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QCD AT LOW ENERGIES

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TROISIEME CYCLE DE LA PHYSIQUE
[illegible]

QCD AT LOW ENERGIES

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SEMESTRE D'HIVER 1988-1989

Abstract

These are notes of 7 lectures which I gave as part of the "Cours du Troisième Cycle de la Physique en Suisse Romande" during January and February 1989 in Lausanne. They provide an introduction to chiral perturbation theory and its application to low energy processes in QCD.

Acknowledgements

I wish to thank the organizers for their kind invitation to give these lectures, and J. - M. Bettems for his generous help in preparing the course.

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1. INTRODUCTION

The standard model

$SU(3)_c \times SU(2) \times U(1)$

describes

- The structure
  - of elementary particles
  - of nuclei (binding energy, energy levels,...)
  - of atoms
- Chemical properties of matter
- Scattering processes

  $p\bar{p} \to W^{\pm}X, Z^{o}X, \ldots$
- Nuclear reactions in sun

  $\vdots$

VERY AMBITIOUS!

Claim:

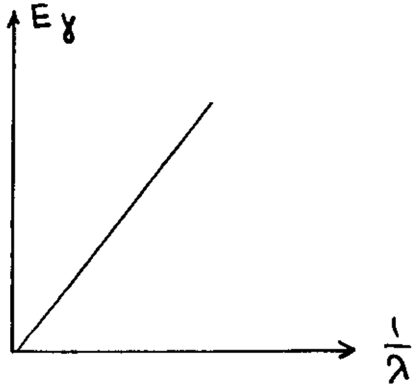
at low energies, Standard model reduces to

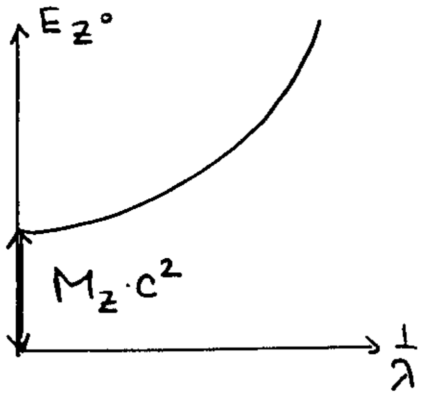
$$SU(3)_c \times \cancel{SU(2)} \times U(1)$$

Reason:

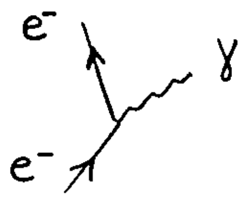
Structure of $|\Omega\rangle$ is complicated
$\uparrow$
ground state (vacuum)

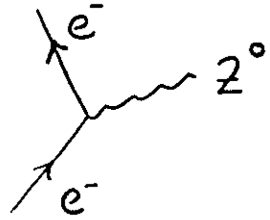
$|\Omega\rangle$ • transparent for γ } if λ
• opaque for $W^\pm, Z^\circ$ } is large





Energy gap





Z° penetrates up to

$$r \sim \frac{1}{M_Z} \simeq 2 \cdot 10^{-18}\,m$$

in vacuum

Model Higgs-particles in vacuum

colourless
neutral

$\swarrow$ $\searrow$

γ, gluons
don't see the Higgs particles

$W^\pm, Z$
scattered $\rightarrow$ Energy gap

Gauge fields with energy gap freeze at low energies

Higgs-particles $\rightarrow m_e, m_\mu, m_\tau$ 4a

$m_u, m_d, m_s, \dots; M_W, M_Z$

and we are left with

$\boxed{SU(3)_c \times U(1)}$ for $E \ll M_{W,Z} c^2$

$\{$ Up to corrections of order $O(E^2/M_W^2) \}$

Further approximation:

$e = 0$

u, d, s quarks only

$\boxed{\mathcal{L}_{eff} = \mathcal{L}_{QCD}}$ world where

leptons, photon $\rightarrow$ free particles

quarks + gluons $\rightarrow$ interact

$M_n - M_p \simeq 2\,\text{MeV}$

Exp. 1.29 MeV

1.1 Parameters in QCD

$$\mathcal{L}_{QCD} = -\frac{1}{2g^2}\langle F_{\mu\nu}F^{\mu\nu}\rangle + \sum_{q=u,d,s} \bar{q}\, i\gamma^\mu D_\mu q - \sum_{q=u,d,s} m_q \bar{q} q$$

$\langle A \rangle \equiv$ trace A for any matrix A

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$$A_\mu = A_\mu^a \frac{\lambda^a}{2} \qquad 8 \text{ gluons } A_\mu^a$$

$$D_\mu = \partial_\mu - i A_\mu$$

See Altarellis lectur [1]

Parameters:

$m_u, m_d, m_s, g^2 \longrightarrow \Lambda$

Λ : int. const. in $\mu \frac{d}{d\mu} g(\mu) = \beta[g(\mu)]$

Spectrum of bound states can be understood with

$m_u \sim 5$ MeV

$m_d \sim 9$ MeV

$m_s \sim 170$ MeV

$\left\{ \begin{array}{l} m_c \sim 1300 \text{ MeV} \\ m_b \sim 5 \text{ GeV} \end{array} \right.$

H. Leutwyler
J. G.
Phys. Rep. 82
[2]

m_u, m_d, m_s are <u>small</u> in comparison with a typical hadronic scale:

$$\frac{m_u}{m_p} = 0.005 \,, \quad \frac{m_d}{m_p} \simeq 0.01 \,, \quad \frac{m_s}{m_p} \simeq 0.18$$

$$\boxed{m_u = m_d = m_s = 0 \ : \ \text{good approximation}}$$

Remark: $M_{proton} = 2m_u + m_d$
$M_{neutron} = m_u + 2m_d$ $\rightarrow m_u \simeq m_d \simeq 300$ MeV

$\boxed{\text{is principally wrong}}$

$E < 1$ GeV ; $e = 0$

$$\mathcal{L}_{eff} = -\frac{1}{2g^2} \langle F_{\mu\nu} F^{\mu\nu} \rangle + \sum_{q=u,d,s} \bar{q} i \gamma^\mu D_\mu q$$

- renormalizable theory with 1 parameter:

$$g^2 \leftrightarrow \Lambda_{\overline{MS}} = 286 \,^{+245}_{-168} \text{ MeV}$$

Altarelli
Uppsala
[3]

- spectrum:

$M_\pi^2 = M_K^2 = M_\eta^2 = 0$; 8 Goldstone particl

$M_\rho^2 = C_\rho \cdot \Lambda^2_{\overline{MS}}$

$M_p^2 = C_p \cdot \Lambda^2_{\overline{MS}}$

$M_{\Omega^-}^2 = C_{\Omega^-} \cdot \Lambda^2_{\overline{MS}}$

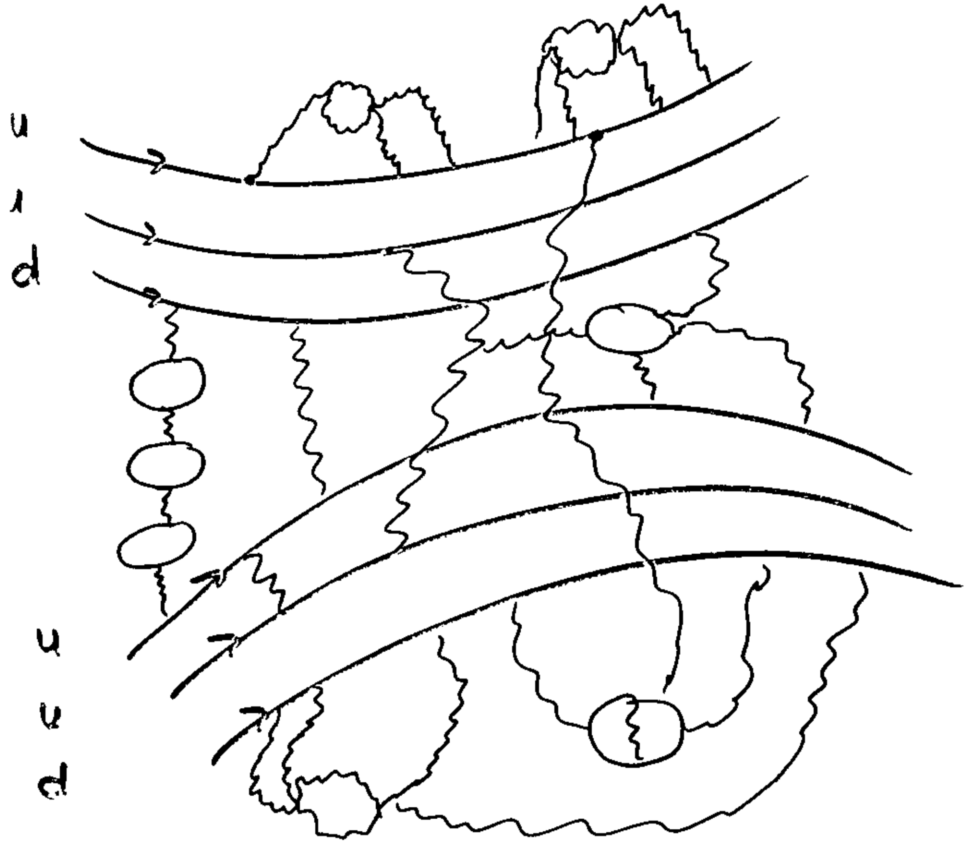
$C_\rho, \ldots$: pure numbers

M_p^2 / M_ρ^2 : Pure numbers
$M_{\Omega^-}^2 / M_\rho^2$ no parameter

$$\frac{d\sigma^{pp\to pp}}{d\Omega} = \frac{1}{M_p^2}\, f\left(\frac{s}{M_p^2}, \frac{t}{M_p^2}\right)$$

MOST REMARKABLE –

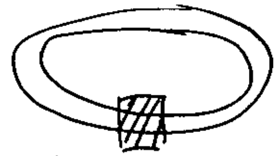
IN VIEW OF



1.2 Calculations ?

QCD is a <u>very beautiful</u> theory – unfortunately calculations are also <u>very difficult</u>:

$\mathcal{L}_{QCD} \longleftrightarrow$?



quarks
gluons

↑

"simple"

Detectors:
Mesons
protons, neutrons

$\longleftrightarrow$

complicated
(Confinement)

Several methods have been developed to cope with the problem:

i) Processes at high energies

At high momentum transfer, coupling constant which is operative becomes small:

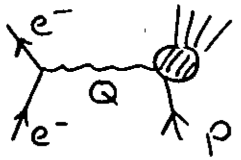
$$\alpha_{QCD}(Q^2) \to 0, \quad Q^2 \to \infty$$

(Asymptotic freedom)

Can calculate

- $R(s) = \sigma^{e^+e^- \to h}(s) / \sigma^{e^+e^- \to \mu^+\mu^-}$

- Q^2-dependence of structure functions



$F_i(x, Q^2)$

Altarellis lect. [1]

But: In general no precision statements, because one needs additional rules to calculate e.g. cross sections

$$\mathcal{L}_{QCD} \xrightarrow{?} \frac{d\sigma}{d\Omega}$$

ii) Lattice calculations

$$\mathcal{L}_{QCD} \longrightarrow M_p/M_\rho,\ M_\Delta/M_\rho,\ \sigma^{pp}_{TOT}, \dots$$

$\uparrow$ Ape, Cray, IBM, ...

no further assumptions

non perturbative

Review: Gupta [5]

- CPU-time needed is huge

- Asymmetries in mass spectrum

$\updownarrow$

Lattice of order $> \frac{1}{M_\pi}$ needed

$\to$ Long time until precision calculations (e.g. $M_n - M_p = 1.29$ MeV ?) will become available

iii) $1/N_c$ expansion

The size of the gauge group is enlarged,

$$SU(3) \longrightarrow SU(N_c) \qquad N_c : \# \text{ of colours}$$

One considers Green functions in the limit

$$N_c \to \infty, \quad g^2 N_c \text{ fixed} \qquad (1.1)$$

Ref. [4]

→ Green functions are proportional to a power of N_c in the limit (1.1)

An infinite # of graphs contribute at each order in the expansion

Example:

$$\langle 0 | T \, \bar{u}_x i\gamma_5 u_x \, \bar{u}_y i\gamma_5 u_y \, | 0 \rangle$$

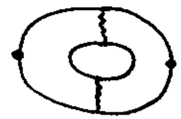
$\bar{u}_y i\gamma_5 u_y$  $\bar{u}_x i\gamma_5 u_x$

In the limit $N_c \to \infty$, $g^2 N_c$ fixed only the following graphs survive

$+ \cdots \quad O(N_c)$

Graphs with 1 quark loop suppressed

e.g.



$O(1)$

Non-planar graphs suppressed:

$O\left(\frac{1}{N_c}\right)$

iv) Chiral perturbation theory (CHPT)

Chiral symmetry of $\mathcal{L}_{QCD}$

$$\downarrow$$

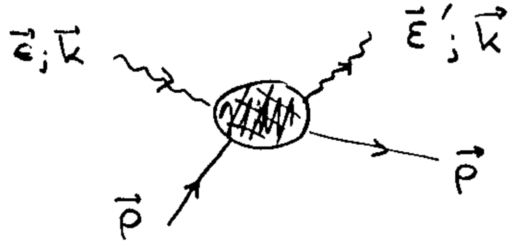
Ward identities for Greenfunctions

CHPT: Solves WI in a perturbative manner, takes into account also unitarity. Ref. [8]

Works at low energies ($E < 1\,\mathrm{GeV}$), because there the expansion parameter can be made small in the laboratory.

Illustration with Comptonscattering

$$\gamma + p \to \gamma + p$$



Low (1954!) [10]

$$\frac{d\sigma}{d\Omega} \sim |T|^2$$

$$\delta =: \frac{|\vec{k}|}{M_p}$$

$$T = C_0\, \vec{\varepsilon}\,'\cdot\vec{\varepsilon} + C_1\cdot\delta\,(\vec{\varepsilon}\,'\times\vec{\varepsilon})\cdot\vec{\sigma} + O(\delta^2) \qquad (1.2)$$

C_0: $-Q^2$

C_1: $i\,\frac{Q^2}{2}\,\varkappa_p^2$

$\varkappa_p = 1.79$ anom. magn. moment of proton

- Taylorseries in δ
- Expansionparameter can be made small in laboratory:

Consider scattering of low energy photons

- for photons of sufficiently small energies, the first few terms in the expansion <u>dominate</u>

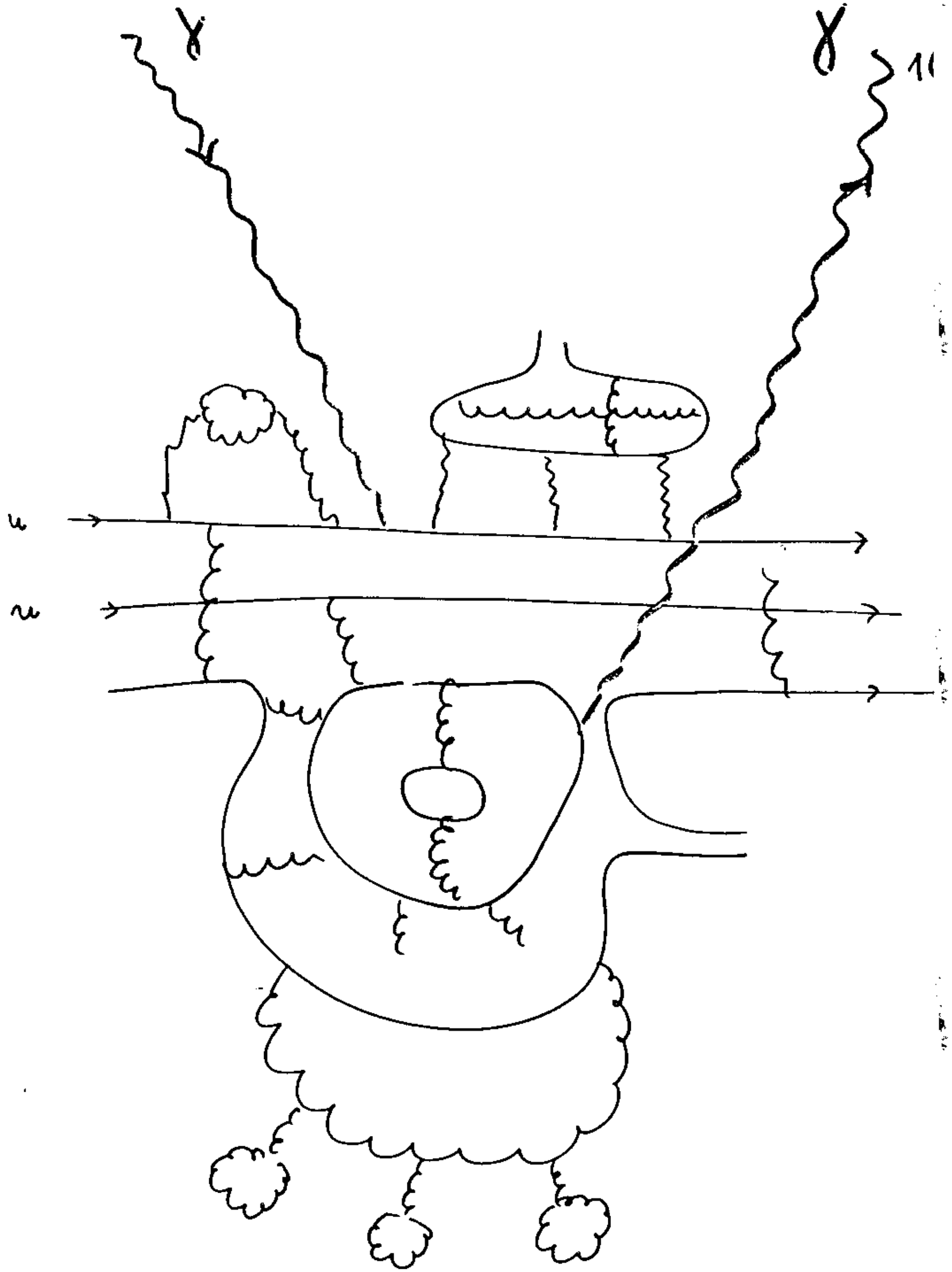
- Expansion 1989 ? (Fig. p.16a)
  Result makes only use of

a) $T = \epsilon'^{\mu}\epsilon^{\nu}\int dx\, e^{ikx}\langle p|T\, j_\mu^{e.m.}(x)\, j_\nu^{e.m.}(0)|p\rangle$

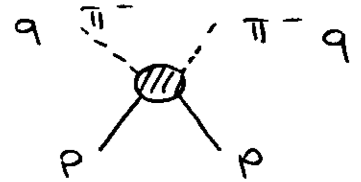
b) $\partial^\mu j_\mu^{e.m.}(x) = 0$
$\uparrow$
Gauge invariance of QED

Both a) and b) true in QCD + QED.

$\to$ Result (1.2) remains valid, although Feynmandiagrams look very complicated.



Back to CHPT: Consider elastic π^--proton scattering ($m_u = m_d = 0$)



a) $T_{\mu\nu} = \int dx\, e^{-iqx} \langle p | T A^{\pi^-}_{\mu}(x) A^{\pi^+}_{\nu}(0) | p \rangle$

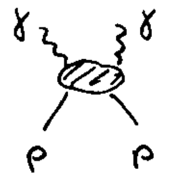
b) $A^{\pi^-}_{\mu}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)$

$\partial^\mu A^{\pi^-}_{\mu} = 0$ $\qquad A^{\pi^+}_{\mu} = \{A^{\pi^-}_{\mu}\}^{+}$

The scattering amplitude is the residue of the pion poles in $T_{\mu\nu}$,

$$T_{\mu\nu} = \frac{q_\mu q_\nu}{F_\pi^2}\left(\frac{1}{q^2}\right)^2 T(p,q) \longrightarrow \frac{d\sigma}{d\Omega} \sim |T|^2$$

Key point: Structure for matrix elements very similar



$\leftrightarrow \langle p | T j^{e.w.}_{\mu} j^{e.w.}_{\nu} | p \rangle ; \ \partial^\mu j^{e.w.}_{\nu} = 0$

$\leftrightarrow \langle p | T A^{\pi^-}_{\mu} A^{\pi^+}_{\nu} | p \rangle ; \ \partial^\mu A^{\pi^\pm}_{\mu} = 0$

Result:

$$\boxed{T^{\pi^- p \to \pi^- p} = \frac{|\vec{q}|}{2F_\pi^2}(1 - g_A^2)(1+\delta)} \qquad (1.3)$$

$F_\pi = 93.3\,\text{MeV}$, $g_A = 1.25$

$\vec{q}$: CM momentum

$\langle 0 | A^-_\mu | \pi^- \rangle \neq 0$ $\qquad$ Weinber 1967 [

- Expansion is done in powers of $\delta = O\left(\frac{|\vec{q}|}{M_p}\right)$: δ can be made small in laboratory

- Strong coupling constant α_s does not appear in the expansion! [Only implicitly, through F_π, g_A]

Calculations of that type were done for hadronic reactions in the late 60's and early 70's: Age of

<u>Current algebra</u>   rev.: Adler/Dashen [6]

How about higher order terms in eq. (1.3)?

- Very difficult with methods of current algebra
- $\boxed{\text{CHPT:}}$ Allows to determine those terms in a <u>systematic</u> manner, including corrections which are due to $m_{quark} \neq 0$ (Remember: Result above for $\pi^- p \to \pi^- p$ true for $m_u = m_d = 0$)
- Method works for any Green function in QCD – in fact also for

!
weak interactions [9]
QCD at low temperature and large volume [11]
production and decay of

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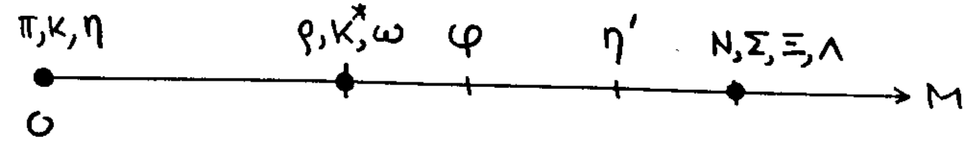
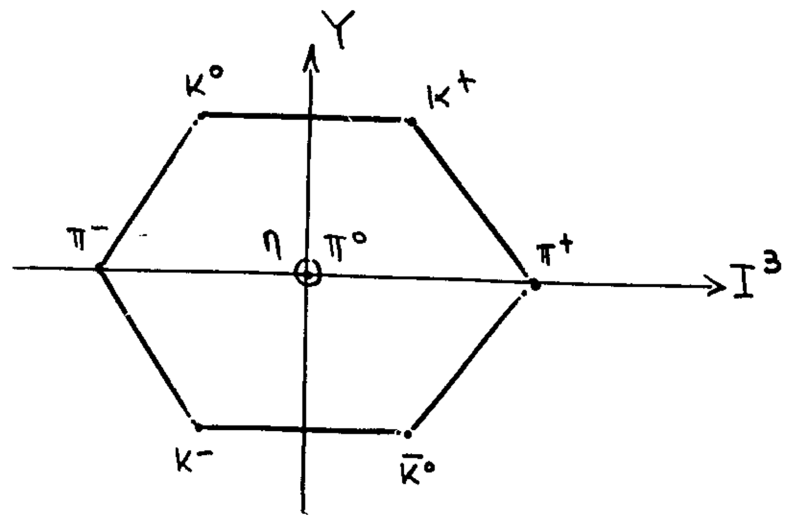
2. CHIRAL SYMMETRY AND ALL THAT

Content



$$\boxed{\text{spectrum at } m_u = m_d = m_s = 0}$$



2.1 QCD-Lagrangian (Notation)

For details see e.g. Altarellis lectures. [1]

$$\mathcal{L}_{QCD} = -\frac{1}{2g^2}\langle F_{\mu\nu}F^{\mu\nu}\rangle + \mathcal{L}_q^{(\ell)} + \mathcal{L}_q^{(h)}$$

$$\mathcal{L}_q^{(\ell)} = \bar{u}i\gamma^\mu D_\mu u + \bar{d}i\gamma^\mu D_\mu d + \bar{s}i\gamma^\mu D_\mu s - m_u\bar{u}u - m_d\bar{d}d - m_s\bar{s}s$$

$$\mathcal{L}_q^{(h)} = \bar{c}i\gamma^\mu D_\mu c + \bar{b}i\gamma^\mu D_\mu b + \ldots - m_c\bar{c}c - m_b\bar{b}b - \ldots \qquad (2.1)$$

$\mathcal{L}_q^{(\ell)}$: light quarks (u,d,s)

$\mathcal{L}_q^{(h)}$: heavy quarks $(c,b,t,\ldots)$

This splitting will be useful in the following discussion. Reason:

$m_u, m_d, m_s \ll m_{Proton}$: light

$m_u = m_d = m_s = 0$ good approximation

$m_c, m_b, m_t \gg m_{Proton}$: heavy

$m_c = m_b = m_t = 0$:

Gluons

$F_{\mu\nu}$: traceless, hermitean 3×3 matri

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$A_\mu = A_\mu^a \frac{\lambda^a}{2}$ Summation over index a

λ^a: Gell-Mann matrices (→notat

A_μ^a $a = 1,\ldots,8$: 8 gluon fields

$\langle M \rangle$: trace M.

Note: $\mathcal{L} = -\frac{1}{2g^2}\langle F_{\mu\nu}F^{\mu\nu}\rangle$: highly non-trivial field-theory [quarks not necessary for mathematical consistency!]

Pure Yang-Mills theory [→Glue-balls]

Quarks

D_μ in (2.1) : covariant derivative
↳ see below!

$$D_\mu = \partial_\mu - i A^a_\mu \underbrace{\frac{\lambda^a}{2}}$$
↳ 3×3, acts on colour indices

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$
← colour

In (2.1) :

$$\bar{u}\, i\gamma^\mu D_\mu u =$$

$$(\bar{u}_1\ \bar{u}_2\ \bar{u}_3)\left\{ i\gamma^\mu \partial_\mu \cdot \mathbb{1} + A^a_\mu \gamma^\mu \frac{\lambda^a}{2} \right\} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$
↑ 3×3

$$m_u \bar{u} u \equiv m_u \{ \bar{u}_1 u_1 + \bar{u}_2 u_2 + \bar{u}_3 u_3 \}$$

Gauge transformations

QCD is <u>gauge</u> theory

$\mathcal{L}_{QCD}$ invariant under

$$A_\mu \longrightarrow U A_\mu U^+ + i U \partial_\mu U^+$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \longrightarrow U \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \qquad q = u, d, s$$

$$U = U(x)\,, \quad U(x) U^+(x) = \mathbb{1}\,, \quad \det U(x) = 1 \qquad (2.2$$

The (space-time dependent) matrices $U(x)$ act on the colour indices.

According to (2.2), the gauge group is

$$SU(3)_c$$
└ colour

[Note: $D'_\mu q' \equiv (\partial_\mu - i A'_\mu) q'$

$= U D_\mu q$: transforms as c

therefore <u>covariant</u> derivative]

2.2 Flavour Symmetry (2 flavours)

Consider contribution of u, d quarks to $\mathcal{L}_{QCD}$ [see (2.1)]

$$\mathcal{L}_{ud} \doteq \bar{u} i\gamma^\mu D_\mu u + \bar{d} i\gamma^\mu D_\mu d - m_u \bar{u} u - m_d \bar{d} d$$

$$= (\bar{u}\ \bar{d}) \begin{pmatrix} i\not{D} - m_u & 0 \\ 0 & i\not{D} - m_d \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} \qquad (2.3)$$

where I introduced

$i\not{D} \equiv i\gamma^\mu D_\mu$ (all indices suppresied)

It is convenient to write $\mathcal{L}_{ud}$ in terms of right- and lefthanded fields:

Let

ψ: Dirac 4-spinor

$$\psi_L = \tfrac{1}{2}(1-\gamma_5)\psi \quad , \quad \psi_R = \tfrac{1}{2}(1+\gamma_5)\psi$$

$$\psi = \psi_L + \psi_R \quad ; \quad \bar{\psi}_{\substack{L \\ R}} = \tfrac{1}{2}\bar{\psi}(1 \pm \gamma_5)$$

2 groups of γ-matrices:

$$\bar{\psi}\Gamma\psi = \begin{cases} \bar{\psi}_L \Gamma \psi_L + \bar{\psi}_R \Gamma \psi_R & ; \Gamma = \gamma_\mu, \gamma_\mu\gamma_5 \\ \bar{\psi}_L \Gamma \psi_R + \bar{\psi}_R \Gamma \psi_L & ; \Gamma = \mathbb{1}, i\gamma_5, \sigma_{\mu\nu} \end{cases}$$

Now we can write for $\mathcal{L}_{ud}$:

$$\mathcal{L}_{ud}(u_L, d_L; u_R, d_R; m_u, m_d) =$$

$$(\bar{u}_L\ \bar{d}_L) \begin{pmatrix} i\not{D} & 0 \\ 0 & i\not{D} \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} - (\bar{u}_L\ \bar{d}_L) \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \begin{pmatrix} u_R \\ d_R \end{pmatrix}$$

$$+ (L \leftrightarrow R) \qquad (2.4)$$

Next we consider a rotation of the Left-handed quark fields:

$$\begin{pmatrix} u_L' \\ d_L' \end{pmatrix} \doteq V_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad (2.5)$$

where V_L is a constant (x-independent) U(2)-matrix

$$V_L \cdot V_L^+ = \mathbb{1}$$

<u>Claim:</u> For $m_u = m_d = 0$, $\mathcal{L}_{ud}$ is invariant under the transformation (2.5):

$$\mathcal{L}_{ud}(u_L', d_L'; u_R, d_R; 0, 0) = \mathcal{L}_{ud}(u_L, d_L; u_R, d_R; 0, 0)$$

<u>Note:</u> Since the gluon fields are not touched by the transformation (2.5), the term $-\frac{1}{2g^2}\langle F_{\mu\nu}F^{\mu\nu}\rangle$ in $\mathcal{L}_{QCD}$ (see (2.1)) is also invariant. Furthermore, the other quark fields $s, c, t, \ldots$ remain the same $\longrightarrow$ therefore,

$$\left.\begin{matrix} \mathcal{L}_{QCD}\big|_{m_u = m_d = 0} \\ \\ \text{is invariant under (2.5)} \end{matrix}\right\} \longrightarrow \text{conserved currents!}$$

<u>Proof:</u>

$$\mathcal{L}_q(u_L', d_L'; u_R, d_R; 0, 0)$$

$$= \underbrace{(u_L', d_L')\begin{pmatrix} i\partial\!\!\!/ & 0 \\ 0 & i\partial\!\!\!/ \end{pmatrix}\begin{pmatrix} u_L' \\ d_L' \end{pmatrix}}_{\circledast} + (\bar{u}_R\, \bar{d}_R)\begin{pmatrix} i\partial\!\!\!/ & 0 \\ 0 & i\partial\!\!\!/ \end{pmatrix}\begin{pmatrix} u_R \\ d_R \end{pmatrix}$$

$$\circledast = (\bar{u}_L\, \bar{d}_L)\, \underbrace{V_L^{+}\begin{pmatrix} i\partial\!\!\!/ & 0 \\ 0 & i\partial\!\!\!/ \end{pmatrix} V_L}_{D'} \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

$$D' = \begin{pmatrix} i\partial\!\!\!/\,[|V_L^{11}|^2 + |V_L^{21}|^2] & i\partial\!\!\!/\,[\bar{V}_L^{11} V_L^{12} + \bar{V}_L^{21} V_L^{22}] \\ i\partial\!\!\!/\,[\bar{V}_L^{12} V_L^{11} + V_L^{21} \bar{V}_L^{22}] & i\partial\!\!\!/\,[|V_L^{12}|^2 + |V_L^{22}|^2] \end{pmatrix},$$

On the other hand we have from

$$V_L^{+} V_L = \mathbb{1}$$

$$|V_L^{11}|^2 + |V_L^{21}|^2 = 1 = |V_L^{12}|^2 + |V_L^{22}|^2$$

$$\bar{V}_L^{11} V_L^{12} + \bar{V}_L^{21} V_L^{22} = \bar{V}_L^{12} V_L^{11} + \bar{V}_L^{22} V_L^{21} = 0$$

and therefore

$$D' = D$$

and

$$\mathcal{L}_{ud}(u_L', d_L'; u_R, d_R; 0, 0) = \mathcal{L}_{ud}(u_L, d_L; u_R, d_R; 0, 0)$$

$\square$

It is obvious that $\mathcal{L}_{ud}$ is also invariant under transformations of the <u>right-handed</u> fields

$$\begin{pmatrix} u_R' \\ d_R' \end{pmatrix} = V_R \begin{pmatrix} u_R \\ d_R \end{pmatrix} \; ; \; V_R^\dagger V_R = \mathbb{1} \tag{2.6}$$

On the other hand, $\mathcal{L}_{ud}$ is <u>not</u> invariant under (2.5) or (2.6) if $m_u, m_d \neq 0$: Consider

$$\begin{pmatrix} u_L' \\ d_L' \end{pmatrix} = V_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \; ; \; \begin{pmatrix} u_R' \\ d_R' \end{pmatrix} = V_R \begin{pmatrix} u_R \\ d_R \end{pmatrix}$$

Then the mass term in $\mathcal{L}_{ud}$ becomes

$$(\bar{u}_L' \; \bar{d}_L') \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \begin{pmatrix} u_R' \\ d_R' \end{pmatrix} + (L \leftrightarrow R)$$

$$= (\bar{u}_L \bar{d}_L) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} u_R \\ d_R \end{pmatrix} + (L \leftrightarrow R)$$

with

$$M_{11} = m_u \bar{V}_L^{11} V_R^{11} + m_d \bar{V}_L^{21} V_R^{21}$$

$$M_{12} = m_u \bar{V}_L^{11} V_R^{12} + m_d \bar{V}_L^{21} V_R^{22}$$

$$M_{21} = m_u \bar{V}_L^{12} V_R^{11} + m_d \bar{V}_L^{22} V_R^{21}$$

$$M_{22} = m_u \bar{V}_L^{12} V_R^{12} + m_d \bar{V}_L^{22} V_R^{22}$$

Invariant, if

- $m_u = m_d = 0$

or

- $m_u = m_d$, $V_L = V_R$ ('diagonal subgroup')

or

- $m_u \neq m_d$: $V_L = V_R = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}$

<u>Summary:</u> $m_u = m_d = 0$

$\mathcal{L}_{QCD}$ invariant under

$$\begin{pmatrix} u \\ d \end{pmatrix}_{L,R} \longrightarrow V_{L,R} \begin{pmatrix} u \\ d \end{pmatrix}_{L,R} \qquad V_{L,R} \in U(2)$$

$$\boxed{U(2) \times U(2)}$$

Only subgroup, if $m_u, m_d \neq 0$

2.3 Flavour symmetry (3 flavours)

We consider now the case where

$$m_u = m_d = m_s = 0$$

in $\mathcal{L}_{QCD}$. [Remember that this is a good approximation to the real world!]

Then $\mathcal{L}_q^{(\ell)}$ in eq. (2.1) is invariant under

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}_{L,R} \to V_{L,R} \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{L,R}$$

where $V_{L,R}$ are now 3×3 unitary, constant (i.e. x-independent) matrices

$$V_{L,R} \cdot V_{L,R}^{+} = \mathbb{1}$$

According to Noether's theorem, there will be a conserved current for each of the continuous parameters which parametrize the symmetry group:

$$U(3) \;:\; 9 \text{ parameters}$$

Hence we expect $2\times 9 = 18$ conserved currents. Due to quantum effects, only 17 currents are conserved. These are

$$L_\mu^a = \bar{q}_L \gamma_\mu \frac{\lambda^a}{2} q_L \;; \qquad \partial^\mu L_\mu^a = 0$$

$$a = 1, \dots, 8$$

$$R_\mu^a = \bar{q}_R \gamma_\mu \frac{\lambda^a}{2} q_R \qquad \partial^\mu R_\mu^a = 0$$

$$V_\mu = \bar{q} \gamma_\mu \cdot \mathbb{1} q \qquad \partial^\mu V_\mu = 0$$

where $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$; $q_{L,R} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{L,R}$ (2.7)

The global symmetry of QCD in the chiral limit $m_u = m_d = m_s = 0$ is thus

$$\boxed{SU(3)_L \times SU(3)_R \times U(1)_V} \qquad (2.8)$$

$$L_\mu^a \qquad R_\mu^a \qquad V_\mu : \; q_{L,R} \to e^{i\alpha} q_{L,R}$$

The nonconserved current is

$$\bar{q}\gamma^\mu\gamma_5 q \,, \quad \partial_\mu \bar{q}\gamma^\mu\gamma_5 q = -\frac{3}{16\pi^2} F_{\mu\nu}^a \tilde{F}^{\mu\nu a}$$

where $\tilde{F}^{a}_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma,a}$

$\varepsilon^{0123} = 1$

$F\tilde{F}$ is itself a divergence → it is possible to write an 18th current which <u>is</u> conserved: U(1) - problem.

Discarded in the following

Nomenclature

- eq. (2.8) is referred to as

$$\left.\begin{array}{l} \mathcal{L}_{QCD} \text{ is "chiral symmetric"} \\ \text{or} \\ \mathcal{L}_{QCD} \text{ is "chiral invariant"} \\ \text{or} \\ \text{chiral symmetry of QCD} \end{array}\right\} \text{for } m_u = m_d = m_s = 0$$

- $m_u = m_d = m_s = 0$ or $m_u = m_d = 0,\ m_s \neq 0$

  "chiral limit"

e.g. " in the chiral limit, there are 17 conserved currents "

For later use we introduce the linear combinations

$$V^{\mu a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q = L^{\mu a} + R^{\mu a}$$

$$A^{\mu a} = \bar{q}\gamma^\mu\gamma_5 \frac{\lambda^a}{2} q = L^{\mu a} - R^{\mu a} \qquad a=1,\dots,8$$

$$\partial_\mu V^{\mu a} = \partial_\mu A^{\mu a} = 0 \qquad (2.9a)$$

We will also include the ninth conserved vector current in this list,

$$V^{\mu a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q \quad , \quad a=0,1,\dots,8$$

$$\underbrace{\lambda^0 = \sqrt{\tfrac{2}{3}}\cdot \mathbb{1}}_{\text{convention}} \qquad \rightarrow \langle \lambda^{0\,2} \rangle = 2$$

As noted in section (2.2), the quark masses break the symmetry (2.8) – the divergence of the currents is proportional to m_u, m_d, m_s:

$$\partial_\mu V^{\mu a} = i\,\bar{q}(x)\left[\mathcal{M}, \frac{\lambda^a}{2}\right] q(x)$$

$$\partial_\mu A^{\mu a} = i\,\bar{q}(x)\left\{\mathcal{M}, \frac{\lambda^a}{2}\right\}\gamma_5 q(x) \qquad (2.10)$$

where

$$\mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

is the quark mass matrix, and

$$[A,B] = AB - BA$$

$$\{A,B\} = AB + BA$$

In the case $m_u = m_d = m_s \neq 0$, the nine vector currents are still conserved. This is most easily seen from eq. (2.10):

$$m_u = m_d = m_s = m \rightarrow \mathcal{M} = m\cdot\mathbb{1}$$

$$[\mathbb{1}, \lambda^a] = 0$$

2.4 Algebra of charges Q^a

In the chiral limit $m_u = m_d = m_s = 0$, the 17 conserved currents in (2.9) give rise to 17 conserved charges

$$Q_V^a = \int d^3x \; V_0^a(\vec{x}, t) \qquad a = 0, 1, \dots, 8$$

$$Q_A^a = \int d^3x \; A_0^a(\vec{x}, t) \qquad a = 1, \dots, 8 \qquad (2.11)$$

with
$$\frac{dQ_V^a}{dt} = 0 = \frac{dQ_A^a}{dt}$$

It does therefore not matter, what value of t is chosen in (2.11)

Proof of time-independence:

$$\frac{dQ_V^a}{dt} = \int d^3x \; \partial_0 V_0^a(\vec{x}, t) \underset{\partial_\mu V^{\mu a} = 0}{=} -\int d^3x \; \partial_i V^{ia}(\vec{x}, t) \underset{\text{partial integration}}{=} 0$$

These charges generate an SU(3)×SU(3)×U(1) algebra:

$$\begin{tikzcd} \mathcal{L}_{QCD} \arrow[r, "SU(3)\times SU(3)\times U(1)"] & \begin{pmatrix} V^{\mu a} \\ A^{\mu a} \end{pmatrix} \arrow[r] & \begin{pmatrix} Q_V^a \\ Q_A^a \end{pmatrix} \arrow[ll, bend left, "\text{generate}"] \end{tikzcd}$$

see e.g. Georgi [2]

Consider the SU(3)×SU(3) part.

Define
$$Q_{\substack{L \\ R}}^a \doteq \frac{1}{2}\left(Q_V^a \pm Q_A^a\right) \qquad a = 1, \dots, 8$$

Then

$$\boxed{\begin{aligned} &[Q_I^a, Q_I^b] = i f^{abc} Q_I^c \qquad I = L, R \\ &[Q_L^a, Q_R^b] = 0 \end{aligned}} \qquad (2.12)$$

Lie-algebra of $SU(3)_L \times SU(3)_R$

f^{abc}: Structure constants of SU(3) (→ p. 59)

The corresponding commutation relations for the vector- and axial vector charges Q_V^a, Q_A^a read

$$[Q_V^a, Q_V^b] = i f^{abc} Q_V^c$$

$$[Q_A^a, Q_A^b] = i f^{abc} Q_V^c$$

$$[Q_V^a, Q_A^b] = i f^{abc} Q_A^c \qquad (2.13)$$

Note that Q_V^1, Q_V^2 and Q_V^3 generate the same Lie-Algebra as angular momentum in Quantum mechanics,

$$[Q_V^i, Q_V^j] = i\,\varepsilon^{ijk} Q_V^k \qquad i,j,k \in \{1,2,3\}$$

$$\varepsilon^{123} = 1 = f^{123}$$

In addition we shall use

$$[Q_V^a, A_\mu^b(\vec{x},t)] = i f^{abc} A_\mu^c(\vec{x},t)$$

The relations (2.12), (2.13) may be proven either

- using the general argument summarized in the figure on top of p. ~~36~~ 41
- or using the definitions (2.11) and going back to the anticomm. relations for fermion fields at equal times.

2.5 Spontaneous breakdown of chiral symmetry

We consider the hamiltonian which corresponds to the chiral limit of QCD:

$$\mathbb{H}_0 \doteq \mathbb{H}_{QCD}\big|_{m_u = m_d = m_s = 0}$$

The heavy quarks c, t, b, .. are included in $\mathbb{H}_0$, together with their nonzero mass terms. Recall that the presence of the heavy quarks does not invalidate the conservation of the vector- and axial-currents (eq. (2.9)).

Claim: The vector- and axial charges (2.11) commute with $\mathbb{H}_0$

$$[Q_V^a, \mathbb{H}_0] = 0 \qquad a = 0, 1, \dots, 8$$

$$[Q_A^a, \mathbb{H}_0] = 0 \qquad a = 1, \dots, 8$$

(2.14)

[Proof is given below]

we shall now argue that the spectrum of $\mathbb{H}_0$ consists of mass degenerate multiplets.

Consider an eigenstate of $\mathbb{H}_0$

$$\mathbb{H}_0 |\Psi\rangle = E |\Psi\rangle$$

Then $|\Psi\rangle_V^a \doteq Q_V^a |\Psi\rangle$ has the same energy as $|\Psi\rangle$:

$$[\mathbb{H}_0, Q_V^a] |\Psi\rangle = \mathbb{H}_0 |\Psi\rangle_V^a - E |\Psi\rangle_V^a = 0$$

$$\boxed{\mathbb{H}_0 |\Psi\rangle_V^a = E |\Psi\rangle_V^a} \qquad a = 0, 1, \dots, 8$$

In general, $Q_V^a |\Psi\rangle \neq |\Psi\rangle$, and because the charges Q_V^a generate the Lie-Algebra of SU(3), one expects that the mass-degenerate multiplets come in multiplets of SU(3). In the three quark sector we expect the ground state multiplet to consist of P, N, Λ, Σ, Ξ - all with the same mass. The decuplet (quark spins aligned) is also degenerate in mass,

$$M_N \neq M_\Delta = M_{\Sigma^*} = M_{\Xi^*} = M_\Omega .$$

There will also be triplets, antitriplets and sextets of baryons. A list of these multiplets can be found e.g. in $\underbrace{\text{[2], Appendix A}}_{\text{Ref. Introd.}}$

There is $\overbrace{\text{even}}^{\text{apparently}}$ more $\overbrace{\text{symmetry}}^{\text{in the spectrum}}$ than we exploited until now: axial charges are also conserved: $[\mathbb{H}_0, Q_A^a] = 0$.

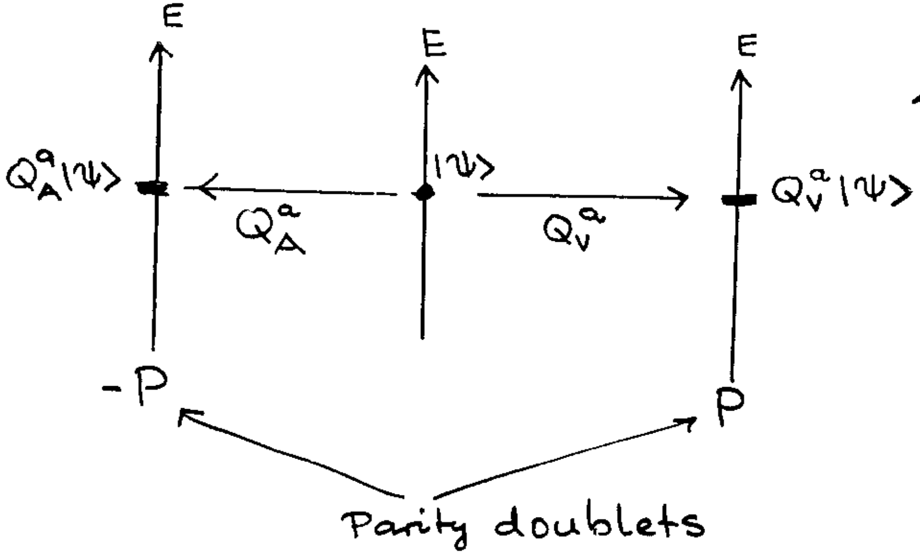
From

$$\mathbb{H}_0 |\Psi\rangle = E |\Psi\rangle$$

we conclude

$$\mathbb{H}_0 Q_A^a |\Psi\rangle = Q_A^a \mathbb{H}_0 |\Psi\rangle = E Q_A^a |\Psi\rangle$$

The axial charges carry negative parity - $\rightarrow$ to a given bound state there seems to exist a mass degenerate state of opposite parity:



<u>No trace</u> of such a symmetry in nature!

<u>Explanation</u> [Nambu, Jona-Lasinio 1960,61]: [3]

Vacuum is non-invariant under Q_A^a:

$$Q_A^a|0\rangle \neq 0\,;\quad a=1,\dots,8 \qquad (2.15)$$

<u>Consequences of (2.15):</u>

1) There are 8 massless, pseudoscalar particles in spectrum of H_0 (Goldstone bosons) [4,1]

2. $Q_A^a|\psi\rangle$ has same energy as $|\psi\rangle$, it is however not a one particle state:

$$Q_A|\psi\rangle = |\psi;\underbrace{G_1,\dots,G_N,\dots}\rangle$$

Goldstone particles
zero momentum
→ zero energy

$Q_A|\psi\rangle$ is therefore not listed in the PDG

→ no contradiction with $[H_0, Q_A^a]=0$

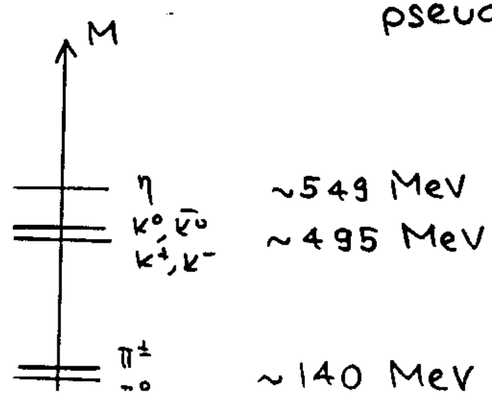
$$\boxed{[H_0, Q_A^a]=0 \;\not\to\; \text{Multiplet structure of states}}$$

"Spontaneously broken symmetry"

"Hidden symmetry"

Where are these massless, pseudoscalar particles?

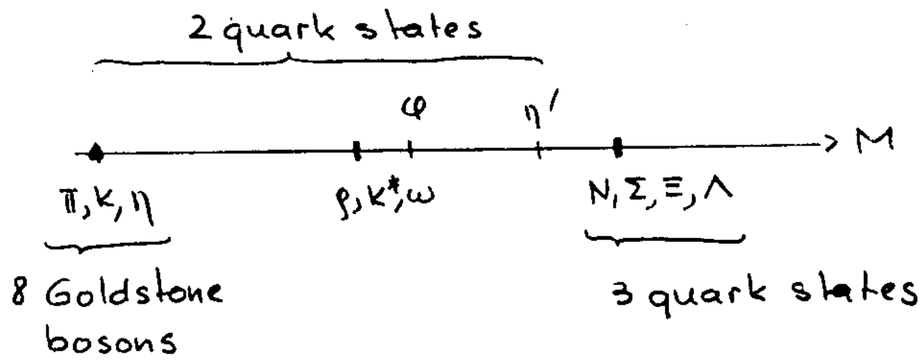
$\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$: lightest hadrons, are pseudoscalars



π, K, η are 'would-be Goldstone bosons': They would be massless in a world where $m_u = m_d = m_s = 0$.
They are not massless, because

$m_u \sim 5$ MeV
$m_d \sim 9$ MeV
$m_s \sim 170$ MeV.

Besides π, K, η there will be more (massive) multiplets which are bound states of a (massless) quark-antiquark pair: (ρ, ω, K^*) [quark spins aligned] etc.



and many more!

• Proof that $[H_0, Q_V^a] = 0$.

$$Q_V^a = \int d^3x \, V_0^a(\vec{x}, t)$$
$$= \int d^3x \, e^{iH_0 t} V_0^a(\vec{x}, 0) e^{-iH_0 t}$$

$$\frac{dQ_V^a}{dt} = 0 = i \int d^3x \, [H_0, V_0^a(\vec{x}, t)] = i [H_0, Q_V^a]$$

And analogous for $[H_0, Q_A^a] = 0$

$\square$

2.6 Goldstone theorem and $\langle 0|\bar{q}q|0\rangle$

Consider a QFT with a conserved current

$$\partial^\mu j_\mu = 0$$

whose charge Q satisfies

$$\langle 0|[Q,\Phi]|0\rangle \neq 0 \;;\; Q = \int d^3x\, j_0(\vec{x},0)$$

for some (scalar or pseudoscalar) field Φ.

Theorem [4] : In this case there is a massless particle in the theory with the quantum numbers of the field Φ. Its coupling to the current j^μ does not vanish,

$$\langle 0|\, j^\mu(0)\,|m\rangle \neq 0$$

↑ Goldstone boson

$$P_\mu|m\rangle = p_\mu|m\rangle \,,\; p^2 = 0$$

- Why Goldstone-boson? (heuristic)

Let $Q_A^a|0\rangle \neq 0$

→ 1) $\exists\ \langle m|$ with

$$\langle m|Q_A^a|0\rangle \neq 0$$

and therefore

$$\int d^3x \langle m|A_0^a(\vec{x},t)|0\rangle \neq 0$$

2) Use translational invariance

$$\langle m|A_0^a(\vec{x},t)|0\rangle = e^{iE_m t - i\vec{p}_m\cdot\vec{x}} \langle m|A_0^a(0)|0\rangle$$

From 1):

$$(2\pi)^3\,\delta^3(\vec{p}_m)\, e^{iE_m t} \langle m|A_0^a(0)|0\rangle \neq 0 \quad \Big|\ \frac{d}{dt}$$

$$E_m\cdot\delta^3(\vec{p}_m)\, \underbrace{\langle m|A_0^a(0)|0\rangle}_{\neq 0} = 0 \qquad (\dot{Q}_A = 0\,!$$

$$\boxed{\lim_{\vec{p}_m\to 0} E_m(\vec{p}_m) = 0}$$

→ $|m\rangle$ massless state

Gell-Mann, Oakes, Renner [6]:

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle \neq 0 \qquad (2.16)$$

signal spontaneous breakdown of chiral symmetry.

Indeed: Use

$$i[Q_A^m, \bar{q}i\gamma_5\lambda^m q] = \begin{cases} \bar{u}u + \bar{d}d & m = 1,2,3 \\ \bar{u}u + \bar{s}s & m = 4,5 \\ \bar{d}d + \bar{s}s & m = 6,7 \\ \frac{1}{3}(\bar{u}u + \bar{d}d + 4\bar{s}s) & m = 8 \end{cases} \qquad (2.17)$$

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

and take matrix elements between the groundstate $|0\rangle$. In view of (2.16), the 8 fields $\bar{q}i\gamma_5\lambda^m q$ have the property

$$\langle 0|[Q_A^m, \bar{q}i\gamma_5\lambda^m q]|0\rangle \neq 0$$

Invoke Goldstone theorem with

$$j_\mu^m = A_\mu^m, \quad \Phi^m = \bar{q}i\gamma_5\lambda^m q$$

$\rightarrow$ || 8 massless pseudoscalar particles

if (2.16) holds

Note: 1) In general, one has

$$\langle 0|\bar{u}u|0\rangle \neq \langle 0|\bar{d}d|0\rangle \neq \langle 0|\bar{s}s|0\rangle \neq \langle 0|\bar{u}u|0\rangle \qquad (2.18)$$

These expected values are equal only for $m_u = m_d = m_s$.

2) The nondiagonal matrix elements

$\langle 0|\bar{u}d|0\rangle$, $\langle 0|\bar{u}s|0\rangle$ all vanish. This can be seen from

$$[Q_V^a, \bar{q}\lambda^b q] = i f^{abc}\bar{q}\lambda^c q$$

and taking matrix elements between $|0\rangle$ using

$Q_V^a|0\rangle = 0$ (otherwise scalar massless particles!)

Shall use

$$\langle 0 | A^a_\mu(0) | \pi^b(p) \rangle = i\, p_\mu\, F_0\, \delta^{ab}$$

Lorentz covariance — convention — group theory

2.7 Isospin, Hypercharge, Strangeness

Consider $|\pi^a\rangle$, $a = 1, \dots, 8$

$$\langle 0 | A^a_\mu | \pi^b \rangle = i p_\mu F_0 \delta^{ab}$$

$$[Q_V^a, Q_V^b] = i f^{abc} Q_V^c$$

$$[Q_V^a, A_\mu^b] = i f^{abc} A_\mu^c \qquad (2.19)$$

Group theory: 2 out of the 8 operators Q_V^a can simultaneously be diagonalized

Choose

$$Q_V^3, \quad Q_V^8$$

'Isospin' 'Hypercharge'

Let $\quad Q_V^a |\pi^b\rangle = i f^{abc} |\pi^c\rangle \qquad (2.20)$

Then

$$[Q_V^a, Q_V^b] |\pi^\gamma\rangle = i f^{abc} Q_V^c |\pi^\gamma\rangle$$

i.e. (2.20) is a realization of the comm. relations in (2.19)

Eigenstates of

$$\mathrm{I}_3 = Q_V^3, \quad Y = \frac{2}{\sqrt{3}} Q_V^8$$

are

$$|K^+\rangle = -\frac{1}{\sqrt{2}}(|\pi^4\rangle + i|\pi^5\rangle)$$

$$|K^0\rangle = -\frac{1}{\sqrt{2}}(|\pi^6\rangle + i|\pi^7\rangle)$$

$$|\pi^+\rangle = -\frac{1}{\sqrt{2}}(|\pi^1\rangle + i|\pi^2\rangle)$$

$$|\pi^-\rangle = \frac{1}{\sqrt{2}}(|\pi^1\rangle - i|\pi^2\rangle)$$

$$|\pi^0\rangle = |\pi^3\rangle$$

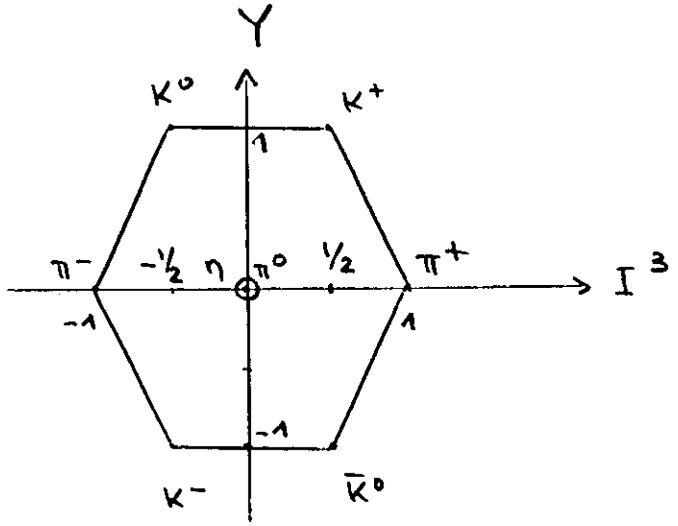
$$|\bar{K}^0\rangle = -\frac{1}{\sqrt{2}}(|\pi^6\rangle - i|\pi^7\rangle)$$

$$|K^-\rangle = \frac{1}{\sqrt{2}}(|\pi^4\rangle - i|\pi^5\rangle)$$

$$|\eta\rangle = |\pi^8\rangle$$

Ref. [5]

with eigenvalues given in the following



Strangeness

$$[Q_V^0, A_\mu^i] = 0 \quad \forall i \qquad Q_V^0 = \sqrt{\tfrac{2}{3}} \int d^3x\, V_0^0(\vec{x}, 0)$$

$$\langle 0 | A_\mu^i Q_V^0 | \pi^k \rangle = 0 \quad \rightarrow \quad Q_V^0 | \pi^k \rangle = 0$$

Eigenvalues of $\cdot \frac{1}{3} Q_V^0$ are called "baryonnumber"

" of $\cdot\, S = Y - \frac{1}{3} Q_V^0$ are called "strangeness"

$$S = -\int d^3x\, s^+(x)\, s(x) \longleftarrow$$

strangeness = hypercharge for mesons

2.8 SUMMARY

1) $\mathcal{L}_{QCD}$ invariant under

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}_{\substack{L\\R}} \longrightarrow V_{\substack{L\\R}} \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{\substack{L\\R}} \qquad (\text{for } m_u = m_d = m_s = c$$

$$V_{\substack{L\\R}} \in SU(3) \quad \text{or} \quad V_L = V_R = e^{i\alpha} \cdot \mathbb{1}$$

$$\leftrightarrow \quad SU(3)_L \times SU(3)_R \times \underbrace{U(1)_V}_{} \quad \text{symmetry}$$

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow e^{i\alpha} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

2)
$$\partial^\mu A_\mu^a = 0 \qquad a = 1, \dots, 8$$
$$\partial^\mu V_\mu^a = 0 \qquad a = 0, 1, \dots, 8$$

$$[Q_V^a, Q_V^b] = i f^{abc} Q_V^c$$
$$[Q_V^a, Q_A^b] = i f^{abc} Q_A^c$$
$$[Q_A^a, Q_A^b] = i f^{abc} Q_V^c$$

SU(3) × SU(3) algebra

3) $[Q^a_{V,A}, H_0] = 0$

4) a) $Q^a_V \to$ Multipletstruktur

$Q^a_V|0\rangle = 0$: "Wigner-Weyl realization of symmetry"

b) $Q^a_A|0\rangle \neq 0$

$Q^a_A \to$ 8 Goldstone bosons

pseudoscalar, massless

"Nambu-Goldstone realization of symmetry"

5) $Q^a_A|0\rangle \neq 0$

$\updownarrow$

$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle \neq 0$

6) $\to$ Multipletstructure in chiral limit

Notation

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_0 \doteq \sqrt{\tfrac{2}{3}} \cdot \mathbb{1}$$

The Nonvanishing Values of f_{ijk} and d_{ijk}

(ijk)	f_{ijk}	(ijk)	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	$\frac{1}{2}$	146	$\frac{1}{2}$
156	$-\frac{1}{2}$	157	$\frac{1}{2}$
246	$\frac{1}{2}$	228	$1/\sqrt{3}$
257	$\frac{1}{2}$	247	$-\frac{1}{2}$
345	$\frac{1}{2}$	256	$\frac{1}{2}$
367	$-\frac{1}{2}$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	$\frac{1}{2}$
678	$\sqrt{3}/2$	355	$\frac{1}{2}$
		366	$-\frac{1}{2}$
		377	$-\frac{1}{2}$
		448	$-1/2\sqrt{3}$
		558	$-1/2\sqrt{3}$
		668	$-1/2\sqrt{3}$
		778	$-1/2\sqrt{3}$
		888	$-1/\sqrt{3}$

$$\left.\begin{aligned} f_{0ik} &\doteq 0 \\ d_{0ik} &\doteq \sqrt{\tfrac{2}{3}}\,\delta_{ik} \end{aligned}\right\} \; i,k = 0,\dots,8$$

$$\left.\begin{aligned} d_{ijk} &\text{ even} \\ f_{ijk} &\text{ odd} \end{aligned}\right\} \text{ under permutation of two indices}$$

$$\left.\begin{aligned} [\lambda_i, \lambda_k] &= 2i f_{ik\ell}\lambda_\ell \\ \{\lambda_i, \lambda_k\} &= 2 d_{ik\ell}\lambda_\ell \\ \mathrm{Tr}\,\lambda_i\lambda_k &= 2\delta_{ik} \end{aligned}\right\} \; \begin{aligned} &i,k,\ell = 0,\dots,8 \\ &(\text{summation over } \ell) \end{aligned}$$

References Chapter 2

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3. QCD AT LOW ENERGIES

(MESONS)

3.1 Generating functional

We wish to discuss the properties of Greenfunctions in QCD at low energies. Consider e.g.

e.m. form factors of mesons	$\langle 0 \vert T A_\mu^i A_\rho^j j_\sigma^{e.m.} \vert 0 \rangle$
$K \to \pi e \nu$	$\langle 0 \vert T A_\mu^l \bar{u}\gamma_\nu s A_\rho^m \vert 0 \rangle$
$\pi\pi \to \pi\pi$	$\langle 0 \vert T A_\mu^i A_\nu^k A_\rho^l A_\sigma^m \vert 0 \rangle$,
$\vdots$	$\vdots$

It is extremely useful to collect all these Greenfunctions in a generating functional constructed as follows:

Add to QCD-lagrangian external fields

$$\mathcal{L} = \mathcal{L}^0_{QCD} + \bar{q}\{\gamma_\mu[v^\mu + \gamma_5 a^\mu] - s + i\gamma_5 p\}q \qquad (3.1$$

v^μ, a^μ, s, p : hermitean c-number q stands for

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$$\mathcal{L}^0_{QCD} = \mathcal{L}_{QCD}|_{m_u = m_d = m_s = 0}$$

Heavy quarks are incorporated in $\mathcal{L}^0_{QCD}$!

The ordinary lagrangian for QCD is recovered from (3.1) by setting

$v^\mu = a^\mu = p = 0$,

$$s(x) = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \equiv \mathcal{M}$$

The <u>generating functional</u> is defined by

$$e^{iZ(v,a,s,p)} =$$

$$\langle 0 | T \exp\{ i \int \bar{q} [\gamma_\mu (v^\mu + \gamma_5 a^\mu) - s + i\gamma_5 p] q \} | 0 \rangle$$

The functional $Z(v,a,s,p)$ contains all the information on the Green functions built from vector, axial vector, scalar and pseudoscalar currents!

[For this to verify, one simply expands the exponential in (3.2)]

As an example, put $v^\mu = a^\mu = p = 0$

The term linear in $s(x)$ is the vacuum expectation value of $\bar{q}q$ in the chiral limit,

$$Z = -\int dx \langle 0 | \bar{q}(x) s(x) q(x) | 0 \rangle$$

$$= -\int dx\, s^{\alpha\beta} \langle 0 | \bar{q}^\alpha q^\beta | 0 \rangle$$

As a second example, consider the term proportional to $a^\mu(x) a^\nu(y)$ in Z,

$$Z = \frac{i}{2} \int dx\, a_i^\mu(x) \langle 0 | T A_\mu^i(x) A_\nu^k(y) | 0 \rangle a_k^\nu(y) dy$$

$$a^\mu(x) = a_i^\mu(x) \frac{\lambda^i}{2}, \quad A_\mu^i = \bar{q} \gamma_\mu \gamma_5 \frac{\lambda^i}{2} q$$

<u>Remark:</u>

The use of generating functionals in the framework of quantum mechanics is discussed in appendix C.

3.2 Invariance versus anomalies

In Chapter 2:

$\mathcal{L}^0_{QCD}$ is invariant under

$$q_{\substack{L\\R}} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{\substack{L\\R}} \longrightarrow V_{\substack{L\\R}} \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{\substack{L\\R}} \;;\quad V^+_{\substack{L\\R}} V_{\substack{L\\R}} = \mathbb{1}$$

with $V_{\substack{L\\R}}$ space-time independent.

From here on: $\langle v^\mu \rangle = \langle a^\mu \rangle = 0$

$V_{\substack{L\\R}} \in SU(3)$ $\quad$ $\det V_{\substack{L\\R}} = 1$

The lagrangian (3.1) is even invariant for local (space-time dependent) transformations

$$q_{\substack{L\\R}} \to V_{\substack{L\\R}}(x)\, q_{\substack{L\\R}}$$

provided the external fields are changed as well,

$$v'_\mu + a'_\mu = V_R (v_\mu + a_\mu) V_R^+ + i V_R \partial_\mu V_R^+$$

$$v'_\mu - a'_\mu = V_L (v_\mu - a_\mu) V_L^+ + i V_L \partial_\mu V_L^+ \qquad (3.3)$$

[illegible]

The generating functional Z remains therefore apparently the same,

$$Z(v', a', s', p') = Z(v, a, s, p) \qquad \text{(naive)}$$

Ultraviolet divergences generate anomalies and destroy the invariance of Z under chiral transformations

$$Z(v', a', s', p') \neq Z(v, a, s, p) \qquad \text{Quantum effects}$$

Consider infinitesimal transformations

$$V_R = \mathbb{1} + i\alpha + i\beta + \ldots$$

$$V_L = \mathbb{1} + i\alpha - i\beta + \ldots$$

with hermitean α, β. The corresponding changes in the spinor fields $q(x)$ are

$$\delta q = -\alpha^a \frac{\lambda^a}{2} q - \beta^a \frac{\lambda^a}{2} \gamma_5 q$$

Note that

$$[\alpha^a Q_V^a, q(\vec{x}, 0)] = -\alpha^a \frac{\lambda^a}{2} q(\vec{x}, 0)$$

$$[\beta^a Q_A^a, q(\vec{x}, 0)] = -\beta^a \frac{\lambda^a}{2} \gamma_5 q(\vec{x}, 0)$$

Theory can be renormalized such that change in Z is independent of α. For $\operatorname{tr}\beta = 0$ one finds [1]

$$\delta Z = \int dx \langle \beta \Omega \rangle$$

$$\Omega = -\frac{N_c}{16\pi^2} \varepsilon^{\alpha\beta\mu\nu} \Big[v_{\alpha\beta} v_{\mu\nu} + \frac{4}{3} \nabla_\alpha a_\beta \nabla_\mu a_\nu + \frac{2i}{3} \{ v_{\alpha\beta}, a_\mu a_\nu \} + \frac{8i}{3} a_\mu v_{\alpha\beta} a_\nu + \frac{4}{3} a_\alpha a_\beta a_\mu a_\nu \Big]$$

where

$$v_{\alpha\beta} = \partial_\alpha v_\beta - \partial_\beta v_\alpha - i[v_\alpha, v_\beta]$$

$$\nabla_\alpha a_\beta = \partial_\alpha a_\beta - i[v_\alpha, a_\beta]$$

$$\varepsilon^{0123} = 1 \tag{3.4}$$

- $\langle \beta \Omega \rangle$: "axial vector anomaly"
- Ω only involves a_μ, v_μ ; no $p(x), s(x)$

The transformation laws (3.3), (3.4) state:

$$Z(v,a,s,p) = Z_A(v,a,s,p) + \bar{Z}(v,a,s,p) \tag{3.5a}$$

where $\bar{Z}(v,a,s,p)$ is invariant under $SU(3)_L \times SU(3)_R$, whereas Z_A is not

$$\delta Z_A = -\int dx \langle \beta \Omega \rangle$$

$$\bar{Z}(v',a',s',p') = \bar{Z}(v,a,s,p) \tag{3.5b}$$

→ Chiral symmetry poses very strong restrictions on the generating functional. Without solving the theory, one can restrict the form of Z at low energies and extract information on

hadron spectrum
scattering amplitudes
decay processes

I show in the next paragraphs how this can be achieved.

3.3 Low energy expansion

In a well defined sense (we) can discard in a first approximation the non-invariant piece Z_A in the generating functional (3.5a) [see below]. We are left with construction of a functional Z which is invariant under

$$SU(3)_L \times SU(3)_R$$

$$Z(v', a', s', p') = Z(v, a, s, p) + \underbrace{\cdots}_{\text{neglected}} \qquad (3.6)$$

Work at low energies.

Expand Green functions in powers of the external momenta: CHPT [2]

Consider e.g.

$$T^{ik}_{\mu\nu}(p) = i \int dx\, e^{ip(x-y)} \langle 0 | T A^i_\mu(x) A^k_\nu(y) | 0 \rangle$$

If there would be no massless particles in the theory, we could write

$$T^{ik}_{\mu\nu} = C^{ik}_1 g_{\mu\nu} + C^{ik}_2 p_\mu p_\nu + C^{ik}_3 p^2 g_{\mu\nu} + O(p^4) \qquad (3.7)$$

C^{ik}_m : pure numbers, independent of p.

The symmetry requirement (3.6) restricts the coefficients C^{ik}_m. This is easily seen by constructing the piece quadratic in a^μ in the generating functional Z:

Instead of the fields v^μ, a^μ we consider

$$F^R_\mu = v_\mu + a_\mu \;;\; F^L_\mu = v_\mu - a_\mu$$

which transform under $SU(3)_L \times SU(3)_R$ as shown in (3.3).

From experience with gauge theories we know that we have to form the field strengths

$$F^I_{\mu\nu} = \partial_\mu F^I_\nu - \partial_\nu F^I_\mu - i[F^I_\mu, F^I_\nu] \;;\; I = R, L$$

which transform as

$$F'^I_{\mu\nu} = V_I F^I_{\mu\nu} V_I^+ \qquad ; \; I = R, L$$

The quadratic piece in Z is thus

$$Z = c_1 \int dx \langle F^R_{\mu\nu} F^{R\mu\nu} \rangle + c_2 \int dx \langle F^L_{\mu\nu} F^{L\mu\nu} \rangle$$

Next we invoke symmetry under parity transformations

$$\langle F^R_{\mu\nu} F^{R\mu\nu} \rangle \xrightarrow{P} \langle F^L_{\mu\nu} F^{\mu\nu L} \rangle \xrightarrow{P} \langle F^R_{\mu\nu} F^{\mu\nu R} \rangle$$

which requires $c_1 = c_2 = c$,

$$Z = c \int dx \{ \langle F^R_{\mu\nu} F^{\mu\nu R} \rangle + \langle F^L_{\mu\nu} F^{\mu\nu L} \rangle \} + \ldots\ldots \qquad (3.8)$$

The dots in (3.8) denote contributions from s, p and/or contributions which involve more powers of the field strengths $F^I_{\mu\nu}$ or more derivatives.

Note that global gauge invariance would allow a term $\sim (F^R_\mu F^{\mu R} + F^L_\mu F^{\mu L})$ in (3.8).

Explicitly, one finds from (3.8)

$$i \int dx\, e^{ip(x-y)} \langle 0 | T\, V^i_\mu(x) V^k_\nu(y) | 0 \rangle = c \cdot \delta^{ik} (p_\mu p_\nu - g_{\mu\nu} p^2) + O(\ldots$$

$$i \int dx\, e^{ip(x-y)} \langle 0 | T\, A^i_\mu(x) A^k_\nu(y) | 0 \rangle = c\, \delta^{ik} (p_\mu p_\nu - g_{\mu\nu} p^2) + O(\ldots \qquad (3.9$$

Therefore:

$$\left.\begin{array}{lll} 3 & \text{constants} & c_1, c_2, c_3 \text{ in } (3.7) \\ 3 & \text{"} & \text{for } \langle 0 | V_\mu V_\nu | 0 \rangle \end{array}\right\} 6$$

$$\downarrow \text{ Chiral symmetry}$$

c : 1 constant!

- c not fixed by symmetry requirements
- chiral symmetry poses very strong constraints on two point function (on Green functions in general)

$T^{ik}_{\mu\nu}$ does <u>not</u> have the structure displayed in (3.7) in the real world, because QCD contains pseudoscalar Goldstone bosons with

$$\langle 0 | A_\mu^i | \pi^k(p) \rangle = i F_0 p_\mu \delta^{ik}$$

$Q_A|0\rangle \neq 0$

$$i\int dx\, e^{ip(x-y)} \langle 0 | T A_\mu^i(x) A_\nu^k(y) | 0 \rangle = \delta^{ik} F_0^2 \frac{p_\mu p_\nu}{-p^2 - i\epsilon} + \cdots$$

dots denote terms not singular at $p^\mu = 0$.

Obviously $T_{\mu\nu}^{ik}$ does not admit a simple Taylor series expansion in the external momenta.

3.4 Counting low energy dimensions

Let

$$A_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp i A_\mu^2)$$

which have the following matrix elements

$$\langle 0 | A_\mu^+(0) | \pi^+(p) \rangle = -i p_\mu F_0$$

$$\langle \pi^+ | A_\mu^-(0) | 0 \rangle = i p_\mu F_0$$

in the chiral limit $m_u = m_d = m_s = 0$

{Recall that

$$|\pi^+\rangle = -\frac{1}{\sqrt{2}}[|\pi^1\rangle + i|\pi^2\rangle] \}$$

According to the Källen-Lehmann representation, one has in general, for $m_u, m_d, m_s \neq 0$

$$i\int dx\, e^{ip(x-y)} \langle 0 | T A_\mu^+(x) A_\nu^-(y) | 0 \rangle = \frac{p_\mu p_\nu}{M_{\pi^+}^2 - p^2} \cdot F_\pi^2 + \cdots \qquad (3.10)$$

Here F_π is the pion decay constant for nonvanishing quark masses,

$$\langle 0 | A_\mu^+ | \pi^+(p) \rangle = -i p_\mu F_\pi$$

[Shall see later that $F_0 = F_\pi |_{m_u = m_d = m_s = 0}$]

The pole-term $(M_{\pi^+}^2 - p^2)^{-1}$ in (3.10) has two effects:

1) At $m_u = m_d = m_s = 0$, the two point functio[n] can not be expanded in a power series in p_μ

2) If one turns on the masses, the positio[n] of the pole moves from $p^2 = 0$ to $p^2 = M_\pi^2$: Big perturbation!

Way out of dilemma:

Treat M_π^2 of same order as p^2.

Since $M_\pi^2 \sim (m_u + m_d)$, one has

$$m_u, m_d = O(p^2) = m_s$$

Keep pole terms!

Leading term in (3.10):

$$O\left(\frac{p_\mu p_\nu}{p^2}\right) = O(1)$$

Extend counting to the external fields

$$v_\mu, a_\mu, s, p$$

as follows.

1) $V_{\substack{R\\L}}$ count as order 1, i.e. $O(p^0)$

→ Ward identities only link Green functions of the same dimension.

2) $\partial_\mu V_{\substack{R\\L}}$ count as $O(p)$

3) a_μ, v_μ " " $O(p)$ see (3.3)

4) $s(x) \ni \mathfrak{M}$ " " $O(p^2)$, because $s \ni \mathfrak{M}$

5) $p(x)$ " " $O(p^2)$, see (3.3)

Note: This counting is different from usual dimensional analysis:

$$[s(x)] = L^{-1}$$

but $s(x) = O(p^2) = L^{-2}$ in low energy expansion

Counting for Z:

Consider term in Z which leads to (3.10).

Let

$$a^\mu = f^\mu \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{pmatrix} + \bar{f}^\mu \frac{1}{\sqrt{2}}\begin{pmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

Then

$$Z = F_\pi^2 \int dx\, dy\, \bar{f}^\mu(x)\, \partial_\mu^x \partial_\nu^y \Delta_c(x-y; M_{\pi^+}^2)\, f^\nu(y) + \dots \tag{3.11}$$

Low energy dimension?

$$\partial_\mu a_\pm^\mu = O(p^2)\ ;\quad \Delta_c = O(p^{-2}) \to Z = O(p^2)$$

Compare (3.8):

$F^I_{\mu\nu} \sim \partial_\mu v_\nu \ldots \quad O(p^2)$

$\rightarrow$ Z in (3.8) is $O(p^4)$!

Now recall (3.4), (3.5)

$$Z = \underset{\substack{\uparrow \\ O(p^4)}}{Z_A} + \bar{Z}$$

Anomalies do not affect leading low energy contribution to Z ! Term of order p^2 must be invariant under $SU(3)_L \times SU(3)_R$

The term (3.11) is obviously not gauge invariant. One has to add terms to (3.11) such that piece of order p^2 becomes invariant.

See below how that is done.

3.5 Generating functional to order p^2

In this paragraph we construct a generating functional with the following properties: FOOTNOTE

i) It has low energy dimension p^2

ii) It is invariant under
$SU(3)_L \times SU(3)_R$

[We disregard singlet vector and axial currents and put
$\mathrm{tr}\, a^\mu = \mathrm{tr}\, v^\mu = 0$]

iii) Z reproduces the 2-point function (3.10).

For this we consider the nonlinear σ-model coupled to external fields. Let

$$U(x) \in SU(3)$$

be a (space-time dependent) 3×3 matrix which may be parametrized by

$$U(x) = \exp \frac{i\phi^a \lambda^a}{F_0}$$

where $\phi^a(x)$ are eight pseudoscalar fields.

U is taken to transform under $SU(3)_R \times SU(3)_L$ as

$$U(x) \to V_R(x)\, U(x)\, V_L^+(x) \quad , V_{L,R} \in SU(3)$$

The covariant derivative

$$\nabla_\mu U \doteq \partial_\mu U - i(v_\mu + a_\mu) U + i U (v_\mu - a_\mu)$$

transforms like U,

$$\nabla'_\mu U' \doteq \partial_\mu U' - i(v'_\mu + a'_\mu) U' + i U' (v'_\mu - a'_\mu) = V_R \nabla_\mu U V_L^+$$

The generating functional at leading order is then given by

$$Z_1 \doteq \frac{F_0^2}{4} \int dx < \nabla_\mu U^+ \nabla^\mu U + 2B_0 \{(s+ip)U^+ + U(s-ip)\}> \tag{3.12}$$

where B_0 is a real constant, and

$$\nabla_\mu U^+ = [\nabla_\mu U]^+$$

As it stands, Z_1 is a functional of the matrices U and of the external fields v^μ, a^μ, s and p [v_μ, a_μ occur in the covariant derivative $\nabla_\mu U$].

The rule is now to solve the equations of motion for the field $U(x)$

$$\nabla_\mu \nabla^\mu U U^+ - U \nabla^\mu \nabla_\mu U^+ + U\chi^+ - \chi U^+ - \frac{1}{3} < U\chi^+ - \chi U^+ > \cdot \mathbb{1} = 0 \tag{3.13}$$

$$\chi = 2B_0(s+ip)$$

This fixes the matrix U in terms of the external fields,

$$U = U(v, a, s, p) \tag{3.14}$$

Evaluating the action (3.12) at the solution (3.13) thus results in a functional $Z_1(v,a,s,p)$ which is $SU(3)_R \times SU(3)_L$ invariant by construction and thus satisfies condition ii) on p.78.

Furthermore, counting $U(x)$ as order one, Z_1 is of order p^2 (condition i)).

We need to check iii): Does Z_1 reproduce the correct value (3.10) for the two point function $<0| A_\mu^i A_\nu^k (0)>$?

3.6 Terms quadratic in a^μ

To verify that Z_1 reproduces (3.10), it suffices to consider terms quadratic in a^μ.

Therefore let

$$S(x) = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \quad ; \quad v^\mu = p = 0$$

To solve the equations of motion (3.13), we expand $U(x)$,

$$U(x) = 1 + i\phi(x) + O(\phi^2) \quad ; \quad \phi = \sum_a \phi^a \lambda^a$$

The equations of motion become

$$\Box\phi + B_0\{m,\phi\} - \tfrac{2}{3} B_0 \langle m\phi\rangle \cdot \mathbb{1} = 2\partial^\mu a_\mu + O(\phi^3, \phi^2 a, \phi a^2) \qquad (3.15)$$

and Z_1 is

$$Z_1 = \frac{F_0^2}{4}\int dx \langle (\partial_\mu\phi - 2a_\mu)(\partial^\mu\phi - 2a^\mu) - B_0\{m,\phi^2\}\rangle$$
$$+ \frac{F_0^2}{4}\int dx \left[4B_0 m + O(\phi^4, \phi^3 a, \phi^2 a^2)\right] \qquad (3.16)$$

The rule is to solve (3.15) with the usual ϵ- prescription for the propagator and to demand that

$$\phi = 0 \quad \text{for } a^\mu = 0$$

According to (3.15), ϕ thus starts at order a^μ. The higher order terms can therefore not contribute at order $O(a^2)$ in the action. Furthermore, the higher order terms in (3.16) as well as the constant piece $\sim \int d^4x \cdot 4B_0 m$ may be droped for our purpose. Using the equation of motion (3.15) the action Z_1 becomes

$$Z_1 = \frac{F_0^2}{2}\int dx \{\langle \phi\, \partial_\mu a^\mu\rangle + 2\langle a_\mu a^\mu\rangle\} + \dots \qquad (3.17)$$

where the dots denote higher order terms in a^μ.

It remains to solve (3.15) for ϕ and to insert in (3.17). To diagonalize the mass term we introduce 8 traceless 3x3 matrices

$\lambda_{\pi^+}, \lambda_{\pi^-}, \dots, \lambda_\eta$ with the properties

$$B_0\{m, \lambda_\rho\} - \tfrac{2}{3} B_0 \langle m \lambda_\rho \rangle = \overset{\circ}{M}{}^2_\rho \lambda_\rho$$

$$\langle \lambda_\rho \lambda^+_{\rho'} \rangle = 2\delta_{\rho\rho'}$$

Explicitly,

$$\lambda_{\pi^+} = -\tfrac{1}{\sqrt{2}}(\lambda^1 + i\lambda^2), \quad \lambda_{\pi^-} = \tfrac{1}{\sqrt{2}}(\lambda^1 - i\lambda^2)$$

$$\lambda_{K^+} = -\tfrac{1}{\sqrt{2}}(\lambda^4 + i\lambda^5), \quad \lambda_{K^-} = \tfrac{1}{\sqrt{2}}(\lambda^4 - i\lambda^5)$$

$$\lambda_{K^0} = -\tfrac{1}{\sqrt{2}}(\lambda^6 + i\lambda^7), \quad \lambda_{\bar{K}^0} = -\tfrac{1}{\sqrt{2}}(\lambda^6 - i\lambda^7)$$

$$\lambda_{\pi^0} = \lambda^3 \cos\varepsilon + \lambda^8 \sin\varepsilon, \quad \lambda_\eta = -\lambda^3 \sin\varepsilon + \lambda^8 \cos\varepsilon$$

with

$$\mathrm{tg}\, 2\varepsilon = \frac{\sqrt{3}}{2} \frac{m_d - m_u}{m_s - \hat{m}}, \quad \hat{m} = \tfrac{1}{2}(m_u + m_d)$$

The eight masses $\overset{\circ}{M}{}^2_\rho$ are given below.

Now write

$$\phi = \sum_\rho \lambda_\rho \phi_\rho, \quad a_\mu = \sum_\rho a^\rho_\mu \lambda_\rho$$

Then the equations of motion become

$$(\Box + \overset{\circ}{M}{}^2_\rho)\phi_\rho = 2\partial_\mu a^\mu_\rho$$

and the action is

$$Z_1 = \sum_\rho \int dx\, \bar{a}^\mu_\rho(x)\, T^\rho_{\mu\nu}(x-y)\, a^\nu_\rho(y)\, dy$$

$$T^\rho_{\mu\nu} = 2F_0^2 \{\partial^x_\mu \partial^y_\nu \Delta_c(x-y; M^2_\rho) + g_{\mu\nu}\delta^4(x-y)\} \tag{3.18}$$

Finally we insert

$$a^\mu = f^\mu \sqrt{2}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \bar{f}^\mu \sqrt{2}^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{f^\mu}{2} \lambda_{\pi^-} - \frac{\bar{f}^\mu}{2} \lambda_{\pi^+}$$

and find

$$Z_1 = \frac{1}{4} \int dx\, \bar{f}^\mu(x) (T^{\pi^+}_{\mu\nu} + T^{\pi^-}_{\mu\nu}) f^\nu(y)\, dy$$

Since $M^2_{\pi^+} = M^2_{\pi^-}$ (see below)

we find that Z_1 indeed reproduces the pole term in (3.10), with

$$F_0^2 = F_\pi^2 \Big|_{m_u = m_d = m_s = 0}$$

We have thus verified iii)

3.7 Quark mass expansion of 0^-

Let us go back to eq. (3.18). It tells us that each of the 8 two point functions

$$\int dx\, e^{ipx} \langle 0 | T\, A^P_\mu(x)\, A^{P+}_\nu(y) | 0 \rangle \quad ; P = 1, \dots, 8$$

has a pole at $\mathring{M}_P^2$. The spectrum of the non-linear σ-model thus contains 8 one-particle states of mass $\mathring{M}_P^2$ (at this order in the low energy expansion). Explicitly, one finds [5,6]

$$\mathring{M}^2_{\pi^\pm} = (m_u + m_d) B_0$$
$$\mathring{M}^2_{K^\pm} = (m_u + m_s) B_0$$
$$\mathring{M}^2_{K^0} = \mathring{M}^2_{\bar{K}^0} = (m_d + m_s) B_0$$
$$\mathring{M}^2_{\pi^0} = (m_u + m_d) B_0 - \tfrac{4}{3}(m_s - \hat{m}) B_0 \cdot \sin^2\varepsilon / \cos 2\varepsilon$$
$$\mathring{M}^2_{\eta} = \tfrac{2}{3}(\hat{m} + 2 m_s) B_0 + \tfrac{4}{3}(m_s - \hat{m}) B_0 \cdot \sin^2\varepsilon / \cos 2\varepsilon \qquad (3.19)$$

[The index o in $\mathring{M}_P$ is to remind us that these formuloe for the masses hold only at leading order. The corrections of order $\mathcal{M}^2$ may be found in [5,6]; ε is given on p. 83]

Remarks

i) The $(\text{mass})^2$ of the 8 pseudoscalars $\pi^\pm, \pi^0, K^\pm, K^0$, $\bar{K}_0$ and η is <u>linear</u> in m_u, m_d, m_s. For $m_u = m_d = m_s = 0$, the 8 pseudoscalars become massless (Goldstone bosons).

ii) Let $m_u = m_d = \hat{m}$. Then one finds from (3.19)

$$\frac{m_s}{\hat{m}} = \frac{2M_K^2 - M_\pi^2}{M_\pi^2} = 25.9$$

[with $M_K = 495$ MeV, $M_\pi = 135$ MeV]

Corrections due to terms of order m_q^2 in the expansion (3.19) were evaluated in [5,6] with the result

$$\frac{m_s}{\hat{m}} = 25.0 \pm 2.5$$

iii) For $m_u \neq m_d$, the energy levels of the isospin partners (π^+, π^0) and (K^+, K^0) split:

$$M_{\pi^+} - M_{\pi^0} = \frac{2}{3} \frac{M_K^2 - M_\pi^2}{M_\pi} \frac{\sin^2\varepsilon}{\cos 2\varepsilon} \qquad \varepsilon \sim (m_u - m_d)$$

$$M_{K^+}^2 - M_{K^0}^2 = (m_u - m_d) B_0 \qquad (3.20)$$

The $\pi^+ - \pi^0$ mass difference is of order $(m_u - m_d)^2$ and thus tiny [using $\varepsilon \sim 10^{-2}$ [6], one has $M_{\pi^+} - M_{\pi^0} = 0.11$ MeV from (3.20)].

The bulk part of the $\pi^+ - \pi^0$ mass difference is due to electromagnetic interactions.

On the other hand, the $(\text{mass})^2$ of the neutral and charged kaon differ by a term linear in $(m_d - m_u)$.

Including higher order terms in the expansion (3.19), the mass splitting for kaons is [6]

$$[M_{K^+} - M_{K^0}]_{QCD} = -5.28 \pm 0.33$$

The contributions from electromagnetic interactions have the 'wrong' sign,

$$[M_{K^+} - M_{K^0}]_{QED} = 1.27 \pm 0.30 \text{ MeV}$$

The combined effect of QCD and QED gives the observed value

$$[M_{K^+} - M_{K^0}]_{exp} = -4.01 \pm 0.13 \text{ MeV.}$$

iv) The ratio m_d/m_u is rather accurately known [6,5],

$$\frac{m_d}{m_u} = 1.76 \pm 0.13$$

v) The constant B_0 in (3.19) is related to the condensate $\langle 0|\bar{u}u|0\rangle$ (in the chiral limit). Indeed, the term linear in $s(x)$ in the action (3.12) is

$$Z_1 = B_0 F_0^2 \int d^4x \, \mathrm{tr}\, s(x)$$

from where (see p. 56 64)

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle = - B F^2$$

3.8 Where are we?

Let me summarize the state of affairs.

1) The generating functional $Z(v,a,s,p)$,

$$e^{iZ(v,a,s,p)} =$$

$$\langle 0 | T \exp i \int dx \{ \bar{q} \gamma_\mu (v^\mu + \gamma_5 a^\mu) q - \bar{q} s q + i \bar{q} \gamma_5 p q \} | 0 \rangle$$

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \; ; \; v^\mu, a^\mu, s, p : \text{ external fields}$$

contains all the information on the Greenfunctions built from vector, axial, scalar and pseudoscalar currents in QCD.

2) Z may be split into a piece which is invariant under $SU(3)_R \times SU(3)_L$ transformations and a piece which is not,

$$Z = \bar{Z} + Z_A$$

with δZ_A given in (3.4)

3) The symmetry property

$$\bar{Z}(v', a', s', p') = \bar{Z}(v, a, s, p) \qquad (3.21)$$

incorporates $\boxed{\text{all}}$ (nonanomalous) War identities of the Green functions!
[v', a', s', p' are defined in (3.3)]

4) The constraints (3.21) are solved in CHPT by expanding Z in powers of the external fields v, a, s, p and their derivatives. The low energy dimension are

$a_\mu, v_\mu \qquad O(p)$

$s(x), p(x) \qquad O(p^2)$

Additional derivatives count as addit. powers of p, e.g. $\partial_\rho a_\mu = O(p^2)$.

5) Any field theory of the type $\mathcal{L}_0 + \mathcal{L}_I$

with

1) $SU(3)_R \times SU(3)_L$ symmetry ($\mathcal{L}_I = 0$)

2) Spont. symm. breakdown $\longrightarrow SU(3)_V$

3) $\langle 0 | A_\mu^a | \pi^b \rangle \neq 0$

4) No other massless particles in theory

5) $\mathcal{L}_I$ same one-particle matrix elements as $-\bar{q}\mathcal{M}q$

$\mathcal{L}_I$ same transformation property as $-\bar{q}\mathcal{M}q$

gives rise to the same leading terms as in QCD. [3]

6) The non linear σ-model coupled to external currents, restricted to tree diagrams, has these properties. The leading term in the low energy expansion of Z is of order p^2 and given by

$$Z_1 = \frac{F_0^2}{4} \int dx \langle \nabla_\mu U \nabla^\mu U^+ + \chi U^+ + U \chi^+ \rangle$$

which has to be evaluated at the solution to the classical equation of motion

$$\nabla_\mu \nabla^\mu U U^+ - U \nabla_\mu \nabla^\mu U^+ + U\chi^+ - \chi U^+ - \frac{1}{3} \langle U\chi^+ - \chi U^+ \rangle \cdot \mathbb{1} \cdot$$

7) Z_1 embodies in a very compact manner the leading terms of <u>all</u> Green functions built from vector, axial vector, scalar and pseudoscalar currents in terms of the 2 constants F_0 and B_0 which are related to the pion decay constant and the condensate through

$$F_0 = F_\pi |_{m_u = m_d = m_s = 0}$$

$$\langle 0 | \bar{u}u | 0 \rangle = \langle 0 | \bar{d}d | 0 \rangle = \langle 0 | \bar{s}s | 0 \rangle = -F_0^2 B_0$$

chiral limit value.

References Chapter 3

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C. Itzykson and J.B. Zuber, *Quantum Field Theory* (Mc Graw Hill, 1980), Chapters 5 and 6.

Appendices to Chapter 3

Appendix A : LSZ-reduction formula

Appendix B : Simple version for sections (3.5, 3.6)

Appendix C : Generating functional for harmonic oscillator

APPENDIX A

LSZ-reduction formula

Consider the perturbative expansion of Green functions in $\lambda\varphi^4$-theory. The Gell-Mann-Low formula

$$\tau(x_1,\dots,x_n) \doteq \langle 0|T\,\varphi(x_1)\dots\varphi(x_n)|0\rangle$$

$$= \langle 0|T\,\varphi^0(x_1)\dots\varphi^0(x_n)\,S\,|0\rangle / \langle 0|S|0\rangle \qquad \text{(A.1)}$$

allows the evaluation of the Green function $\tau(x_1,\dots,x_n)$ in a power series in the coupling constant λ. (φ^0 denotes a free field. I disregard all complications due to renormalization, like Z - factors and the like.)

How does one calculate the scattering matrix element for e.g. the elastic process $p_1 + p_2 \to p_3 + p_4$? According to the LSZ-reduction formula (Lehmann, Symanzik and Zimmermann)

$$\langle p_4, p_3|S-1|p_1,p_2\rangle = i(2\pi)^4\delta^4(P_f - P_i)\,T(p_4,p_3;p_1,p_2)$$

$$= \lim_{p_i^2\to m^2} i^4 \prod_{i=1}^{4}(m^2-p_i^2)\int dx_4\,dx_3\,dx_2\,dx_1\, e^{i(p_4x_4+p_3x_3-p_2x_2-p_1x_1)}$$

$$\times \langle 0|T\,\varphi(x_1)\dots\varphi(x_4)|0\rangle \qquad \text{(A.2)}$$

Therefore

$$i^3\int dx_4\,dx_3\,dx_2\, e^{i(p_4x_4+p_3x_3-p_2x_2-p_1x_1)}\langle 0|T\varphi(x_1)\dots\varphi(x_4)|0\rangle$$

$$= \frac{R(s,t,u;p_4^2,p_3^2,p_2^2,p_1^2)}{\prod_{i=1}^{4}(m^2-p_i^2)}\,; \qquad \begin{aligned} s &= (p_1+p_2)^2 \\ t &= (p_3-p_1)^2 \\ u &= (p_4-p_1)^2 \end{aligned} \qquad \text{(A.3)}$$

The scattering amplitude is obtained by putting all momenta on the mass shell:

$$T(p_4,p_3;p_1,p_2) = R(s,t,u;p_4^2,p_3^2,p_2^2,p_1^2)\Big|_{p_i^2=m^2} \qquad \text{(A.4)}$$

According to this result, the Fourier transform of the Green function $\langle 0|T\,\varphi(x_4)\ldots\varphi(x_1)|0\rangle$ has a fourfold pole - the residue of this pole is the T-matrix element.

Note that we need not take φ as the interpolating field to calculate the T-matrix element - any field with

$$\langle 0|\varphi|p\rangle \neq 0$$

does the job (see e.g. Araki and Haag 1967). We might consider for instance

$$i^3\int dx_4\,dx_3\,dx_2\, e^{i(p_4x_4+p_3x_3-p_2x_2-p_1x_1)}\langle 0|T\,\partial^\mu\varphi(x_1)\varphi(x_2)\varphi(x_3)\partial^\nu\varphi(x_4)|0\rangle$$

$$= \frac{p_1^\mu\, p_4^\nu}{\prod_{i=1}^{4}(m^2-p_i^2)}\, R(s,t,u;p_4^2,p_3^2,p_2^2,p_1^2)$$

where R is the same function as before.

How does one calculate e.g. elastic $\pi\pi$-scattering in QCD ? The pions carry isospin, and we define

$$\langle\pi(p_4)^m\,\pi(p_3)^\ell|S-1|\pi(p_2)^k\,\pi(p_1)^i\rangle = i(2\pi)^4\delta^4(P_f-P_i)$$

$$\{\delta^{ik}\delta^{\ell m}A(s,t,u)+\delta^{i\ell}\delta^{km}A(t,s,u)+\delta^{im}\delta^{k\ell}A(u,t,s)\} \qquad \text{(A.5)}$$

From chapter II we know that

$$\langle 0|A^i_\mu(0)|\pi^k\rangle = i p_\mu\,\delta^{ik}F_\pi$$

see p. 54 . The Fourier transform of the four-point Green function has therefore a pole in each leg,

$$i^3\int dx_4\,dx_3\,dx_2\, e^{i(p_4x_4+p_3x_3-p_2x_2-p_1x_1)}\times$$

$$\langle 0|T\,A^m_\mu(x_4)A^\ell_\nu(x_3)A^k_\rho(x_2)A^i_\sigma(x_1)|0\rangle$$

$$= \frac{(p_4)_\mu(p_3)_\nu(p_2)_\rho(p_1)_\sigma\, F_\pi^4}{\prod_{i=1}^{4}(M_\pi^2-p_i^2)}\Big\{\delta^{ik}\delta^{\ell m}A(s,t,u;p_1^2,\ldots,p_4^2)$$

$$+\,\delta^{i\ell}\delta^{km}A(t,s,u;p_1^2,\ldots,p_4^2)$$

$$+\,\delta^{im}\delta^{k\ell}A(u,t,s;p_1^2,\ldots,p_4^2)\Big\} \qquad \text{(A.6)}$$

The scattering amplitude is easily obtained from this equation by putting all momenta on the mass shell,

$$A(s,t,u) = A(s,t,u;p_1^2,\ldots,p_4^2)\Big|_{p_i^2=M_\pi^2} \qquad \text{(A.7)}$$

References

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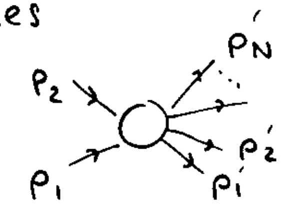
Appendix B

Simple version of sections (3.5, 3.6)

Instead of proceeding as described on p. 78-84, one may do a short cut for the evaluation of scattering matrix elements and of masses:

Scattering matrix elements for the processes



pseudoscalar mesons only

or masses of pseudoscalar mesons may be evaluated at leading order in the low energy expansion (CHPT) by evaluating tree diagrams in the nonlinear sigma model.

B1. Nonlinear σ-model

The Lagrangian is given

$$\mathcal{L}(x) = \frac{F_0^2}{4}\langle \partial_\mu U \partial^\mu U^+ + 2B_0 \mathcal{M}(U+U^+)\rangle$$

where $U(x)$ is a space-time dependent unitary matrix with determinant one,

$$U(x)\,U^+(x) = \mathbb{1}, \quad \det U(x) = 1$$

We can parametrize $U(x)$ as

$$U(x) = e^{i\phi(x)/F_0}, \quad \phi = \phi^+, \quad \langle\phi\rangle = 0$$

$\mathcal{L}$ is invariant under (for $\mathcal{M}=0$)

$$U \to U' = V_R\, U\, V_L^+, \quad V_{R,L} \in SU(3)$$

$\to$ 16 conserved currents

$$L_\mu^a = \frac{1}{i}\frac{F_0^2}{2}\langle \frac{\lambda^a}{2} U^+ \partial_\mu U\rangle$$

$$R_\mu^a = \frac{1}{i}\frac{F_0^2}{2}\langle \frac{\lambda^a}{2} U \partial_\mu U^+\rangle \qquad a = 1,\dots,8$$

B2 Expanding in ϕ

Expand the Lagrangian in powers of the field ϕ,

$$U = 1 + i\phi/F_0 - \frac{1}{2F_0^2}\phi^2 - \frac{i}{6F_0^3}\phi^3 + \frac{1}{24F_0^4}\phi^4 + \dots$$

$$\mathcal{L} = \frac{1}{4}\langle \partial_\mu\phi\,\partial^\mu\phi - 2B_0\mathcal{M}\phi^2\rangle$$

$$+ \frac{1}{4F_0^2}\langle -\frac{1}{3}\partial_\mu\phi\,\partial^\mu\phi^3 + \frac{1}{4}\partial_\mu\phi^2\,\partial^\mu\phi^2 + \frac{B_0\mathcal{M}}{12}\phi^4\rangle$$

$$+ \text{ terms with } \phi^6, \phi^8, \dots$$

- Lagrangian $\mathcal{L}$ contains an infinite number of terms!
- describes interactions of mesons

ϕ^4

$2 \to 2$ $\qquad$ $2 \to 4$ $\qquad$ $2 \to 6$ $\qquad \dots$

What can one do with it?

B3. Calculations

Introduce the fields

$$\phi(x) = \sqrt{2}\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2\eta}{\sqrt{6}} \end{pmatrix}(x)$$

with $\pi^+ = (\pi^-)^+$, $K^+ = (K^-)^+$, $K^0 = (\bar{K}^0)^+$

$\rightarrow \phi = \phi^+$, $\langle \phi \rangle = 0$ ✓

Write

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{int}$$

$\mathcal{L}_{kin}$ contains terms quadratic in the fields,

$$\begin{aligned}\mathcal{L}_{kin} = {} & \partial_\mu \pi^+ \partial^\mu \pi^- - M^2_{\pi^+} \pi^+ \pi^- \\ & + \partial_\mu K^+ \partial^\mu K^- - M^2_{K^+} K^+ K^- \\ & + \partial_\mu K^0 \partial^\mu \bar{K}^0 - M^2_{K^0} K^0 \bar{K}^0 \\ & + \tfrac{1}{2}\left(\partial_\mu \pi^0 \partial^\mu \pi^0 - M_1^2 \pi^{0\,2}\right) + \tfrac{1}{2}\left(\partial_\mu \eta \partial^\mu \eta - M_2^2 \eta^2\right)\end{aligned}$$

where

$$M^2_{\pi^+} = (m_u + m_d) B_0$$

$$M^2_{K^+} = (m_u + m_s) B_0$$

$$M^2_{K^0} = (m_d + m_s) B_0$$

$$M_{\pi^0\eta} \sim (m_u - m_d) B_0$$

The interacting piece $\mathcal{L}_{int}$ is

$$\mathcal{L}_{int} = \mathcal{L}^{(4)} + \underset{\substack{\uparrow \\ \text{6 fields}}}{\mathcal{L}^{(6)}} + \dots$$

$$\mathcal{L}^{(4)} = \frac{1}{6F_0^2}\Big\{ \partial_\mu \pi^- \partial^\mu \pi^- \pi^+ \pi^+ - 2\,\partial_\mu \pi^- \pi^- \partial^\mu \pi^+ \pi^+ + \pi^- \pi^- \partial_\mu \pi^+ \partial^\mu \pi^+ + \dots \Big\}$$

+ terms with kaons, η.

B4. What do we learn?

Calculate everything at tree level.

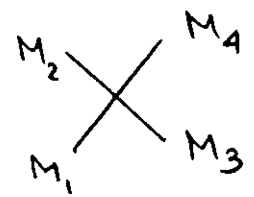
From $\mathcal{L}_{kin}$

Read off masses, e.g.

$M^2_{\pi^+} = M^2_{\pi^-} = (m_u + m_d) B_0$, see p. 85 ff

From $\mathcal{L}^{(2)}$

Describes



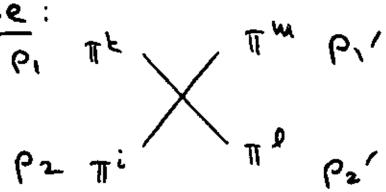
$M_1 + M_2 \to M_3 + M_4$ elastic scattering of mesons π, K, η.

Note: $\mathcal{L}^{(2)}$ contains 2 derivatives

$\to$ Scattering matrix element $\sim p^2$

$\to$ small at low energies.

Example:



$$T^{ik \to lm} = \delta^{ik}\delta^{lm} A(s,t,u) + \delta^{il}\delta^{km} A(t,s,u) + \delta^{im}\delta^{kl} A(u,t,s)$$

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2} \qquad F_\pi = F_0$$

$s = (p_1 + p_2)^2$

$t = (p_1' - p_1)^2$

$$T^{I=0} = 3A(s,t,u) + A(t,u,s) + A(u,s,t)$$

$$= \frac{2s - M_\pi^2}{F_\pi^2}$$

$$T^{I=0}\Big|_{\text{threshold}} \equiv 32\pi a_0^0$$ scattering length

$$\to a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16 \qquad \text{Exp. } 0.26 \pm 0.05$$

S. Weinberg, PRL 17 (1966) 616

Example

$\eta \to \pi^+\pi^-\pi^0$

$$A(s,t) = \frac{\sqrt{3}}{4}\,\frac{m_u - m_d}{m_s - \hat{m}}\,\frac{s - \frac{4}{3}M_\pi^2}{F_\pi^2}$$

$$\Gamma_{\eta \to \pi^+\pi^-\pi^0} = 66\,\text{eV} \qquad \text{Exp. } 197 \pm 29\,\text{eV}$$

$317 \pm 20\,\text{eV}$

$\longleftrightarrow$?

Higher order terms?

APPENDIX C

Harmonic oscillator

To illustrate the usefulness of generating functionals, consider the harmonic oscillator in the framework of quantum mechanics.

$$\mathbb{H} = \frac{\mathbb{P}^2}{2m} + \frac{1}{2} m\omega^2 \mathbb{x}^2 \qquad [\mathbb{P}, \mathbb{x}] = \frac{1}{i}$$

$$(\hbar = 1)$$

1) For analogy with QFT, introduce operators in the Heisenbergpicture

$$\mathbb{x}(t) \doteq e^{i\mathbb{H}t}\, \mathbb{x}\, e^{-i\mathbb{H}t}, \quad \mathbb{p}(t) \doteq e^{i\mathbb{H}t}\, \mathbb{p}\, e^{-i\mathbb{H}t}$$

which satisfy

$$\left(\frac{d^2}{dt^2} + \omega^2\right) \mathbb{x}(t) = 0$$

$$\left(\frac{d^2}{dt^2} + \omega^2\right) \mathbb{p}(t) = 0$$

Compare:

$$\left(\frac{d^2}{dt^2} - \Delta + m^2\right) \varphi(t, \vec{x}) = 0$$

free fields

2) Consider ground state of the system

$|0\rangle$ with $H|0\rangle = \frac{\omega}{2}|0\rangle$

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-\frac{m\omega}{2}x^2}$$

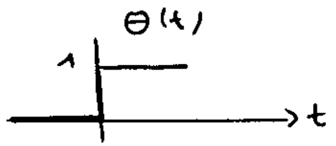
3) Define two-point function

$$\langle 0|T\, x(t)\, x(t')|0\rangle = \begin{cases} \langle 0|x(t)\,x(t')|0\rangle & t>t' \\ \langle 0|x(t')\,x(t)|0\rangle & t'>t \end{cases}$$

Explicit formula:

$$\frac{1}{i}G_F(t-t') \doteq \langle 0|T\, x(t)\, x(t')|0\rangle$$

$$G_F(t) = \frac{i}{2m\omega}\left\{\Theta(t)\,e^{-i\omega t} + \Theta(-t)\,e^{i\omega t}\right\}$$



4) How does one evaluate n-point functions

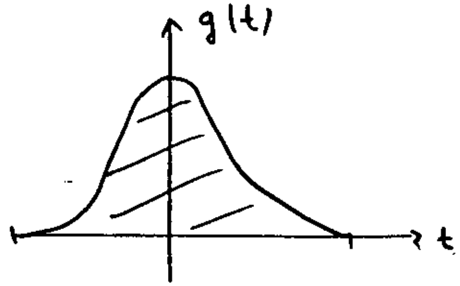
$\langle 0|T\, x(t_1)\dots x(t_n)|0\rangle$?

"n-point function"

Introduce 'generating functional'

$$e^{iZ(g)} \doteq \langle 0|T\, e^{i\int_{-\infty}^{\infty} g(t)\,x(t)\,dt}|0\rangle \qquad (C.1)$$

generating functional — ordinary c-number function — Operator



5) Why "generating"?

Because $Z(g)$ generates all n-point functions

$$Z(g) \text{ known} \longrightarrow \underset{\text{known}}{\langle 0|T\, x(t_1)\cdots x(t_n)|0\rangle}$$

see below

Why "functional"?

Because it produces a number from the function $g(t)$. In the present case

$$Z(g) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\, dt'\, g(t)\, G_F(t-t')\, g(t') \qquad (C.2)$$

[Only quadratic in $g(t)$, because potential is quadratic]

6) How is knowledge of $Z(g)$ used?

Expand left- and righthand side in eq. (C.1).

Left-hand side:

$$1 + i Z(g) - \frac{1}{2!} Z(g)^2 + \cdots$$

$$= 1 + \frac{i}{2}\int dt\, dt'\, g(t)\, G_F(t-t')\, g(t') - \frac{1}{8}\left\{\int dt_1 dt_1'\, g(t_1)\, G_F(t_1-t_1')\, g(t_1')\right\} \times \left\{\int dt_2 dt_2'\, g(t_2)\, G_F(t_2-t_2')\, g(t_2')\right\} + O(g^6)$$

Right-hand side

$$\langle 0|T e^{i\int dt\, g(t)\, x(t)}|0\rangle = \langle 0|\mathbb{B}|0\rangle$$

$$\mathbb{B} = 1 + i\int dt\, g(t)\, x(t) - \frac{1}{2}\int dt\, dt'\, T\{g(t)x(t)\, g(t')x(t')\} + \cdots$$

$$T\, g(t)x(t)\,g(t')x(t') = \begin{cases} g(t)x(t)\,g(t')x(t') & t>t' \\ g(t')x(t')\,g(t)x(t) & t'>t \end{cases}$$

$$= g(t)g(t')\,T\,x(t)x(t')$$

Conclusions:

1) No term linear on LHS

$\rightarrow \langle 0|x(t)|0\rangle = 0$

2) Compare term quadratic in g

$$\rightarrow \langle 0|T\,x(t_1)x(t_2)|0\rangle = \frac{1}{i}G_F(t_1-t_2) \quad \checkmark$$

3) $\langle 0|T\,x(t_1)\dots x(t_4)|0\rangle = \left(\frac{1}{i}\right)^2\left\{G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23}\right\}$

$$G_{12} \doteq G_F(t_1-t_2)$$

4) $\langle 0|T\,x(t_1)\dots x(t_{2N+1})|0\rangle = 0$

5) $\boxed{\text{It is useful to know } Z(g)\,!}$

(Not only for harmonic oscillator)

6) Can generate other Green-functions:

Suppose we wish to know

$$\langle 0|T\,x^2(t_1)\,x^2(t_2)\,x(t_3)\dots|0\rangle$$

$\curvearrowright$ Consider

$$e^{iZ(g,h)} = \langle 0|T\,e^{i\int dt\{g(t)x(t) + \underbrace{h(t)x^2(t)}_{\text{new piece}}\}}|0\rangle$$

etc.

4. GENERATING FUNCTIONAL AT ORDER p^4

$$e^{iZ(v,a,s,p)} = \langle 0 | T\, e^{i\int \bar{q}[\gamma_\mu v^\mu + \gamma_\mu \gamma_5 a^\mu - s + ip\gamma_5] q} | 0 \rangle$$

$$\boxed{Z = \underbrace{Z^{(2)}}_{O(p^2)} + \underbrace{Z_A + Z^{(4)}}_{O(p^4)} + \underbrace{Z^{(6)}}_{O(p^6)} + \cdots}$$

Order p^2: Green functions obtained from

Z_1 in (3.12), (3.13)

$$\boxed{Z^{(2)} \equiv Z_1 \equiv \int dx\, \mathcal{L}^{(2)}}$$

Order p^4: Contributions from

Z_A , $Z^{(4)}$

- Z_A : not invariant under chiral transform., see (3.4). Explicit construction by Wess and Zumino [1]. Z_A does not contain any constants of nature except $\hbar$ and c.

$$\boxed{Z_A \equiv \int dx\, \mathcal{L}_{WZ}(x)}$$

- $Z^{(4)}$: Chirally invariant, of order p^4.

One class of contributions to $Z^{(4)}$ is obtained by evaluating tree graphs at order p^4.

4.1 Tree graphs

One adds terms of order p^4 to the classical action $Z^{(2)}$:

$$Z^{(4)} = \int dx\, \mathcal{L}^{(4)}(x)$$

$$\mathcal{L}^{(4)} = \sum_{i=1}^{12} L_i P_i$$

where

$$\boxed{\chi = 2B_0(s+ip)}$$

$$
\begin{aligned}
P_1 &= (\langle \nabla_\mu U^+ \nabla^\mu U \rangle)^2 \\
P_2 &= \langle \nabla_\mu U^+ \nabla_\nu U \rangle \langle \nabla^\mu U^+ \nabla^\nu U \rangle \\
P_3 &= \langle \nabla^\mu U^+ \nabla_\mu U \, \nabla^\nu U^+ \nabla_\nu U \rangle \\
P_4 &= \langle \nabla^\mu U^+ \nabla_\mu U \rangle \langle \chi^+ U + \chi U^+ \rangle \\
P_5 &= \langle \nabla^\mu U^+ \nabla_\mu U (\chi^+ U + U^+ \chi) \rangle \\
P_6 &= \langle \chi^+ U + \chi U^+ \rangle^2 \\
P_7 &= \langle \chi^+ U - \chi U^+ \rangle^2 \\
P_8 &= \langle \chi^+ U \chi^+ U + \chi U^+ \chi U^+ \rangle \\
P_9 &= -i \langle F^R_{\mu\nu} \nabla^\mu U \nabla^\nu U^+ + F^L_{\mu\nu} \nabla^\mu U^+ \nabla^\nu U \rangle \\
P_{10} &= \langle U^+ F^R_{\mu\nu} U F^{\mu\nu L} \rangle \\
P_{11} &= \langle F^R_{\mu\nu} F^{\mu\nu R} + F^L_{\mu\nu} F^{\mu\nu L} \rangle \\
P_{12} &= \langle \chi^+ \chi \rangle
\end{aligned}
\qquad (4.1)
$$

A few remarks are in order:

1) Each term P_i in this list is chirally invariant. $P_1 - P_{12}$ exhaust the chiral invariant counterterms allowed by Lorentz invariance, P and C symmetry [4]. [In the evaluation of $Z_1^{(4)}$, the field $U(x)$ is needed only to leading order - therefore the equations of motion (3.13) have been used to reduce the number of terms. Total divergences are discarded.]

2) At leading order, two constants F_0, B_0 suffice to determine the low energy behaviour of the Green functions. At first non leading order one needs 12 additional low energy coupling constants $L_1, \dots, L_{12}$.

3) $L_1, \dots, L_{12}$ can be divided into three different classes [7]: 7 on-shell couplings, 3 off-shell couplings and 2 contact terms.

i) The contact terms (L_{11} and L_{12}) are not directly accessible to experiment. The value of L_{12} if of theoretical interest because it occurs in the quark mass expansion of the condensates $\langle 0|\bar{q}q|0\rangle$ [4].

ii) The 3 off-shell couplings L_5, L_9 and L_{10} determine the ratio F_K/F_π, the meson charge radii and the decaywidth $\pi \to e\nu\gamma$ respectively. On the other hand, the experimental information on these quantities allows to pin down the value of L_5, L_9 and L_{10} within rathe small errors.

iii) The remaining 7 on-shell coupling constants contribute at order p^4 in the low energy expansion of scattering matrix elements. [For $v_\mu = a_\mu = p = 0$, $s(x) = \mathcal{M}$, only $P_1 - P_8$ survive in the list (4.1). Furthermore, the equations of motion imply that $6P_5 - P_7 + 3P_8$ is a total divergence: only 2 of the three coupling constants L_5, L_7 and L_8 are linearly independent for on-shell processes.]

4) The coupling constants $L_1, \ldots, L_{12}$ are needed to renormalize the divergences which occur in the loop calculations discussed below.

As an example consider the contribution of $\mathcal{Z}_1^{(4)}$ to the quark mass expansion of M_π^2, M_K^2 and M_η^2 for $m_u = m_d = \hat{m}$:

$$M_\pi^2 = \underline{2\hat{m} B_0} \{ 1 + 2\hat{m} K_3 + K_4 \}$$

$$M_K^2 = \underline{(\hat{m} + m_s) B_0} \{ 1 + (\hat{m} + m_s) K_3 + K_4 \}$$

$$M_\eta^2 = \underline{\tfrac{2}{3} (\hat{m} + 2m_s) B_0} \{ 1 + \tfrac{2}{3} (\hat{m} + 2m_s) K_3 + K_4 \} + K_5$$

where

$$K_3 = \frac{8B_0}{F_0^2} (2L_8 - L_5)$$

$$K_4 = (2\hat{m} + m_s) \frac{16 B_0}{F_0^2} (2L_6 - L_4)$$

$$K_5 = (m_s - \hat{m})^2 \frac{128}{9} \frac{B_0^2}{F_0^2} (3L_7 + L_8)$$

Note that $M_\eta^2 \neq \frac{1}{3} (4 M_K^2 - M_\pi^2)$

[illegible]

4.2 Loops

Up to here:

$\left.\begin{matrix} \mathcal{Z}^{(2)} \\ \mathcal{Z}^{(4)} \end{matrix}\right\}$ Tree graphs with nonlinear σ-model.

What about graphs with loops?

Two viewpoints:

i) Disregard loops → Unitarity violated

or

ii) Include loops → contributions not well defined, because nonlinear σ-model is not renormalizable!!

$\boxed{\text{DILEMMA}}$

Solution [Weinberg[2]]:

Consider one-loop graphs. Use dimensional regularization which preserves chiral symmetry. Counterterms needed are of order p^4.

→ They must have the structure of the terms listed in (4.1)!

→ With suitable renormalization of $L_1, \ldots, L_{12}$ in $\mathcal{L}^{(4)}$ one gets finite Green-functions at one loop level.

The necessary renormalization of L_i was determined in ref. [4]:

$$L_i = L_i^r + \Gamma_i \lambda \qquad i = 1, \ldots, 12$$

$$\lambda = (4\pi)^{-2} \mu^{d-4} \left\{ \frac{1}{d-4} - \frac{1}{2} \left[\ln 4\pi + \Gamma'(1) + 1 \right] \right\}$$

$$\Gamma_1 = \frac{3}{32}, \; \Gamma_2 = \frac{3}{16}, \ldots \tag{4.2}$$

For the complete list of Γ_i's see [4].

μ is the scale introduced by dimensional regularization. The divergent pieces $\Gamma_i \lambda$ are cancelled by corresponding divergencies in one-loop integrals. Note that the renormalized couplings L_i^r are scale-dependent:

$$\mu \frac{d}{d\mu} L_i^r = -\Gamma_i \mu \frac{d}{d\mu} \lambda = -\frac{1}{16\pi^2} \Gamma_i \quad \text{at } d=4. \tag{4.3}$$

Physical quantities are scale independent. The scale dependence of the contributions from L_i^r is compensated by the scale dependence of loop contributions. As an example consider the electric charge radius of the pion,

$$\langle r^2 \rangle_V^\pi = \frac{12 L_9^r}{F_0^2} - \frac{1}{32\pi^2 F_0^2} \left\{ 2 \ln \frac{M_\pi^2}{\mu^2} + \ln \frac{M_K^2}{\mu^2} + 3 \right\}$$

Using $\Gamma_9 = \frac{1}{4}$ it is easy to see that $\langle r^2 \rangle_V^\pi$ is scale independent

The first two terms in the low energy expansion of the generating functional are obtained by evaluating

i) Tree and one loop graphs with the nonlinear σ-model lagrangian
$$\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \nabla_\mu U \nabla^\mu U^+ + \chi U^+ + \chi^+ U \rangle$$

ii) Tree graphs which contain one vertex of $\mathcal{L}^{(4)}$ or $\mathcal{L}_{WZ}$

Compact description:

Path integrals

4.3 Path integrals

Very compact description in terms of path integral. The generating functional is evaluated as follows. One starts with an effective lagrangian
$$\mathcal{L}^{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}_{WZ} + \dots$$
where dots denote terms of order $\geq p^6$. Z at first nonleading order is then obtained from
$$\underline{e^{iZ}}_{QCD} = \underbrace{\int [dU]\, e^{i\int dx\, \mathcal{L}^{eff}}}_{\text{Pion-field theory}} \quad (4.4)$$
where integral over the field $U(x)$ is calculated in one loop approximation: Denote solution to classical equations of motion (3.13) by U_c and expand $U(x)$ around this solution,

$$U = u e^{i\xi} u \,, \quad u^2 = U_c$$

with $\xi = \xi^+$, $\langle\xi\rangle = 0$. U_c is an extremum of $\int dx \mathcal{L}^{(2)}$, and therefore

$$\int dx \mathcal{L}^{(2)} = \int dx \mathcal{L}^{(2)}\big|_{U_c} - \frac{1}{2}\int dx \langle \xi D \xi \rangle + O(\xi^3)$$

D : second order differential operator which contains the external fields.

Result of Gaussian integration in (4.4) is

$$e^{iZ} = e^{i\int dx \mathcal{L}^{eff}|_{U_c}} [\det D]^{-\frac{1}{2}}$$

and thus

$$\boxed{Z = \int dx \mathcal{L}^{eff}\big|_{U_c} + \frac{i}{2} \ln \det D}$$

det D contains UV divergences which are regularized by going to d dimensions. The poles which occur as $d \to 4$ are removed by renormalizing L_i as shown in (4.2)

Very economic way to determine the counterterms which are necessary to render the generating functional finite at one loop. I refer to [4] where the evaluation of det D is discussed in some detail.

Summary of all this:

$$\langle 0 | T e^{i\int dx \{\bar{q}\gamma_\mu[v^\mu + \gamma_5 a^\mu]q - \bar{q}(s - i\gamma_5 p)q\}} | 0 \rangle$$

$$\|$$

$$\int [dU] e^{i\int dx \mathcal{L}^{eff}[U, v, a, s, p]}$$

$$\mathcal{L}^{eff} = \underset{\substack{\uparrow \\ \sigma\text{-model}}}{\mathcal{L}^{(2)}} + \mathcal{L}^{(4)} + \mathcal{L}_{WZ} + \cdots$$

4.4 Values of low energy constants

To evaluate numerically physical quantities with above machinery we need the values of the low energy coupling constants $L_1, \dots, L_{10}$ at some scale. Value at any other scale from (4.3). L_{11}, L_{12} "high-energ constants", convention dependent.

In principle $L_i(\mu)$ fixed by dynamics of QCD in terms of $\Lambda, m_c, m_b, m_t, \dots$. Not possible to calculate with present technique. For several attempts see [8].

In absense of calculational scheme: determine $L_i(\mu)$ from information on low energy data, together with large N_c-arguments:

$O(N_c^2)$: L_7

$O(N_c)$: $L_1, L_2, L_3, L_5, L_8, L_9, L_{10}, H_1, H_2$

$O(1)$: $\underbrace{2L_1 - L_2,\ L_4,\ L_6}_{\text{assumption: also suppressed at } N_c \to \infty}$

→ Table.

Shall see in next chapter how the value of $L_i^r(\mu)$ can be explained.

TABLE I

Values of the low-energy coupling constants (running scale taken at $\mu = M_\eta$)

	Value	Source	Γ_i
L_1^r	$(0.9 \pm 0.3) \cdot 10^{-3}$	$\pi\pi$ D-waves, Zweig rule	$\frac{3}{32}$
L_2^r	$(1.7 \pm 0.7) \cdot 10^{-3}$	$\pi\pi$ D-waves	$\frac{3}{16}$
L_3	$(-4.4 \pm 2.5) \cdot 10^{-3}$	$\pi\pi$ D-waves, Zweig rule	0
L_4^r	$(0 \pm 0.5) \cdot 10^{-3}$	Zweig rule	$\frac{1}{8}$
L_5^r	$(2.2 \pm 0.5) \cdot 10^{-3}$	$F_K : F_\pi$	$\frac{3}{8}$
L_6^r	$(0 \pm 0.3) \cdot 10^{-3}$	Zweig rule	$\frac{11}{144}$
L_7	$(-0.4 \pm 0.15) \cdot 10^{-3}$	Gell-Man-Okubo, L_5, L_8	0
L_8^r	$(1.1 \pm 0.3) \cdot 10^{-3}$	$K^0 - K^+$, R, L_5	$\frac{5}{48}$
L_9^r	$(7.4 \pm 0.7) \cdot 10^{-3}$	$\langle r^2 \rangle^\pi_{\text{c.m.}}$	$\frac{1}{4}$
L_{10}^r	$(-6.0 \pm 0.7) \cdot 10^{-3}$	$\pi \to e\nu\gamma$	$-\frac{1}{4}$

Ref. [4]

4.5 Some applications

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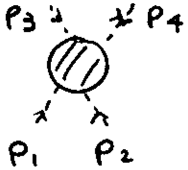
4.51 Elastic ππ-scattering

$\pi\pi \to \pi\pi$



$$s = (p_1+p_2)^2 \qquad t = (p_1+p_3)^2 \qquad u = (p_1+p_4)^2$$

Described in terms of one invariant amplitude $A(s,t,u)$

$$A(s,t,u) \leftrightarrow \text{F.T.}\ \langle 0|T[A_\mu A_\nu A_\rho A_\sigma]|0\rangle$$

Expansion around

$$p_i^\mu = 0,\ m_u = m_d = 0,\quad m_s \neq 0 \text{ fixed}$$

$$(SU(2)\times SU(2))$$

Result:

$$A(s,t,u) = \underbrace{\frac{1}{F_\pi^2}(s-M_\pi^2)}_{} + \underbrace{B(s,t,u)}_{} + \underbrace{C(s,t,u)}_{} + O(p$$

$Z^{(4)}$

$\mathcal{L}^{(4)}$

$O(p^2)$	$O(p^4)$	$O(p^4)$
Tree [9]	Loop	Polyn.

$$B(s,t,u) = P^{(1)}(s)\,\bar{J}(s) + P^{(2)}(t,u)\,\bar{J}(t) + P^{(2)}(u,t)\,\bar{J}(u)$$

$$\bar{J}(s) = (16\pi^2)^{-1}\{\sigma \ln[(\sigma-1)/(\sigma+1)] + 2\}$$

$$\sigma = (1-4M_\pi^2/s)^{1/2}$$

"Loop"

$$C(s,t,u) = (96\pi^2F_\pi^4)^{-1}\{2(\bar{\ell}_1-\tfrac{4}{3})(s-2M_\pi^2)^2 + (\bar{\ell}_2-\tfrac{5}{6})[s^2+(t-u)^2] - 12M^2s + 15M^4\}$$

"Polynomial

$F^2, M^2, \bar{\ell}_1, \bar{\ell}_2$

4 low energy constants

$$F^2, M^2, \bar{l}_1, \bar{l}_2 \longrightarrow \{F_\pi^2, M_\pi^2, \underbrace{\bar{l}_1, \bar{l}_2}_{a_2^0,\, a_2^2}, \bar{l}_3, \underbrace{\bar{l}_4}_{\langle r^2\rangle_S^\pi}\}$$

where

$$\langle p' | \bar{u}u + \bar{d}d | p \rangle = F_S(0)\left[1 + \frac{1}{6}\langle r^2\rangle_S^\pi \cdot t\right]$$

Scattering lenghts and effective ranges:

Isospin amplitudes

e.g. $T^0(s,t) = 3A(s,t,u) + A(t,u,s) + A(u,s,t)$

⋮

Partial waves:

$$T^I(s,t) = 32\pi \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\Theta)\, t_\ell^I(s)$$

$$t = -2q^2(1-\cos\Theta) \; ; \; s = 4(M_\pi^2 + q^2)$$

$$\mathrm{Re}\, t_\ell^I = q^{2\ell}\{a_\ell^I + q^2 b_\ell^I + O(q^4)\}$$

"scattering length" "effective range"

S- and P-wave threshold parameters:

$I=0$, $\ell=0$

$$a_0^0 = (7/32\pi)(M_\pi^2/F_\pi^2)[1 + \tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi - (M_\pi^2/672\pi^2F_\pi^2)(15\bar{l}_3 - 353)] + \tfrac{25}{4}M_\pi^4(a_2^0 + 2a_2^2) + O(M_\pi^6),$$

$$b_0^0 = (1/4\pi F_\pi^2)(1 + \tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi + \tfrac{39}{64}M_\pi^2/\pi^2F_\pi^2) + 10M_\pi^2(a_2^0 + 5a_2^2) + O(M_\pi^4),$$

$$a_0^2 = -(M_\pi^2/16\pi F_\pi^2)[1 + \tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi + (M_\pi^2/480\pi^2F_\pi^2)(15\bar{l}_3 - 107)] + \tfrac{5}{2}M_\pi^4(a_2^0 + 2a_2^2) + O(M_\pi^6),$$

$$b_0^2 = -(1/8\pi F_\pi^2)(1 + \tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi - \tfrac{89}{320}M_\pi^2/\pi^2F_\pi^2) + 10M_\pi^2(a_2^0 + \tfrac{1}{2}a_2^2) + O(M_\pi^4),$$

$I=1$, $\ell=1$

$$a_1^1 = (1/24\pi F_\pi^2)(1 + \tfrac{1}{3}M_\pi^2\langle r^2\rangle_S^\pi + \tfrac{19}{576}M_\pi^2/\pi^2F_\pi^2) + \tfrac{10}{3}M_\pi^2(a_2^0 - \tfrac{5}{2}a_2^2) + O(M_\pi^4),$$

$$b_1^1 = 7/2160\pi^3F_\pi^4 + \tfrac{10}{3}(a_2^0 - \tfrac{5}{2}a_2^2) + O(M_\pi^2).$$

Numerical result: Table Ref [15]

Table 1
Comparison of the predicted threshold parameters with experiment.

	Soft pions	Experiment	Improved low energy theorems	Size of correction 3:1	Chiral logs only $\mu = 500$ MeV	$\mu = 1$ GeV
> a_0^0	0.16	0.26 ± 0.05	0.20 ± 0.01	1.29	0.19	0.20
b_0^0	0.18	0.25 ± 0.03	0.24 ± 0.02	1.37	0.23	0.26
a_0^2	−0.045	−0.028 ± 0.012	−0.042 ± 0.002	0.95	−0.042	−0.041
b_0^2	−0.089	−0.082 ± 0.008	−0.075 ± 0.005	0.84	−0.079	−0.073
$2a_0^0 - 5a_0^2$	0.534	0.657 ± 0.052	0.614 ± 0.018	1.15	0.58	0.60
> a_1^1	0.030	0.038 ± 0.002	0.037 ± 0.003	1.26	0.031	0.032
b_1^1			(5 ± 3) × 10⁻³		0.5 × 10⁻³	0.5 × 10⁻³
a_2^0		(17 ± 3) × 10⁻⁴	input		7 × 10⁻⁴	15 × 10⁻⁴
a_2^2		(1.3 ± 3) × 10⁻⁴	input		3 × 10⁻⁴	6 × 10⁻⁴

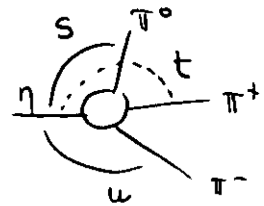
Ref. [15]

4.52 $\eta \to 3\pi$

$\eta \to \pi^0 \pi^+ \pi^-$, $\eta \to 3\pi^0$

Kinematics

$$\langle \pi^0 \pi^+ \pi^- \text{ out} | \eta \rangle = i(2\pi)^4 \delta^4(P_f - P_i) A(s,t,u)$$



$s = (p_\eta - p_{\pi^0})^2$
etc

The 3 pions are in $I=1$ state

$$A(s,t,u) = A_{QCD} + e^2 A_{QED} + O(e^4)$$

$$A_{QCD} = (m_u - m_d)\, f(p^2, \Lambda, m_u, m_d, m_s, \dots)$$

$$f = f_0 + f_1 + \dots$$

$$f_0 = \frac{B_0}{3\sqrt{3}} \frac{1}{F_\pi^2} \left\{ 1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2} \right\} \qquad s_0 = M_\pi^2 + \frac{1}{3} M_\eta^2$$

$$f_1 = m_s \cdot f' \quad , \quad f' = O(1)$$

$$f_2 = m_s^2 \cdot f'' \qquad f'' = O(1) \qquad \text{etc}$$

$$A_{QED} = g(p^2, \Lambda, m_u, m_d, \dots)$$

$$g = g_0 + g_1 + \cdots$$

$$g_0 = O(1) \;,\; g_1 = m_s \cdot \underbrace{g'}_{O(1)}$$

Sutherland's theorem:

$$g_0 = 0$$

Compare: $\left.\begin{array}{l}(M^{\gamma}_{\pi^+})^2 \simeq \frac{3\alpha}{4\pi} M_\rho^2 \left(\frac{f_\rho}{f_\pi}\right)^2 \ln \frac{f_\rho^2}{f_\rho^2 - f_\pi^2} \\ M^{\gamma}_{\pi^0} = 0\end{array}\right\}$ [1]

$M^{\gamma}_{K^+} = M^{\gamma}_{\pi^+} \;,\; M^{\gamma}_{K^0} = 0$ [2]

$M^{\gamma}_{\pi^+} - M^{\gamma}_{\pi^0} = (4.9 \pm 0.2)$ MeV exp. 4.6 MeV

$M^{\gamma}_{K^+} - M^{\gamma}_{K^0} = (1.27 \pm 0.3)$ MeV exp −4.01 MeV

[1] Das et al., PRL 18 (1967) 507

[2] Dashen PRev 183 (1969) 1245.

$\eta \to 3\pi$: One loop result

Still a puzzle.

$\Gamma_{\eta \to \pi^0\pi^+\pi^-} = 66$ eV Tree result.

Corrections of order p^4 have been evaluated [6]. Several low energy constants from $\mathcal{L}^{(4)}$ contribute. Except for L_3, they can be expressed in terms of F_π, F_π, M_K, M_π and M_η. For large N_c, L_3 can be related to $\pi\pi$ scattering lengths.

→ Parameter free prediction:

$$\underline{\Gamma_{\eta \to \pi^+\pi^-\pi^0} = 160 \pm 50\,\text{eV}} \qquad (4.5)$$

Experiment: Normalized with $\eta \to 2\gamma$.

Old data on $\eta \to 2\gamma$ (Primakoff effect)

$\Gamma_{\eta \to 2\gamma} = 324 \pm 46$ eV [11]

$$\Gamma_{\eta \to \pi^+\pi^-\pi^0} = 197 \pm 29\ \text{eV} \qquad (4.6)$$

Data on photoproduction

$$\Gamma_{\eta\to 2\gamma} = 520 \pm 30\,\mathrm{eV} \quad [12]$$

$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = 317 \pm 20\,\mathrm{eV} \qquad (4.7)$$

(4.6) is consistent with the prediction (4.5), the more recent experimental result (4.7) is not consistent.

Prediction (4.5) based on low energy expansion to order p^4. The main correction of order p^4 stem from $\pi\pi$- final state interactions:

$$\frac{A_{CD}^{(4)}}{A_{CD}^{(2)}} = 1.5 = 1 + \overbrace{\underbrace{0.33}_{\substack{\text{pion}\\\text{loops}}} \underbrace{+0.06}_{\substack{K,\eta\\\text{loops}}} \underbrace{-0.03}_{\substack{\text{chiral}\\\text{logs}}} \underbrace{+0.14}_{\substack{\text{coupling}\\\text{constants}}}}^{O(p^4)}$$

Uncertainty in (4.5) based on estimates of $O(p^6)$ from pion loops.

Did we underestimate the size of higher order contributions?

Note:

$$317\,\mathrm{eV} = 66\,\mathrm{eV}\cdot\underbrace{[1 + 3.8]}_{[\underset{\text{Tree}}{1} + \underset{\text{1 Loop}}{0.5} + \underset{\text{2 loops}}{0.69}]^2}$$

Contributions from 2 loops would have to be larger than contributions of order p^4 which we have calculated!

Remark:

$$\Gamma_{\eta\to 3\pi^0} / \Gamma_{\eta\to\pi^+\pi^-\pi^0}$$

contain experimental discrepancies. Is a pity, because is test of isospin structure.

Low energy theorem:

$$\Gamma_{\eta \to 2\gamma} = \frac{\alpha^2 M_\eta^3}{192 \pi^3 F_\pi^2} \left\{ 1 + O(m_q) \right\}$$

Ignoring higher order corrections:

$$\Gamma_{\eta \to 2\gamma} = 170\,\text{eV} \leftrightarrow Z_A$$

Analysis of corrections $O(m_q)$ taken up recently [10].

Leading term in $\Gamma_{\eta \to 2\gamma}$ stems from Wess–Zumino Lagrangian $\to$ Corrections are of $O(p^6)$. New Coupling constants appear. Maybe that estimates of $L_1, \ldots, L_{10}$ (see next chapter) can be extended to the relevant coupling constant of $\mathcal{L}^{(6)}$.

$$\Gamma_{\eta \to 2\gamma} = 170\,\text{eV} \underset{?}{\leftrightarrow} [520 \pm 30]\,\text{eV}$$

Exp.

Branching ratio $\dfrac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$

$$r = \frac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$$

Current algebra $r = 1.51$

CHPT $r = 1.43$

PDG (82)

indirect measurements $r = 1.36 \pm 0.06$

direct " :

inconsistent { all data $r = 1.28 \pm 0.14$; old data excl. 1.46 ± 0.13 }

] 1.344 ± 0.053

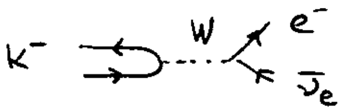
CHPT agrees with $r = 1.36 \pm 0.06$ at one standard deviation level

Current algebra $(r = 1.51)$: $2.5\,\sigma$

4.53 Cabibbo angle, F_K/F_π

Leutwyler and Roos [13] have analyzed the matrix element

$\langle K | V^\mu | \pi \rangle$ responsible for $K^- \to \pi^0 e^- \bar\nu_e$



Isospin breaking effects (due to $m_u \neq m_d$) are of same size as SU(3) breaking effects due to $m_s \neq \hat m$!

Result:
$$V_{us} = .220 \pm 0.002 \qquad (4.8)$$

Semileptonic hyperon decay are in perfect agreement with (4.8), see [14]

With (4.8), Leutwyler and Roos obtain

$$F_K/F_\pi = 1.22 \pm 0.01$$

Leutwyler, Roos
1984

4.54 Quark mass ratios

CHPT implies low energy theorem:

$$\frac{m_d - m_u}{m_s - \hat m} \cdot \frac{2\hat m}{m_s + \hat m} = \frac{K^{0\,2} - K^{+\,2} - \pi^{0\,2} + \pi^{+\,2}}{K^2 - \pi^2} \cdot \frac{\pi^2}{K^2} \{1 + O(m^2) + O(\frac{\alpha m}{m_s - \hat m}$$

Coupling constants from $\mathcal{L}^{(4)}$ do not occur!

Old values [ref. 6 in chapter 3]:

$$\frac{m_s - \hat m}{m_d - m_u} = 43.5 \pm 2.2 \quad ; \quad \frac{m_s}{\hat m} = 25 \pm 2.5$$

One gets with these values

$$(1.77 \pm 0.19) \cdot 10^{-3} \overset{?}{=} 1.73 \cdot 10^{-3}$$
$$\checkmark$$

4.55 Formfactors of vetor currents

$$\left.\begin{array}{l}\langle \pi^+ | j_\mu^{e.m.} | \pi^+ \rangle \\ \langle K^+ | j_\mu^{e.m.} | K^+ \rangle \\ \langle K^0 | j_\mu^{e.m.} | K^0 \rangle\end{array}\right\} \text{ charge radii } \underline{\checkmark}$$

- $\langle K^+ | \bar{u}\gamma_\mu s | \pi^0 \rangle \leftrightarrow K_{\ell 3}$ decay

$$\langle K^+ | \bar{u}\gamma_\mu s | \pi^0 \rangle = \sqrt{\tfrac{1}{2}}\left\{(p'+p)_\mu f_+(t) + (p'-p)_\mu f_-(t)\right\}$$

$$f_0(t) \doteq f_+ + \frac{t}{M_K^2 - M_\pi^2} f_-$$ "scalar form factor"

$$f_0(t) = f_0(0)\left[1 + \frac{1}{M_\pi^2}\lambda_0 \cdot t + O(t^2)\right]$$

CHPT: parameter free prediction:
(Use F_K/F_π)

$$\boxed{\lambda_0 = 0.017 \pm 0.004}$$

old high statistic SLAC experiment:

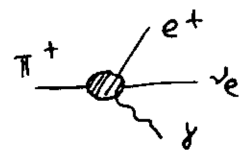
$\lambda_0 = 0.019 \pm 0.004$ [16]

More recent [17]

$$\boxed{\lambda_0 = 0.046 \pm 0.006}$$

CONTRA-DICTION

4.56 $\pi^+ \to e^+ \nu \gamma$



$$\langle 0 | J_\mu^{e.m.} \begin{array}{c}V\\A\end{array} | \pi \rangle$$

$$d\Gamma/dx\,dy = \frac{\alpha}{2\pi}\Gamma_{\pi\to e\nu}\left\{IB(x,y) + \frac{1}{4}\frac{M_\pi^2}{M_e^2}\cdot|F_V|^2/f_\pi^2 \,*\right.$$
$$\left.*\left[(1+\gamma)^2 SD^+(x,y) + (1-\gamma)^2 SD^-(x,y)\right]\right\}$$

$x = E_\gamma/2M_\pi$, $y = E_e/2M_\pi$

IB, SD^+, SD^- known

$\theta(\gamma, e^+) \sim 180°, 135°$

$\to \underline{\gamma = 0.52 \pm 0.06}$

A. Bay et al., Phys. Lett. B174 (1986) 445

Confirmed by $\pi^+ \to e^+ \nu e^+ e^-$ S. Egli et al. Phys. Lett. B175 (1986) 97

$\underline{\gamma = 0.7 \pm 0.5}$

CHPT : $\gamma = 32\pi^2 [L_9^r + L_{10}^r]$

$L_9 \leftrightarrow$ charge radius of pion

$L_{10} \leftrightarrow$ Weinberg sum rules

Das et al.
PRL 18 (1967) 761
19 (1967) 859

$\underline{\gamma = 0.42}$

References Chapter 4

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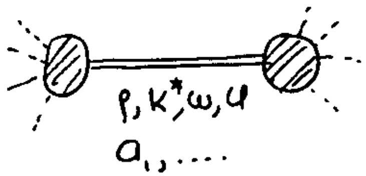
5. The role of resonances



Discussion according to

G. Ecker, A. Pich, E. de Rafael, J.G.

"The role of resonances in chiral perturbation theory"

Nucl. Phys. B321 (1989) 311

6. BARYONS

6.1 Technique

Introduce baryon fields in the lagrangian.

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)}_{\pi\pi} + \mathcal{L}^{(4)}_{\pi\pi} + \mathcal{L}^{(6)}_{\pi\pi} + \cdots$$

$$+ \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \mathcal{L}^{(3)}_{\pi N} + \cdots \quad \Leftarrow \text{new}$$

To evaluate elastic πN-scattering amplitude at low energies, calculate 1-loop graphs with

$$\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi}\{i\gamma^\mu \partial_\mu - \frac{1}{8F_\pi^2}[\phi, \partial_\mu \phi] - M_N\}\Psi$$

$$- \frac{g_A}{2} \bar{\Psi}\gamma^\mu \gamma_5 \partial_\mu \phi \Psi \qquad (5.1)$$

Figs.

A. Svarč, M. Sainio
J.G., Nucl. Phys. B 307 (1988) 779

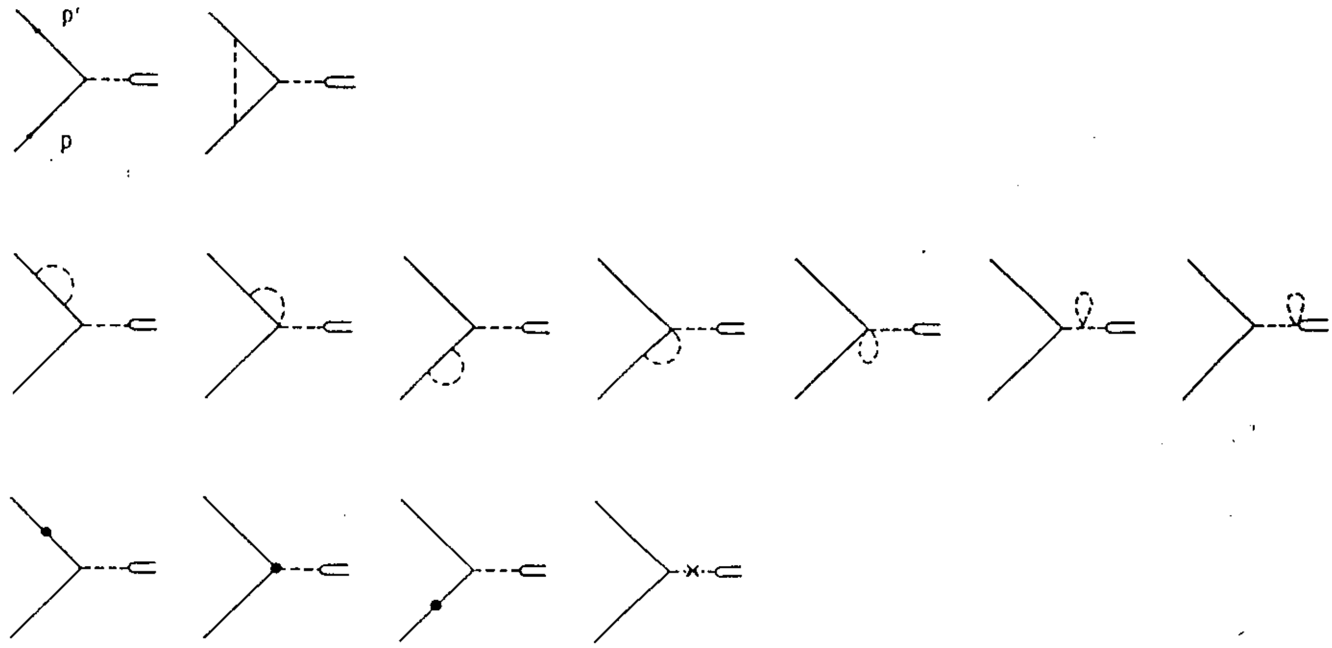


Fig. 6.

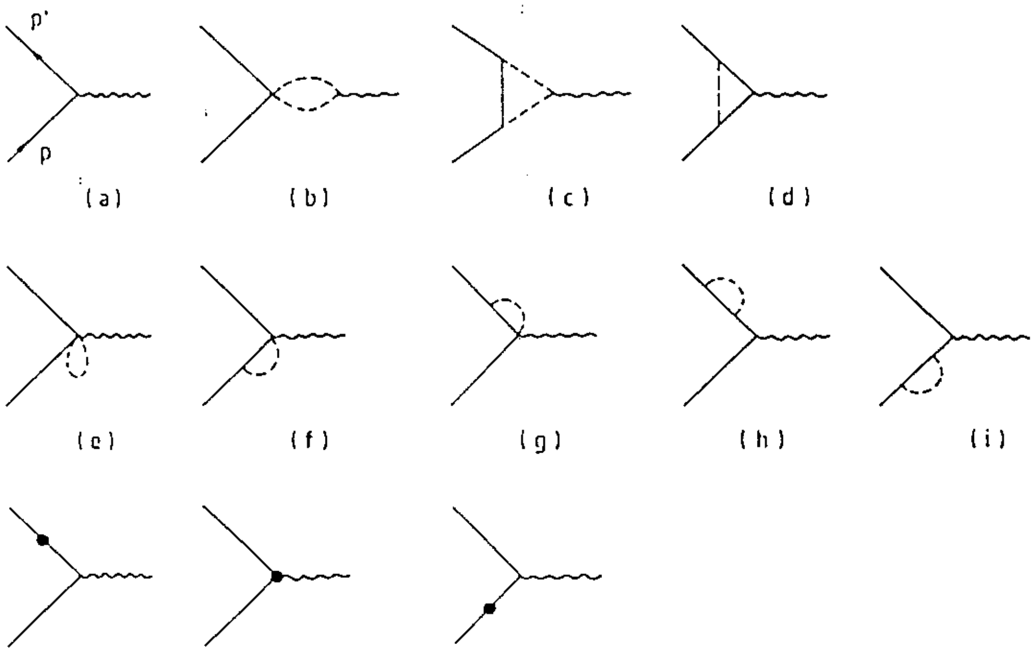


Fig. 7.

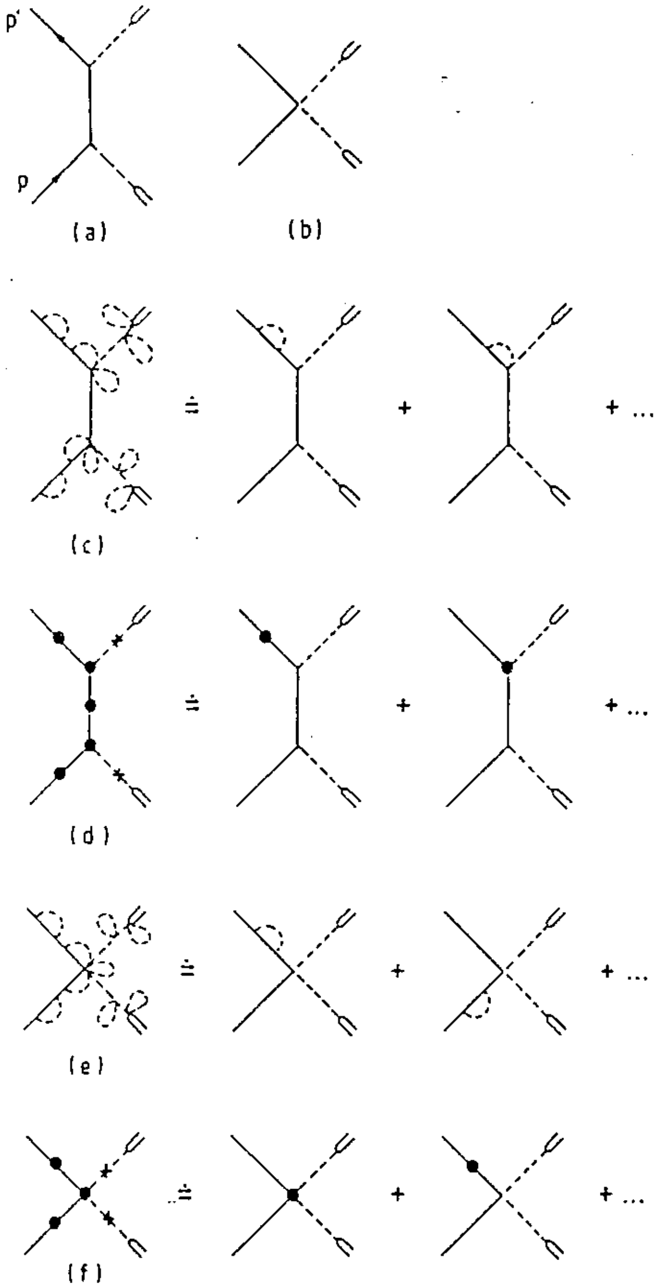


Fig. 8.

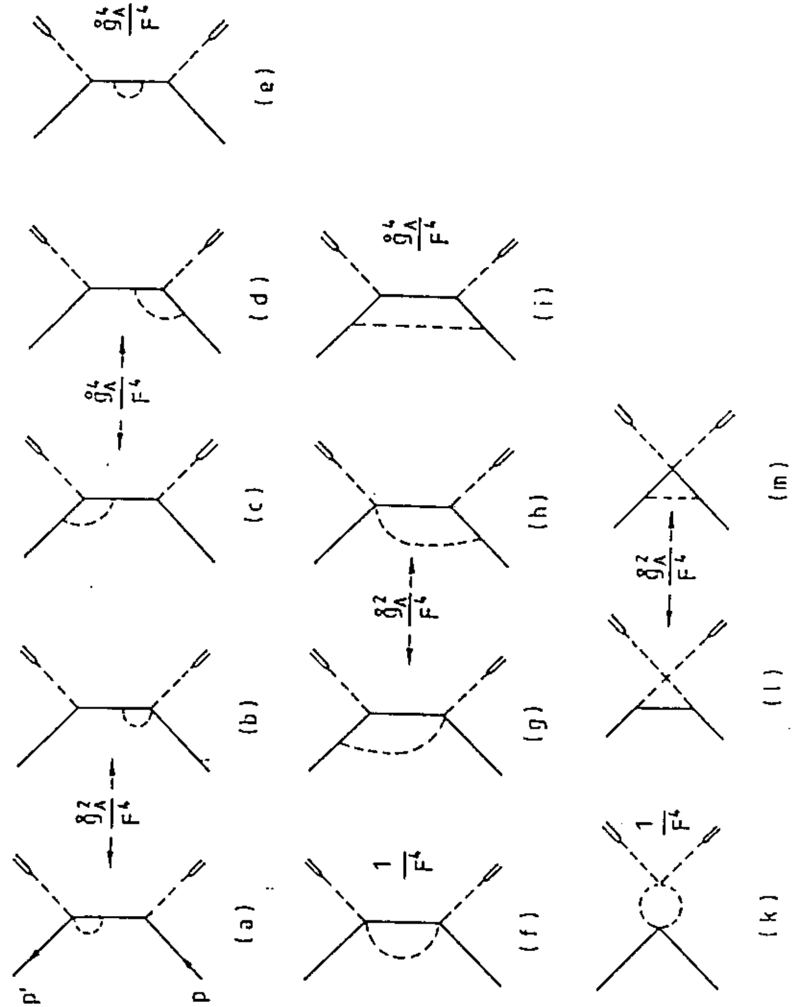
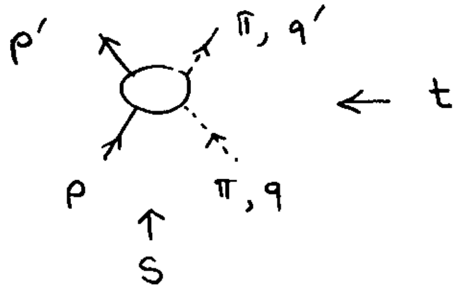


Fig. 9.

6.2 Kinematics

Basic's:



$$\nu = \frac{s-u}{4m} \underset{t=0}{=} \sqrt{M_\pi^2 + k_{LAB}^2}$$

4 amplitudes

$D^\pm(s,t)$, $B^\pm(s,t)$

$c_1, \dots, c_8$

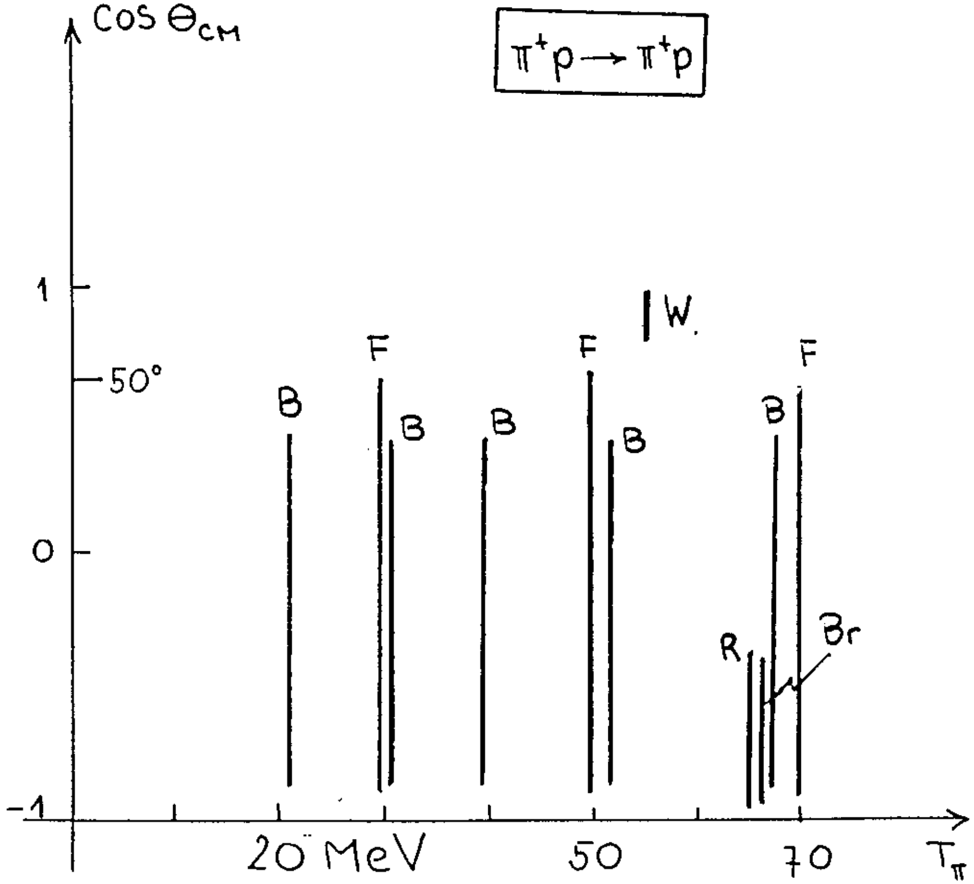
$$\boxed{\begin{array}{l} 8 \text{ parameters at one loop} \\ + \, m, g_A, F_\pi, M_\pi \end{array}}$$

6.3 Available data

No claim for completeness

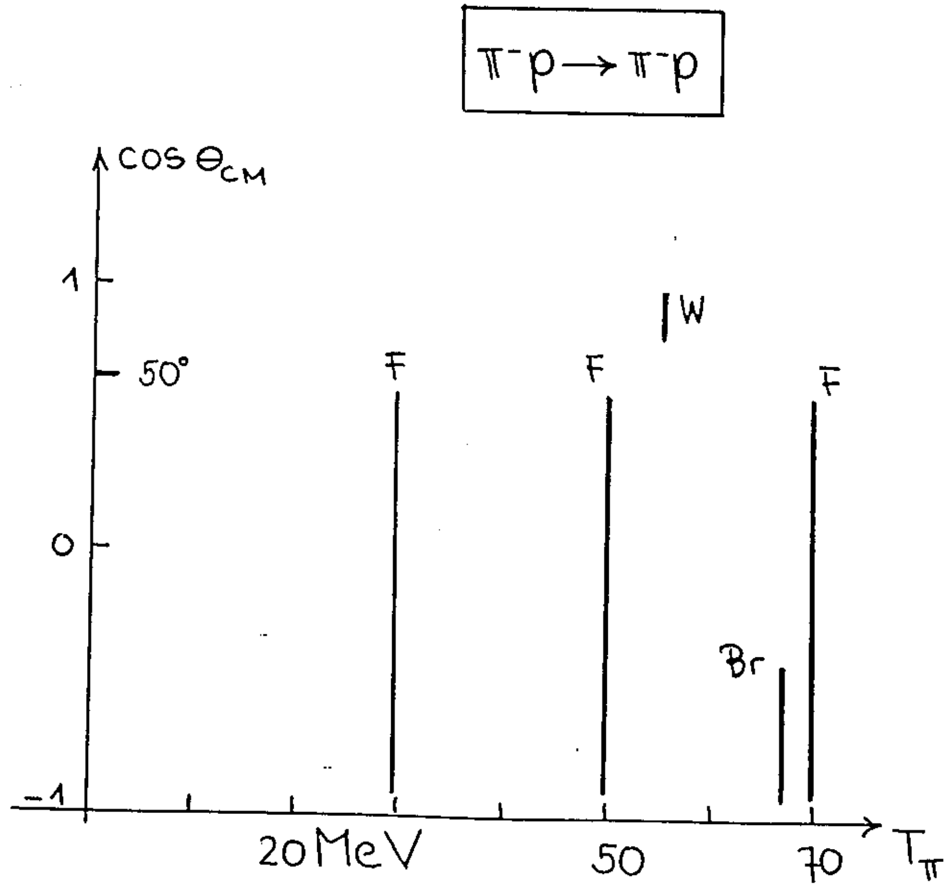
See e.g. M. E. Sadler
Proceedings πN(2) workshop
Los Alamos 1987

B : Bertin et al., 1976
F : Frank et al., 1983
W: Wiedner et al., 1987
R : Ritchie et al., 1983
Br : Brack et al., 1986



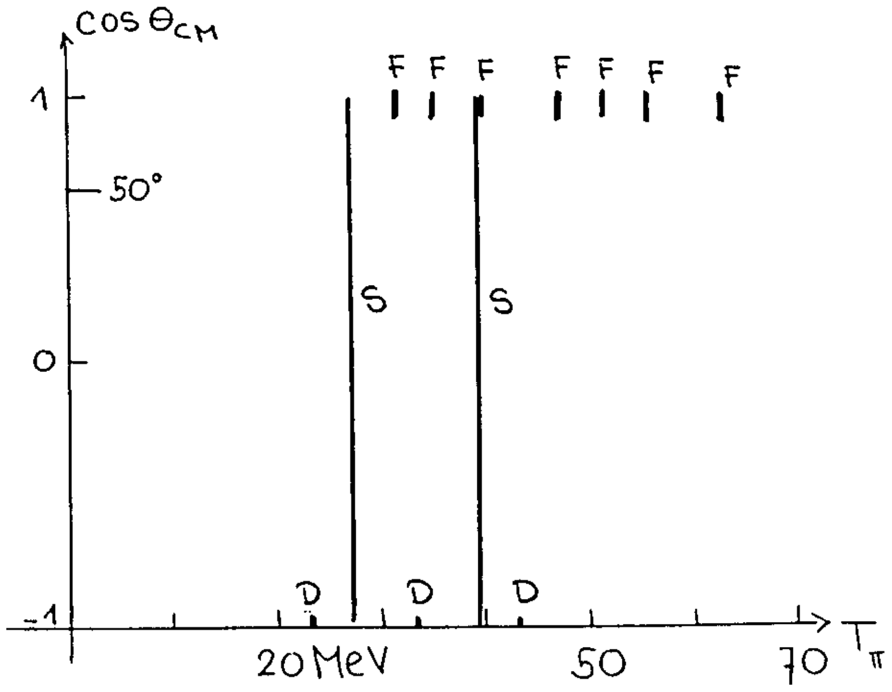
$$T_\pi = \sqrt{M_\pi^2 + k_{LAB}^2} - M_\pi$$

F : Frank et al., 1983
W : Wiedner et al., 1987
Br : Brack et al., 1986

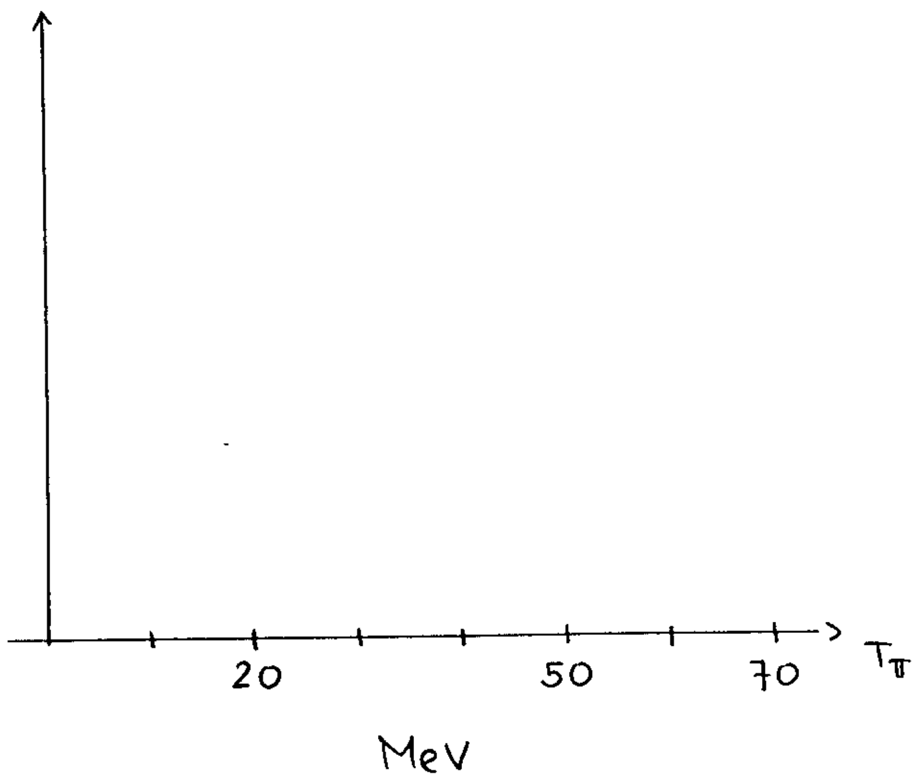


F : Fitzgerald et al. 1986
S : Salomon et al. 1983
D : Duclos et al. 1976

$\boxed{\pi^- p \to \pi^0 n}$

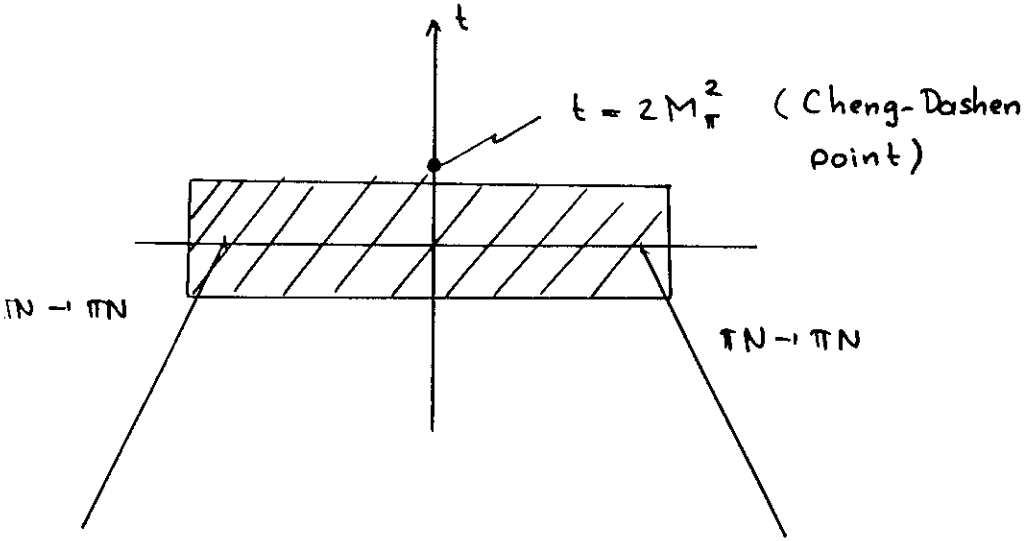


$\boxed{\sigma_{TOT}^{\pi^\pm p}}$



6.4 Range of validity of CHPT

Unfortunately, it turns out that corrections which arise from two-loop graphs are large already near threshold.



outside : 2-loop corrections are large

6.5 Results.

6.51 D^--amplitude

3 free parameters $d_{00}^-, d_{10}^-, d_{01}^-$

→ relations; e.g.

$$\left(2+3\frac{M_\pi}{m}\right)a_{1+}^- + a_{1-}^- = -0.211\left(1-\frac{\varepsilon}{2}\right) \qquad (5.2$$

There are reasons (not detailed here), to believe that $|\varepsilon| < 5\%$

May compare with determination of left hand side in phase shift analyses (no data at threshold!)

Karlsruhe-Helsinki	1980	−0.211	$\varepsilon = 0$
Carter et al.	1973	−0.211	
Zidell et al.	1980	−0.238	$\varepsilon = -0.25$

↑ Contradict CHPT

6.52 D^+-amplitude

3 free parameters d_{00}^+, d_{10}^+, d_{01}^+

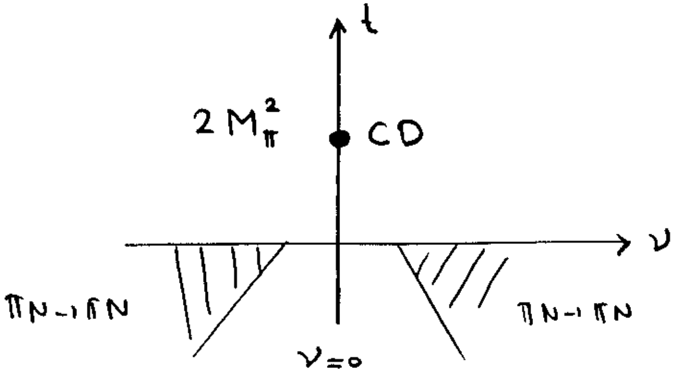
1. Sum rule

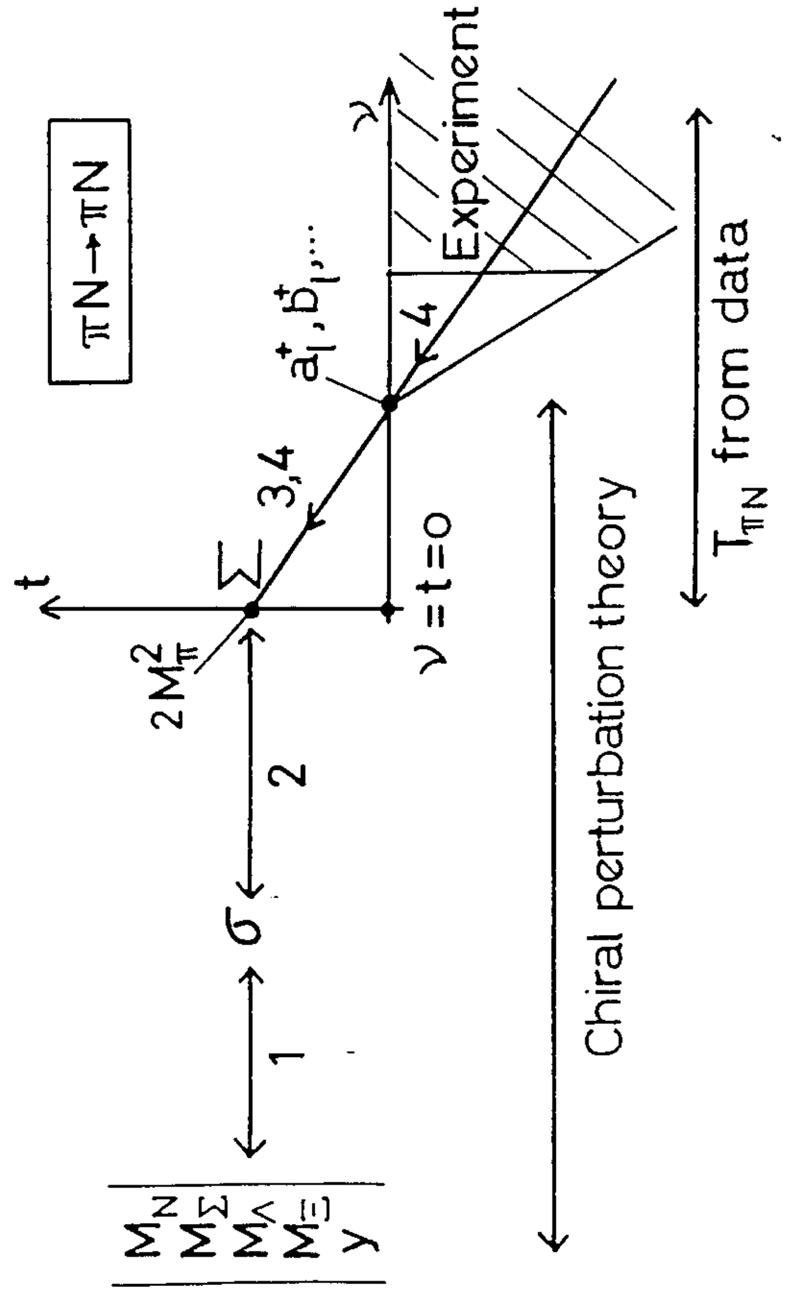
$$\underbrace{\frac{M_\pi^2}{2m} a_{0+}^+ + M_\pi^3 (b_{0+}^+ + 2a_{1+}^+ + a_{1-}^+)}_{0.165} = \underbrace{\int \sigma^{TOT} + \{ \; + \; + \; + \cdots \}}_{0.164} \tag{5.3}$$

Koch 1986

2. Amplitude at Cheng-Dashen point

The amplitude at the Cheng-Dashen point $s = u$, $t = 2M_\pi^2$ is particularly interesting, because the value of D^+ at that point is related to the baryon-spectrum, i.e. to $M_N, M_\Sigma, M_\Xi, M_\Lambda$, see below.





$\Sigma \doteq F_\pi^2 \bar{D}^+$ at CD point

The amplitude Σ has been evaluated very carefully from data by the Karlsruhe group with the result

$$\underline{\Sigma = (64 \pm 8)\,\text{MeV}}$$

Koch 1982
Höhler 1983

Note: New data (≥ 1982) not included in this analysis.

In CHPT, Σ can not be evaluated – it can however be related to other quantities, e.g. the scattering lengths:

$$\Sigma = \pi F_\pi^2 \left[(4 + 2x + x^2) a_{0+}^+ - 4\mu^2 b_{0+}^+ + 12 x \mu^2 a_{1+}^+ \right] + \Sigma_0 \qquad (5.4)$$

$\uparrow$ CHPT

$x = \frac{M_\pi}{m}$

$\Sigma_0 = -12.6\,\text{MeV}$

$$\underline{\Sigma = 60\,\text{MeV}}$$

J.G. 1987

Relations of the type (5.4) had been proposed earlier by

Altarelli et al. 197
Olsson, Osypowski 1980
(No loops)

6.53 The σ-term

The pion-nucleon sigma term is defined by the matrixelement

$$\sigma = \langle p | \hat{m} (\bar{u}u + \bar{d}d) | p \rangle (2M_p)^{-1} \; ; \; \hat{m} = \frac{m_u + m_d}{2} \qquad (5.5)$$

where $|p\rangle$ is a one-proton state of momentum $\vec{p}$, spin index suppressed.

CHPT allows to relate σ with the amplitude Σ evaluated in πN-scattering:

$$\Sigma = \sigma + (4-5)\,\text{MeV} \qquad (5.6)$$

This result is valid to 1 loop in CHPT.

[J.G., 1981, H. Leutwyler, J.G. 1982 ;

It appears that 2 loop corrections to this result may be substantial (i.e. of the same order as the one-loop contribution of (4–5) MeV). The problem is under study.

From (5.6) we get

$$\underline{\sigma \sim 60 \text{ MeV}}$$

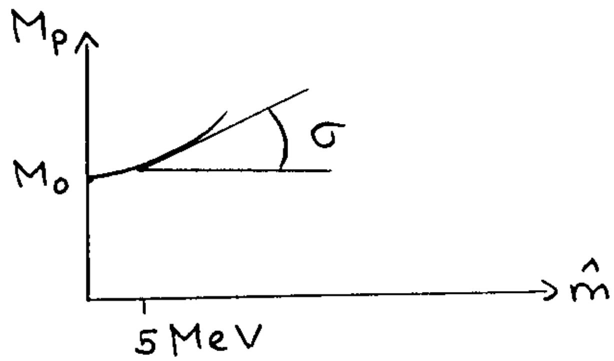
Karlsruhe-Helsinki phase shifts 1980 + CHPT

Why is this result interesting (and hard to understand)?

6.54 $\sigma \leftrightarrow M_N, M_\Sigma, M_\Xi, M_\Lambda$

As mentioned above, σ is also related to the masses of the baryon octet. The reason for this is the Feynman-Hellman-theorem:

$$\sigma = \hat{m}\,\frac{\partial M_p}{\partial \hat{m}} \quad ; \quad \hat{m} = \frac{m_u + m_d}{2}$$



Explicitly:

$$\sigma = \frac{2\hat{m}}{2\hat{m} + m_s}\left\{M_N - M_0 + \frac{3}{2}\,\frac{m_s}{m_s - \hat{m}}\left(M_\Xi - M_\Lambda\right)\right\} + O(m_q^3)$$

The ratio $m_s/\hat{m} = 25 \pm 2.5$ is rather well known from the meson spectrum (H. Leutwyler, J.G. 1982). The parameter M_0 is not fixed by CHPT alone – it is the nucleon mass in the chiral limit $m_u = m_d = m_s = 0$.

The rather surprising consequence of

the value $\sigma = 60$ MeV is the following result:

$$\boxed{\sigma = 60\,\text{MeV} \longrightarrow M_0 = 440\,\text{MeV}}$$

The nucleon mass in the chiral limit would have to be surprisingly small, if σ indeed turns out to be 60MeV.

!!

The one-loop corrections have been worked out: J.G. 1981, H. Leutwyler, J.G. 82

$$\boxed{\begin{array}{ccc} \sigma = 60\,\text{MeV} & & M_0 < 600\,\text{MeV} \\ \pi N \to \pi N & \longrightarrow & \text{or} \\ \text{KH 80} & & \dfrac{2\langle p|\bar{s}s|p\rangle}{\langle p|\bar{u}u + \bar{d}d|p\rangle} = 0.4 \end{array}}$$

Where could we fail?

1. Two-loops in (5.6) under study
2. Data on $\pi N \to \pi N$ $\curvearrowright \Sigma$

This second step (extrapolation of data to the unphysical CD point) is a highly nontrivial procedure, see Fig. on next page. In particular the discussion of error propagation is very difficult.

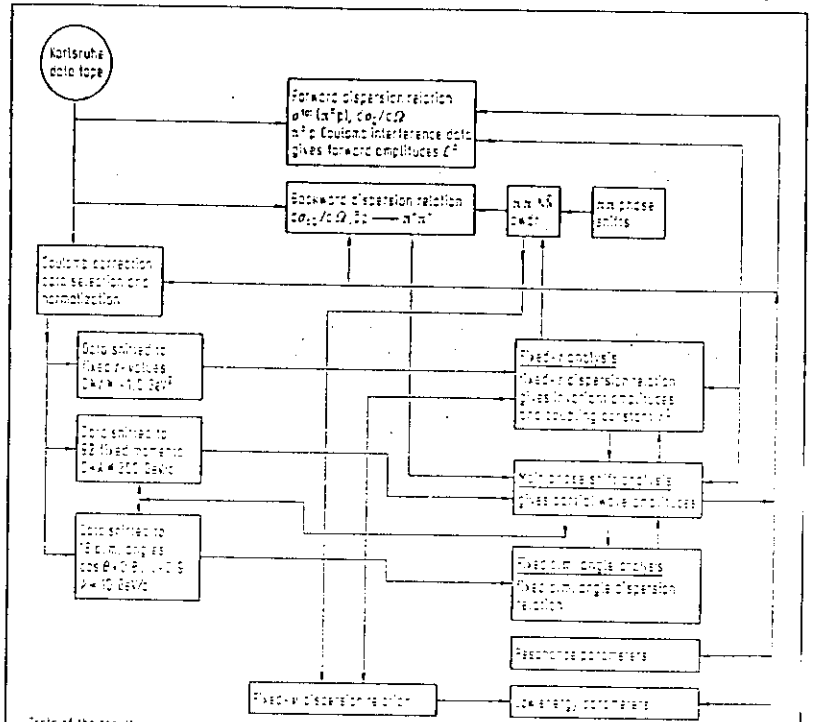
Note:

Extrapolation data $\to$ threshold can not be done on the basis of an effective range formula

$$\mathrm{Re}\, f_\ell = \frac{\sin 2\delta_\ell}{2q} = q^{2\ell}(a_{\ell\pm} + q^2 b_{\ell\pm} + \dots)$$

because this formula fails before the experimental region is reached.

2.1.7 2.1 Methods of partial wave and amplitude analysis Ref. p. 405



Tests of the results:

i) Compatibility with the s-channel partial wave dispersion relations

ii) Compatibility with dispersion relations along hyperbolas in the Mandelstam plane (for instance at fixed c.m. scattering angle and at u=0)

iii) Reasonable behaviour of zero trajectories of invariant and transversity amplitudes

Fig. 1.7.1. Flow diagram of the KH 78 partial wave analysis.

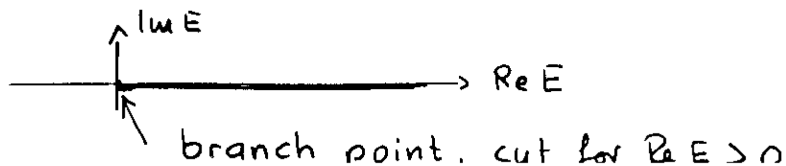
From: G. Höhler, 'Pion-Nukleon Streuung', Landolt-Börnstein Vol. 9b2 (1983) p. 11

6.55 πN partial waves near threshold

H. Leutwyler, M.P. Locher
M. Sainio, J.G. PL B213 (1988) 85

We propose a method which allows to relate (forthcoming) experimental low energy information with the amplitude at the Cheng-Dashen point, with the σ-term and with S-, P-waves.
Basic analyticity constraints are incorporated.

How does it work?
The keywords are analyticity and dispersion relations.
Consider a function $f(E)$ of the single (complex) variable E, and the analyticity domain



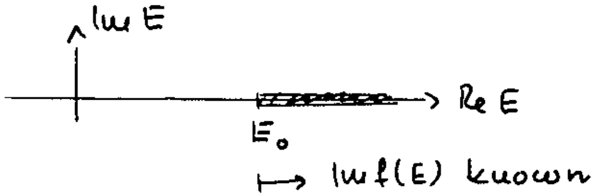
If $f(E)$ falls off as $|E| \to \infty$, one has the following relation between real- and imaginary part of $f(E)$,

$$\mathrm{Re}\, f(E) = \frac{1}{\pi} \mathop{⨏}_{0}^{\infty} \frac{dE'}{E'-E} \, \mathrm{Im}\, f(E') \qquad (5.7)$$

Let's assume furthermore that $f(E)$ fulfills a 'unitarity-type' relation

$$\mathrm{Im}\, f(E) = |f(E)|^2 \qquad (5.8)$$

Claim: If $\mathrm{Im}\, f(E)$ is known for $E > E_0$, we can reconstruct $f(E)$ completely.



We do the reconstruction by iteration:

1st Step: Set $\mathrm{Im}\, f(E) = 0$, $E < E_0$

From (5.7), we can obtain a first approximation to $\mathrm{Re}\, f(E)$ for all E:

$$\mathrm{Re}\, f_1(E) = \frac{1}{\pi} \mathop{⨏}_{E_0}^{\infty} \frac{dE'}{E'-E} \, \mathrm{Im}\, f(E')$$

The condition (5.8) gives then an improved imaginary part for all E:

$$\mathrm{Im}\, f_1(E) = |f(E)|^2$$

$$\to \left(\mathrm{Re}\, f_1(E), \mathrm{Im}\, f_1(E) \right) \quad \forall E$$

2nd Step Evaluate $\mathrm{Re}\, f_2$, $\mathrm{Im}\, f_2$ from improved imaginary part in first step:

$$\mathrm{Re}\, f_2(E) = \frac{1}{\pi} \mathop{⨏}_{0}^{\infty} \frac{dE'}{E'-E} \, \mathrm{Im}\, f_1(E')$$

$$\mathrm{Im}\, f_2(E) = |f(E)|^2$$

Continue until procedure converged.

Application to $\pi N \to \pi N$.

The experimentally well explored region, where both the differential and the total cross sections are measured, begins at

$$k_{LAB} \sim 170\ \mathrm{MeV}/c\ ,\quad T_\pi \sim 80\ \mathrm{MeV}$$ Ampl. $\pi N \to \pi N$

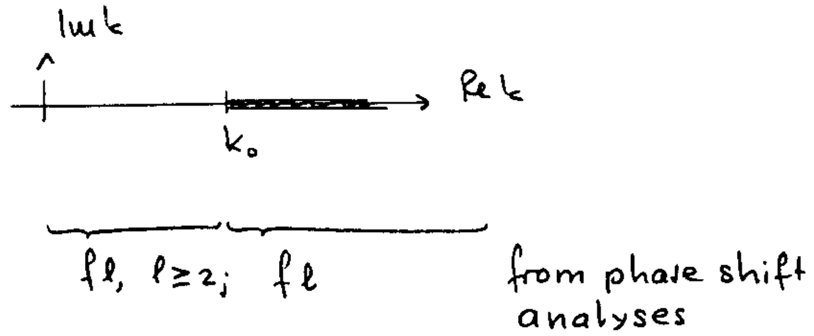
In the region below this energy, $T_{\pi N}$ is dominated by 6 partial waves:

$$f^I_{0+}\ ,\ f^I_{1+}\ ,\ f^I_{1-}\quad ;\ I = \pm \qquad (5.9)$$

We therefore need to apply the procedure [explained above for a single function $f(E)$] to the 6 partial waves $f^I_{\ell\pm}$.

For this to achieve it is best to consider 6 invariant amplitudes with simple analytic properties, and which can be expressed in terms of $f^I_{\ell\pm}$. We have chosen

$$\left.\begin{array}{l} B^\pm\ ,\ D^\pm \\ E^\pm = \dfrac{\partial}{\partial t}\,(A^\pm + \omega B^\pm) \end{array}\right\}\ \text{at } t=0 \qquad \omega = \sqrt{M_\pi^2 + k_{LAB}^2}$$



The analytic properties of $B^\pm$, $D^\pm$ and $E^\pm$ lead to 6 nonlinear integral equations for the 6 partial waves (5.9). They can be solved by iteration in terms of two subtraction constan[ts]

$$\boxed{a^+_{0+},\ a^+_{1+}}\ .$$

Check:

Using amplitudes KH80 $\quad k_{LAB} \geq 170$ MeV

$f_{\ell\pm},\ \ell \geq 2$ from KH80

$a^+_{0+},\ a^+_{1+}$ from KH80

$$\downarrow$$

reproduce $a^-_{0+},\ b^\pm_{0+},\ a^\pm_{1-},\ a^-_{1+}$ within one standard deviation.

$$\boxed{a^+_{0+},\, a^+_{1+} \longrightarrow f_\ell^I \longrightarrow \Sigma \longrightarrow \sigma}$$

Example:

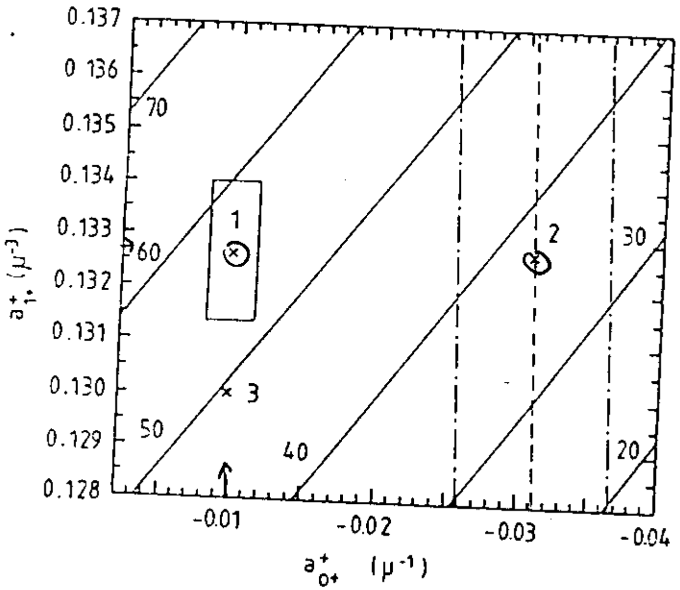


Fig. 1. The value of the Σ-term as a function of the parameters a^+_{0+} and a^+_{1+} using eq. (10) and $\Sigma_1 = 8$ MeV. The rectangle on the left shows the range in the scattering lengths quoted by the Karlsruhe group [3]. The error band on the right reflects the result of the π^-p atomic measurement $a^{\pi^-p}_{0+} = a^+_{0+} + a^-_{0+} = 0.059 \pm 0.006\mu^{-1}$ [24]. The crosses with numbers refer to the curves in fig. 2.

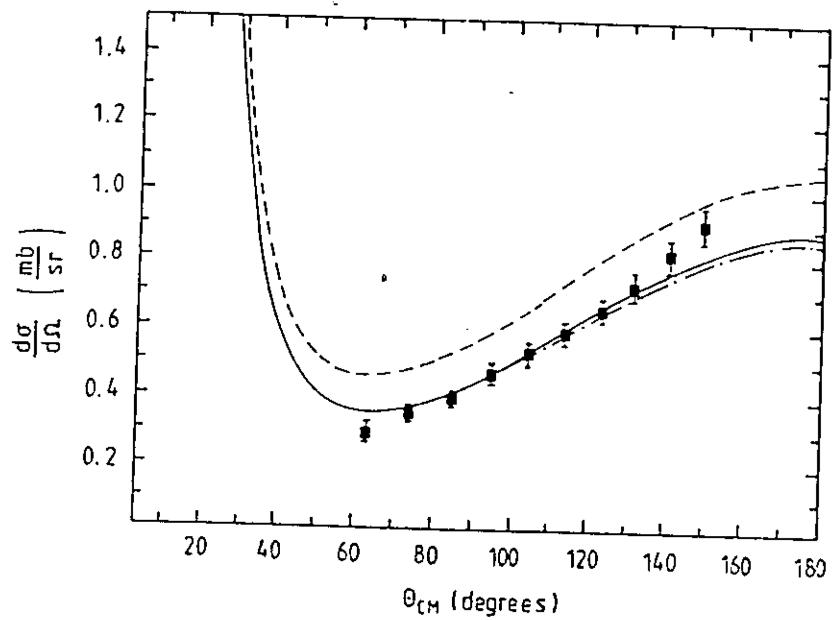


Fig. 2. The differential cross section for π^+p scattering at $k_{LAB} = 79$ MeV$/c$ ($T_\pi = 20.8$ MeV). The solid line gives the Karlsruhe solution (KH.80) (number 1 in fig. 1), the dashed line corresponds to the cross 2 in fig. 1 and the dash-dotted line to the cross 3. The experimental points are from ref. [26].

Under study (in particular by M. Sainio):

$$a^+_{0+}, a^+_{1+} \to f_\ell \to \frac{d\sigma}{d\Omega} \leftrightarrow \text{data?}$$

change

Very convenient machinery to incorporate new data on low energy $\pi N \to \pi N$ or on the value of a^+_{0+} (see below).

6.56 New data

I mention two (out of many others) new measurements of low energy parameters.

a) Elastic $\pi^\pm p \to \pi^\pm p$ below 100 MeV.

W. Gyles et al., Karlsruhe-Tübingen-Heidelberg-Triumf collaboration

$\pi^\pm p \to \pi^\pm p$ at

$T_\pi = 30$-50 MeV at PSI

Measurements done with magnetic spectrometer

Difficulties:

- Contamination of beam with μ, e
- Absolute normalization of $d\sigma/d\Omega$

Normalize with electromagnetic cross section $\mu p \to \mu p$, $e p \to e p$

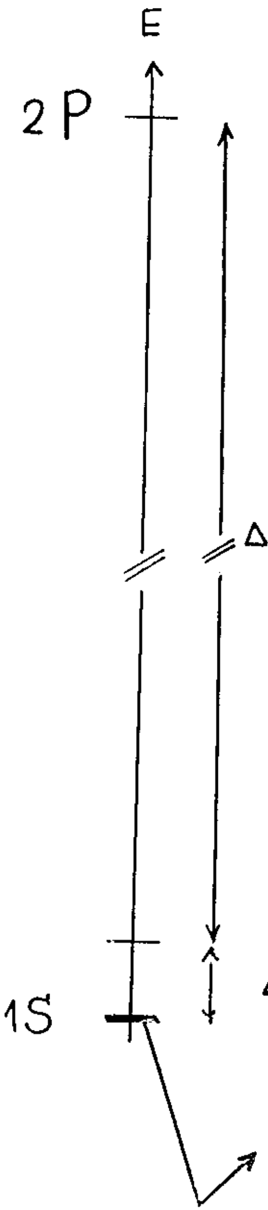
Status: Analysis of data being done presently.

b) Pionic hydrogen

D. Bovet et al., Neuchâtel-ETHZ-PSI collaboration

$\pi^- p$-atom

Deser et al. 1954
Trueman 1961
Rasche and Woolcock 1982



$\Delta E_{QED} = 2429.5$ eV

$\Delta E_{QCD} = f_1(a^+_{0+} + a^-_{0+}) \sim 5$ eV

$\Gamma_{QCD} = f_2(a^-_{0+})^2 \sim 0.8$ eV

f_i known

Aim:

$\left.\begin{array}{l}\Delta E_{QCD} \text{ to } \pm 1\% \\ \Gamma_{QCD} \text{ to } \pm 10\%\end{array}\right\} \Rightarrow \begin{array}{l}\pm 1\% \text{ in } a^+_{0+} + a^-_{0+} \\ < 5\% \text{ in } a^-_{0+}\end{array}$

$\boxed{\text{Determination of } a^\pm_{0+} \text{ independent of phase shift analysis}}$

Checks

(*) $3(a^+_{0+} + a^-_{0+}) = 0.178 \pm 0.019 \; M_\pi^{-1}$

obtained in earlier experiment
(Neuchâtel-SIN-Pasadena,
E. Bovet et al., Phys. Lett. 153B (1985) 231)

Note: (*) + a^-_{0+} from FDR Koch 86

$\rightarrow$ $\sigma \sim 37$ MeV

Would solve the problem with

$\sigma_{Baryon} \leftrightarrow \sigma_{\pi N}$

Technique:

New reflection-type crystal spectrometer

Instrumental width $\sim 0.5\,\text{eV}$

Difficulties:

- Doppler effect ($\pi^- p$ atoms not at rest) $\leftrightarrow \Gamma_{QCD}$
- Background (5 events 2P–1S per 1′)

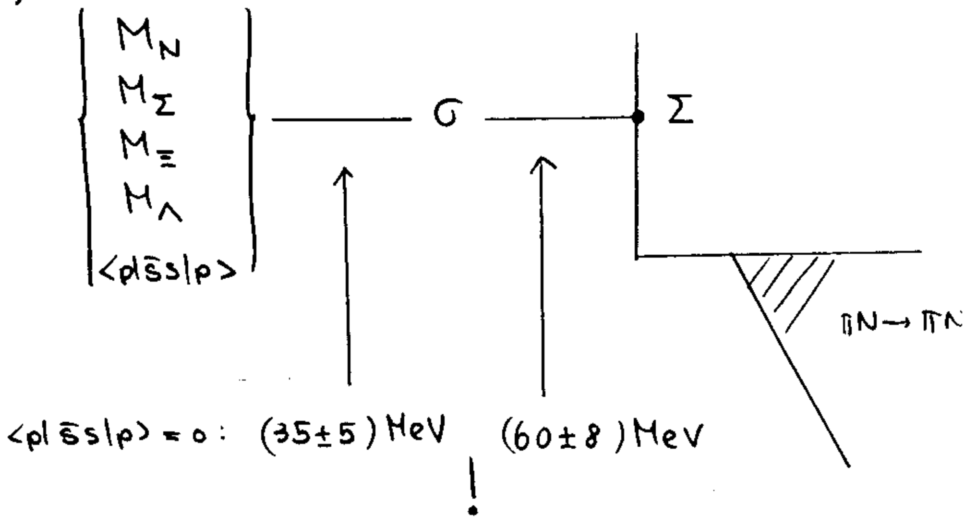
6.6 Summary

1) $\mathcal{L}_{QCD} \longrightarrow \mathcal{L}_{eff} = \mathcal{L}^{(2)}_{\pi\pi} + \cdots + \mathcal{L}^{(1)}_{\pi N} + \cdots$

at low energies

2)  $T_{\pi N}$ to one loop

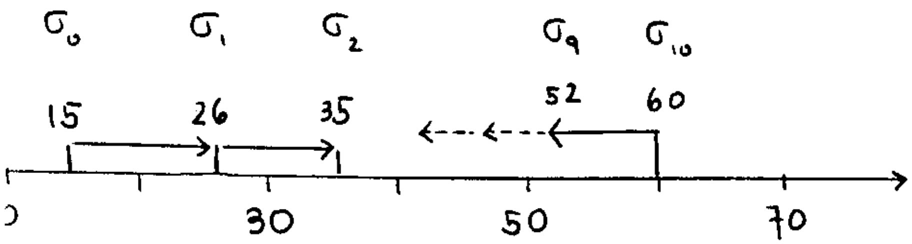
3)



4) $\sigma = 60\,\text{MeV} \longrightarrow M_0 < 600\,\text{MeV}$

$\langle p|\bar{s}s|p\rangle$ large

Note: It appears that the puzzle may finally be solved thanks to many small effects which all work in the same direction:



σ_0 : $\langle p|\hat{m}(\bar{u}u+\bar{d}d)|p\rangle$: $3\times 5 = 15$ MeV

σ_1 : CHPT, lowest order $\rightarrow \langle p|\bar{s}s|p\rangle = 0$

σ_2 : " , 1 loop $\rightarrow \langle p|\bar{s}s|p\rangle \neq 0$

σ_{10} : $\pi N \rightarrow \pi N$, $a_0^0 = 0.26 \rightarrow \pi\pi$-scattering

σ_9 : FDR, $a_0^0 = 0.20$

σ_8 : New data at low energies

σ_7 : 2 loop CHPT

References Chapter 6

References for this chapter are given in the text.

7. LIGHT QUARKS AT LOW TEMPERATURE

7.1 Introduction

At low temperature:

Colour confined

chiral symmetry spontaneously broken

At high temperature:

The interaction among quarks and gluons is weak, such that colour is liberated and chiral symmetry is restored [1].

Low temperature hadronic phase $\longrightarrow$ High temperature quark-gluon-plasma phase

Estimates of critical temperature at which the transition occurs:

$$T_c = 100\,\text{MeV} - 300\,\text{MeV} \qquad [2]$$

Here we consider the hadronic phase at low temperature.

7.2 The condensate at $T \neq 0$

E.g. lattice calculations provide us with temperature-dependent condensate $\langle \bar{q} q \rangle$, defined by

$$\langle \bar{q}q \rangle \doteq \frac{\mathrm{Tr}\, e^{-H_{QCD}/T}\, \bar{q}q}{\underbrace{\mathrm{Tr}\, e^{-H_{QCD}/T}}_{= Z}} \qquad (7.1)$$

where T is identified with inverse euclidean time, $T = 1/x_4$. The trace in (7.1) is to be carried over all states available,

$$\mathrm{Tr}\, A = \sum_n \langle n | A | n \rangle$$

where $|n\rangle = |0\rangle, |\pi\rangle, |2\pi\rangle, \ldots |p\rangle, \ldots, |J/\psi\rangle, \ldots$

What is the connection with the vacuum expectation value ('condensate at zero temperature') $\langle 0|\bar{q}q|0\rangle$ considered before?

Claim: $\langle \bar{q}q \rangle \xrightarrow[T\to 0]{} \langle 0|\bar{q}q|0\rangle$.

The proof is easy: $e^{-H_{QCD}/T}$

$$\mathrm{Tr}\, \bar{q}q\, e^{-H_{QCD}/T} = \langle 0|\bar{q}q|0\rangle + \underbrace{\sum_{n \neq |0\rangle} \langle n|\bar{q}q|n\rangle e^{-E_n/T}}_{\to 0,\ T \to 0}$$

and (for) the partition function

$$Z = \mathrm{Tr}\, e^{-H_{QCD}/T} \xrightarrow[T\to 0]{} 1$$

for the same reason.

From (7.1) we therefore find that indeed

$$\langle \bar{q}q \rangle \xrightarrow[T\to 0]{} \langle 0|\bar{q}q|0\rangle$$

7.3 Chiral symmetry and $\langle\bar{q}q\rangle$

For small quark masses and low temperatures, chiral symmetry constrains the temperature dependence of $\langle\bar{q}q\rangle$, Z:

Example:

$$\langle\bar{q}q\rangle = \sum_{r,s=0,1,2,\dots} c_{rs}\,(T^2)^r\,(T^2\ln T)^s \quad \Big|_{m_u=m_d=0}$$

$$\langle\bar{u}u\rangle = \langle 0|\bar{u}u|0\rangle\Big\{1-\frac{T^2}{8F^2}-\frac{T^4}{384F^4}-\frac{T^6}{288F^6}\ln\frac{\Lambda_q}{T} + O(T^8)\Big\} \tag{7.2}$$

[3,4,6]

The coefficients c_{rs} can be worked out from chiral perturbation theory, using the same lagrangian as was used before:

$$\mathrm{Tr}\, e^{-H_{QCD}/T} = \int [dU]\, e^{-\int^{1/T} d^4x\, \mathcal{L}_{eff}} \tag{7.3}$$

- Integration extends over all pion field configurations which are periodic in the euclidean time direction,

$$U(\vec{x}, x_4 + \tfrac{1}{T}) = U(\vec{x}, x_4).$$

- The lagrangian $\mathcal{L}_{eff}$ in (7.3) is the <u>same</u> lagrangian as discussed in Chapt. 1-5

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots$$

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial_\mu U^+ - M^2(U+U^+)\rangle$$

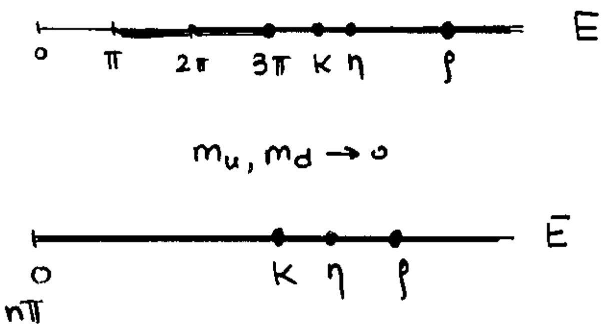
$$U \in SU(2)\ ;\ M^2 = (m_u+m_d)B$$

[continued to the euclidean region] [5]

- The low-energy theorem (7.2) can be derived on the same level of rigour as the current algebra results in Chapt. 1-

Why does (7.3) hold, i.e., why does CHPT constrain the low temperature behaviour?

Spectrum of QCD:



Now consider

$$e^{-E_n/T} |n\rangle$$

$$\sum_n \langle n | \bar{q} q \, e^{-H_{QCD}/T} | n \rangle \qquad (7.4)$$

1) At temperature T, only states with $E \lesssim T$ contribute, i.e., only pions, if T sufficiently small

2) Pions have energy $E \sim T$.

Need matrix elements

$\langle \pi | \bar{q} q | \pi \rangle$, $\langle 2\pi | \bar{q} q | 2\pi \rangle$, ...

at low energies ⟶ CHPT

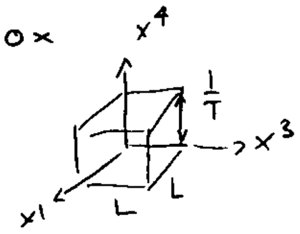
7.4 Results

The 3-loop result (7.2) has been extended recently to account for nonzero quark masses and for the exponentially small contributions generated by the massive states [6]. I refer to this reference for a detailed discussion of the results. [In that reference, the leading coefficients in the low temperature expansion of the pressure are worked out as well.]

7.5 Other applications

The same technique may be used to evaluate the partition function, the condensate, ... in a finite box

$$L \times L \times L \times \frac{1}{T} \quad ; \quad T = \frac{1}{x^4}$$



This is in particular relevant for lattice calculations, which are done in a finite box ab initio. CHPT may thus be used to extrapolate to $L \to \infty$.

I refer to the articles by Leutwyler [4] for a recent account of the state of the art.

References Chapter 7

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[3] J. Gasser and H. Leutwyler, Phys. Lett. **184B** (1987) 83.

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