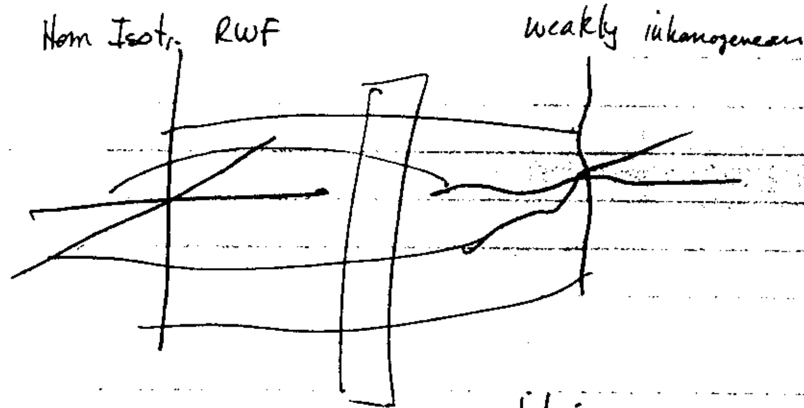
$\tau_{age} \simeq 10^7 \frac{1}{10^{19}\,GeV} \simeq \frac{1}{10^{12}\,GeV}$!

if $\lambda \simeq 10^{-5}$, $\simeq 100\,\tau_{pl}$

$\lambda \sim 10^{-14}$, $\simeq 10^7\,\tau_{pl}$

$\boxed{\Delta T/T}$

fluctuation in temperature are closely related to fluctuation in density. In order to calculate $\frac{\Delta T}{T}$, we will need to know $\frac{\delta\rho}{\rho}$. Since fluctuation on scale of the horizon ~~have~~ never been measured, we need to use g-i calculation in order to compare prediction accurately. Let me elaborate on gauge invariant variable a bit



gauge condition

example : $\frac{\delta\rho}{\rho}$

Keep to flat space: discussion is simpler + infl

In one coordinate system we have

$$\rho = -(\rho^{(0)} + \delta\rho)$$

const background — small deviation

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$\to (g^{(0)}_{\mu\nu} + \delta g_{\mu\nu})\, dx^\mu dx^\nu$$

Suppose in one coordinate system I set $\delta g_{00} = 0$ so that the time in perturbed coord is "synchronized".

C3

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= \left(g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}\right) \ldots$$

One gauge choice is ... $= 0$, ...

the time coordinate in the perturbed

model is chosen to ...

time in the background ...

We do not need to make this ...

suppose $h_{00} \neq 0$. Then

$$t \to t + T$$

where $T \ll t$

$$\rho(t+T) = \rho^{(0)}(t+T) + \delta\rho(t+T)$$

$$= \rho^{(0)}(t) + T\dot{\rho}^{(0)}(t) \ldots$$

$$= \rho^{(0)}(t) + \left[\delta\rho - 3T\frac{\dot{a}}{a}(\ldots)\right]$$

C4

$$\left.\frac{\delta\rho}{\rho}\right|_{NS} = \frac{\rho - \rho^{(0)}}{\rho} = \left.\frac{\delta\rho}{\rho}\right|_{\text{synch}} - 3T\frac{\dot{a}}{a}\left(1+\frac{p}{\rho}\right)$$

$$= \left.\frac{\delta\rho}{\rho}\right|_{\text{synch}} - 3TH\left(1+\frac{p}{\rho}\right)$$

Need to consider some combination of $\frac{\delta\rho}{\rho}$ and some other ~~variables~~ perturbation quantities which combine to eliminate the gauge dependence.

to do a full treatment:

(for flat space) 1) work out perturbation for a plane wave mode - any arbitrary pert is a superposition of planewaves

2) go to conformal time:

$$ds^2 = -dt^2 + a^2(t)(d\vec{x}\cdot d\vec{x})$$

looks much more symmetric

$$ds^2 = -a^2(\eta)\left[d\eta^2 - d\vec{x}\cdot d\vec{x}\right]$$

$$a^2(\eta)\, d\eta^2 = dt^2$$

$$\frac{\dot a}{a} = \frac{1}{a}\frac{da}{dt} = \frac{1}{a}\frac{d\eta}{dt}\frac{da}{d\eta}$$

$$= \frac{1}{a^2}\frac{da}{d\eta} = \frac{a'}{a^2}$$

M. D. $\left(\frac{\dot a}{a}\right)^2 = \frac{1}{a^2}\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho = \ldots$

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a}$$

$$(a')^2 = \frac{8\pi G}{3}\rho_0 a_0^3\, a$$

$$a' = \frac{8\pi G}{3}\rho_0 a_0^3\, a^{1/2}$$

$$a = \left(\frac{\eta}{\eta_0}\right)^2$$

$$\boxed{\left(\frac{a'}{a}\right) = \frac{2}{\eta}}$$

M.D

perturbed metric

$$ds^2 = a^2(\eta)\left[\eta_{\mu\nu} + h_{\mu\nu}\right] dx^\mu dx^\nu$$

$\eta_{\mu\nu}$ = Minkowski $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

~~expand~~ consider the effect of a ~~single~~ scalar perturbation which behaves like a plane wave $e^{i\vec{k}\cdot\vec{x}}$ → an arbitrary perturbation can be obtained by looking at a superposition of plane waves

Bardeen 1980 Kodama & Sasaki 1984

$h_{00} = -2A(\eta)\, Q(x)$ $\qquad Q = e^{i\vec{k}\cdot\vec{x}}$

$h_{0j} = -B(\eta)\, Q_j(x)$ $\qquad Q_j = -\frac{1}{k} Q_{,j}$

$h_{ij} = 2H_L(\eta)\, Q\,\delta_{ij}$ $\qquad Q_{ij} = \frac{1}{k^2} Q_{,ij} - \frac{1}{3}\ldots$

$+\, 2H_T\, Q_{ij}$

energy momentum tensor $T^{\mu}_{\nu} = T^{(0)\mu}_{\nu} + \delta T^{\mu}_{\nu}$

$$T^0_0 = -\rho\left[1 + \delta(\eta)\, Q(\vec{x})\right]$$

$$T^j_0 = -(\rho + p)\, v\, Q^j$$

$$T^i_j = p\left[\delta^i_j + \pi_L Q \delta^i_j + \pi_T Q^i_j(x)\right]$$

~~spatial~~ indicies raised and lowered with

Kronecker δ_{ij}

(in conformal time cons eq does not change)

$$\frac{d\rho}{d\eta} = \rho' = -3\,\frac{a'}{a}(\rho + p)$$

Infinitesimal gauge transformation

$$\bar{\eta} = \eta + TQ$$
$$\bar{x}^i = x^i + L^i Q$$

$$\bar{g}_{\mu\nu}(\eta, x^i) = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\, g_{\alpha\beta}(\eta - TY, x^i - LY)$$

$$\Rightarrow \bar{A} = A - T' - \frac{a'}{a}T$$

$$\bar{B} = B + L' + kT$$

$$\bar{H}_L = H_L - \frac{k}{3}L - \frac{a'}{a}T$$

$$\bar{H}_T = H_T + kL$$

$$\bar{T}_{\mu\nu}(\eta, x^i) = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\, T_{\alpha\beta}(\eta - TY, x^i - L^i Y)$$

$$\Rightarrow \bar{\delta} = \delta + 3\left(1 + \frac{p}{\rho}\right)\frac{a'}{a}T$$

$$\bar{v} = v + L'$$

$$\bar{\pi}_L = \pi_L + 3\frac{dp}{d\rho}\left(1 + \frac{p}{\rho}\right)\frac{a'}{a}T$$

$$\bar{\pi}_T = \pi_T$$

Take combinations of the variables to subtract off gauge pieces

e.g. δ

take $\bar{v} - \bar{B} = v - B - kT$

skip $\left(\frac{1}{k}(\bar{v}-\bar{B})\right)' = \frac{1}{k}(v-B)' - T'$

$$3\left(1+\frac{P}{\rho}\right)\frac{a'}{a}\frac{1}{k}(\bar{v}-\bar{B}) = 3\left(1+\frac{P}{\rho}\right)\frac{a'}{a}\frac{1}{k}(\ldots$$

$$- 3\left(1+\frac{P}{\rho}\right)\ldots$$

$$\delta + 3\left(1+\frac{P}{\rho}\right)\frac{a'}{a}\frac{1}{k}(\bar{v}-\bar{B}) = \delta + 3\left(1+\frac{P}{\rho}\right)\frac{a'}{a}\frac{1}{k}(\ldots$$

$$+ \cancel{3\left(1+\frac{P}{\rho}\right)\frac{a'}{a}T} - 3\left(1+\ldots\right.$$

$$\Delta_c$$

in comoving gauge $v = B = 0$ $\Delta_c = \delta$

Other combinations are possible

$$\Delta_s \equiv \delta + 3\left(1+\frac{P}{\rho}\right)\frac{a'}{a}\frac{1}{k}\left(\frac{1}{k}H_T' - B\right)$$

Δ_s → Mukhanov, Feldman + Brandenberger
Linde, Turner + Kolb

Δ_c – Kodama & Sasaki, Gouda, me

Note $\frac{a'}{a}\frac{1}{k} = \left(\frac{1}{a}\frac{a'}{a}\right)\left(\frac{a}{k}\right)$

$$= H\lambda_{phys}$$

$$= \ldots$$

Note in eq the combination $\frac{a'}{a}\frac{1}{k}$ appears often

when $\lambda_{phys} \ll$ Horizon size

$$\Delta_c = \Delta_s = \delta$$

when $\lambda_{phys} \gg$ Horizon size

$$\Delta_c \neq \Delta_s \neq \delta$$

They don't have to be

Other gauge inv combination Φ = g.i. generalization of the Newtonian potential

D1

~~resting Δ_c except for minus sign~~

~~$\nabla^2_x \Phi = -4\pi G \Delta_c \rho a^2$ ... just ... ~~

~~$-\nabla^2 \phi = 4\pi G \delta \rho a^2$~~

Now taking $\delta R^4{}_\nu - \delta^4_\nu \delta R = 8\pi G \delta T^4{}_\nu$

for cold matter $p \simeq 0$ + plane wave

$\Delta_c' = -kV$ (cons of mass)

$V' + \frac{a'}{a} V = -k\left[4\pi G\left(\frac{a^2}{k^2}\right)\rho\right]\Delta_c = -k\Phi$ (cons of mnt)

$\Phi = +4\pi G \frac{a^2}{k^2}\rho\Delta_c$ Fourier version of Newtonian

$\Delta_c'' = -kV'$

$-\frac{1}{k}\Delta_c'' + \frac{a'}{a}\left(\frac{-1}{k}\Delta_c'\right) = -k\left[4\pi G \frac{a^2}{k^2}\rho\right]\Delta_c$

$$\boxed{\Delta_c'' + \frac{a'}{a}\Delta_c' - 4\pi G a^2\rho\,\Delta_c = 0}$$

$\frac{3}{2}\left(\frac{a'}{a}\right)^2$

$\rho \propto \frac{\rho_0 a_0^3}{a^3}$ soln is $\Delta_c \propto \tau^2 \propto a$

So when $\frac{1}{a}\frac{a'}{a} = \frac{k}{...}$, $\Delta_c = \epsilon_H$ (const)

$\Delta_c \propto a = \Delta_c = \frac{1}{\epsilon_H}\left(\frac{k\tau}{2}\right)^2 ... \left(\frac{a}{a_0}\right)$

D2

$-k\,\nabla\Phi = 4\pi G\,\Delta_c$

$\Phi = +4\pi G \frac{a^2}{k^2}\frac{\rho a_0^3}{a^3}\Delta_{c0}\frac{a}{a_0}$

$= 4\pi G \frac{\rho a_0^2}{k^2}\Delta ... = \text{const} ...$

Now if $\Delta_c = \left(\frac{\tau^2 k^2}{4}\right)\epsilon_H$

when $\frac{a}{a'}\frac{2}{\tau} = \frac{1}{H}\frac{a}{a} = \frac{2\lambda ...}{H ...}$

$\Delta_c = \epsilon_H ...$ Horizon

perfect for infl. on large scale

$\Delta_c = \epsilon_H$ independent of ...

diff $\Rightarrow$ weak log dependence $\to$ ignore ...

Now $\Phi = 4\pi G \frac{\rho_0 a_0^2}{k^2}\frac{k^2\tau_0^2}{4}\epsilon_H = \frac{3}{2}\left(\frac{a_0'}{a_0}\right)^2\frac{\tau_0^2}{4}\epsilon_H$

$= \frac{3}{2}\left(\frac{4}{\tau_0}\right)^2 \frac{\tau_0^2}{4}\epsilon_H = \frac{3}{2}\epsilon_H$

$\Rightarrow \Phi$ is constant on all scales we choose the form

$$\Delta_c = \epsilon_H \left(\frac{k\eta}{2}\right) \ldots$$

when $\frac{k}{a} = \frac{1}{a}\frac{a'}{a} = H \Rightarrow k = \frac{2}{\eta}$

$$\Delta_c \big|_H = \epsilon_H \, \hat{a}(k)$$

ϵ_H is a dimensionless ...

and $\langle \hat{a}^{+}(k) \, \hat{a}(k) \rangle = \ldots$

↳ see $\frac{SM}{M}$ argument

look at Correlation function

$$\langle \tilde{\Delta}_c(\vec{x}, \eta) \, \tilde{\Delta}(\ldots) \rangle$$

$$= \langle \int d^3k' \, \hat{\Delta}^{+}_c(\vec{k}', \eta) e^{\ldots} \ldots$$

$$e^{ik\cdot(\vec{x}+\vec{r})}$$

$$= \int d^3k' d^3k \; \epsilon_H^2 \frac{k^2 \ldots}{2^4} \ldots$$

$$\langle \hat{\Delta}(x,\eta) \Delta(\vec{x}+\vec{r}, \eta) \rangle = \int d^3k \, e^{i\vec{k}\cdot\vec{r}} \epsilon_H^2 \frac{k^4 \eta^4}{2^4} \frac{1}{k^3}$$

$$\xi(r) = \text{F.T. Power Spectrum } P(k)$$

$$\xi(r) = \int d^3k \, e^{i\vec{k}\cdot\vec{r}} P(k)$$

$$P(k) = \epsilon_H^2 \frac{k^4 \eta^4}{2^4} \frac{1}{k^3} \propto k^n$$

$$= \epsilon_H^2 \left(\frac{T}{2}\right)^4 k \Rightarrow n = 1$$

$\vec{r} \to 0$

$$\langle |\Delta(\vec{x},\eta)|^2 \rangle = \int \frac{d^3k}{k^3} \epsilon_H^2 \left(\frac{kT}{2}\right)^4$$

$$\langle |\Delta(\vec{x},\eta)|^2 \rangle \Big|_{k=\frac{2}{T}} = 4\pi \epsilon_H^2 \int \frac{dk}{k} \left(\frac{kT}{2}\right)^4$$

$$= 4\pi \epsilon_H^2 \int d(\ln k)$$

contributes to $\Delta(\vec{x})$ from ... by k interval in ...

To show this is in fact what you want for a scale invariant spectrum consider a mass fluctuation

$$\frac{\delta M(R)}{M(R)} = \frac{\int d^3x\, \rho \int d^3k\, \Delta_c(\vec{k}, \eta)\, e^{i\vec{k}\cdot\vec{x}}}{\int d^3x\, \rho_v}$$

$$= \frac{\int d^3x}{V} \int d^3k\, \Delta_c(\vec{k}, \eta)\, e^{i\vec{k}\cdot\vec{x}}$$

$$= \int d^3k\, \Delta_c(\vec{k}, \eta) \int \frac{d^3x}{V} e^{i\vec{k}\cdot\vec{x}}$$

$$= \int d^3k\, \Delta_c(\vec{k}, \eta)\, W(kR)$$

$$W(kR) = \int \frac{d^3x}{V} e^{i\vec{k}\cdot\vec{x}} = \frac{2\pi}{V}\int_0^R r^2 dr \int d\mu\, e^{ikr\mu}$$

$$= \frac{4\pi}{V}\int_0^R dr\, r^2 \frac{\sin kr}{kr}$$

$$= \frac{3}{(kR)^3}\left(\sin kR - kR \cos kR\right)$$

$W(kR) \sim 1 \quad kR \ll 1$
$\quad\to 0 \quad kR \gg 1$

$$\left\langle \left|\frac{\delta M}{M}\right|^2_{(R)} \right\rangle = \int d^3k'\, d^3k \left\langle \Delta_c^+(k', \eta)\, \Delta_c(k, \eta)\right\rangle W^2(kR)$$

$$= \int d^3k\; \epsilon_H^2 \left(\frac{k\tau}{2}\right)^2 \frac{1}{k^3} W^2(kR)$$

$$= \int d^3k\, P(k)\, W^2(kR)$$

$W \approx \theta(kR)$

$$\approx k^3 P(k)\Big|_{k=\frac{1}{R}}$$

$$= \epsilon_H^2 \left(\frac{k\tau}{2}\right)^2\Big|_{k=\frac{1}{R}}$$

when $\frac{1}{R} \sim k = \frac{2}{\tau}$ $\quad \left(\frac{k}{a} = H\right)$

$$\left(\frac{\delta M}{M}\right)^2\Big|_{H} \simeq \epsilon_H^2 = \text{constant}$$

$\frac{\delta M}{M}$

D7

$$\Phi = \frac{4\pi G}{k^2}\rho \Delta_c a^2$$
$$= \frac{3}{2}\left(\frac{8\pi G}{3}\rho\right)\Delta_c a^2 \frac{1}{k^2}$$
$$= \frac{3}{2}\left[\frac{1}{a^2}\left(\frac{a'}{a}\right)^2\right]\Delta_c a^2 \frac{1}{k^2}$$
$$= \frac{3}{2}\left(\frac{2}{\tau}\right)^2 \epsilon_H \left(\frac{k\tau}{2}\right)^2 \hat{a}(k)\frac{1}{k^2}$$
$$= \frac{3}{2} k^2 \epsilon_H \frac{1}{k^2}\hat{a}(k) = \frac{3}{2}\epsilon_H \hat{a}(k)$$

Φ = const amplitude $\sim \frac{3}{2}\epsilon_H \hat{a}(k)$ on all scales

(not just horizon size)

Infl. ⇒ Φ = constant ⇒ temp. fluctuation

hotter γ, temp diff, cooler γ

Φ

D8

Temperature fluctuation Sachs-Wolfe effect

today, when the universe has cooled to a temp. after $e + p \to H$, Un. becomes transparent, called ~~[illegible]~~ time of matter-radiation decouple

$$\delta T_0 + T_0 = \frac{T_e + \delta T_e}{1+z}$$

$$1+z = 1 + z_{exp} + z_{grav} = \cancel{\frac{T_e}{T_0}} \ldots$$

Typ decrease due to expansion

plus grav redshift

when $t = t_{dec}$

ϕ

z_{grav} depends only on diff in potentials

~~$\delta T_0 + T_0 = \frac{T_e + \delta T_e}{T_0 \ldots}$~~

$$\frac{\delta T_0}{T_0} + 1 = \left[1 + \frac{\delta T_e}{T_e}\right]\left(1 - \frac{T_0}{T_e} z_{grav}\right)$$

$$\frac{\delta T_0}{T_0} + 1 = \left[1 + \frac{\delta T_e}{T_e} \frac{T_0}{T_e} z_{grav}\right]$$

D9

U^α = 4 velocity of fluid

$$1+z = \frac{K_\alpha(\eta_d)\,U^\alpha(\eta_d)}{K_\alpha(\eta_0)\,U^\alpha(\eta_0)}$$

K^α = tangent vector to null geodesic in perturbed spacetime

path: $\vec{x} = \vec{e}\eta$

$$1+z = \frac{a(\eta_0)}{a(\eta_d)}\left[\, 1 - e^i v_i\, Q(\vec{e}\eta)\right]_{\eta_d}^{\eta_0}$$

$$+\frac{1}{2}\int_{\eta_d}^{\eta_0} d\eta \left[\, 2\, h_{00} \ldots\right.$$

We will change this into formula for use with CDM in which the pert are present at η_d ... simple growth law. Notice $1+z$ is not gauge invariant. When we combine with $\frac{\delta T_d}{T_d} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}$ ... which is also not gauge inv. we will

D10

~~a form~~ a simple gauge-invariant formula as the gauge pieces cancel in the final analysis

take $\Delta_c = \delta^{\delta_\gamma}$

$$\frac{\delta T_0}{T_0}(k,\eta_0) = \left[\frac{1}{4}\Delta_{c\gamma} + \frac{1}{3}\Phi\right]Q\Big|_{\eta=\eta_d} + \frac{1}{k}\left(V_b\, e^i\partial_i\right)Q\Big|_{\eta_d}^{\eta_0}$$

Note only for baryon fluct — not true for ...

Intrinsic fluct

$\boxed{\text{Sachs Wolfe effect}}$

doppler shifts due to motions of observer & emitter

note $\Delta_{c\gamma}(\eta_d) \propto \Phi\left(\frac{k\eta_d}{4}\right)^2 \propto \Phi\left(\frac{k}{H}\right)^2\Big|_{\eta_d}$

$$V = -\frac{1}{k}\Delta_c' \;\propto\; \left(\frac{k\eta_d}{2}\right) \propto \left(\frac{k}{H}\right)\Big|_{\eta_d}$$

$$\Phi \propto \frac{3}{2}G_H \ldots \propto \left(\frac{k}{H}\right)^0\Big|_{\eta_d}$$

for $\lambda \gg \frac{1}{H(\eta_d)}$ $\quad \frac{\delta T}{T} \simeq \frac{1}{3}\Phi Q\Big|^{\eta_d}$

$$\frac{\delta T_0}{T_0} = -\frac{1}{2}\, \epsilon_H\, \hat{a}(\vec{k})\, Q(r_d\, \vec{e})$$

$$Q(\vec{k}) = e^{i \vec{k}\cdot\vec{e}\, r_d}$$

$$= 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell\, Y^*_{\ell m}(\theta_k\, \phi_k)\, j_\ell(kr)\, Y_{\ell m}(\theta, \phi)$$

$$\left(\frac{\delta T_0}{T_0}\right)\Bigg|_{\ell, m} = -2\pi\, \epsilon_H\, \hat{a}(\vec{k})\, i^\ell\, Y^*_{\ell m}(\theta_k\, \phi_k)\, j_\ell(kr)$$

$$\int d\Omega\, \frac{\delta T_0}{T_0} = \sum a_{\ell m} Y_{\ell m}(\theta, \phi)$$

Summing over all k modes

$$a_{\ell m} = \frac{\delta T_0}{T_0}\Bigg|_{TOT\ \ell, m} = -2\pi\epsilon_H \int d^3k\ \epsilon_H\, \hat{a}(\vec{k})\, i^\ell\, Y^*_{\ell m}(\theta_k, \phi_k)\, j_\ell(kr)$$

$$\left\langle \left| a_{\ell m} \right|^2 \right\rangle = 4\pi^2 \epsilon_H^2 \iint d^3k'\, d^3k\, \langle \hat{a}^*(\vec{k}')\, \hat{a}(\vec{k}) \rangle \quad \left(= \tfrac{1}{k^3}\delta^3(\vec{k}-\vec{k}')\right)$$

$$(-i)^\ell (+i)^\ell\, Y^*_{\ell m}(\theta_{k'}, \phi_{k'})\, Y_{\ell m}(\theta_k, \phi_k)$$

$$j_\ell(k'r)\, j_\ell(kr)$$

$$\langle |a_{\ell m}|^2 \rangle = 4\pi^2 \epsilon_H^2 \int \frac{d^3k}{k^3}\, Y^*_{\ell m}(\theta_k, \phi_k)\, Y(\theta_k, \phi_k)\, j_\ell^2(kr)$$

$$= 4\pi^2 \epsilon_H^2$$

$$\int d\Omega_k\, Y^*_{\ell m}(\theta_k\, \phi_k)\, Y_{\ell m}(\theta_k\, \phi_k) = 1$$

$$= 4\pi^2 \epsilon_H^2 \int \frac{dk}{k}\, j_\ell^2(kr)$$

$$\langle |a_{\ell m}|^2 \rangle = 4\pi^2 \epsilon_H^2 \frac{1}{2\ell(\ell+1)}$$

each $|a_{\ell m}|$ is gaussian & independent

rotationally invariant

$$\langle a_\ell^2 \rangle = \sum_{m=-\ell}^{\ell} \langle |a_{\ell m}|^2 \rangle = 4\pi^2 \epsilon_H^2 \frac{2\ell+1}{2\ell(\ell+1)}$$

$\langle a_\ell^2 \rangle$ is a multivariate $(2\ell+1)$ Gaussian

$\Rightarrow \chi^2$ distribution

It is therefore dangerous to determine ϵ_H from the COBE quadrupole. [illegible] much better to use a quantity which uses information over the range of scales, e.g. the [illegible] autocorrelation funct.

$$\frac{\delta T}{T}(\vec{e}_{1,2}) = -\int d^3k \, \frac{\epsilon_H}{2} \hat{a}(k) \ldots$$

$$C(\theta) \equiv \left\langle \frac{\delta T}{T}^*(\vec{e}_1) \frac{\delta T}{T}(\vec{e}_2) \right\rangle \quad \text{where } \vec{e}_1 \cdot \vec{e}_2 = \ldots$$

$$C(\theta) = \frac{1}{4\pi} \sum_{\ell=2}^{\infty} \langle a_\ell^2 \rangle P_\ell(\cos\theta)$$

$$= \epsilon_H^2 \sum_{\ell=2}^{\infty} \frac{(2\ell+1)}{\ell(\ell+1)} P_\ell(\cos\theta)$$

You can also predict the variance of $C(\theta)$ ~~[illegible]~~ Scaramella Vittorio [illegible] p. 374

Fluctuations in density

$\Rightarrow$ fluctuations in temp. of microwave background.

fluctuations of d.M density $\Rightarrow$

$$\frac{\delta T}{T_0} \approx -\frac{1}{3} \Phi(x_{ls}) \ldots$$

Positive $\frac{\delta T}{T} \Rightarrow \frac{\delta\rho}{\rho} \Rightarrow \Delta_c$

& tells us something about the spectrum of density fluctuations

which perhaps gives rise to all structure in the universe.

Generic Infl gives rise to very definite predictions about structure which can be tested

$\frac{\Delta T}{T}$ spectrum of COBE is consistent with inflation. What about amplitude?

from particle physics point of view - tells you about potential - cottage industry of theories for potentials - what about

Δ not very strong constraints on ~~amplitude~~ spectral index ~~[illegible]~~ ~~[illegible]~~ could allow ~~[illegible]~~ non-infl. spectrum.

Is amplitude of scale-inv. spectrum consistent with

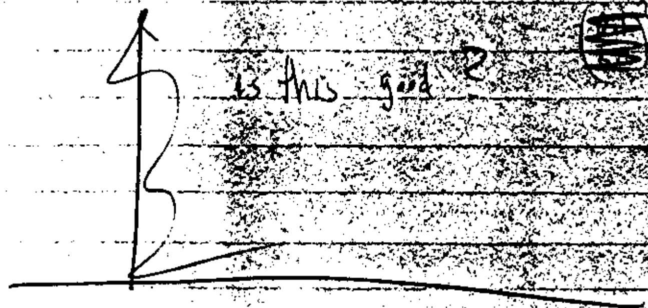
What is needed to explain structure? ~~possibly~~ [illegible] depends on ~~[illegible]~~ some interesting details.

COBE $\left(\frac{\Delta T}{T}\right)_\ell = \frac{\langle a_\ell^2 \rangle}{4\pi}$ $\left(Y_{00} = \frac{1}{\sqrt{4\pi}} \right)$

$$\frac{(2\ell+1)\pi C_\ell^2}{2\ell(\ell+1)} = \left(\frac{Q_{rms-ps}}{T_0}\right)^2 = \text{[illegible]}$$

$$\left(\frac{16.7\mu K}{2.735K}\right)^2 = \left(6.1\times10^{-6}\right)^2 = \frac{6\pi}{12} C_2$$

$\boxed{C_2 = 5.3\times10^{\text{[illegible]}}}$



Norm & bias.

$$\left\langle \left(\frac{\delta M}{M}\right)^2 (8h^{-1}Mpc) \right\rangle = \frac{1}{b^2}\left\langle \left(\frac{\delta N_{gal}}{N_{gal}}\right)^2 \right\rangle$$

light does not trace mass if $b \neq 1$

Dark matter rui infl mdl

Standard CDM models $b = 1.5 - 3$

HDM mdl $b = \frac{1}{2} - 1$

~~Mixed HD~~

ρCDM + ρHDM = pair $b = 1 - 1.5$

COBE measurement

$\Rightarrow$ CDM $b = 0.8$

HDM = $b = 1.6$

$\boxed{\left(\frac{1}{4}\right) \text{CPHDM} \quad b = 1.3}$

other LSS. QDOT Power spectrum

d. corl ; Sa a - Sa a V(bulk)