INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.626 - 10
(Part I)

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

15 June - 31 July 1992

INTRODUCTION TO FUNCTIONAL METHODS, GAUGE THEORIES AND QUANTIZATION

Roberto IENGO
SISSA, Trieste
Italy

Please note: These are preliminary notes intended for internal distribution only.

Green Functions and Scattering Amplitudes.

$$\phi(x) \xrightarrow[t\to\mp\infty]{} \sqrt{Z}\,\phi_{\substack{in\\out}}(x) \qquad \phi_{\substack{in\\out}} = \text{free field}$$

$$(\Box + m^2)\,\phi_{\substack{in\\out}} = 0$$

physical mass

$$\Rightarrow \quad \phi_{\substack{in\\out}}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a_{\substack{in\\out}}(k)\, e^{-ikx} + a^{+}_{\substack{in\\out}}(k)\, e^{+ikx} \right)$$

building $|in\rangle$, $|out\rangle$ Fock space.

Typically $\quad |k_1 \cdots k_n\ in\rangle = a^{+}_{in}(k_1) \cdots a^{+}_{in}(k_n)|0\rangle$

by en. mom. cons. $\quad |0_{in}\rangle = |0_{out}\rangle$ also 1 particle states $|p_{in}\rangle = |p_{out}\rangle$

Normalization: $\quad \langle 0|\,\phi_{\substack{in\\out}}(x)\,|p\rangle = \dfrac{1}{\sqrt{(2\pi)^3 2\omega_p}}\, e^{-ipx}$

S-matrix: $\quad \phi_{out} = S^{-1}\,\phi_{in}\,S$

Scattering Amplitude:

$$\langle f_{out} | i_{in} \rangle = \langle f_{in} | S | i_{in} \rangle$$

Reduction formula:

$$\langle \underbrace{k_1' \cdots k_n' \, out}_{n} | \underbrace{k_1 \cdots k_m \, in}_{m} \rangle = \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \int \prod_1^m d^4 y_\ell \, f(y_\ell) (\Box_{y_\ell} + m^2) \cdot$$

$$\cdot \prod_1^n d^4 x_{\ell'} \, f^*(x_{\ell'}) (\Box_{x_{\ell'}} + m^2) \langle 0 | T \varphi(y_1) \cdots \varphi(y_m) \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle_c$$

$$f(y) = \frac{e^{-iky}}{\sqrt{(2\pi)^3 2\omega_k}} \qquad k^2 = m^2 \quad (\text{mass shell})$$

Define the <u>connected</u> (c) Green Functions

$$(2\pi)^4 \delta^4\left(\sum_i^N p\right) G^N(p_1 \cdots p_N) = \int \prod_1^N d^4 x_i \, e^{-ip_i x_i} \langle 0 | T \varphi(x_1) \cdots \varphi(x_N) | 0 \rangle_c$$

Scattering Amplitude $(N = m+n)$ (factorizing $(2\pi)^4 \delta^4(\Sigma p)$)

$$A(m \to n) = \prod_1^N \frac{1}{\sqrt{(2\pi)^3 2\omega}} \cdot M \,, \qquad M = \lim_{p_i^2 = m^2} \prod_1^N \frac{p_i^2 - m^2}{i\sqrt{Z}} \, G^{(N)}(p_1 \cdots p_N)$$

(phase space $\frac{d^3 p}{(2\pi)^3 2\omega}$)

M is finite for $p^2 \to m^2$, since $G^{(N)}$ has poles for $p^2 \to m^2$.

The Green Functions can be evaluated by Path Integral:

$$\langle 0|T\varphi(x_1)\cdots\varphi(x_N)|0\rangle = N\int \mathcal{D}\varphi(x)\, e^{iS(\varphi)}\, \varphi(x_1)\cdots\varphi(x_N)$$

example

$$S(\varphi) = \int d^4x \left\{ \tfrac{1}{2}\left((\partial\varphi)^2 - m_0^2\varphi^2\right) - \frac{\lambda_0}{4!}\varphi^4 \right\}$$

$$(m_0^2 \to m_0^2 - i\varepsilon)$$

Actually the Euclidean Path Integral is better defined: $\quad dx_0 \to -i\,dx_4$

$$d^4x = d^3x\, dx_0 \to -i\, d^3x\, dx_4 = -i\, d^4x_E$$

$$(\partial\varphi)^2 = \partial_\mu\varphi\,\partial^\mu\varphi = (\partial_0\varphi)^2 - (\vec\partial\varphi)^2 \to -\left((\partial_4\varphi)^2 + (\vec\partial\varphi)\right)^2 = -(\partial_E\varphi)^2$$

$$e^{iS(\varphi)} \to e^{-S_E(\varphi)}$$

$$S_E(\varphi) = \int d^4x_E \left\{ \tfrac{1}{2}\left((\partial_E\varphi)^2 + m_0^2\varphi^2\right) + \frac{\lambda_0}{4!}\varphi^4 \right\} > 0$$

From Path Integral $\Longrightarrow$ Feynman graphs

Euclidean Path Integral $\Longleftrightarrow$ "Wick rotation"

$$k_0 \to i\,k_4$$

$$dk_0 \to i\,dk_4$$

this is the way loops are computed

Define the renormalized field

$$\varphi_R(x) \equiv \frac{\varphi(x)}{\sqrt{Z}} \xrightarrow[t\to\mp\infty]{} \varphi_{\substack{in\\out}}(x)$$

and renormalized Green functs

$$(2\pi)^4\delta^4(\Sigma p)\, G_R^{(N)}(p_1\cdots p_N) = \int \prod_1^N d^4x_i\, e^{-ip_ix_i} \langle 0|T\varphi_R(x_1)\cdots\varphi_R(x_N)|0\rangle$$

By Källen-Lehmann repr:

$$G^{(2)}(p) = \int d^4x\, e^{-ipx}\langle 0|T\varphi(x)\varphi(0)|0\rangle = \frac{iZ}{p^2-m^2} + \int_{4m^2}^{\infty}\frac{d\sigma\,\rho(\sigma)}{p^2-\sigma}$$

therefore

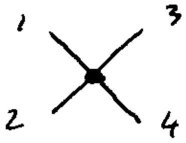
$$G_R^{(2)}(p) \xrightarrow[p^2\to m^2]{} \frac{i}{p^2-m^2}$$

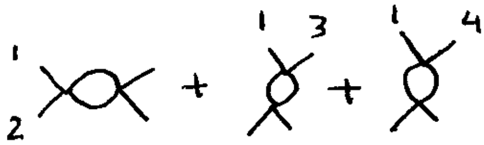
Thus we have the scattering amplitude

$$M = \lim_{p_i^2\to m^2} \prod_1^N \left(G_R^{(2)}(p_i)\right)^{-1} \cdot G_R^{(N)}(p_1\cdots p_N)$$

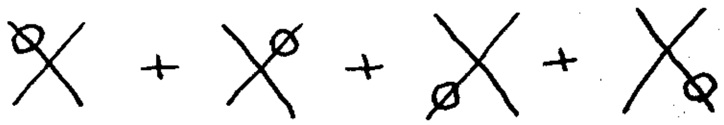
Notice that

$$G_R^{(N)} = Z^{-\frac{N}{2}}\, G^{(N)}$$

$G^{(4)} =$  $= \underbrace{\left(\frac{i}{p_1^2-m_0^2}\cdots\frac{i}{p_4^2-m_0^2}\right)}\times(-i\lambda_0)$

$+$  $+ \left(\text{same}\right)\times(-i\lambda_0)^2\times\left(F((p_1+p_2)^2) + F((p_1+p_3)^2) + F((p_1+p_4)^2)\right)$

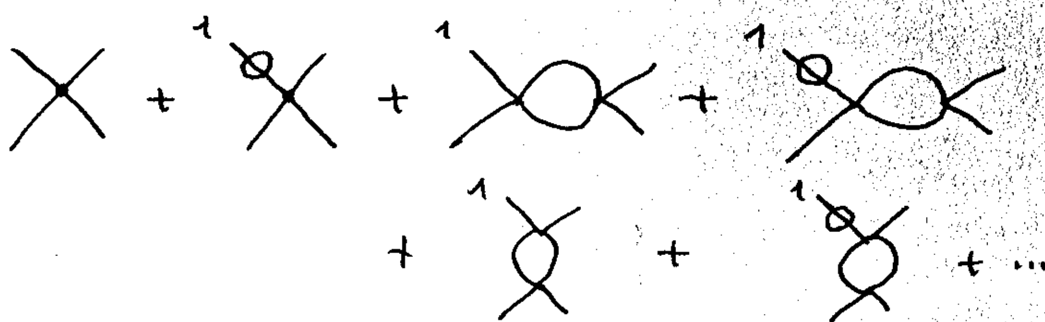
$+$

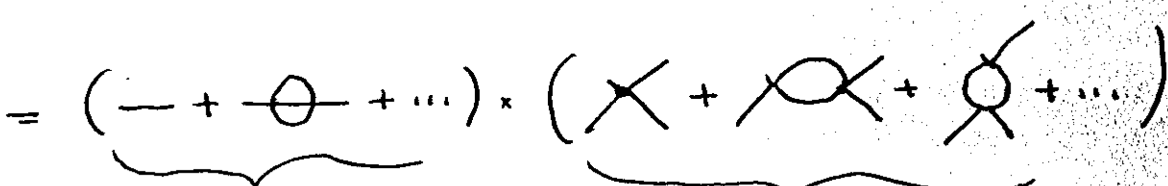
 $\ldots$

$+$

$\ldots +$  $+ \cdots$ $\ldots$

Look in particular at the leg # 1

$G^{(4)} =$  $+ \ldots$

$=$ 

$= \underset{\text{complete propagator}}{G^{(2)}(p_1)} \times G^{(4)\hat{1}}(p_1 \cdots p_4)$

<u>complete</u> propagator

without propagators etc on the leg # 1.

It can be done for <u>every</u> leg:

$$G^{(4)}(p_1 \dots p_4) = \left(G^{(2)}(p_1) \dots G^{(2)}(p_4) \right) \cdot \tilde{\Gamma}^{(4)}(p_1 \dots p_4)$$

↑ <u>complete</u> propagators

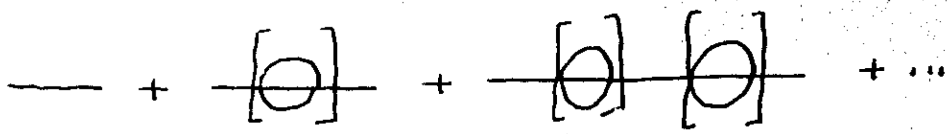
$\tilde{\Gamma}^{(4)}$ = "amputated" = <u>without</u> propagators on external legs

thus

$$M = \lim_{p_i^2 \to m^2} \tilde{\Gamma}_R^{(N)}$$

$$\tilde{\Gamma}_R^{(N)} = \prod_1^N \left(G_R^{(2)}(p_i) \right)^{-1} G_R^{(N)} = Z^{\frac{N}{2}} \, \tilde{\Gamma}^{(N)}$$

Remember

$G^{(2)} =$ 

the computation becomes exact by taking

$[\ ] =$  $+ \dots =$ sum of all 1PI graphs

$$= i \Sigma(p^2)$$

$$\Rightarrow \quad G^{(2)}(p^2) = \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2} \, i\Sigma(p^2) \, \frac{i}{p^2 - m_0^2} + \dots$$

$$= \frac{i}{p^2 - m_0^2 + \Sigma(p^2)}$$

$$\Gamma^{(2)}(p^2) = i\left(G^{(2)}(p^2)\right)^{-1} = \cancel{i}\left(p^2 - m_0^2 + \Sigma(p^2)\right)$$

$$\Gamma_R^{(2)} = Z\,\Gamma^{(2)} = Z\,(p^2 - m_0^2) + Z\,\Sigma(p^2) \xrightarrow[p^2 \to m^2]{}$$

$$\to \quad p^2 - m^2 + O\left((p^2 - m^2)^2\right)$$

therefore $\quad m_0^2 = m^2 + \delta m^2$

$$Z\,\Sigma(m^2) = Z\,\delta m^2 \quad = O(\lambda^2)$$

$$Z \frac{\partial}{\partial p^2}\Sigma\Big|_{p^2 = m^2} = 1 - Z \quad = O(\lambda^2)$$

define $\quad \Sigma_R(p^2) = Z\cdot\Sigma(p^2) - Z\,\delta m^2 - (1 - Z)(p^2 - m^2)$

$\left(\text{of course } \quad \Sigma_R(m^2) = \frac{\partial}{\partial p^2}\Sigma_R\Big|_{m^2} = 0\right)$

$$\Rightarrow \quad \Gamma_R^{(2)}(p^2) = p^2 - m^2 + \Sigma_R(p^2)$$

$\Sigma_R(p^2)$ is expressed in terms of loops (1PI)

can be computed in perturbation theory

<u>It is U.V. finite</u>

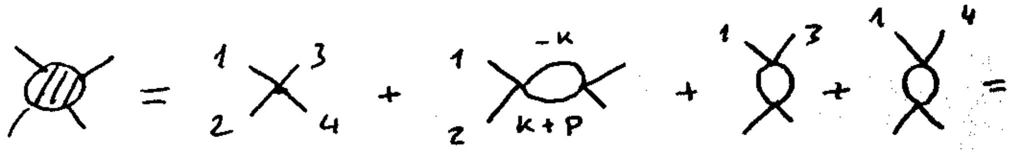
1 PI = one Particle Irreducible

Exercise: compute  at order λ^2

F. rules $= \dfrac{i}{k^2-m_0^2+i\varepsilon}$  $= -i\lambda_0$

loop integration $= \dfrac{d^dk}{(2\pi)^d}$ (in d dimensions)

$d = 4-\varepsilon$

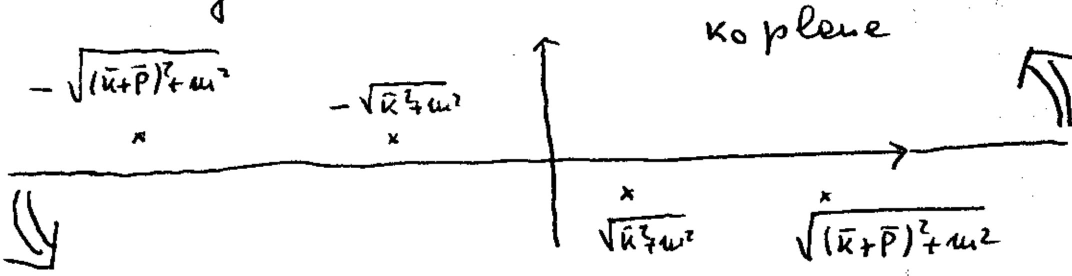


apart from external leg propagators

$$= -i\lambda_0 + (-i\lambda_0)^2 \frac{1}{2!}\int \frac{d^dk}{(2\pi)^d} \frac{i}{(k^2-m^2+i\varepsilon)} \frac{i}{((k+P)^2-m^2+i\varepsilon)} + \text{two more terms}$$

$\underline{P} = p_1+p_2$ take P spacelike and $P_0 = 0$
(then analytic cont. to P timelike too)

Integration over k_0



rotate to imaginary axis

Wick rotation $\quad k_0 \to i k_4$

$$k_0^2 - \bar{k}^2 \to -(k_4^2+\bar{k})^2 = -k_E^2$$

$$k_0^2 - (\bar{k}+\bar{P})^2 \to -(k_4^2 + (\bar{k}+\bar{P})^2) = -(k+P)_E^2$$

$$= -i\lambda_0 + i\frac{\lambda_0^2}{2}\int\frac{d^dK_E}{(2\pi)^d}\underbrace{\frac{1}{(K_E^2+m^2)((K+P)_E^2+m^2)}}_{\int_0^1 dx\,\frac{1}{\left((1-x)(K_E^2+m^2)+x((K+P)_E^2+m^2)\right)^2}} + \cdots$$

$$= \int_0^1 dx\,\frac{1}{\left(\underbrace{(K+xP)}_{K'}{}_E^2 + x(1-x)P_E^2+m^2\right)^2}$$

$$= -i\lambda_0 + i\frac{\lambda_0^2}{2}\int_0^1 dx\,\left(x(1-x)P_E^2+m^2\right)^{\frac{d-4}{2}}\cdot\underbrace{\int\frac{d^dK'}{(2\pi)^d}\frac{1}{(K'^2+1)^2}}_{\int_0^\infty d\alpha\,\alpha e^{-\alpha}\underbrace{\int\frac{d^dK}{(2\pi)^d}e^{-\alpha K^2}}_{\alpha^{-d/2}\frac{\pi^{d/2}}{(2\pi)^d}}}$$

$$= \frac{1}{(4\pi)^{d/2}}\int_0^\infty d\alpha\,\alpha^{1-d/2}e^{-\alpha} = \frac{1}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}=\frac{\varepsilon}{2}\right)$$

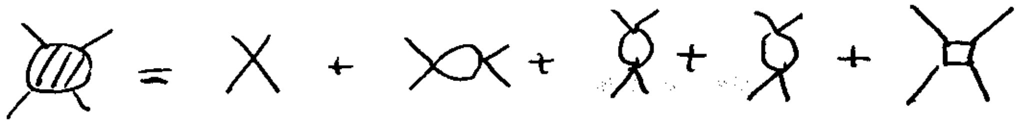
$$= -i\lambda_0\left(1-\frac{\lambda_0}{2}\int_0^1 dx\, e^{-\frac{\varepsilon}{2}\ln(x(1-x)P_E^2+m^2)}\left(\frac{1}{16\pi^2}\frac{2}{\varepsilon}+\text{regular}\right)+\right.$$

$$\left. + \text{two more terms}\right)$$

Put $\lambda_0 = \lambda_R\,\mu^\varepsilon Z_1$ $\qquad Z_1 = 1 + a\lambda_R$

expand in λ_R

Thus really



$-i(Z_1-1)\lambda_R\mu^\varepsilon$

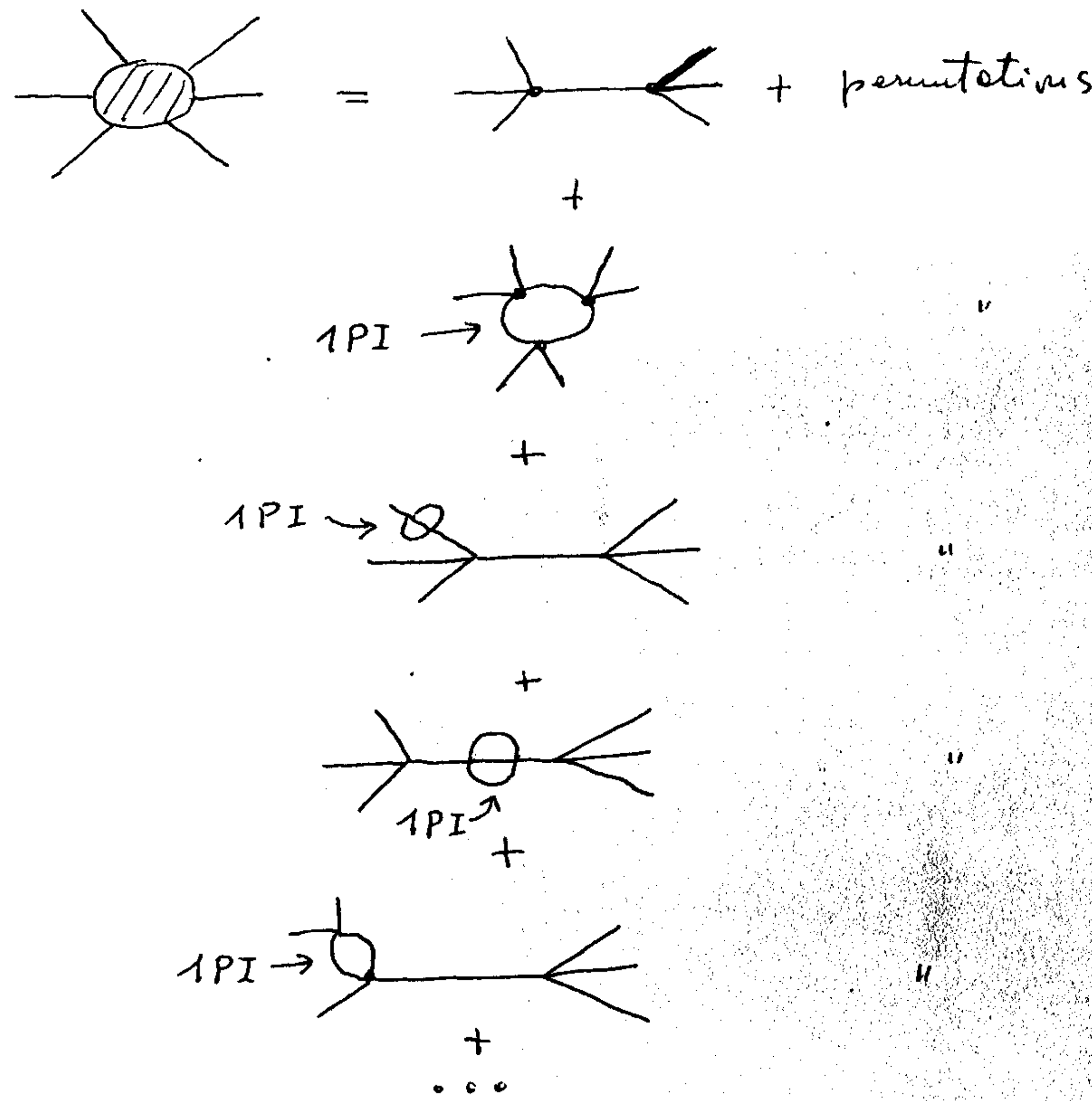
$$= -i\lambda_R\mu^\varepsilon\Big(1 - 3\frac{\lambda_R}{16\pi^2}\frac{1}{\varepsilon} + \frac{\lambda_R}{32\pi^2}\int_0^1 dx\Big(\ln\frac{x(1-x)p_E^2+m^2}{\mu^2} + \text{two other terms} + \text{const}\Big) + a\lambda_R\Big)$$

$\Rightarrow \quad a = \frac{3}{16\pi^2}\frac{1}{\varepsilon}$ that is $Z_1 = 1 + \frac{3}{16\pi^2}\frac{1}{\varepsilon}\lambda_R + \dots$

it cancels the divergence.

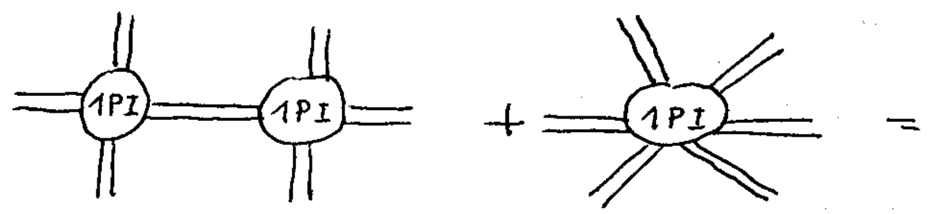
More in general organize the computation

$G^{(6)} =$



Thus

$G^{(6)} =$



(+ permutations)

$$= \left(G^{(2)}G^{(2)}G^{(2)}\right)\Gamma^{(4)}\cdot G^{(2)}\cdot\Gamma^{(4)}\left(G^{(2)}G^{(2)}G^{(2)}\right) +$$

$$+ \quad G^{(2)}G^{(2)}G^{(2)}\cdot\Gamma^{(6)}\cdot G^{(2)}G^{(2)}G^{(2)}$$

<u>defining</u> $\quad \Gamma^{(N)} = \tilde{\Gamma}^{(N)}(1PI)$

(some phase see later)

We wrote

$$G^{(6)} = \prod_{1}^{6} G^{(2)} \cdot \tilde{\Gamma}^{(6)}$$

thus $\quad \tilde{\Gamma}^{(6)} = \Gamma^{(4)} \cdot G^{(2)} \cdot \Gamma^{(4)} + \Gamma^{(6)}$

$\Rightarrow$ Every Green Function can be written as a tree diagram using

$G^{(2)}$ (the <u>complete</u> propagator)

and $\Gamma^{(N)}$ the effective interaction vertices (1PI)

Compare with the Action:

$$S = \int \frac{1}{2!} \varphi (-\Box + m_0^2) \varphi - \frac{\lambda_0}{4!} \varphi^4$$

define the <u>generating functional</u> $\Gamma(\varphi)$

$$\Gamma(\varphi) = \sum_N \frac{\varphi^N}{N!} \Gamma^{(N)}$$

here we mean

$$\varphi^N * \Gamma^{(N)} = \int \prod_1^N d^4x_i \, \varphi(x_1) \dots \varphi(x_N) \Gamma^{(N)}(x_1 \dots x_N)$$

$$\Gamma(\varphi) = \frac{1}{2!} \varphi \cdot \Gamma^{(2)} \cdot \varphi + \frac{1}{4!} \Gamma^{(4)} \varphi^4 + \dots + \frac{1}{N!} \Gamma^{(N)} \varphi^N + \dots$$

$$\Gamma^{(2)} = p^2 - m_0^2 + \Sigma(p^2) \qquad \text{(in Fourier transf)}$$

$$\Gamma^{(4)} = -\lambda_0 + F(p_1 p_2 p_3 p_4) \qquad F = O(\lambda_0^2)$$

$$\Gamma^{(6)} = \Gamma^{(6)}(p_1 \cdots p_6) \qquad \Gamma^{(6)} = O(\lambda_0^3)$$

etc.

Γ is called the "effective Action"

it is to be used for tree diagrams, built by

$\Gamma^{(2)^{-1}}$ = complete propagators and effective vertices = $\Gamma^{(N)}$

The task of perturbation theory: compute Γ.

Renormalization:

$$\Gamma^{(N)} = Z^{-\frac{N}{2}} \cdot \Gamma_R^{(N)}$$

Physical quantities (ex. scattering amplitudes)

are expressed in terms of $\Gamma_R^{(N)}$.

$$\Gamma(\varphi) = \sum \frac{1}{N!} \varphi^N \cdot \Gamma^{(N)} = (\varphi = Z^{1/2} \varphi_R) =$$

$$= \sum \frac{1}{N!} \varphi_R^N \, \Gamma_R^{(N)}$$

Generating functionals for Green Functions.

Define $Z^{(n)} = \langle 0 | T\, \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle$

also including disconnected

let $Z(J) = \sum_n \frac{1}{n!} J^n Z^{(n)}$

We have defined

$$G^{(n)} = Z^{(n)}_{\text{connected}}$$

let $W(J) = \sum_\ell \frac{1}{\ell!} J^\ell G^{(\ell)}$

It can be seen that $Z(J) = e^{W(J)}$

Check:

$$e^{W(J)} = \sum_n \frac{1}{n!} \left(\sum_\ell J^\ell \frac{G^{(\ell)}}{\ell!} \right)^n =$$

$$= \sum_n \frac{1}{n!} \sum_p \sum_{s_1} \cdots \sum_{s_p} \left(\frac{J}{1!} G^{(1)} \right)^{s_1} \cdots \left(\frac{J^p}{p!} G^{(p)} \right)^{s_p} \frac{n!}{s_1! \cdots s_p!} =$$

$$s_1 + s_2 + \cdots + s_p = n$$

$$= 1 + \sum_p \sum_{s_1} \cdots \sum_{s_p} \frac{1}{s_1! \cdots s_p!} \left(\frac{J}{1!} G^{(1)} \right)^{s_1} \cdots \left(\frac{J^p}{p!} G^{(p)} \right)^{s_p}$$

Now, having m objects and connecting them:

s_1 groups of 1 object
s_2 " " 2 " s
$\vdots$
s_p " " p "

$s_1 + 2s_2 + \dots + p s_p = m$

$$\text{multiplicity} = \frac{m!}{s_1! \dots s_p!} \left(\frac{1}{1!}\right)^{s_1} \left(\frac{1}{2!}\right)^{s_2} \cdots \left(\frac{1}{p!}\right)^{s_p}$$

$$\Rightarrow Z(J) = \sum_m \frac{J^m}{m!} Z^{(m)} =$$

$$= \sum_m \frac{1}{m!} \underbrace{\sum_p \sum_{s_1} \cdots \sum_{s_p}}_{s_1+2s_2+\dots+ps_p=m} \underbrace{\frac{m!}{s_1! \dots s_p!} \left(\frac{J}{1!} G^{(1)}\right)^{s_1} \cdots \left(\frac{J^p}{p!} G^{(p)}\right)^{s_p}}_{J^m Z^{(m)}} =$$

making the sum over m: $\sum_m \delta(m - s_1 - 2s_2 - \dots - p s_p) = 1$

we see that

$$Z(J) = e^{W(J)}$$

In general $\quad Z(J) = \int d\xi \, P(\xi) \, e^{J\xi} = \sum_m \frac{1}{m!} J^m \langle \xi^m \rangle$

$P(\xi)$ = probability distribution

$$Z^{(n)} = \langle \xi^n \rangle$$

$G^{(n)}$ = correlations $\quad$ example $\quad G^{(2)} = \langle \xi^2 \rangle - \langle \xi \rangle^2$

Path integration formula

$$Z(J) = N \int \mathcal{D}\varphi \, e^{iS(\varphi) + \int J\varphi}$$

remember $\quad S(pq) = \int dt \, (p\dot{q} - H(pq))$

thus $\quad iS(\varphi) - i\int J\varphi \quad$ is like $\quad H(\varphi) \to H(\varphi) + J\cdot\varphi = H_J(\varphi)$

thus let us consider

$$Z(-iJ) = \sum_n \frac{1}{n!} (-iJ)^n Z^{(n)} =$$

$$= \lim_{T\to\infty} \mathrm{Tr}\left(e^{-iH_J \cdot T}\right) = e^{-i E_0(J)\cdot T}$$

(for instance periodic b.c.s in the path integral)

$E_0(J)$ = lowest energy of H_J

$H_J |0_J\rangle = E_0(J)\, |0_J\rangle \qquad |0_J\rangle$ lowest energy state of H_J.

Let us thus define

$$Z(-iJ) = e^{-iW(J)} \qquad \Longrightarrow \qquad W(J) = E_0(J)\cdot T$$

in Terms of the Green Functions

$$G^{(n)} = \langle 0 | T\varphi \cdots \varphi | 0 \rangle \qquad W^{(n)} = i(-i)^n G^{(n)}$$

examples $\quad W^{(1)} = \langle 0 | \varphi | 0 \rangle \qquad W^{(2)} = -iG^{(2)} = \dfrac{1}{p^2 - m_0^2 + \Sigma(p^2)}$

Now define $\quad -\Gamma(\phi) = W(J) - \phi J$

where $\quad \phi = \dfrac{\delta W}{\delta J} = \langle \varphi \rangle_J \quad$ (<u>before</u> taking $J=0$)

$$\Rightarrow \quad \frac{\delta \Gamma}{\delta \phi} = J \quad \longleftrightarrow \quad J = J(\phi)$$

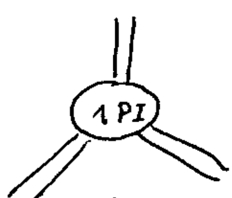
Comparing W and Γ: $\quad W(J) = -\Gamma(\phi(J)) + \phi J$

$$\frac{\delta W}{\delta J} = \phi \qquad \frac{\delta^2 W}{\delta J^2} = W^{(2)} = \frac{\delta \phi}{\delta J} = \left(\frac{\delta J}{\delta \phi}\right)^{-1} = \left(\frac{\delta^2 \Gamma}{\delta \phi^2}\right)^{-1} = \frac{1}{\Gamma^{(2)}}$$

thus $\quad \Gamma^{(2)} = p^2 - m_0^2 + \Sigma(p^2)$

$$\frac{\delta^3 W}{\delta J^3} = W^{(3)} = -\left(\frac{1}{\frac{\delta^2 \Gamma}{\delta \phi^2}}\right)^2 \cdot \frac{\delta^3 \Gamma}{\delta \phi^3} \cdot \left(\frac{\delta \phi}{\delta J} = \frac{1}{\Gamma^{(2)}}\right) = -\left(\frac{1}{\Gamma^{(2)}}\right)^3 \Gamma^{(3)}$$

$$i(-i)^3 G^{(3)} =$$

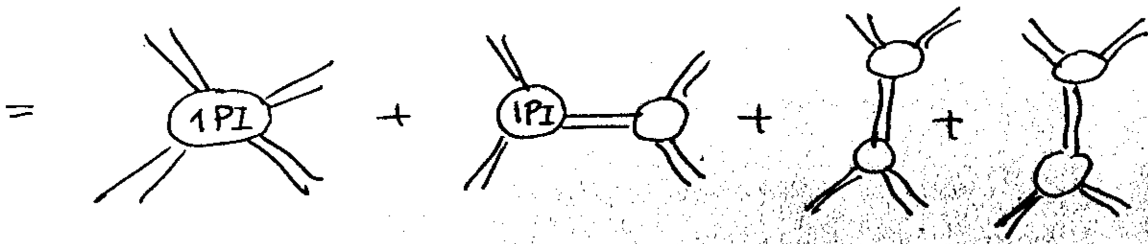


$$= \left(-iG^{(2)}\right)^3 i\tilde{\Gamma}^{(3)}(1PI)$$

(here we assume in general $\Gamma^{(3)} \neq 0$)

thus $\quad \Gamma^{(n)} = -i\tilde{\Gamma}^{(n)}(1PI)$

$$\frac{\delta^4 W}{\delta J^4} = W^4 = \left(\frac{1}{\Gamma^{(2)}}\right)^4 \cdot \left(-\Gamma^{(4)}\right) + 3\left(\frac{1}{\Gamma^{(2)}}\right)^2 \left(-\Gamma^{(3)}\right) \frac{1}{\Gamma^{(2)}} \left(-\Gamma^{(3)}\right)\left(\frac{1}{\Gamma^{(2)}}\right)^2$$



thus $\Gamma(\phi) = \int \frac{1}{2}\left((\partial\phi)^2 - m_0^2\phi^2 + \Sigma\phi^2\right) - \frac{1}{4!}(\lambda_0 + F)\phi^4 - \frac{1}{6!}\Gamma^{(6)}\phi^6 + \dots$

the previously introduced effective Action.

Physical meaning:

Consider a variational problem:

find $|a\rangle$ such that $\langle a|H|a\rangle$ = minimum with $\langle a|a\rangle = 1$

$$\Rightarrow \frac{\delta}{\delta a}\langle a|(H - E_0)|a\rangle = 0 \Rightarrow H|a\rangle = E_0|a\rangle$$

E_0 : multiplier

require an extra constraint $\langle a|\varphi|a\rangle = \phi$ fixed

$\Rightarrow$ another multiplier J

$$\frac{\delta}{\delta a}\langle a|(\underbrace{H + J\varphi}_{H_J} - E)|a\rangle = 0 \qquad H_J|a\rangle = E_0(J)|a\rangle$$

$$\frac{\delta E_0(J)}{\delta J} = \langle a|\varphi|a\rangle = \phi$$

$\Rightarrow \langle a|H|a\rangle = E_0(J) - J\phi = -\Gamma(\phi)$

Thus: $-\Gamma(\phi)$ = minimal energy for configuration such that $\langle\varphi\rangle = \phi$

$\langle 0|\varphi|0\rangle = ?$ look for the min. of $\Gamma(\phi_m)$. Then $\langle 0|\varphi|0\rangle = \phi_{min}$

Loop expansion and powers of $\hbar$

$$e^{i\frac{S}{\hbar}} \Rightarrow \text{propagator} = \frac{i\hbar}{p^2-m^2+i\varepsilon} \qquad \text{Vertices} \sim \hbar^{-1}$$

thus 1PI graph $\sim \hbar^{I-V} = \hbar^{L-1}$

$L=0$ leading for $\hbar\to 0$: classical theory = tree diagrams

Example :

$$Z(J) = N\int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}(S(\varphi)-J\varphi)} \qquad \hbar\to 0 \text{ saddle point}$$

dominated by the classical eqs $\quad \frac{\delta S}{\delta\varphi} = J \Rightarrow \varphi = \varphi_c$

put $\varphi = \varphi_c + \eta \qquad \mathcal{D}\varphi = \mathcal{D}\eta$

$$S(\varphi)-J\varphi = S(\varphi_c) - J\varphi_c + \eta\underbrace{\left(\frac{\delta S}{\delta\varphi}-J\right)\Big|_{\varphi_c}}_{=0} + \frac{1}{2}\frac{\delta^2 S}{\delta\varphi^2}\Big|_{\varphi_c}\eta^2 + O(\eta^3)$$

$$Z(J) = e^{i\frac{S(\varphi_c)-J\varphi_c}{\hbar}} \cdot \underbrace{N\int \mathcal{D}\eta'\, e^{\frac{i}{2}\frac{\delta^2 S}{\delta\varphi^2}\big|_{\varphi_c}\eta'^2 + O(\hbar\,\eta'^3)}}_{\text{const}\times\det\left(\frac{\delta^2 S}{\delta\varphi^2}\big|_{\varphi_c}\right)^{-1/2}}$$

more in detail : $\displaystyle\int \prod d\eta_n\, e^{-\eta_n\Lambda_{nm}\eta_m} = \text{cst}\prod \lambda_e^{-1/2} = \text{cst}(\det\Lambda)^{-1/2}$

$$= \text{cst}\cdot\exp -\tfrac{1}{2}\Sigma\ln\lambda$$

λ_e = eigenvalues of Λ

$$= \text{cst}\cdot\exp -\tfrac{1}{2}\,\mathrm{tr}\ln\Lambda$$

In our case for instance

$$\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi_c} = -\partial^2 - m_0^2 - \frac{\lambda_0}{2}\varphi_c^2$$

$$\det\left(\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi_c}\right)^{-1/2} = \exp -\frac{1}{2}\,\mathrm{tr}\ln \left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi_c} =$$

$$= \exp -\frac{1}{2}\int d^4x \underbrace{\int \frac{d^4k}{(2\pi)^4}\ln\left(m_0^2 + \frac{\lambda_0}{2}\varphi_c^2 - k^2\right)}_{\text{Wick rotating } = i\int \frac{d^4k_E}{(2\pi)^4}\ln\left(m_0^2 + k_E^2 + \frac{\lambda_0}{2}\varphi_c^2\right)}$$

Therefore $\left(\det\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi_c}\right)^{-1/2} = \exp -\frac{i}{2}K(\varphi_c)$

$$\Rightarrow \quad W(J) = -S(\varphi_c) + J\varphi_c + \frac{\hbar}{2}K(\varphi_c) = -\Gamma(\phi) + J\phi$$

where $\phi = \frac{\delta W}{\delta J} = \varphi_c + \frac{\hbar}{2}K'\frac{\delta\varphi_c}{\delta J}$

at order $\hbar$: $\Gamma(\phi) = S(\phi) - \frac{\hbar}{2}K(\phi) =$

= classical $S(\phi)$ +  +  +  + ...

$K(\phi)$ = one loop correction (quantum)

Fermions path Integral

functional integration over anticomm. variables

$$\{a, b\} = 0 \implies a^2 = 0$$

$$\implies f(a) = c_0 + c_1 a$$

requiring $\int da\, f(a+b) = \int da\, f(a)$

$$\implies \int da \text{ const} = 0 \qquad \int da\, a = 1 \text{ (normalization)}$$

$$\int da\, f(a) = -c_1 \qquad (\text{if } da\, c_1 = -c_1\, da)$$

Functional int: $\psi(x)$ anticomm. funct. of x

Expanding on a basis: $\psi(x) = \sum_n \xi_n(x)\, a_n$

$\xi_n(x)$: a basis of ordinary functions

a_n : anticom variables

Typically the Fermion Action is $\bar{\psi} \Lambda \psi$

$$\Lambda = i\{ i\gamma(\partial + ieA) - m\}$$

take Λ eigenfs as a basis $\qquad \Lambda \xi_n = \lambda_n \xi_n$

Euclidean funct. integration :

expand $\bar{\psi}(x) = \sum_n \bar{a}_n \xi_n^*(x) \qquad \bar{a}_n$ is indep. variable

$$\bar{\psi}\Lambda\psi = \sum_n \lambda_n \bar{a}_n a_n \qquad \mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_n (da_n\, d\bar{a}_n)$$

$$\int \mathcal{D}\bar\Psi\mathcal{D}\Psi \exp \bar\Psi \Lambda \Psi =$$

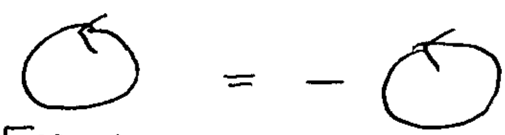
$$= \lim_{N\to\infty} \int \prod_1^N (da_n\, d\bar a_n)\left(1 + \frac{\sum_1^N \lambda_n \bar a_n a_n}{1!} + \dots + \frac{\left(\sum_1^N \lambda_n \bar a_n a_n\right)^N}{N!}\right) =$$

$$= \prod_1^N \lambda_n \bar a_n a_n$$

$$= \lim \prod_1^N \lambda_n = \det\Lambda = \exp \operatorname{tr}\ln\Lambda$$

compare Bosonic $\int \mathcal{D}\phi\mathcal{D}\bar\phi \exp \bar\phi\Lambda\phi = (\det\Lambda)^{-1} = \exp - \operatorname{tr}\ln\Lambda$

$\Rightarrow$ in the Fermionic case every loop a relative (−) sign with respect to the Bosonic case



Fermions Bosons

$\Lambda = \Lambda(eA_\mu)$ by expanding in eA_μ:

Fermions Bosons

Closed loops for Fermions: extra (−) sign

For open lines

Fermions Bosons