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SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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N=2 SUPERSYMMETRIC INTEGRABLE MODELS AND TOPOLOGICAL FIELD THEORY

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Please note: These are preli[illegible]

[illegible] now on $\mathcal{R} \equiv$ (chiral chiral) ring

Landau - Ginzburg formulation of $N=2$ SCFT's

Simple definition: If there are a set of generators, x_i and a superpotential, $W(x_i)$, such that

$$\mathcal{R} \equiv \frac{\mathcal{P}(x)}{\left\{\frac{\partial W}{\partial x_i}\right\}} \qquad \mathcal{J} = \left\{\sum_j p_j(x_j)\frac{\partial W}{\partial x_j}\right\}$$

ideal generated by the $\frac{\partial W}{\partial x_i}$

One also requires quasihomogeneity of W.

$$W(\lambda^{\omega_i} x_i) \equiv \lambda W(x_i)$$

$$\omega_i = q(x_i) = 2h(x_i)$$

E.g. $W(x) = x^{k+2}$ for minimal models.

$$h(\phi_l) = \frac{1}{2} q(\phi_l) = \frac{l}{2(k+2)}$$

Physical motivation: If the $N=2$ superconformal theory can be obtained as the ~~I~~ infra-red renormalization group fixed point of some action that has $W(x_i)$ as its superpotential,

Bosonic potential: $V = |\nabla W|^2$

then the chiral ring of the $N=2$ SCFT will be characterized in precisely this manner. The quasihomogeneity is required because of the [illegible] scaling properties at the superconformal point.

A very good analogy is the way one can think of $c = \frac{1}{2}$ ~~ordinary~~ Virasoro minimal model as arising from the IR limit of the Landau-Ginzburg description of the spin field of the Ising model. In $N=2$ superconformal field theories, the Landau Ginzburg theory gives exact information about the conformal theory. Another amusing point: one can construct exactly solved lattice models in which the chiral primary fields are indeed the order parameters of the lattice model. The standard LG description of these models will therefore characterize the chiral primary fields as claimed above

— Additional comment about $\rho(z)$ — the maximal chiral primary field

Since $J(z)$ is a $U(1)$ current, it can be written:

$$J(z) = i\sqrt{c/3}\,\partial X(z)$$

for some boson $X(z)$, with $X(z)X(w) = -\ln(z-w)$.

Consider
$$T_{pf}(z) = T_{N=2}(z) - \left(-\tfrac{1}{2}(\partial X)^2\right)$$
$$= T_{N=2}(z) - \frac{3}{2c}\left(J(z)\right)^2$$

This is simply the energy momentum tensor of the $N=2$ theory with the $U(1)$ field chopped out. (it is called a parafermion theory)

$$h_{pf}(\rho) = h_{N=2}(\rho) - \frac{3}{2c}\left(q(\rho)\right)^2$$

but
$$q_{N=2}(\rho) = \frac{c}{3} = 2h_{N=2}(\rho)$$

So
$$h_{pf}(\rho) = \frac{c}{6} - \frac{3}{2c}\left(\frac{c}{3}\right)^2 \equiv 0$$

So ρ is pure $U(1)$. Explicitly

$$\rho = e^{i\sqrt{c/3}\,X(z)}$$

Similarly
$$\bar{\rho}(z) = e^{-i\sqrt{c/3}\,X(z)}$$

Topological Twisting (Witten & Eguchi & Yang)

$$T_{top} = T + \tfrac{1}{2}\partial J\,, \qquad h_{top} = h - \tfrac{1}{2}q$$

The effect: G^+ now has conformal wt 1
G^- now has conformal wt 2

Define
$$Q = \oint G^+(z)\,dz$$

$$T_{top}(z) = 2\{Q, G^-(z)\}$$

$$Q^2 = 0$$

— Topological Hilbert space is the BRST cohomology of Q.

If $Q|\psi\rangle = 0$

then (Hodge decomposition)

$$|\psi\rangle = |\phi\rangle + Q|\psi_1\rangle$$

chiral primary

$$\mathcal{H}_{top} \cong \{\mathcal{R}\}$$

Dijkgraaf Verlinde & Verlinde

Topological correlation functions on a Sphere

Key observation : – for chiral primary fields

$$\frac{\partial}{\partial z}\phi(z) = L_{-1}\phi(z) = \left(\{G^+_{-1/2}, G^-_{-1/2}\}\phi\right)(z)$$

$$\equiv \left(G^+_{-1/2}\, G^-_{-1/2}\,\phi\right)(z)$$

since $G^+_{-1/2}\phi = 0$

– In topological language

$$\frac{\partial}{\partial z}\phi(z) = Q\left(G_{-1}\phi\right)$$

Consider the correlation function.

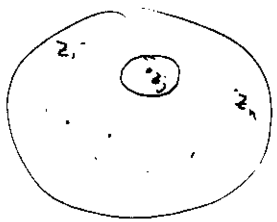
These should all be functions of z & $\bar{z}$.

$$\frac{\partial}{\partial z_j}\langle\phi_1(z_1)\cdots\cdots\phi_n(z_n)\rangle_{top}$$

$$= \langle\phi_1(z)\cdots\left(Q(G^-_{-1}\phi)(z_j)\right)\cdots\cdots\phi_n(z_n)\rangle_{top}$$

$$\equiv \oint_{z_j} d\zeta\,\langle G^+(\zeta)\,\phi_1(z_1)\cdots\cdots(G^-_{-1}\phi)(z_j)\cdots\phi_n(z_n)\rangle$$

small contour around z_j



Deform contour

$$\oint_{z_j} d\zeta = -\int_{\text{around all other punctures}} d\zeta$$

But $\int_{z_k} d\zeta\,\langle G^+(z)\,\phi_1(z_1)\cdots\cdots\phi_n(z_n)\rangle$

$\equiv 0 \quad (k\neq j)$ since $Q\phi(z_k) = 0$

Hence

$$\frac{\partial}{\partial z_j}\langle\phi_1(z_1)\cdots\cdots\phi_n(z_n)\rangle_{top} \equiv 0$$

– Topological correlators are independent of positions of insertions.

Ghost number anomaly

$J(w)$ is not a conserved current in top. theory

$$T_{top}(z)\,J(w) = \frac{-c/6}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w}$$

$$\Rightarrow [L_n, J_m] = -m\,J_{m+n} - \tfrac{c}{6}\,n(n+1)\,\delta_{m+n,0}$$

So $J_0 = [J_1, L_{-1}] \Rightarrow J_0^+ \equiv -[J_{-1}, L_1]$

$$= J_0 + c/3$$

So

$$J_0|0\rangle = 0 \Rightarrow \langle 0|J_0^+ = 0$$

$$\Rightarrow \boxed{\langle 0|J_0 = -c/3\,\langle 0|}$$

So for

$$\langle 0|\, \phi_1(z_1) \;\cdots\; \phi_n(z_n)|0\rangle_{top} \neq 0$$

one must have $\displaystyle\sum_{i=1}^{n} q_i = c/3$.

— and $\langle 0|$ must have charge $-c/3$

$\Leftrightarrow$ there is a charge of $-c/3$ at infinity.

N=2 counterpart of topological correlators

There is no U(1) anomaly so one must include the field of charge $-c/3$, so consider

$$\langle \phi_1(z_1) \cdots\cdots \phi_n(z_n)\, \bar{\rho}(\xi)\rangle_{N=2}$$

then

$$\langle \phi_1(z_1) \;\cdots\; \phi_n(z_n)\, \bar{\rho}(\xi)\rangle_{N=2} = \left[\prod_{i=1}^{n} (z_i - \xi)^{-q_i}\right] \langle \phi_1(z_1) \cdots\cdots \phi_n(z_n)\rangle_{top}$$

constant.

If one puts $\bar{\rho}$ at infinity one means that one considers the limit

$$\boxed{\lim_{\xi\to\infty} \xi^{+2h_\rho} \langle \phi_1(z_1) \;\cdots\; \phi_n(z_n)\, \bar{\rho}(\xi)\rangle \equiv \langle \phi_1(z_1) \;\cdots\; \phi_n(z_n)\rangle_{top}}$$

since $\xi^{2h_\rho} = \xi^{q_\rho} = \xi^{c/3} = \xi^{\Sigma q_i}$

so $\displaystyle\lim_{\xi\to\infty} \xi^{2h_\rho} \prod_{i=1}^{n} (z_i - \xi)^{-q_i} \equiv 1$

If one is careful, the foregoing is precisely what one means by computing $\langle \quad \rangle_{top}$ with a charge of $-c/3$ at infinity

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Solution to Exercise — a short digression

Consider an operator $Y(z)$ of conformal dimension h such that

$$\oint_0 z^n\, Y(z)\, dz\, |0\rangle \equiv 0 \quad \text{for } n \geq \ldots$$

e.g. $Y(z) = T(z),\ G^{\pm}(z),\ J(z) \ldots$

— question: what is the maximum value of n in $\oint z^n Y(z)\, dz$ such that the contour integral can be pulled of infinity.

i.e. $\oint_\infty z^n Y(z)\, dz \equiv 0$?

Under the conformal transformation $\zeta = 1/z$

$$Y(z) \equiv \left(\frac{\partial \zeta}{\partial z}\right)^h Y(\zeta) = -\zeta^{2h} Y(\zeta)$$

so

$$\oint_\infty z^n Y(z)\, dz = \oint_0 \zeta^{2h-n} Y(\zeta) \left(-\tfrac{1}{\zeta^2}\right) d\zeta$$

$$= \oint_0 \zeta^{2h-2-n} Y(\zeta)\, d\zeta$$

Acting on the vacuum (or identity operator) at infinity, this produces zero if and only if

$$2h - 2 - n \geq 0$$

$$\Rightarrow \boxed{n \leq 2h-2}$$

So if $v(z)$ a function that satisfies

$|v(z)| \leq A\,|z|^{2h-2}$ then $\oint_\infty Y(z)\, v(z)\, dz \equiv 0$.

So consider

$$Q = \oint (z-\zeta)\, G^+(z)\, dz \qquad h = 3/2 \text{ — so this can be pulled off at } \infty$$

$$Q\,\phi_j(z) \equiv (z_j - \zeta)\, G^+_{-1/2}\phi + G^+_{1/2}\phi \equiv 0$$

since ϕ is c.p.

$$Q\,\bar\rho(\zeta) \equiv G^+_{1/2}\bar\rho \equiv 0$$

$$Q(G^-_{-1/2}\phi_i)(z_i) = \oint_{z_i} (z - z_i + z_i - \zeta)\, G^+(z)\, (G^-_{-1/2}\phi_i)(z_i)\, dz$$

$$\equiv \underbrace{(G^+_{1/2} G^-_{-1/2})}_{\{G^+_{1/2},\, G^-_{-1/2}\} \equiv 2L_0 + J_0}\phi_i(z_i) + (z_i - \zeta)\underbrace{(G^+_{-1/2} G^-_{-1/2}\phi_i)(z_i)}_{\{G^+_{-1/2},\, G^-_{-1/2}\} = 2L_{-1}}$$

since $G^+_{1/2}\phi = G^+_{-1/2}\phi = 0 \longrightarrow$

$$\equiv (2h_i + q_i)\,\phi_i + 2(z_i - \zeta)\,\partial_{z_i}\phi_i$$

$$= 2\left[q_i + (z_i - \zeta)\partial_{z_i}\right]\phi_i$$

So

$$0 \equiv \left\langle \oint_\infty dz\,(z-\zeta)\, G^+(z)\, \phi_1(z_1) \cdots (G^-_{-1/2}\phi_i(z_i)) \cdots \phi_n(z_n)\, \bar\rho(\zeta)\right\rangle$$

$$\equiv -\sum_{j=1}^{n} \left\langle (z-\zeta)\, G^+(z)\, \phi_1(z_1) \cdots (\ \ ) \cdots \phi_n(z_n)\, \bar\rho(\zeta)\right\rangle$$

$$= -2\left[q_i + (z_i - \zeta)\partial_{z_i}\right] \langle \phi_1(z_1) \cdots \phi_i(z_i) \cdots \phi_n(z_n)\rangle$$

Solve the D.E & one gets

$$\langle \phi_1(z_1) \cdots \phi_i(z_i) \cdots \phi_n(z_n)\rangle \equiv \text{Const} \prod_{j=1}^{n} (z_i - \zeta)^{-q_i}$$

Properties of topological correlation functions

Let $\phi_i(z,\bar z)$ be a basis of $\mathcal{R}$

Define

$$\eta_{ij} \equiv \langle \phi_i \phi_j \rangle_{top}$$

$$C_{ijk} \equiv \langle \phi_i \phi_j \phi_k \rangle_{top}$$

$$\eta^{kl} \equiv (\eta_{kl})^{-1}$$

Then

a) $(\phi_i \phi_j) \equiv C_{ij}{}^k \phi_k$ ← product in chiral ring

where $\underline{C_{ij}{}^k \equiv \eta^{kl} C_{ijl}}$.

structure constants of the chiral ring

More generally

b) $\langle \phi_{i_1}(z_1) \cdots\cdots \phi_{i_n}(z_n) \rangle$

$$\equiv C_{i_1 i_2}{}^k \langle \phi_k(z) \phi_{i_3}(z_3) \cdots\cdots \phi_{i_n}(z_n) \rangle$$

Proof : Insert a complete set of states, but recall that any state has a hodge decomposition:

$$|\psi\rangle = |\phi\rangle + Q|\psi_1\rangle + Q^+|\psi_2\rangle$$

Note that

$$|\phi\rangle, \quad Q|\psi_1\rangle \quad \text{and} \quad Q^+|\psi_2\rangle$$

are all mutually orthogonal, so we can choose a complete set of states that respects this decomposition. Hence

$$\langle \phi_{i_1} \cdots\cdots \phi_{i_n} \rangle_{top} \equiv \sum_{\substack{|\psi\rangle = |\phi\rangle \\ \text{or } Q|\psi_1\rangle \\ \text{or } Q^+|\psi_2\rangle}} \langle \phi_{i_1} \phi_{i_2} | \psi \rangle \langle \psi | \phi_{i_3} \cdots \phi_{i_n} \rangle$$

$$= \sum_{|\phi_k\rangle} \langle \phi_{i_1} \phi_{i_2} | \phi_k \rangle \eta^{kl} \langle \phi_l | \phi_{i_3} \cdots \phi_{i_n} \rangle$$

$$\left(\text{as} \quad \langle \phi_{i_1} \phi_{i_2} Q|\psi_1\rangle = 0 \quad \text{and} \quad \langle \psi_2 | Q \phi_{i_3} \cdots \phi_{i_n} \rangle = 0 \right)$$

$$= C_{i_1 i_2}{}^k \langle \phi_k \phi_{i_3} \cdots\cdots \phi_{i_n} \rangle_{top}$$

N.B. $\displaystyle\sum_{k,l} |\phi_k\rangle \eta^{kl} \langle \phi_l| = \mathbb{1}$ on the chiral ring

Perturbed $N=2$ SCFT

Usually one considers

$$S = \underset{\phi}{S_0} + \int d^2z \sum_i \Big(t_i \overbrace{G^-_{-1/2}\tilde{G}^-_{-1/2}\phi_i}^{\Psi_i} + \bar{t}_i \overbrace{G^+_{-1/2}\tilde{G}^+_{-1/2}\bar{\phi}_i}^{\bar{\Psi}_i} \Big)$$

action of the $N=2$ SCFT.

The t_i are coupling constants
ϕ_i are chiral primaries
$\bar{\phi}_i$ " anti-chiral "

Usually one restricts to $h_i \leq \frac{1}{2}$.

Why $t G^-_{-1/2} \tilde{G}^-_{-1/2} \phi(z,\bar z)$ as a perturbation?

— Answer it preserves supersymmetry.

This follows from rather general properties of $N=2$ supersymmetric theories, but, as a simple exercise in techniques in perturbed conformal field theory: recall that if

$\Theta(z)$ is a holomorphic field, then to first order in perturbation theory one has

$$\partial_{\bar z}\, \Theta(z) \equiv t\, X(z,\bar z) \quad \text{where one}$$

obtains $X(z,\bar z)$ by looking at the coefficient of $\dfrac{1}{(z-w)}$ in

$$\Theta(z)\ \Psi(w,\bar w) \equiv \sum_{n=-M}^{\infty} (z-w)^n X_n(z,\bar z)$$

Note expand about z not w.

where

$\Psi(z,\bar z)$ is the perturbing operator.

So

$$G^+(z) \sum_i \Big(t_i\, G^-_{-1/2}\tilde{G}^-_{-1/2}\phi_i(w,\bar w) + \bar t_i\, G^+_{-1/2}\tilde{G}^+_{-1/2}\bar\phi_i(w,\bar w)\Big)$$

$$\| \qquad \sum_{n=-\infty}^{\infty} G^+_{n+1/2}(z-w)^{-n-2}$$

and $G^{\pm}_{n+1/2}\big(G^+_{-1/2}\tilde{G}^+_{-1/2}\bar\phi\big) \equiv 0$ for $n \geq -1$ (remember $(G^+_{-1/2})^2 = 0$)

$$G^+_{n+1/2}\big(G^-_{-1/2}\tilde{G}^-_{-1/2}\phi\big) \equiv 0 \qquad n \geq 1$$

$$\begin{aligned}
G^+_{\frac12}\big(G^-_{-1/2}\tilde{G}^-_{-1/2}\phi\big) &= (2L_0 + J_0)\big(\tilde{G}^-_{-1/2}\bar\phi\big) \\
&= (2h_\phi + q_\phi)\,\tilde{G}^-_{-1/2}\bar\phi \\
&\equiv 4h_\phi\, \tilde{G}^-_{-1/2}\bar\phi
\end{aligned}$$

$$\begin{aligned}
G^+_{-1/2}\big(G^-_{-1/2}\tilde{G}^-_{-1/2}\phi\big) &\equiv 2L_{-1}\,\tilde{G}^-_{-1/2}\bar\phi \\
&\equiv 2\,\partial_w\big(\tilde{G}^-_{-1/2}\bar\phi(w,\bar w)\big)
\end{aligned}$$

$$G^+(z)\,\left(G^-_{-1/2}\,\tilde G^-_{-1/2}\,\phi\right)(w,\bar w)$$

$$\equiv \frac{2\partial_w\left(\tilde G^-_{-1/2}\,\phi(w,\bar w)\right)}{(z-w)} + 4h_\phi\,\frac{(\tilde G^-_{-1/2}\,\phi)(w,\bar w)}{(z-w)^2}$$

$$\equiv \frac{2(1-2h_\phi)}{(z-w)}\,\partial_z(\tilde G^-_{-1/2}\,\phi)(z,\bar z) + \frac{4h_\phi}{(z-w)^2}(\tilde G^-_{-1/2}\,\phi)(z,\bar z)$$

where I have expanded about z instead of w.

$$\Rightarrow\quad \partial_{\bar z}\,G^+(z) \equiv \partial_z\left[2\sum_i t_i(1-2h_i)(\tilde G^-_{-1/2}\,\phi_i)(z,\bar z)\right]$$

So $\quad \left(G^+(z),\ 2\sum_i t_i(1-2h_i)(\tilde G^-_{-1/2}\,\phi_i)(z,\bar z)\right)$

Similarly $\quad \left(G^-(z),\ 2\sum_i \bar t_i(1-2h_i)(\tilde G^+_{-1/2}\,\bar\phi_i)(z,\bar z)\right)$

$$\left(2\sum_i t_i(1-2h_i)(G^-_{-1/2}\,\phi_i)(z,\bar z),\ \tilde G^+(\bar z)\right)$$

$$\left(2\sum_i \bar t_i(1-2h_i)(\tilde G^+_{-1/2}\,\bar\phi_i)(z,\bar z),\ \tilde G^-(\bar z)\right)$$

— Supercurrents are modified —

but it is still supersymmetric.

Perturbed topological correlation functions — perturbed chiral ring

Consider

$$F(z_i,\bar z_i; t_j,\bar t_j) \equiv \left\langle \phi_{i_1}(z_1,\bar z_1)\cdots\phi_{i_n}(z_n,\bar z_n)\; e^{-\sum_i\int d^2z\,(t_i\Psi_i(z,\bar z)+\bar t_i\bar\Psi_i(z,\bar z))}\right\rangle_{top}$$

where

$$\Psi_i \equiv G^-_{-1/2}\,\tilde G^-_{-1/2}\,\phi_i(z,\bar z)$$

$$\bar\Psi_i \equiv G^+_{-1/2}\,\tilde G^+_{-1/2}\,\bar\phi_i(z,\bar z)$$

First point: everything of the form

$$\int d^2z\ G^+_{-1/2}\,\tilde G^+_{-1/2}\,\bar\Phi(z,\bar z)$$

completely decouples since

$$G^+_{-1/2}\,\tilde G^-_{-1/2}\,\bar\Phi \equiv G^+_{-1/2}(\tilde G^-_{-1/2}\,\bar\Phi)$$

and

$$\because\quad G^+_{-1/2}\phi_i = 0$$

$$G^+_{-1/2}\left(G^+_{-1/2}\,G^+_{-1/2}\,\bar\Phi(w,\bar w)\right) \equiv 0$$

$$G^+_{-1/2}\left(G^-_{-1/2}\,\tilde G^-_{-1/2}\,\phi(w,\bar w)\right) \equiv \partial_z\left(\tilde G^-_{-1/2}\,\phi(w,\bar w)\right)$$

and this vanishes when $\int d^2z$ is performed*

* This requires careful treatment of divergences.

So F is independent of $\bar{t}_i$.

FROM NOW ON DROP THE $\bar{\Psi}_i$ terms.

F is also independent of z_i and $\bar{z}_i$.

— the proof is identical to before, and works because

$$\int d^2z \; G^-_{-1/2} \; \tilde{G}^-_{-1/2} \; \phi(z,\bar{z})$$

is annihilated by $Q = G^+_{-1/2}$

So this integrated insertion is "chiral primary".

So we in fact have F is only a function of t_j (and not even $\bar{t}_j$)

In addition, most of what I showed earlier about topological correlators remains true.

* Remember total ghost # must be $c/3$ and one must include any insertions of $\oint d^2z \; G^-_{-1/2} \tilde{G}^-_{-1/2} \phi$ in this counting.

I Define

$$\eta_{ij}(t) = \langle \phi_i(z_1,\bar{z}_1) \; \phi_j(z_2,\bar{z}_2) \; e^{-\sum t_i \int d^2z \, \Psi_i(z,\bar{z})} \rangle$$

$$c_{ijk}(t) = \langle \phi_i \; \phi_j \; \phi_k \; e^{-\sum t_i \int d^2z \, \Psi_i(z,\bar{z})} \rangle$$

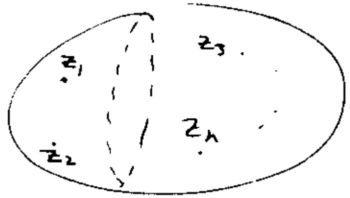
etc.

Then

$$\langle \phi_{i_1} \cdots \phi_{i_n} \; e^{-\sum t_i \int d^2z \cdots} \rangle$$

$$\equiv \; c_{i_1 i_2}{}^{k} \; \langle \phi_k \; \phi_{i_3} \cdots \phi_n \; e^{-\sum t_i \int d^2z \cdots} \rangle$$

There is one subtlety here — where one inserts the complete set of states, it is the same as cutting the sphere along a circle and inserting the states on the Hilbert space on the circle



states inserted here

But the $\int d^2z$ in the perturbation is done over the whole sphere, so one must split this integration up — The whole point is that $e^{-\sum t_i \int d^2z \, \Psi_i}$ has the exactly correct structure so that indeed

$$\langle \phi_{i_1} \cdots \phi_{i_n} e^{-\sum t_j \int d^2z \Psi_j} \rangle$$

$$= \sum_{|s\rangle} \langle \phi_{i_1} \phi_{i_2} e^{-\sum t_i \int_{H_-} d^2z \Psi_i} |s\rangle \langle s| \phi_{i_3} \cdots \phi_{i_n} e^{-\sum t_i \int_{H_+} d^2z \Psi_i} \rangle$$

$H_\pm$ are right & left parts of sphere

So factorization works nicely
(and of course only chiral primaries contribute in intermediate states)

These correlation functions also satisfy several other remarkable properties

(i) $\eta_{ij}(t)$ is constant (i.e. $\partial_{t_\ell} \eta_{ij} = 0$)

(ii) $C_{ijk} = C_{(ijk)}$ (Obvious)

$C_{ij}{}^{k} C_{\ell m k} = C_{(ij}{}^{k} C_{\ell m) k}$ $\quad [C_i, C_j] = 0$ as matrices $C_i = (C_i)_j{}^k$

(iii) $\partial_i C_{jk\ell} = \partial_{(i} C_{jk\ell)}$

where $\partial_i = \dfrac{\partial}{\partial t_i}$

cf. GAUSSIAN (NORMAL) COORDS.

Proof (ii) is a trivial consequence of the manifest symmetry of

$$\langle \phi_i \phi_j \phi_k \phi_\ell \; e^{-\sum \int \cdots} \rangle$$

$$= C_{ij}{}^{m} \langle \phi_m \phi_k \phi_\ell \; e^{-\sum \int \cdots} \rangle$$

To establish the others we have to work on the supersymmetry Ward identities:
We will work carefully in the $N=2$ language
Consider

$$\partial_\ell C_{ijk}$$

$$= \int d^2\zeta \; \langle \phi_i(z_1) \phi_j(z_2) \phi_k(z_3) G^-_{-1/2} \bar{G}^-_{-1/2} \phi_\ell(\zeta) e^{-\sum \int \cdots} \rangle$$

Consider the vector field

$$v(z) = \frac{(z - z_1)(z - z_2)}{(z - \zeta)} \quad \text{— grows linearly at } \infty$$

So

$$\oint_\infty G^-(z)\, v(z)\, dz = 0$$

in correlation function

Also, because $G^-_r \bar{\rho} \equiv 0 \quad r \geq -3/2$

one has

$$\oint_{\xi} G^-(z)\, v(z)\, \bar{\rho}(\xi)\, dz = 0$$

The fact that

$$G^-_r \phi_i \equiv G^-_r \phi_j \equiv 0 \quad r \geq 1/2$$

means that

$$\oint_{z_1} G^-(z)\, v(z)\, \phi_\ell(z_1) = 0$$

and similarly for $\phi_j(z_2)$

One can also see that when $\oint G^-(z) v(z)\, dz$ encounters a perturbation the $G^-_{-1/2}(G^-_{-1/2}\tilde{G}^-_{-1/2}\phi)$ generates a total derivative, which integrates to zero.

Finally

$$\oint_{z_3} v(z)\, G^-(z)\, \phi_k(z_3)\, dz$$

$$= \frac{(z_3 - z_1)(z_3 - z_2)}{(z_3 - \xi)}\, G^-_{-1/2}\, \phi_k(z)$$

and

$$\oint_{z_4} v(z)\, G^-(z)\, \tilde{G}^-_{-1/2}\phi_\ell(\xi) = \frac{(\zeta - z_1)(\xi - z_2)}{(\zeta - \xi)} G^-_{-1/2}\tilde{G}^-_{-1/2}\phi_\ell$$

The net effect of this is that one gets

$$\frac{(\zeta - z_1)(\zeta - z_2)}{(\zeta - \xi)} \langle \phi_i(z_1)\, \phi_j(z_2)\, \phi_k(z_3) (G^-_{-1/2}\tilde{G}^-_{-1/2})\, \phi_\ell)(\zeta)\, e^{-\int \Sigma} \cdots$$

$$= \frac{(z_3 - z_1)(z_3 - z_2)}{(z_3 - \xi)} \langle \phi_i(z_1)\, \phi_j(z_2)\, (G^-_{-1/2}\phi_k)(z_3) (\tilde{G}^-_{-1/2}\phi_\ell(\zeta)) \cdots e^{-\int \Sigma} \cdots \bar{\rho} \rangle$$

In the same manner one can move the $\tilde{G}^-_{-1/2}$ off the ϕ_ℓ and onto the $G^-_{-1/2}\phi_k$

Next one uses the $SL(2, \mathbb{C})$ invariance of the conformal theory to realize that this correlation function only depends upon (z_1, z_2, z_3, ζ) only through the cross ratios of the quantities. So integrating $\int d^2\zeta$ is equivalent to $\int d^2 z_3$, up to a change of variables involving the cross ratios. The factors of $\frac{(\zeta - z_1)(\zeta_1 - z_2)}{(z_3 - z_1)(z_3 - z_2)}$ generate precisely the right factor in the change of variables in the $d\zeta \to dz_3$. The factors $(z_3 - \xi)$ and $(\xi - \xi)$ are Koba Nielsen corrections mentioned earlier that go away when $\xi \to \infty$.

This shows that

$$\partial_\ell C_{ijk} = \partial_k C_{ij\ell}$$

Now repeat the argument with

$$\phi_k \equiv \mathbb{1}$$

Then $C_{ij\mathbb{1}} \equiv \eta_{ij}$

and $G^-_{-1/2}\mathbb{1} \equiv 0$

So that when one moves the $G^-_{-1/2}$ from ϕ_ℓ to ϕ_k one sees that the integrand (and not just integral)

Hence $\boxed{\partial_\ell \eta_{ij} \equiv 0}$ & IS EQUAL TO ITS CRITICAL ($t=0$) VALUE

THE IMPORTANT THING ABOUT THE PARAMETRIZATION IN TERMS OF CONFORMAL PERTURBATION THEORY IS THAT IN THESE "COORDINATES" THE TOPOLOGICAL METRIC IS FLAT

— the t_i are thus canonical flat coordinates for the theory.

Summarizing

In the perturbed model

$$(*)\begin{cases} \eta_{ij}(t) = \eta_{ij}(t=0) \\ C_{ij}{}^{m} C_{k\ell m} = C_{(ij}{}^{m} C_{k\ell)m} \\ \partial_i C_{jk\ell} = \partial_{(i} C_{jk\ell)} \end{cases}$$

This last identity means that

$$C_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}$$

$\mathcal{F}(t)$ is called the free energy of the model and completely characterizes all perturbed correlation functions (on the sphere).

Quasihomogeneity: scale any field or operator according to $X \to \lambda^q X$

i.e. $\phi_i \to \phi_i \lambda^{q_i}$, $G^\pm \to \lambda^{\pm 1} G^\pm$

$$\int \overline{\rho} \to \lambda^{-c/3} \int \overline{\rho}$$

and scale $t_i \to \lambda^{1-q_i} t_i$.

Then from the definition of the correlation fu...

$$C_{ijk} \longrightarrow \lambda^{q_i+q_j+q_k - c/3}$$
$$\parallel$$
$$\partial_i \partial_j \partial_k \mathcal{F}$$

and so

$$\boxed{\mathcal{F}(\lambda^{1-q_i} t_i) = \lambda^{3-c/3} \mathcal{F}(t_i)}$$

This fact, the equations (*) and simple charge content of the chiral ring are usually sufficient to determine $\mathcal{F}(t)$ (up to constant, quadratic and linear terms).

$\Rightarrow$ One knows the dimensions of the t_i, one makes an ansatz for $\mathcal{F}(t)$, ensuring that it has the correct scaling. One then realizes that the normalizations of the t_i are a matter of choice – so you fix these by fixing some terms in $\mathcal{F}(t)$, and then one can determine the rest of the coefficients in the ansatz by using (*).

Effective Landau - Ginzburg Potentials

If the $N=2$ superconformal model has a Landau Ginzburg potential, $W_0(x_a)$, and since the perturbations preserve the supersymmetry, the resulting theory has an effective Landau-Ginzburg potential: $W(x_a; t_i)$.

This means that if x_a are some generators of the perturbed ring, and $P_i(x_a)$ is some basis, then the multiplication is defined by:

$$P_i(x_a)\, P_j(x_b) = f_{ij}{}^{k}\, P_k(x_b) \quad \underline{\text{mod}} \left(\frac{\partial W}{\partial x_a}\right)$$

structure constants

There is, however, a natural basis induced from the conformal point namely ϕ_i where

$$\phi_i\, \phi_j \equiv C_{ij}{}^{k}\, \phi_k$$

where $C_{ij}{}^{k}(t)$ are the structure constants computed by conformal perturbation theory. One can, of course realize the basis, ϕ_i, as polynomials in its generators, x_a, but since the structure constants have become functions of t, the polynomials of x_a that define ϕ_i are now functions of t: i.e.

$\phi_i \equiv \phi_i(x_a; t)$. (E.g. if one had $\phi_2 = \phi_1^2$ for $t = 0$, one might find that $C_{11}{}^2 = 1$ and $C_{11}{}^0 = t$ i.e. $\phi_2 = \phi_1^2 + t = x_1^2 + t$).

The statement that there is now an effective Landau – Ginzburg potential is therefore that the under polynomial multiplication –

$$\phi_i(x_a;t)\,\phi_j(x_a;t) \equiv C_{ij}{}^k(t)\,\phi_k(x_a,t) \mod \left\{\frac{\partial W}{\partial x_a}\right\}$$

where $C_{ij}{}^k(t)$ are the structure constants defined earlier.

Since the perturbation theory involves:

$$e^{-\int \sum t_i G_{-1/2}\bar{G}_{-1/2}\phi_i} \qquad \text{—— } (*)$$

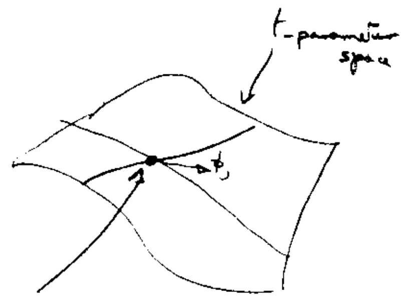
one has, at each point in parameter space:

$$\delta W \equiv \sum \delta t_j \phi_j$$

$$\Rightarrow \quad \phi_j(x_i;t) = \left(\frac{\partial W}{\partial t_j}\right)(x_a;t) \qquad (**)$$

Given the form of (*), one might have been tempted to conclude that $W \equiv W_0 + t_j \phi_j(x_a)$ — but $\phi_j \equiv \phi_j(x_a;t)$ is a function of t and so (**) means that $W(x_a,t)$ is, in general, a non-linear function of t

Analogy:



ϕ_j's as conformal fields, represent tangent vectors at the N=2 Superconformal point. The t_j are gaussian (Normal) coordinates. So we can parameterize the perturtheory in terms of ϕ_j and t, and in this sense, the ϕ_j can be thought of as being independent of t. However, at any point in the t parameter space, we can ask about the relationship of ϕ_i to a subset of generators, x_a, of the ring. The polynomials $\phi_i(x_a;t)$ are function of t. Moreover, the vanishing relations now also depend upon t_i the latter being characterized by $W(x_a;t)$ or through $C_{ij}{}^k(t)$.

Finally note that because of scaling, one must have

$$W(\lambda^{q_a} x_a; \lambda^{1-q_i} t_i) = \lambda W(x_a;t)$$

We now have a huge number of consistency conditions that must be satisfied by $C_{ij}{}^k(t)$ and $W(x_a;t)$.

Once one identifies the generators x_a from amongst the ϕ_i, one can obtain the $\phi_i(x_a;t)$ from the $C_{ij}{}^k(t)$. One can then compute $W(x_a;t)$ using $\frac{\partial W}{\partial t_i} = \phi_i$. (Integrability of this also implies constraints on the $C_{ij}{}^k$). One then has the further constraints that

$$\left(\frac{\partial W}{\partial t_i}\right)\left(\frac{\partial W}{\partial t_j}\right) = C_{ij}{}^k\left(\frac{\partial W}{\partial t_k}\right) \quad \text{mod} \left(\frac{\partial W}{\partial x_a}\right)$$

All of this well and truly overdetermines W.

In practice the solving of this system can prove very laborious. In the last few years, a lot of sophisticated technology has been developed for determining $C_{ij}{}^k(t)$ and $W(x_a;t)$ — with t_i being the flat perturbation parameters.

Example: Minimal Models

The basis for the unperturbed (t=0) chiral ring can be taken to be

$$\phi_\ell = x^\ell \quad \text{at } t=0 \quad , \ell = 0, \ldots, n$$

and I will normalize $W_0(x)$ to

$$W_0(x) = -\frac{1}{(n+2)} x^{n+2} .$$

With the basis chosen above one has, for $t=0$,

$$C_{ij}{}^k \equiv \delta_{i+j,k} \qquad 0 \le i,j,k \le n$$

To first order in the perturbation parameters,

$$W \equiv W_0 + A \left(\sum_{k=0}^{n} t_k x^k \right)$$

where A is an overall normalization of the perturbation — we will fix this later.

The perturbed superpotential, W, has the scaling property:

$$W(\lambda^{\frac{1}{n+2}} x ; \lambda^{\frac{n+2-k}{n+2}} t_k) = \lambda W(x; t_k).$$

at a general point in t-parameter space. The scaling

dimensions of the field, and the coupling constants, are deduced from the U(1) charges.

The topological metric η_{ij} can be computed at $t=0$

$$\eta_{ij} = \langle x^i x^j \rangle_{top}$$
$$= \langle x^{i+j} \rangle_{top}$$

Normalize x so that

$$\langle x^n \rangle_{top} = 1$$

Hence

$$\eta_{ij} = \delta_{i+j,n}$$

This is the metric at all values of t since $\partial_\ell \eta_{ij} = 0$

To deduce the C_{ijk} & $W(x,t)$ from consistency conditions alone is a painful procedure, and I will take a short cut here. Consider the field

$$(G^-_{-3/2} G^-_{-1/2} \phi_1)(z)$$

Then

$$G^+_{3/2}\, G^-_{-3/2}\, G^-_{-1/2}\, \phi_1(z) = \{G^+_{3/2}, G^-_{-3/2}\}\, G^-_{-1/2}\, \phi_1$$

$$- \, G^-_{-3/2}\{G^+_{3/2}, G^-_{-1/2}\}\, \phi_1$$

since $G^+_{3/2}\, \phi_1 = 0$

$$= \left(2L_0 + 3J_0 + \tfrac{2c}{3}\right)\left(G^-_{-1/2}\, \phi_1\right)$$

$$- \, G^-_{-3/2}\, \underbrace{[2L_1 + 2J_1]\, \phi_1}_{=0}$$

$$= \left[2\left(h_{\phi_1} + \tfrac{1}{2}\right) + 3\left(q_{\phi_1} - 1\right) + \tfrac{2c}{3}\right] G^-_{-1/2}\, \phi$$

Now

$$c = \frac{3n}{n+2}, \quad h_{\phi_1} = \frac{1}{2(n+2)}, \quad q_\phi = \frac{1}{(n+2)}$$

So

$$G^+_{3/2}\, G^-_{-3/2}\, G^-_{-1/2}\, \phi_1 = \frac{1}{n+2}\left[\cancel{1} + \cancel{(n+2)} + \cancel{3} - 3\cancel{(n+2)} + \cancel{2n}\right] G^-_{-1/2}\phi$$

$$= 0$$

But

$$|\, G^-_{-3/2}\, G^-_{-1/2}|\phi_1\rangle|^2 = \langle\phi_1|\, G^+_{1/2}\, \underbrace{G^+_{3/2}\, G^-_{-3/2}\, G^-_{-1/2}\, |\phi_1\rangle}_{=0}$$

$$= 0$$

So

$$\boxed{G^-_{-3/2}\, G^-_{-1/2}\, \phi_1 \equiv 0}$$

i.e. $G^-_{-3/2}\, G^-_{-1/2}\, \phi_1$ is a null state

It therefore follows that

$$\boxed{G^-_r\, G^-_{-1/2}\, \phi_1 = 0 \qquad r \geq -3/2} \quad (*)$$

Now consider the following

$$\oint dz\, v(z)\, \langle G^-(z)\, \phi_i(z_1)\, \phi_j(z_2)\, \phi_k(z_3)\, \phi_{i_0}(\zeta_0)\, (G^-_{-1/2}\phi_1)(\zeta)$$

$$\left[\prod_{n=1}^{M} (G^-_{-1/2}\, \phi_{i_n})(\zeta_n)\right] \bar{\rho}(\xi)\rangle$$

$$(M \geq 0)$$

with

$$v(z) = \frac{(z - z_1)(z - z_2)(z - z_3)}{(z - \zeta)(z - \xi)}.$$

Take $\oint dz$ around the point at ∞, and the integral will be zero. Now rewrite the integral as one running around all the punctures. By construction of $v(z)$, and because of $(*)$, the only contribution come from $\oint dz$ around ζ_0, which gives $v(\zeta_0)(G^-_{-1/2}\, \phi_{i_0})(\zeta_0)$.

Hence

$$\langle \phi_i(z_1)\,\phi_j(z_2)\,\phi_k(z_3)\,(G^-_{-1/2}\phi_\ell)(\xi)\prod_{j=0}^{M}(G^-_{-1/2}\phi_{i_j})(\xi_j)\rangle_{top}$$

$$\equiv 0 \qquad M \geq 0$$

This means that

$$\frac{\partial}{\partial t_\ell}\left(\frac{\partial}{\partial t_1} C_{ijk}\right) \equiv 0$$

or

$$\partial_{t_1} C_{ijk} \equiv \text{Constant}$$

$$\parallel\!|$$

$$\partial_i C_{1jk}$$

Therefore

$C_{1j}{}^{k}$ is at most linear in t.

Consider

$$\frac{\partial}{\partial t_k} C_{1ij}, \equiv \langle \phi_1\,\phi_i\,\phi_j \int d^2z\, G^-_{-1/2}\bar{G}^-_{-1/2}\phi_k\rangle_{top}$$

By charge conservation

$$q_1 + q_i + q_j + q_k - 1 = c/3$$

$$\Leftrightarrow \quad i+j+k = 2n+1 .$$

In particular,

$$C_{1nn} \propto t_1 .$$

Now, indirectly fix the normalization constant, A, on p. 45 by setting:

$$C_{1nn} = t_1 .$$

$$\Rightarrow \quad \boxed{C_{1n}{}^{0} = t_1} \quad \text{and} \quad \boxed{C_{1n}{}^{n-1} = t_n}$$

Now recall that at $t=0$

$$C_{ij}{}^{k} = 1 \qquad k = i+j$$
$$C_{ij}{}^{k} = 0 \qquad k = i+j-1$$

Observe that this remains unchanged for $t \neq 0$

since $\dim(t_\ell) = \frac{n+2-\ell}{n+2} \geq \frac{2}{n+2}$

and so there is no coupling of low enough dimension to modify them.

Now use

$$C_{1i}{}^{j}\, C_{jk}{}^{l} = C_{1k}{}^{j}\, C_{ij}{}^{l}$$

to determine the terms in $C_{ij}{}^{k}$ that are linear in t_1.

E.g.

$$C_{11}{}^{j}\, C_{jh}{}^{l} = C_{1k}{}^{j}\, C_{1j}{}^{l}$$

$$\Rightarrow \quad C_{2j}{}^{k} + \underbrace{C_{11}{}^{0}\, C_{0k}{}^{l}}_{O(t_n)\times I_d} = \underbrace{C_{1k}{}^{j}\, C_{1j}{}^{l}}_{(C_1)^2 \text{ as a matrix}}$$

As matrices

$$C_2 = C_1^2 - \text{const} \times t_n \times I$$

If one now squares C_1 and keeps only terms linear in C_1, one finds:

$$C_{2\,n-1+l}{}^{l} = t_1 \qquad l = 0, 1$$

Iterating:

$$C_{3j}{}^{k} + C_{12}{}^{1}\, C_{1k}{}^{l} + C_{12}{}^{0}\, C_{0k}{}^{l} = C_{1k}{}^{j}\, C_{2k}{}^{l}$$

gives the terms in $C_{3j}{}^{k}$ that are linear in t_1.

One finds:

$$C_{3\,n-2+l}{}^{l} = t_1 \qquad l = 0, 1, 2$$

In general

$$C_{j\,(n+1-j+l)}{}^{l} = t_1 \qquad l = 0, \dots, j-1$$

Hence

$$\frac{\partial}{\partial t_1} C_{j\,(n+1-j+l)\,(n-l)} = 1$$

$$\Rightarrow \quad \frac{\partial}{\partial t_j} C_{1\,(n+1-j+l)}{}^{l} = 1$$

Hence

$$\boxed{C_{1\,(n+1-j+l)}{}^{l} = t_j} \qquad l = 0, \dots, j-1$$

in matrix language:

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & & & 0 \\ t_n & 0 & 1 & 0 & \cdots & & 0 \\ t_{n-1} & t_n & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \ddots & & \vdots \\ \vdots & & & & 0 & 1 & 0 \\ t_2 & & & & t_n & 0 & 1 \\ t_1 & t_2 & \cdots & & t_{n-1} & t_n & 0 \end{pmatrix}$$

It is now elementary to compute $W(x)$, since $W'(x) = 0$ must be the vanishing relation, and this must also be the characteristic equation of C_1. Hence

$$W'(x) \propto \det\left(x\mathbb{1} - C_1\right)$$

To completely determine $W(x)$ one needs the constant term, which can be obtained from

$$\frac{\partial W}{\partial t_j} \equiv \phi_j(x,t).$$

The ϕ_j are now elementary to determine since we now know C_1 completely. One has $\phi_1 = x$

$$\begin{aligned} x\phi_j &= C_{1j}{}^{k}\,\phi_k \\ &= \phi_{j+1} + \sum_{l=0}^{j} C_{1j}{}^{l}\,\phi_l \\ &= \phi_{j+1} + \sum_{l=0}^{j-1} t_{n+1+l-j}\,\phi_l \end{aligned}$$

which determines ϕ_{j+1} completely.

I now wish to concentrate on two special cases:

a) $t_1 = t$, $t_j = 0$, $j \geq 2$

b) $t_n = t$, $t_j = 0$, $j \leq n-1$

a) is trivial:

$$\begin{aligned} W'(x) &\propto \det(x - C_1) \\ &\propto x^{n+1} \pm t_1 \qquad (\pm = (-1)^n) \end{aligned}$$

$$W(x) = \frac{1}{n+2}x^{n+2} \pm t_1 x$$

b) Let Δ_n be the $(n+1)\times(n+1)$ determinant $\det(x - C_1)$. Then

$$\Delta_n = x\,\Delta_{n-1} - t\,\Delta_{n-2}$$

$$\Delta_1 = x^2 - t, \quad \Delta_2 = x^3 - 2tx$$

This is the recurrence relation for the Chebyshev polynomials. However the first two Δ's are not quite Chebyshevs.

The solution to the recurrence relation can be written:

$$\Delta_n = (t)^{\frac{1}{2}(n+1)}\,\frac{\sin(n+2)\theta}{\sin\theta} = W'(x)$$

$$x = 2\sqrt{t}\,\cos\theta$$

One can integrate these, and indeed

$$W(x) = 2\, t^{\frac{1}{2}(n+2)} \cos(n+2)\theta$$

$$W'(x) = t^{\frac{1}{2}(n+1)} \frac{\sin(n+2)\theta}{\sin\theta}$$

$W(x)$ is Chebyshev

$$x = 2\sqrt{t}\,\cos\theta$$

Note. the critical points of W occur at

$$\theta = \frac{j\pi}{n+2} \qquad j = 1, \ldots, n+1$$

at which points

$$= 2\, t^{\frac{1}{2}(n+2)} \cos(j\pi)$$

$$= (-1)^j\, 2 (t)^{\frac{1}{2}(n+2)}$$

The fact that W takes only two values at its critical points will be of importance later.

N=2 integrable models

It is known that many of the N=2 superconformal models have one (and sometimes more) perturbations that lead to N=2 supersymmetric quantum integrable field theories. There are several methods for seeing this — one can use conformal perturbation theory as Mussardo has described, but this approach is unsystematic since one often does not know which perturbation to consider, and what spin the conserved current should have. On the other hand, there are more sophisticated Toda or free field methods that give one families of integrable models (Nemeschansky will describe one of these). The rule of thumb is that if there is a W-algebra somewhere, and if the W algebra generators are top components of a superfield, then there is usually a special $N=2$ supersymmetric relevant perturbation that leads to an $N=2$ supersymmetric integrable model.

For example, in the minimal series, ~~there are~~ three known perturbations that lead to an integrable model. These perturbations are given by ϕ_1, ϕ_2 and ϕ_n in the $c = \frac{3n}{n+2}$ model. For the ϕ_1 perturbation the holomorphic parts of the first ~~few~~ (of spin ≥ 3) conserved currents (or at least their corresponding states are):

$$G^+_{-1/2}\, G^-_{-1/2}\, J_{-1}^2 |0\rangle \Leftrightarrow G^+_{-1/2} G^-_{-1/2} :J^2:(z)$$

$$G^+_{-1/2}\, G^-_{-1/2} \left[J_{-1}^3 + \tfrac{1}{2}(c-3) J_{-1} L_{-2}\right] |0\rangle$$

$$G^+_{-1/2}\, G^-_{-1/2} \left[J_{-1}^4 + (c-3) L_{-2} J_{-1}^2 + \tfrac{2}{9}(c^2-3c+18) L_{-2}^2\right] |0\rangle$$

The new feature of the $N=2$ supersymmetric quantum integrable models is that we have an exact effective Landau-Ginzburg potential (for a lot of the models), and it is easy to compute.

Since the bosonic potential of the theory is $V = |\nabla W|^2$, it follows that zero energy ground states of the theory correspond to critical points of W. If $x_a^{(\alpha)}$ and $x_a^{(\beta)}$ are the values of x_a at two of these vacuum states, we can seek out minimum energy soliton configurations interpolating between these Landau-Ginsburg vacua. That is, we now consider a Lorentzian two dimensional field theory on $\mathbb{R}^2$ – (σ, τ) with

$$x_a = x_a^{(\alpha)} \quad \text{at} \quad \sigma = -\infty$$
$$x_a = x_a^{(\beta)} \quad \text{at} \quad \sigma = +\infty$$

One can then establish the following facts.

Define $\Delta W = W(x_a^{(\beta)}) - W(x_a^{(\alpha)})$

m = mass of soliton configuration

a) $m \geq |\Delta W|$

b) $m = |\Delta W|$ if and only if the soliton is a fundamental soliton ~~that~~ is killed by two of the four supercharges.

c) The classical trajectory, $x_a(\sigma)$, of a fundamental soliton ~~maps~~ at rest, maps to a straight line in the W-plane

(i.e. the ~~phase~~ complex phase of $W(x_a(\sigma))$ is constant).

Thus: W gives you quantum exact an impressive amount of information about the solitons of the integrable model.

Example: — the perturbation ϕ_n has

Naive $W(x;t) = \frac{1}{n+2}x^{n+2} + t\,x^n$ ← wrong

$W(x;t)$ = Chebyshev.

all the minima lie on the real line at

$$x = 2\sqrt{t}\cos\frac{j\pi}{n+2}$$

$|\Delta W|$ between nearest neighbours

$$= 4\,t^{\frac{1}{2}(n+2)} = \text{const}$$

all fundamental solitons have the same mass.

— the perturbation ϕ_1 leads to

$$W(x,t) = \frac{1}{n+2}x^{n+2} - t\,x$$

The ground states lie at $x = e^{\frac{2\pi i j}{n+1}}(t)^{\frac{1}{n+2}}$, $j = 0, 1, \dots, n$

the vertices of an $(n+1)$-gon

Let m_p be the length of a diagonal that subtends p sides - Elementary high school (the sine rule) geometry shows that

$$\frac{m_p}{m_1} = \frac{\sin\left(\frac{\pi p}{n+1}\right)}{\sin\left(\frac{\pi}{n+1}\right)}$$

— which are the Toda mass ratios.
(The reason for this will probably become evident from Nemeschansky's lectures)

Where do facts a), b) and c) come from?
they can all be obtained from a semi-classical argument
(See: Olive & Witten, phys Lett B 78 (1978) 97
(see Fendley, Mathur, Vafa & Warner phys Lett B 243 (1990) 257)

Fact a) can also be derived from the quantum commutators of the perturbed supersymmetry.

The outline proof is as follows:

Before you perturb, in Lorentz space the ~~non-zero~~ superalgebra commutators are

$$\{Q_+, Q_-\} = 2p_+ \quad , \quad \{\tilde{Q}_+, \tilde{Q}_-\} = 2p_-$$

where $p_\pm$ are the light-cone momentum components

After one perturbs, one finds that

$$\{Q_+, \tilde{Q}_+\} = 2\Delta W$$
$$\{Q_-, \tilde{Q}_-\} = 2\overline{\Delta W}$$

Let $A = Q_+ - \tilde{Q}_-\left(\frac{\Delta W}{m^2}\right)p_+$

Then if $|s\rangle$ is the soliton state one has

$$\langle s|\{A, A^\dagger\}|s\rangle \geq 0$$

$$\Rightarrow \quad m^2 \geq |\Delta W|^2$$

with equality if and only if $A|s\rangle = A^\dagger|s\rangle = 0$

The fact that

$\{Q_+, \tilde{Q}_+\} = 2\Delta W$ can be computed from the formulae on p. 29, and the fact that

$\phi_i \equiv \frac{\partial W}{\partial t_i}$. Then using quasihomogeneity of W, one

Hence $W(\lambda^{\frac{1}{n+2}} x, \lambda^{1-j/n+2} t_j) = \lambda W(x)$

$$\Rightarrow \quad W(x) = \frac{1}{(n+2)} \frac{\partial W}{\partial x} + \sum_j \left(1 - \frac{j}{n+2}\right) \frac{\partial W}{\partial t_j} .$$

This formula can be used to rewrite the

$$2 \sum t_i (1 - 2h_i)\, \phi_i(z,\bar z) = 2\left[W(x) - \frac{1}{(n+2)} \frac{\partial W}{\partial x} \right]$$

and when one does the calculation, one finds

$$\{ Q_+, \tilde Q_+ \} = 2\left[W(x) - \frac{1}{(n+2)} \frac{\partial W}{\partial x} \right]_{x_a^{(0)}}^{x_a^{(1)}}$$

and $\frac{\partial W}{\partial x}(x_a^{(0)}) = \frac{\partial W}{\partial x}(x_a^{(1)}) = 0$ by construction.

By this time I hope I have convinced you that the close relationship of $N=2$ SCFT to a topological field theory provides an extremely valuable computational tool for obtaining exact quantum information in $N=2$ theories.

I have only had time to describe a small part of this subject, and I have restricted myself to one particular aspect. There are plenty of related and highly active areas of research - some of which will soon be described by other lecturers.

REFERENCES

Disclaimer:

Since I only included very few references in my notes, I will try to rectify this here. The following list is not meant to be complete, it will contain (inadvertent) egregious omissions, but hopefully it will provide sufficient information to be of help.

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Fusion rings, topological G/G and N = 2 SCFT

) D. Gepner — Fusion rings and Geometry preprint NSF-ITP-90-184

) M. Spiegelglass & S. Yankielowicz — "G/G Topological Field Theories by cosetting G_k." Technion preprint PH-34-90 ; "Fusion Rules as Amplitudes in G/G Theories" Technion preprint PH-35-90

iii) M. Spiegelglass "Setting Fusion Rings in Topological Landau Ginzburg" Technion preprint PH-8-91

) K. Intriligator "Fusion Residues" — Harvard preprint ~ 1991

) D. Nemeschansky & N.P Warner "Topological Matter, Integrable Models and Fusion rings", USC preprint USC-91/031. Nucl Phys B to appear

Topological - Landau Ginzburg

(i) C. Vafa Mod. Phys. A 6 (1991) 337.

(ii) R. Dijkgraaf, E. Verlinde & H. Verlinde, Nucl. Phys. B352 (1991) 52 ; "Also Cargese / Trieste Lectures

* (iii) Keke Li Nucl. Phys. B354 (1991) 711 and 725.

* (iv) Dijkgraaf — Cargese Lectures 1991.

(v) E. Verlinde & N.P. Warner, Phys. Lett. B, ~ (1991)

(vi) D.-J. Smit, W. Lerche & N.P. Warner, Nucl. Phys. B ~ 1992

* (vii) Very Recent papers of Dubrovin and of Krichever.

(viii) B. Blok & A. Varchenko, "Topological Conformal Field Theories and Flat Coordinates" preprint IASSNS-HEP-91/5 (1991)
— This should be published by now

(ix) K. Saito references — get them from

* These papers are directed towards coupling topological gravity to topological matter.

Mirror Symmetries / Calabi-Yau

The Basic idea Mirror symmetries is discussed in Lerche, Vafa & Warner and also in work of Dixon. There is a vast body of literature on the subject of Calabi-Yaus, Mirrors & $N=2$. I have no intention of trying to provide a list. There are, however, many recent review/new papers coming out of the Conference at MSRI, Berkeley 1991. — Authors whose collaborators cover a significant part of the subject are: B. Greene, P. Candelas, E. Witten, C. Vafa, R. Schimmrigk (spelling), Aspinwall, Plesser, Morrison ---- and their collaborators, friends to whom I apologise from not mentioning here.

Integrable $N=2$ Field Theories

(i) P. Fendley, S. Mathur, C. Vafa & N.P. Warner, Phys. Lett. B243 (1990) 257.

(ii) P. Fendley, S. Mathur, W. Lerche & N.P. Warner Nucl. Phys. B348 (1991) 66.

(iii) W. Lerche & N.P. Warner. Nucl. Phys. B. ~ 1991 ~ 1992

(iv) See also D. Nemeschansky & N.P. Warner above.

(v) P. Fendley & K. Intriligator — two recent Harvard preprints

(vi) T. Maassarani, D. Nemeschansky & N.P. Warner — "Lattice Analogues of $N=2$ Superconformal Models" — USC preprints ~ 1992

(vii) A. Le Clair, D. Nemeschansky & N.P. Warner, "S-matrices for Perturbed $N=2$ Superconformal Field Theory and Quantum Groups".

— There are also extensive papers on minimal $N=2$ models on lattices - "

Other papers

— H. Saleur ~ 1991 preprints on $N=2$ & polymers

— H. Saleur & P. Fendley ~ 1992 — polymers & $N=2$

— Boucher Friedan & Kent ~ 1987 (Physics Letters) — classification of Unitary $N=2$ reps.

— There are also many, many papers on $N=2$ and superfield technology from 1980s e.g. Gates, Hull & Rocek.

— P. Howe & P. West — Feynman diagrams and ϵ- expansion methods in $N=2$ Landau Ginzburg.