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INTRODUCTION TO SUPERSYMMETRY

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Please note: These are preliminary notes intended for internal distribution only.

Introduction to Supersymmetry

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1. Why Supersymmetry ?

In field theory, the masses of particles with spin $\geq \frac{1}{2}$ are "protected" by symmetries. This means that the lagrangian has a bigger symmerty in the absence of mass terms. In particular, spin-1 particles are protected by local gauge symmetries, and spin-$\frac{1}{2}$ particles are protected by chiral symmetries.

The presence of a symmetry for $m=0$ implies that, once masses are given, quantum corrections to them will be proportional to mg^2 where g is a coupling constant. Thus, the quantum corrections are small.

The situation is different for scalar fields. Consider the lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2 - \lambda (|\phi|^2)^2 \tag{1.1}$$

This lagrangian has a global $U(1)$ symmetry, and setting $m=0$ does not give rise to any new symmetry.

Consider the first quantum correction to the scalar mass. It is given by the following Feynmann diagram

$$-i\Delta m^2 = \quad$$  $$= -4i\lambda \int^{\Lambda} \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \tag{1.2}$$

We have introduced an ultraviolet cutoff Λ. On dimensional grounds

$$\Delta m^2 \approx \lambda \Lambda^2 \tag{1.3}$$

We see that if the cutoff is large, then $\Delta m^2 \gg m^2$.

This behaviour is responsible for one of the unresolved problems of the standard model. In the standard model, the breaking of $SU(2)_L \times U(1) \to U(1)_{EM}$ is achieved by postulating a Higgs sector. The purely scalar part of the standard-model is very similar to the lagrangian (1.1). The main difference is that we take $m^2 < 0$. This gives rise to a "Mexican hat" potential. The Higgs field develops an expectation value

$$\langle \phi \rangle = v = \sqrt{\frac{-m^2}{2\lambda}} \tag{1.4}$$

The Higgs scale v is determined experimentally in terms of the known masses of the W and Z and the two coupling constants of the electroweak theory.

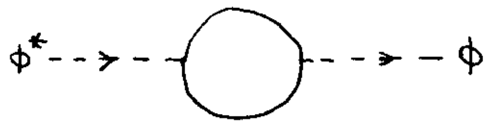
Since v is proportional to $|m|$, it receives at the one loop level a correction similar to (1.3). This must be cancelled by adding a counter-term $c\lambda\Lambda^2|\phi|^2$. In order that v retains its correct value, the numerical coefficent c must be determined with an accuracy of $|\frac{m}{\Lambda}|^2$. In order to get an idea of how small this number is, we have to associate Λ with some physical scale. In the real world, the relevant cutoff could be as high as 10^{19} GeV, the scale at which gravitational interactions become strong. Since $v \approx |m| \approx 100$ GeV we have $|\frac{m}{\Lambda}|^2 \approx 10^{-34}$. This unpleasant situation is know as the "fine tuning problem".

The fine tuning problem could be avoided if we find a way to cancell the quadratically divergent contributions to scalar masses. One can easily convince himself that this cannot be done in a theory containing only bosons. The reason is that the one loop contribution (1.2) is always proportional to the quartic couplings, and the latter always have the same sign because of the requirement that the classical energy should be bounded from below.

Still proceeding on a trial and error basis, let us see whether we can do any better in the presence of fermions. As a first guess, let us try the introduction of a Dirac fermion and the new lagrangian

$$\mathcal{L}' = \mathcal{L} + i\bar{\psi}\not{\partial}\psi - m'\bar{\psi}\psi - y(\bar{\psi}\psi\phi + h.c.) \qquad (1.5)$$

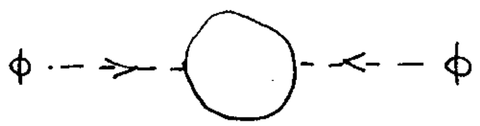
we now have a new diagram



(1.6)

This diagram will also give rise to a quadratically divergent contribution. Let us check its sign. We have i^2 from the propagators, $(-iy)^2$ from the vertices and an all important minus sign from the fermion look. Altogether, the contribution of (1.6) has the opposite sign from (1.2). In order that the two will cancell each other, y^2 must be related to λ.

Although the quadratic divergence of (1.2) can be cancelled by (1.6), the lagrangian (1.5) gives rise to new quadratic divergences which are absent from the original theory. One such divergence is



(1.7)

The reason for this new divergence is that in (1.5) we coupled only the real part of ϕ to the fermion. Indeed, the lagrangian (1.5) does not even have the global $U(1)$ invariance of (1.1). Something more sophisticated must be done in order to cancell the quadratic divergences. This "something" is supersymmetry. We will introduce it in section 3 but first we have to discuss Weyl fermions.

also dign sign look

2. Weyl fermions

A Dirac fermion is the direct sum of two irreducible representations of the lorentz group $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. This is most easily seen in the chiral representation for the γ-matrices. We take

$$\gamma_\mu = \begin{pmatrix} & \bar{\sigma}_\mu \\ \sigma_\mu & \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{2.1}$$

$$\sigma_\mu = (1, \sigma^a), \quad \bar{\sigma}_\mu = (1, -\sigma^a) \tag{2.2}$$

The Lorentz generators $\Sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu \gamma_\nu]$ then take the diagonal form

$$\Sigma_{\mu\nu} = \frac{i}{2}\begin{pmatrix} \bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu & \\ & \sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu \end{pmatrix} \equiv \begin{pmatrix} \bar{\sigma}_{\mu\nu} & \\ & \sigma_{\mu\nu} \end{pmatrix}. \tag{2.3}$$

If we write

$$\psi_{Dirac} = \begin{pmatrix} \psi_L \\ \psi_R' \end{pmatrix} \tag{2.4}$$

we see that ψ_L and ψ_R' transform independently. In Minkowsky space, the operation of charge conjugation

$$\psi_R \equiv i\sigma_2 \psi_L^* \tag{2.5}$$

takes a fermion transforming under $(\frac{1}{2},0)$ to one transforming under $(0,\frac{1}{2})$. We can therefore describe any spin-$\frac{1}{2}$ field in terms of left handed - or Weyl fermions - and their charge conjugates. As we will see below, Weyl fermions occur naturally in $N=1$ supersymmetric theories. The hermitian lagrangian for a Weyl fermion is

$$\mathcal{L}_W = \frac{i}{2}\left(\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\, \bar{\sigma}^\mu \psi \right) \tag{2.6}$$

We also note that the following inner product is a Lorentz scalar

$$\psi\psi \equiv i\psi^T\sigma_2\psi \equiv \psi^T \epsilon \psi \tag{2.7}$$

$$\bar{\psi}\bar{\psi} \equiv -\bar{\psi}^T \epsilon \bar{\psi} = (\psi\psi)^+ \tag{2.8}$$

3. The Wess-Zumino model

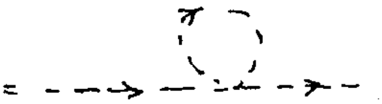
3.1 massless case

The quadratic divergence we encountered in section 1 can be eliminate by "supersymmetrizing" the lagrangian (1.1). We consider first the case $m=0$. The SUSY version of the lagrangian (1.1) with $m=0$ is known as the massless Wess-Zumino model. The lagrangian is

$$\mathcal{L}' = |\partial_\mu \phi|^2 - g^2 (\phi^* \phi)^2 + \frac{i}{2} \bar\psi \bar\sigma^\mu \overleftrightarrow{\partial_\mu} \psi - g(\psi\psi\phi + \bar\psi\bar\psi\phi^*) \qquad (3.1)$$

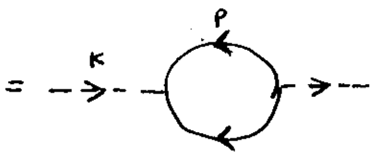
Notice that the quartic coupling is the square of the Yukawa coupling. At the one loop level we find two contributions to the scalar self-energy

$$\Sigma^{(b)} = \qquad (3.2)$$



$$= 4g^2 \int^{\Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + i\epsilon}$$

$$\Sigma^{(f)} = \qquad (3.3)$$



$$= -2g^2 \int^{\Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{tr}(\bar\sigma^\mu p_\mu)\epsilon[(\bar\sigma^\nu)^T (p-K)_\nu]\epsilon^T}{(p^2 + i\epsilon)[(p-K)^2 + i\epsilon]} = \qquad (3.4)$$

$$= -2g^2 \int^{\Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{tr}(\bar\sigma^\mu \sigma^\nu)\; p_\mu (p-K)_\nu}{(p^2 + i\epsilon)[(p-K)^2 + i\epsilon]} = \qquad (3.5)$$

$$= -4g^2 \int^{\Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{p^2 - p\cdot K}{(p^2 + i\epsilon)[(p-K)^2 + i\epsilon]} \qquad (3.6)$$

In eq. (3.4) the minus sign is due to the fermion loop and 2 is a symmetr factor. (A given ψ from the $\psi\psi\phi$ vertex can be contracted with one of the two $\bar\psi$-s in the $\bar\psi\bar\psi\phi^*$ vertex). ϵ and ϵ^T come respectively from the $\psi\psi\phi$ and $\bar\psi\bar\psi\phi^*$ vertex. We have used

$$\text{F.T. } \langle 0 | T\{\bar\psi(x)\, \psi(y)\} | 0 \rangle = \frac{i\bar\sigma^\mu p_\mu}{p^2 + i\epsilon} \qquad (3.7)$$

$$\text{F.T. } \langle 0 | T\{\psi(x)\, \bar\psi(y)\} | 0 \rangle = \qquad (3.8)$$

$$= \frac{i(\bar\sigma^\mu)^T p_\mu}{p^2 + i\epsilon}$$

and the identities

$$\epsilon (\bar{\sigma}^{\mu})^{T} \epsilon^{T} = \sigma^{\mu} \tag{3.9}$$

$$\mathrm{tr}\, \bar{\sigma}^{\mu} \sigma^{\nu} = 2 \eta^{\mu\nu} \tag{3.10}$$

The arrows in eq. (3.3) lead from $\bar{\Psi}$ (or ϕ^*) to Ψ (or ϕ). The two-component notation we use here makes manifest the Weyl character of the fermions. Its disadvantage is that one has to keep track of ϵ-matrices and to use both σ^{μ} and $\bar{\sigma}^{\mu}$. An alternative formulation using four component Majorana fermions is also available. In the four-component notation, the Feynman rules are very similar to the familiar Feynman rule for Dirac fermions except that symmetry factors are different. Its disadvantage is that Fiertz identities (to be encountered below) become more complicated.

Comparing eqs. (3.2) and (3.6) we see that the quadratic divergence cancels. We are left with a ~~logarithmi~~ logarithmic divergence. Furthermore, on dimensional grounds the logarithmic divergence must take the form

$$\Sigma^{(b)} + \Sigma^{(f)} \approx g^2 K^2 \log \frac{\Lambda^2}{K^2} \quad . \tag{3.11}$$

It is therefore cancelled by <u>wave-function</u> renormalization.

We have therefore achieved more then just cancelling the quadratically divergent contribution to Δm^2. In fact, we find that Δm^2 vanishes! This statement remains true to all orders in perturbation theory.

This is an example of how SUSY protects scalar masses. In the lagrangian (3.1) the fermion is protected from aquiring a mass by a chiral <u>R-symmery</u>. The charges under this U(1) symmetry are

$$\begin{aligned} Q(\Psi) &= -1 \qquad & Q(\bar{\Psi}) &= +1 \\ Q(\phi) &= +2 \qquad & Q(\phi^*) &= -2 \end{aligned} \tag{3.12}$$

SUSY requires the equality of the scalar and fermion masses. Consequently, the scalar remains massless as long as the fermion does.

3.2 massive case

The massive version of the Wess-Zumino model is defined by the lagrangian

$$\mathcal{L} = \mathcal{L}' - m^2|\phi|^2 - \frac{m}{2}(\psi\psi + h.c.) - mg(\phi^2\phi^* + h.c.) \qquad (3.13)$$

Notice that $\mathcal{L}$ is no longer invariant under the R-symmetry. This is to be expected – a fermion mass always reduces the symmetry of the lagrangian. Another important observation is that $\mathcal{L} - \mathcal{L}'$ contains not only mass terms but also an interaction term. Both mass and interaction terms must be related in a particular way in order to have SUSY.

Let us examine some one loop diagrams in the massive theory. First we need the propagators:

$$\text{F.T.} \quad \langle 0| T\{\phi^*(x)\ \phi(y)\}|0\rangle = \frac{i}{p^2-m^2+i\epsilon} \qquad (3.14)$$

$$\text{F.T.} \quad \langle 0| T\{\bar\psi(x)\ \psi(y)\}|0\rangle = \frac{i\bar\sigma^\mu p_\mu}{p^2-m^2+i\epsilon} \qquad (3.15)$$

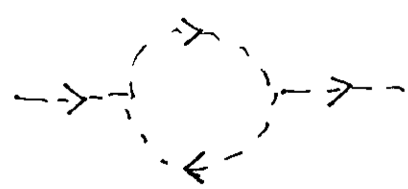
$$\text{F.T.} \quad \langle 0| T\{\psi(x)\ \psi(y)\}|0\rangle = \qquad (3.16)$$

$$= \text{F.T.} \quad \langle 0| T\{\bar\psi(x)\ \bar\psi(y)\}|0\rangle = \frac{im}{p^2-m^2+i\epsilon}$$

Taking $m=0$ we go back to the propagators of the massless theory. In particular the $\langle\psi\psi\rangle$ and $\langle\bar\psi\bar\psi\rangle$ propagators vanish in the massless limit.

We start we the two point function $\Gamma^{(\phi^*\phi)}$. In addition to the diagrams (3.2) and (3.3) we have two more, coming from the new trilinear vertex:

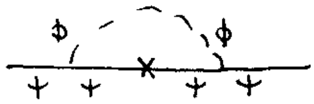

(3.17)


(3.18)

The quadratic divergence cancells as before, but now we have a logarithmic divergence proportional to m^2, which must be cancelled by a mass counter-term

$$\Delta m^2 \approx g^2 m^2 \log \Lambda \quad . \tag{3.19}$$

Next consider the fermion self-mass. This is given by the two point function $\Gamma^{(\psi\psi)}$. The only possible diagram is



(3.20)

but (3.20) vanishes identically because there is no $\langle\phi\phi\rangle$ propagator.

The vanishing of (3.20) is related to the analytic property of the ~~interaction~~ fermionic interaction: it contains only fields $(\psi\psi\phi)$ or only their complex conjuges $(\bar\psi\bar\psi\phi^*)$ but not both. A similar statement applies to the fermion mass term: $(\psi\psi$ or $\bar\psi\bar\psi)$. On the other hand, the scalar potential must be positive definite, hence it ~~depends on~~ contains terms involving ϕ and ϕ^* simultaneously.

At this point, we have already raised our expectations from SUSY. Since SUSY relates bosons and fermions, we anticipate their masses be renormalized in the same way. As we shall see, it is possible to reformulate the massive Wess-Zumino model such that counter-terms for bosons and fermions are equal, by introducing auxiliary fields. This will be discussed in section 6.

The last two-point function we want to examine is $\Gamma^{(\phi\phi)}$. The reader can check that at the one-loop level there are two diagrams which contribute to $\Gamma^{(\phi\phi)}$, and that their sum vanishes identically. At higher orders, $\Gamma^{(\phi\phi)}$ does not vanish but remains finite - there is no need for a $\phi\phi$ counter-term.

4. The Supersymmetry Algebra

We now want to introduce SUSY in a more systematic way. In field theory, symmetries have <u>generators</u> which satisfy an <u>algebra</u>. We distinguish in particular between space-time symmetries which act on both coordinates and fields, and internal symmetries which act only on fields. The space-time symmetries include translations and Lorentz transformations with generators P_μ and $M_{[\mu,\nu]}$ respectively. They satisfy the Poincare algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\,(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho}) \tag{4.1}$$

$$[M_{\mu\nu}, P_\lambda] = i\,(\eta_{\lambda\nu} P_\mu - \eta_{\lambda\mu} P_\nu) \tag{4.2}$$

$$[P_\mu, P_\nu] = 0 \tag{4.3}$$

Internal symmetries satisfy some Lie algebra of the form

$$[Q^a, Q^b] = i f^{abc}\, Q^c \tag{4.4}$$

In addition, they commute with both $M_{\mu\nu}$ and P_λ.

What is common to all the above symmetries is that their generators are <u>bosonic</u>. The result of their action on a given field depends only on the field and its derivative (in the case of space-time symmetries).

SUSY is a symmetry which relates bosons and fermions, i.e. particles of different spin. This gives SUSY a unique status among the symmetries of relativistic field theory. Let us denote the SUSY generator by Q. When acting on a scalar field it should give us a fermion

$$[Q, \phi(x)] \sim \psi(x) \quad . \tag{4.5}$$

In order that a relation like (4.5) should hold, Q must have the following properties. First, its dimension must be $\frac{1}{2}$. Second, it must carry a spinorial index, i.e.

$$[\, Q_\alpha ,\ \phi(x)\,] \sim \psi_\alpha(x) \qquad . \qquad (4.6)$$

As a result, Q_α transforms in a non-trivial way under the Lorentz group

$$[\, M_{\mu\nu}\ Q_\alpha\,] = i(M_{\mu\nu})_{\alpha\beta}\, Q_\beta \qquad (4.7)$$

We will shortly determine the representation (n,m) to which Q_α belong. Since Q_α is spinorial, $m \neq n$ and so the operation of charge conjugation will define a new object which we denote $\bar{Q}_{\dot\alpha} \equiv (Q_\alpha)^{*}$

The SUSY generators Q_α and $\bar{Q}_{\dot\alpha}$ take bosons, i.e. fields which satisfy canonical commutation relations, to fermions i.e. fields which satisfy anticommutation relations. This indicates that the Q-s should satisfy anticommutations relations among themselves. We will postulate

$$\{ Q_\alpha , Q_\beta \} = \{ \bar{Q}_{\dot\alpha} , \bar{Q}_{\dot\beta} \} = 0 \qquad (4.8)$$

A generalization of (4.8) will be discussed below. Relations (4.8) are reminicent of the anticommutation relations $\{\psi_\alpha, \psi_\beta\} = 0$ for fermions.

Next we have to determine the ~~rela~~ anticommutator $\{\bar{Q}_{\dot\alpha}\ Q_\alpha\}$. Notice that $\delta_{\alpha\dot\alpha}\{\bar{Q}_{\dot\alpha}, Q_\alpha\}$ is a positive definite operator, hence we cannot postulate that this anticommutator vanish. Since Q_α and $\bar{Q}_{\dot\alpha}$ commute with H, the same is true for $\{\bar{Q}_{\dot\alpha}, Q_\alpha\}$. Consequently, we must identify $\{\bar{Q}_{\dot\alpha}, Q_\alpha\}$ with some of the generators of space-time symmetries. Furthermore, these generators must belong to the symmetric representation (k,k) ${}_{,\,k \le m+n}^{|m-n| \le k}$. The only candidates are the translation generators P_μ which belong to the $(\frac{1}{2},\frac{1}{2})$ represantation. This in Turns, imply that Q_α must belong to the $(0,\frac{1}{2})$ represantation, i.e. it transforms as a Weyl spinor .

We thus postulate

$$\{ Q_\alpha , \bar{Q}_{\dot\alpha} \} = 2\, \sigma^\mu_{\alpha\dot\alpha} P_\mu \qquad (4.9)$$

The relations (4.8) and (4.9) define the <u>N=1</u> <u>SUSY</u> <u>algebra</u>. It can be shown that they imply

$$[Q_\alpha , P_\mu] = 0 \qquad (4.10)$$

Taken together, the relations (4.1)-(4.3), (4.7)-(4.10) define a <u>graded</u> <u>Lie</u> <u>algebra</u> - the super-Poincare algebra.

By taking a trace in eq. (4.9) we obtain

$$\mathrm{tr}\, \{ Q , Q^\dagger \} = \bar\sigma^{0}_{\dot\alpha\alpha} \{ Q_\alpha , \bar{Q}_{\dot\alpha} \} = 4H \qquad (4.11)$$

In a SUSY theory, <u>the hamiltonian</u> <u>is</u> <u>the</u> <u>square</u> <u>of the SUSY</u> <u>generators</u>. This extremely important property has several consequences.

(1) In a SUSY theory the energy of all states in non-negative. As we will see shortly, this is a meaningful statement even for the vacuum energy. (→ eq. (4.9))

(2) The hamiltonian is the generator of time evolution, and it depends on all mass parameters and coupling constants needed to define a theory. The same must be true for the SUSY generator. Consequently the transformation law for some fields must depend on those mass parameters and coupling constants. As we shall see, these are always the fermions.

(3) Consider the two subspaces consisting of all bosonic or fermionic states with a given energy $E > 0$. Apriori, the action of Q and $\bar{Q}$ takes us from U_B <u>into</u> $U_F' \subseteq U_F$. Acting again we go from U_F' to $U_B' \subseteq U_B$:

(4.12)

but the result of these two steps is equal to the action of H, which for $E>0$ is a one-to-one mapping of U_B onto itself. Therefore there is a one-to-one correspondence between bosonic and fermionic states with the same energy: The action of Q and $\bar{Q}$ together maps U_B onto U_F and vice versa.

(4) A state with $E=0$ need not be paired. Since $E=0$ is the lowest possible energy, the state with $E=0$ is the vacuum state. We will assume that this state is unique. (This may not be true for non-abelian SUSY theories).

(5) In ordinary field theories, the vacuum energy is a divergent quantity. At the oneloop level, for example, one has

$$E^{(1)}_{vac} = \langle 0 | H | 0 \rangle = \sum_i E_i^B - \sum_j E_j^F \tag{4.13}$$

The r.h.s. of (4.13) is the difference $H - :H:$. In SUSY theories thanks to the one-to-one correspondence between bosons and fermions

$$E^{(1)}_{vac} = 0 \tag{4.14}$$

In fact $E_{vac}=0$ holds as an exact statement to all orders in perturbation theory.

(6) A symmetry is broken spontaneously iff its generator does not annihilate the vacuum. For SUSY

$$Q|0\rangle \neq 0 \text{ or } \bar{Q}|0\rangle \neq 0 \iff E_{vac} > 0 \tag{4.15}$$

The vacuum energy is an order parameter for SUSY breaking. The statements we made in (5) are true provided $E_{vac}=0$ at the classical level. In a spontaneously broken SUSY theory, the masses of bosons and fermions are not equal. The vacuum energy is no longer finite. It diverges, but only logarithmically.

5. On-Shell Representation

In the previous section we studied the abstract algebraic properties of the SUSY generators and some of their physical consequences. We now want to construct a representation of the SUSY algebra on the field operators of a quantum field theory. Because of the relation (4.11) these transformation rules must be given with reference to a specific lagrangian. This is different from the case of internal symmetries, where the transformation rule

$$[Q^a\ \phi_A] = T^a_{AB}\phi_B \tag{5.1}$$

is independent of any specific lagrangian. (and need not be a symmetry).

We will now give the action of Q_α and $\bar{Q}_{\dot\alpha}$ on the fields ϕ and ψ_α of the Wess-Zumino model. First we introduce some terminology which allows us to write the transformation rules in a way which is easily generalizable to other SUSY theories. We define a superpotential

$$W = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3 \tag{5.2}$$

Notice that W is an analytic polynomial of degree three. This allows us to write the lagrangian (3.13) in a compact form

$$\mathcal{L} = |\partial_\mu\phi|^2 + \frac{i}{2}\bar\psi\bar\sigma^\mu \overleftrightarrow{\partial_\mu}\psi - \left|\frac{\partial W}{\partial\phi}\right|^2 - \frac{1}{2}\left(\psi\psi\frac{\partial^2 W}{\partial\phi^2} + h.c.\right) \tag{5.3}$$

The transformation rules are: (chiral rep.)

$$\begin{aligned} i[Q_\alpha\ \phi] &= \sqrt{2}\,\psi_\alpha & [\bar{Q}_{\dot\alpha}\ \phi] &= 0 \\ i\{Q_\alpha\ \psi_\beta\} &= -\sqrt{2}\,\epsilon_{\alpha\beta}\left(\frac{\partial W}{\partial\phi}\right)^* & i\{\bar{Q}_{\dot\alpha}\ \psi_\beta\} &= i\sqrt{2}\,\sigma^\mu_{\beta\dot\alpha}\partial_\mu\phi \end{aligned} \tag{5.4}$$

Since we work in the canonical framework, the last relation should really be written as

$$i\{\bar{Q}_{\dot\alpha}\ \psi_\beta\} = i\sqrt{2}\left(\sigma^0_{\beta\dot\alpha}\pi^* + \sigma^k_{\beta\dot\alpha}\partial_k\phi\right) \tag{5.5}$$

These are realized with

$$Q_\alpha = \sqrt{2}\int d^3x \left\{ \psi\sigma^0(\bar\sigma^0\Pi + \bar\sigma^k\partial_k\phi^*) + i\bar\psi\bar\sigma^0\left(\frac{\partial W}{\partial\phi}\right)^* \right\} \qquad (5.6)$$

It is customary to introduces anticommuting parameters $\xi_\alpha, \bar\xi_{\dot\alpha}$. Define

$$\delta_\xi X = i[\xi Q, X]$$
$$\delta_{\bar\xi} X = i[\bar\xi\bar Q, X] \qquad (5.7)$$

Here X is either ϕ or ψ. We have only commutators on the r.h.s. of (5.6) since, by assumption, ξ and $\bar\xi$ anticommute with ψ and $\bar\psi$. We rewrite the transformation rules as

$$\delta_\xi\phi = \sqrt{2}\,\xi\psi \qquad \delta_{\bar\xi}\phi = 0 \qquad (5.8a)$$
$$\delta_\xi\psi = -\sqrt{2}\,\xi\left(\frac{\partial W}{\partial\phi}\right)^* \qquad \delta_{\bar\xi}\psi = -i\sqrt{2}\sigma^\mu\bar\xi\,\partial_\mu\phi \qquad (5.8b)$$

The form (5.8) is more convenient for the lagrangian path integral formalism. We define a set of transformation rules without yet specifying what is the operator which generates them.

The lagrangian (5.3) cannot be invariant under the SUSY transformations (5.8), because the result of two successive transformations should be a derivative of $\mathcal{L}(x)$. But it transforms into a total derivative and so the action is invariant. Using the Neother method we find the conserved current

$$S^\mu = \sqrt{2}\left\{ \psi\sigma^\mu(\bar\sigma^\nu\partial_\nu\phi^*) + i\bar\psi\bar\sigma^\mu\left(\frac{\partial W}{\partial\phi}\right)^* \right\} \qquad (5.9)$$

as expected,

$$Q_\alpha = \int d^3x\, S_{0\alpha}(x) \qquad (5.10)$$

In terms of the transformations δ_ξ, the SUSY algebra takes the form

$$\delta_\eta\delta_\xi - \delta_\xi\delta_\eta = 2i(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta)\partial_\mu \qquad (5.11)$$

The Term in parenthesis can be thought of as the magnitude of a translation in space generated by two successive "translations" in a "fermionic direction". This observation is the starting point for the superspace formulation of SUSY theories, which will be introduced in section 7.

We can check explicitly the ~~operatorial~~ relation (5.11) on the fields ϕ and ψ, using the explicit form (5.8). For ϕ we find the expected result, but for ψ we find (use Fierz)
$(\psi\phi)\bar{\chi}_{\dot\beta} = -\frac{1}{2}(\phi\sigma^\mu\bar{\chi})(\psi\sigma_\mu)_{\dot\beta}$

$$\begin{aligned}(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\psi_\alpha &= 2i(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta})\partial_\mu\psi_\alpha \\ &\quad + \xi_\alpha\bar{\eta}\left(i\bar{\sigma}^\mu\partial_\mu\psi - \left(\frac{\partial^2 W}{\partial\phi^2}\right)^*\bar{\psi}\right) - (\xi\leftrightarrow\eta) \qquad (5.12)\end{aligned}$$

The algebra does not close on ψ! Notice that the additional terms are proportional to the ψ-field equation. The algebra closes only if ψ satisfy its field equation - it closes only on-shell. Notice that we did not have this problem in the hamiltonian formalism. The assumption in the canonical framework that the time translation is generated by a commutator with H implies automatically that all fields satisfy their equations of motion.

In the path integral formalism, on the other hand, one integrates over a functional space, ~~without ass~~ which implies that $\phi(x)$ and $\psi(x)$ can be arbitrary. The domination of particular paths is found only after the path integration is performed.

The reason for the non-closure is the following. The field $\phi(x)$, whether taken as a canonical field or a function, has two degrees of freedom. For $\psi(x)$, on-shell it carries two degrees of freedom but off-shell it has four (the naive number of real components). This is a completely general property of fermions - imposing their field equation reduces the number of degrees of freedom by half. We can now understand why the algebra fails on $\psi(x)$: we are mapping the space of all functions $\{\psi(x)\}$ into a space $\{\phi(x)\}$ which has only half the number of independent elements. Upon acting again we necessarily end up with a sub-space of $\{\psi(x)\}$.

On the other hand, the action of $\bar{\sigma}^m \partial_m$ is invertible - hence it ~~map~~ ~~mapp~~ maps $\{\Psi(x)\}$ onto a space with the same dimension.

We conclude with a summary of the disadvantages of the on-shell formulation:

1. The SUSY transformations of the fermions are non-linear.
2. " " " " " closes only on-shell.
3. Counter-terms are not manifestly ~~SUSY~~ supersymmetric.

The basic objects that one constructs in perturbation theory are off-shell Green functions, and the basic tool used by physicists ~~#~~ is the path integral formalism. It is therefore desirable to reformulate SUSY theories such that the above difficulties will be absent. For the Wess-Zumino model, all these problems are overcomed by the introduction of <u>auxiliary field</u>, which will be discussed in the next section. As we shall see, in order to avoid similar problems in gauge theories one has to take a more drastic action.

The basic property of auxiliary fields is that they have <u>algebraic</u> field equations. Therefore, one can always eliminate them by solving their field equations. This operation will take us back to the on-shell formulation. The introduction of auxiliary fields simplifies the structure of perturbation theory and makes SUSY manifest already off-shell, but the price is that additional steps must be taken in order to translate these off-shell quantities into predictions on physical observables.

Finally, we note that in the canonical framework there is no room (and no need) for auxiliary fields. Therefore, whenever one wants to make a statement regarding the <u>operator</u> Q_α or the Hilbert space of a SUSY theory, the on-shell formulation is more appropriate.

6. Off-shel reperesentations and auxiliary fields

The algebra does not close on ψ_α because, off-shell, the nuber of its degrees of freedom doubles from 2 to 4, whereas ϕ has 2 degrees of freedom both on- and off-shell. To remedy this we have to invent a new bosonic field which has two degrees of freedom and which exists as independent field only off shell. For the Wess Zumino model what we need is a complex scalar field denoted $F(x)$. The transformation rules for the <u>supermultiplet</u> (ϕ, ψ, F) are

$$\delta_\xi \phi = \sqrt{2}\, \xi \psi \qquad (6.1a)$$
$$\delta_\xi \psi = -i\sqrt{2}\, \sigma^\mu \bar{\xi}\, \partial_\mu \phi + \sqrt{2} F \xi \qquad (6.1b)$$
$$\delta_\xi F = -i\sqrt{2}\, \bar{\xi}\, \bar{\sigma}^\mu \partial_\mu \psi \qquad (6.1c)$$

The algebra (5.11) now closes on all three fields. Notice that the dimension of F is 2, ~~[illegible]~~ and that (6.1) is a <u>linear</u> Transformation. Comparing the ψ-transformation laws (5.8b) and (6.1b) suggests the identification

$$F = -\left(\frac{\partial W}{\partial \phi}\right)^* \qquad (6.2)$$

Indeed, eq. (6.2) terns out to be the F-fields equation. The F-transformation law is compatible with the field equation (6.2). Using the ψ-field equation we find

$$\delta_\xi F = -\sqrt{2}\, \bar{\xi}\, \bar{\psi} \left(\frac{\partial^2 W}{\partial \phi^2}\right)^* \quad \text{on-shell}, \qquad (6.3)$$

which is the result obtained by applying the ~~[illegible]~~ (complex conjugation of the) ϕ-transformation law to the r.h.s. of eq. (6.2).

The lagrangian of the Wess-Zumino model now takes the form

$$\mathcal{L} = |\partial_\mu \phi|^2 - \frac{i}{2}\bar{\Psi}\bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu \Psi + F^* F$$
$$+ \left(F \frac{\partial W}{\partial \phi} - \frac{1}{2}\Psi\Psi \frac{\partial^2 W}{\partial \phi^2} + h.c. \right) \qquad (6.4)$$

The reader will easily verify that the F-field equation that follows from the lagrangian (6.4) is eq. (6.2). Substituting the F-field equation in (6.4) we obtain (5.3) again. Alternatively, we can establish the equality

$$Z = \int D\phi D\Psi D\bar{\Psi}\, e^{iS(\phi,\Psi)} = \int D\phi D\Psi D\bar{\Psi} DF\, e^{iS(\phi,\Psi,F)} \qquad (6.5)$$

by performing the gaussian integration ~~and~~ over F(x).

The lagrangian (6.4) naturally splittes into two parts. The terms on the first row are real (and positive definite for bosons in the euclidean path integral). This is the kinetic part of the lagrangian. For F, the "kinetic" term is just F^*F as follows on dimensional grounds.

The terms on the second row are analytic. This is now true for both bosonic and fermionic terms. They are ~~all~~ expressible in terms of the superpotential W, which contains all mass parameters and coupling constants of the theory.

The lagrangian (6.4) gives rise to manifestly SUSY Green functions. The same is true for counterterms, provided the SUSY Pauli-Villars regularization is used. (We will return to the issue of regularization when we discuss gauge theories).

Let us reexamine the two point functions ~~that~~ as they follow from the lagrangian (6.4). The ϕ and Ψ propagator are the same as before (see (3.14-16)). In addition we have

$$\text{F.T. } \langle 0 | T\{F^*(x)\ F(y)\} | 0 \rangle = i \frac{p^2}{p^2 - m^2 + i\epsilon} \qquad (6.6)$$

and a mixed propagator

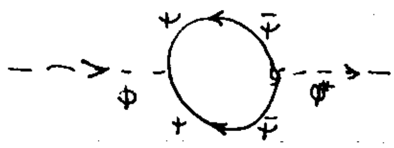
$$\text{F.T. } \langle 0 | T\{\phi(x)\ F(y)\} | 0 \rangle = -im \frac{1}{p^2 - m^2 + i\epsilon} \qquad (6.7)$$

The reader is invited to check that

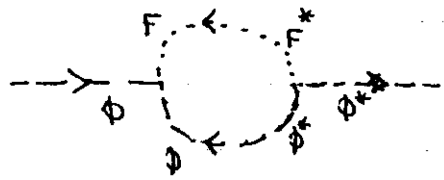
$$\begin{pmatrix} \Box & -m \\ -m & -1 \end{pmatrix} \begin{pmatrix} \langle \phi^* \phi \rangle & \langle \phi^* F^* \rangle \\ \langle F \phi \rangle & \langle F F^* \rangle \end{pmatrix} = \delta^4(x-y) \qquad (6.8)$$

The bosonic propagators (3.14), (6.6) and (6.7) are the inverse of the ~~operator~~ differential operator defined by the quadratic part of the bosonic lagrangian.

The two diagrams contributing to $\Gamma^{(\phi^*\phi)}$ are



(6.9)



(6.10)

which sum to

$$\Gamma^{(\phi^*\phi)} = 4g^2 \int \frac{d^4p}{(2\pi)^4} \frac{p \cdot K}{(p^2 - m^2 + i\epsilon)[(p-K)^2 - m^2 + i\epsilon]} \quad . \qquad (6.11)$$

Therefore, the only divergence in $\Gamma^{(\phi^*\phi)}$ is proportional to K^2. Furthermore, the kinetic divergences in all two point function arise with the same coefficient.

Thus, at the one loop level the counter-terms lagrangian is manifestly SUSY, and is proportional to the kinetic part only

$$\mathcal{L}^{c.t.} = Z\left(|\partial_\mu \phi|^2 + \frac{i}{2} \bar{\psi} \bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu \psi + |F|^2 \right) \qquad (6.12)$$

The SUSY form of the counter-term lagrangian is an expected result of the use of a SUSY regularization for the "manifestly" SUSY lagrangian (6.4). The absence of mass and ~~coupling~~ interaction counter-term is an additional and unexpected benefit. The absence of mass and interaction counter-terms persists to all orders in perturbation theory and is known as "non-renormalization theorems". The full renormalizable lagrangian of the ~~most~~ Wess-Zumino model is the sum of the tree lagrangian (6.4) and the counterterm lagangian (6.12). The renormalized F-field equation is

$$(1+Z)F = -\left(\frac{\partial W}{\partial \phi}\right)^{*} \tag{6.13}$$

The renormalizable lagrangian for the on-shell formulation is obtained by substituting (6.13) in (6.4) + (6.12). The reader can check that the resulting counter-term lagrangian is not manifestly supersymmetric.

The absence of mass ~~and~~ interaction counter-terms does not imply that masses and couplings do not run. They run, but ~~At~~ their momentum dependence is entirely due to the wave function renormalization. This is similar to what happens in ~~gauge theories, where QE~~ QED, where the renormalization of electric charge arises only from the photon two-point function. Indeed, one way to prove the non-renormalization theorems is to re-write the Wess-Zumino lagrangian in terms of a bigger set of fields which has ~~an additional~~ a local symmetry.

Finally, we note that, for models containing only scalar supermultiplets, the lagrangian (6.4) or (5.3) is the most general one. The different models are completely determined by the superpotential whose most general form for a renormalizable theory is

$$W = a_i \phi_i + \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{3} g_{ijk} \phi_i \phi_j \phi_k \tag{6.14}$$

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