INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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MINIWORKSHOP ON
STRONGLY CORRELATED ELECTRON SYSTEMS

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"SOME EXPLICITLY SOLVABLE ONE-DIMENSIONAL MODELS"

F. GEBHARD
Phillipps University
Physics & Material Sciences Center
Renthof 5
Marburg D-3550
Germany

Some explicitly solvable one-dimensional models (1)

Florian Gebhard

Talk's plan: 14 transparencies

I. Introduction: continuum models with $1/r^2$-interaction
(Calogero-Sutherland)

II. Spin-½ lattice model with $1/r^2$-interaction
(Haldane-Shastry)

III. Supersymmetric t-J model with $1/r^2$-exchange
(Kuramoto-Yokoyama)

IV. Hubbard model with $1/r$-hopping (linear dispersion)
(Gebhard-Ruckenstein)

V. Conclusions / Summary / Bibliography

I. Introduction: continuum models with $1/r^2$-interaction

Calogero 1969: one-dimensional continuum model for N particles:

$$H = \frac{\hbar^2}{2m}\left[-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j=2}^{N} V(x_i - x_j)\right]$$

$$V(x) = \underbrace{\left(\frac{m\omega}{\hbar}\right)^2 \frac{x^2}{2}}_{\text{harmonic potential to keep particles together}} + \underbrace{\frac{g}{x^2}}_{\text{"zentrifugal" potential}} \quad ; \quad g \geq -\frac{1}{2} \text{ for stability}$$

Results:

- all eigenenergies and their degeneracy found
- ground state wave function

$$\psi_0(x_1, \dots, x_N) \sim \prod_{i<j=2}^{N}\left[(x_i - x_j)^\lambda e^{-\gamma\omega(x_i - x_j)^2}\right]$$

$$\lambda = \frac{1}{2}\left(1 + \sqrt{1+2g}\right) \; ; \; \gamma = \text{const},$$

- excited states:

$$\psi(x_1, \dots, x_N) = \underbrace{\phi(x_1, \dots, x_N)}_{\text{polynomial in } (x_i - x_j)} \psi_0(x_1, \dots, x_N)$$

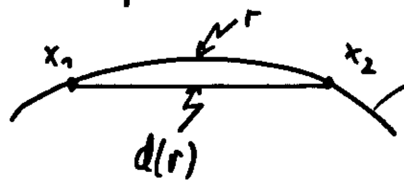
Problems:
- eigenenergies essentially trivial (related to $g=0$ case)
- no thermodynamical limit for $\omega \to 0$ possible

Sutherland 1970 - 1972: Calogero model on a ring of length L

- no harmonic term in $V(r)$ necessary to bring particles close
- $1/r^2$ - interaction has to be summed over multiple interactions.

$$V(r) = g \cdot \sum_{n=-\infty}^{+\infty} \left(\frac{1}{r+nL} \right)^2 = g \cdot \left[\frac{L}{\pi} \sin \frac{\pi}{L} r \right]^{-2} ; \; g \geq -\frac{1}{2}$$

"two - dimensional" interpretation :



arc of a circle of length L

$$d(r) = \frac{L}{\pi} \sin \frac{\pi}{L} r$$ chord distance between x_1 and x_2

Results :

- ground state given by

$$\psi_0 (x_1, x_2, \dots, x_N) = \prod_{i<j=2}^{N} \left| \sin \frac{\pi}{L} (x_i - x_j) \right|^{\lambda}$$

$$\lambda = \frac{1}{2} \left[1 + \sqrt{1+2g} \right]$$

Jastrow product form

- excited states given by

$$\psi (x_1, x_2, \dots, x_N) = \underbrace{\phi(x_1, \dots, x_N)}_{\substack{\text{polynomials in } (z_i - z_j) \\ \text{with } z_i = \exp(2\pi i x_i / L)}} \psi_0 (x_1, \dots, x_N)$$

Laughlin-wavefunction

- spectrum and thermodynamics exactly known:

$$E = E_0 + \sum_k k^2 n_k + \frac{\lambda}{2} \sum_{k,k'} |k-k'| \, n_k n_{k'}$$

k: integers; n_k: occupation numbers $(0, 1, 2, \dots)$
(used to label the polynomials $\Phi(x_1, \dots, x_N)$)

- ground state correlation functions exactly known from random matrix theory (Dyson, Mehta) for the values

$\lambda = 1$ $(g=0)$: free fermions
$\lambda = 1/2$ $(g = -1/2)$: most attractive case
$\lambda = 2$ $(g = 4)$: repulsive case

especially for $\lambda = 2$:

$$\Psi_0(x_1, \dots, x_N) = \prod_{i<j=2}^{N} \left| \sin \frac{\pi}{L}(x_i - x_j) \right|^2 \equiv \text{Product of two Slater determinants}$$

One-particle density matrix:

$$\rho(r) = \frac{\mathrm{Si}(2\pi \rho_0 r)}{2\pi r} \qquad (\rho_0 : \text{particle density})$$

$$\left(\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt \right)$$

II. Spin-1/2 lattice model with $1/r^2$-interaction

1. Motivation and ground state:

$$|\Psi_0\rangle = \underbrace{\prod_{l=1}^{L}\left(1-\hat{n}_{l,\uparrow}\hat{n}_{l,\downarrow}\right)}_{\substack{\text{Gutzwiller projector}\\ P_{D=0}}} \underbrace{\prod_{|k|<\pi/2} \hat{c}^{+}_{k,\uparrow}\hat{c}^{+}_{k,\downarrow}|vac\rangle}_{\text{Fermi-sea at half-filling } (d=1)}$$

- $|\Psi_0\rangle$ is totally equivalent to the $\lambda=2$ Sutherland wave function
- spin-spin correlation function in $d=1$

$$C_l^{S^zS^z} = \frac{1}{L}\sum_{m=1}^{L}\langle \hat{S}^z_{l+m}\hat{S}^z_m\rangle = (-1)^l \frac{Si(\pi l)}{4\pi l}$$

(Gebhard and Vollhardt, 1987)

Haldane - Shastry 1988: spin-1/2 Heisenberg lattice model on a ring of length L

$$\hat{H}_{HS} = \sum_{l<m=2}^{L} J_{l,m}\,\hat{\vec{S}}_l\,\hat{\vec{S}}_m\,;\quad J_{l,m} = \left[\frac{L}{\pi}\sin\frac{\pi}{L}(l-m)\right]^{-2} J$$

- $|\Psi_0\rangle$ is ground state for $J>0$ (antiferromagnetic state)
- $|\Psi_0\rangle$ is one of the simplest RVB-states (product of singlett pairs)

Remark: model can be extended to $SU(N)$ (in the continuum for general interaction strength):
Kawakami, 1992; Ha and Haldane, 1992;

2. Spectrum and thermodynamics: Haldane 1991

- Sutherland wavefunctions (Jastrow-Laughlin-type states) provide eigenstates for maximum spin
- problem: not enough states – how about lower spin states?

Haldane 1991: all eigenenergies are given by (numerically checked up to $L=N=12$)

$$E = E_0 + \sum_{k\sigma} (k^2 - k_0^2)\, n_{k\sigma} + \sum_{\substack{k\sigma \\ k'\sigma'}} \left[k_0 - |k-k'| \right] n_{k\sigma} n_{k'\sigma'}$$

(Sutherland formula with spin indices attached)

Conditions: (1) available are "orbitals" from $(|k| \le k_0)$
$-k_0, -k_0 + 2\pi/L, \dots, k_0 \;;\; k_0 = \frac{2\pi}{L} \cdot M$

(2) number of orbitals restricted by
$$2M = L - \sum_{k\sigma} n_{k\sigma}$$

(3) $n_{k\sigma} = 0, 1, 2, \dots$: occupation numbers for spin-$\frac{1}{2}$ "bosons" ("spinons")

Thermodynamics: free energy of a spin-$\frac{1}{2}$ Ising model in k-space with k-dependent interaction

$$f = \text{const} - T/2\pi \int_{-\pi}^{\pi} dk \, \ln \left[1 + e^{-\beta/4 [k^2 - \pi^2]} \right]$$

Examples: ($J>0$: antiferromagnetic case) (6)

ground state: $\sqcup\ \sqcup \ldots \sqcup\ \sqcup$: $M = L/2$; $L/2+1$ empty orbitals

lowest excitation: $\underbrace{\sqcup\ \sqcup \ldots \sqcup \overset{\sigma}{\bullet} \sqcup\ \sqcup \ldots \sqcup \overset{\sigma'}{\bullet} \sqcup\ \sqcup \ldots \sqcup\ \sqcup}_{S=1 \text{ or } S=0}$: $M = L/2 - 1$, $L/2$ orbitals, 2 spinons

highest excitation: ferromagnetic state $S = L/2$


L spinons in a single orbital ($M=0$

Statistics: 2 spinons "block" 1 orbital
(Fermions: 1 Fermion blocks 1 orbital
Bosons: no blocking of orbitals)

Haldane: spinons have half-valued (anyon-) statistics: "spinons are semions"

III. Supersymmetric t-J model with $1/r^2$ - exchange

generalization of the Haldane-Shastry model to include charge degrees of freedom (holes).
Kuramoto and Yokoyama:

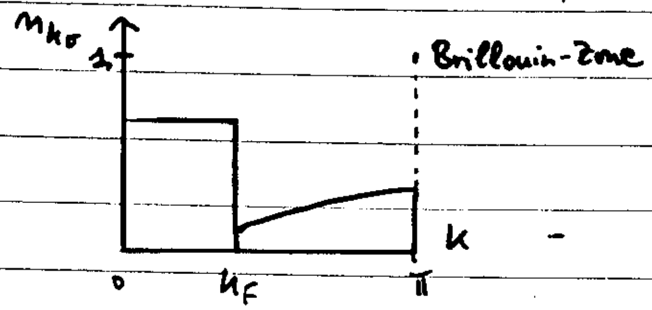
$$\hat H_{t-J} = \hat P_{D=0}\left[\sum_{l<m=2}^{L} J_{l,m}\left(\sum_\sigma \hat c^{+}_{l,\sigma}\hat c_{m,\sigma} + \hat{\vec S}_l\cdot\hat{\vec S}_m - \frac{1}{4}\hat n_l \cdot \hat n_m\right)\right]\hat P_{D=0}$$

1. ground state properties:

- ground state wave function: Gutzwiller projected Fermi-sea

$$|\psi_0\rangle = \prod_{l=1}^{L}\left(1-\hat n_{l,\uparrow}\hat n_{l,\downarrow}\right)\prod_{|k|<k_F}\hat c^{+}_{k,\uparrow}\hat c^{+}_{k,\downarrow}|\mathrm{vac}\rangle$$ (Kuramoto, Yokoyama, 1991)

- ground state correlation functions exactly known (Metzner and Vollhardt, 1987; Gebhard and Vollhardt, 1987); e.g., momentum distribution $\langle \hat n_{k,\sigma}\rangle = n_{k,\sigma}$



- low temperature physics derived from Sutherland's asymptotic Bethe Ansatz assumption by Kawakami (1992):
  charge-spin separation in the system: $v_c \neq v_s$
  (no Fermi liquid but a "noninteracting" Luttinger liquid)

2. Thermodynamics: Wang, Liu, Coleman (1992) (8)

- maximum spin states constructed (Sutherland states), proof that asymptotic Bethe Ansatz idea works for the supersymmetric t-J model

- spin degeneracy determined numerically up to $L=10$ sites; degeneracy scheme in analogy to Haldane

- free energy:

$$\underline{f = -\mu - T \int_{-\pi}^{\pi} \frac{dp}{2\pi} \ln\left(1 + e^{-\beta \varepsilon_s(p)}\right)}$$

$$\varepsilon_s(p) = \varepsilon_0(p) - T \ln\left[\frac{1}{2} + \sqrt{\frac{1}{4} + 2\exp\left[\beta(\varepsilon_0(p)+a)\right]\cosh\beta a}\,\right]$$

$$\varepsilon_0(p) = \frac{1}{2}\left(p^2 - \pi^2/3\right); \quad a = \pi^2/6 + \mu/2$$

at half-filling: $\mu \to \infty$ (no charge excitations)

$$\underline{f = \text{const} + f_{HS} = \text{const}' - \frac{T}{2\pi}\int_{-\pi}^{\pi} dp \ln\left[1 + \exp\left(-\frac{\beta}{4}(p^2 - \pi^2)\right)\right]}$$

Haldane's result

IV. Hubbard model with linear dispersion (1/r-hopping)

Hubbard model with general hopping amplitudes $t_{l,m}$:

$$\boxed{\hat{H}_{\text{Hubbard}} = \hat{T} + U\hat{D} = \sum_{\substack{l\neq m=1\\ \sigma}}^{L} t_{l,m}\, \hat{c}^{+}_{l,\sigma} \hat{c}_{m,\sigma} + U\cdot \sum_{l=1}^{L} \hat{n}_{l,\uparrow} \hat{n}_{l,\downarrow}}$$

$d=1$
L even

- We seek a Hubbard model which reduces to the Haldane-Shastry spin model for $U\to\infty$, $n=N/L=1$:

$$\hat{H}_{\text{Hubbard}} \longmapsto \hat{H}_{HS} = \sum_{l\neq m=1}^{L} 2\,\frac{|t_{l,m}|^2}{U}\left(\vec{\hat{S}}_l\cdot\vec{\hat{S}}_m - \frac{1}{4}\right)$$

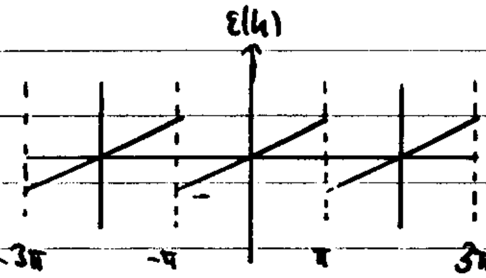
with
$$\frac{|t_{l,m}|^2}{U} \overset{!}{=} \frac{t^2}{U}\left[\frac{L}{\pi}\sin\frac{\pi}{L}(l-m)\right]^{-2}$$

- Choice: $t_{l,m} = it(-1)^{l-m}\left[\frac{L}{\pi}\sin\left(\frac{\pi}{L}(l-m)\right)\right]^{-1} = t^{*}_{m,l}$

- Result:

$$\boxed{\hat{T} = \sum_{k\sigma} \varepsilon(k)\, \hat{c}^{+}_{k,\sigma}\hat{c}_{k,\sigma}}$$
$$\boxed{\varepsilon(k) = t\cdot k\,;\quad |k|<\pi}$$

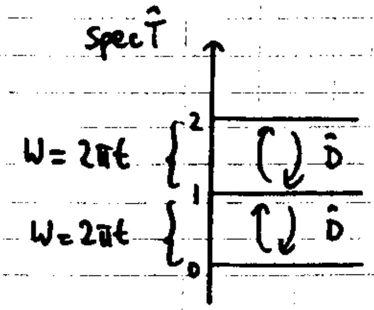
viz.:

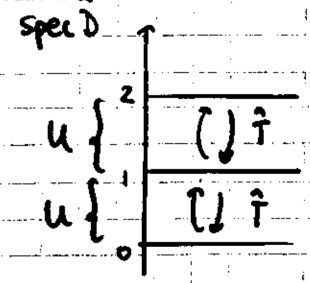


- $\hat{T}$ has negative, $\hat{D}$ has positive parity (in general, $\hat{H}$ has no definite parity)

- $\hat{H}$ may be considered a "Luttinger model on a lattice", however: only right-moving particles!

1. Spectra of $\hat{T}$ and $\hat{D}$: (fixed total momentum $\hat{Q}$)





highly degenerate levels, separated by W

highly degenerate levels, separated by U

2. Representation of the spectrum and its degeneracies

usually:

spec $\hat{T}$ can easily be determined by counting of configurations in k-space

spec $\hat{D}$ " " " " " counting of configurations in position space

here: many simultaneous eigenstates of $\hat{T}$ and $\hat{D}$!

introduce: (i) hard-core bosons for

charge: $x \equiv \hat{d}^{+}$ double occ. spin: $\uparrow \equiv \hat{s}_{\uparrow}^{+}$ $\uparrow$-spin

$o \equiv \hat{e}^{+}$ empty site $\downarrow \equiv \hat{s}_{\downarrow}^{+}$ $\downarrow$-spin

(ii) configuration space $\{\varkappa\} \equiv \{k\}$: $-\pi < \varkappa < \pi$

(iii) distribute one of $x, o, \uparrow, \downarrow$ on each site of the configuration space

(iv) $\hat{D}$ and $\hat{T}$ can be determined by just counting the number of x, and the $\varkappa$-values of occupied orbitals, respectively.

Result: simultaneous eigenstates of $\hat{T}$ and $\hat{D}$ described by

$$\boxed{\hat{H}_0^{\,eff} = \sum_{\chi}^{\text{no } \boxed{\uparrow\downarrow}\, \boxed{\times 0}} \left(\frac{t\chi}{2}\right)\left(\hat{s}^{+}_{\chi,\uparrow}\hat{s}_{\chi,\uparrow} + \hat{s}^{+}_{\chi,\downarrow}\hat{s}_{\chi,\downarrow}\right) - \left(\frac{t\chi}{2}\right)\left(\hat{d}^{+}_{\chi}\hat{d}_{\chi} + \hat{e}^{+}_{\chi}\hat{e}_{\chi}\right) + U\hat{d}^{+}_{\chi}\hat{d}_{\chi}}$$

however: $[\hat{T}, \hat{D}]_- \neq 0$!

some configurations do mix: they can be described by doubletts as $(\Delta = 2\pi/L)$

$$\boxed{\begin{aligned}\hat{H}_1^{\,eff} = \sum_{\chi}\Big\{&\left[\frac{U}{2} - (\mathrm{sgn}\,\chi)\left(\pi t - \frac{U}{4\pi}(2\chi+\Delta)\right)\right]\boxed{\times 0}_{\chi} \\ &+\left[\frac{U}{2} + (\mathrm{sgn}\,\chi)\left(\pi t - \frac{U}{4\pi}(2\chi+\Delta)\right)\right]\boxed{\uparrow\downarrow}_{\chi} \\ &+ \frac{U}{4\pi}\sqrt{(2\pi)^2 - (2\chi+\Delta)}\left[\boxed{\times 0}_{\chi} \to \boxed{\uparrow\downarrow}_{\chi} + \text{h.c.}\right]\end{aligned}}$$

Without proof: $\hat{H}^{eff} = \hat{H}_0^{\,eff} + \hat{H}_1^{\,eff}$ is always correct!

3. Free energy:

Diagonalization of $\hat{H}^{eff}$ (2×2 matrix) $\to$ 1d Ashkin-Teller model
$\to$ free energy density f for $(N, L \to \infty)$, $n = N/L$ fixed; $\beta = \frac{1}{T}$

$$\boxed{f = -\frac{1}{\beta}\int_{-\pi}^{\pi}\frac{d\chi}{2\pi}\ln\left\{\frac{1}{2}\left[X_\chi + \sqrt{X_\chi^2 - 4\left[S_{\chi,\uparrow}S_{\chi,\downarrow}(1-P_\chi^{-1}) + D_\chi\cdot E_\chi(1-P_\chi)\right]}\right]\right\}}$$

where: $S_{\chi,\sigma} = \exp\left[(-\beta)\left(\frac{t\chi}{2} - \mu + \sigma\frac{g\mu_B H_0}{2}\right)\right]$

$D_\chi = \exp\left[(-\beta)\left(-\frac{t\chi}{2} - 2\mu + U\right)\right]$ $\qquad X_\chi = S_{\chi,\uparrow} + S_{\chi,\downarrow} + D_\chi + E_\chi$

$E_\chi = \exp\left[(-\beta)\left(-t\chi/2\right)\right]$

$P_\chi = \exp\left[(-\beta/2)\left(2t\chi - U + \sqrt{(2\pi t)^2 + U^2 - 4tU\chi}\right)\right]$

4. Support for the effective Hamiltonian:

(1a)

(i) small systems: • $L = 2, 4, 6$ checked at $N = L$
($6, 70, 924$ states)

(ii) perturbation theory: • ground state energy checked
to order U^3 (all n)
to order n^3 (all U)

• free energy checked to order $\beta = 1/T$

(iii) comparison to Haldane's work: • $U \to \infty$, $n = 1$; $J = 4t^2/U$

Here: $\underline{\hat{H}^{\text{eff}} \doteq (-J) \sum_x \left[(\pi/2)^2 - (\varrho/2)^2 \right] \tilde{n}^{s}_{x-0,\uparrow} \tilde{n}^{s}_{x,\downarrow}}$

Ising model with $\tilde{n}^{s}_{0x,\sigma} = \frac{1}{2} + \sigma S^z_x$

• complete agreement with Haldane's free energy!

• new interpretation of "orbitals" and "spinons":

ground state: $\sqcup (\uparrow\downarrow) \sqcup (\uparrow\downarrow) \sqcup \cdots \sqcup (\uparrow\downarrow) \sqcup (\uparrow\downarrow) \sqcup$: $L/2$ $S = 0$ $(\uparrow\downarrow)$-pairs
$\Leftrightarrow$ $L/2 + 1$ empty orbitals $\sqcup$

elementary excitation $\equiv$ broken $(\uparrow\downarrow)$-pair $\equiv$ 2 spinons

$\sqcup (\uparrow\downarrow) \sqcup (\uparrow\downarrow) \sqcup \cdots \sqcup (\uparrow\downarrow) \overset{\sigma}{\bullet} (\uparrow\downarrow) \sqcup \cdots \sqcup (\uparrow\downarrow) \overset{\sigma'}{\bullet} (\uparrow\downarrow) \sqcup \cdots \sqcup (\uparrow\downarrow) \sqcup (\uparrow\downarrow) \sqcup$

$(L/2 - 1)$ $(\uparrow\downarrow)$-pairs $\Leftrightarrow$ $L/2$ orbitals

2 spinons •

semions are not really necessary ...

etc.

5. ground state properties:

a) Metal - Insulator transition: MIT

ground state energy density ($W = 2\pi t$: bandwidth)

$$\boxed{e_0(n \le 1) = \frac{Un - Wn(1-n)}{4} - \frac{1}{24WU}\left[(W+U)^3 - \left[(W+U)^2 - 4WUn\right]^{3/2}\right]}$$

$$e_0(n \ge 1) = e_0(2-n) + U(n-1)$$

Chemical potential at half-filling: $\mu_- = \left.\frac{\partial e_0(n \le 1)}{\partial n}\right|_{n=1} = \frac{1}{4}\left[W + U - |W-U|\right]$

$$\mu_+ = U - \mu_-$$

For $U \le W$: $\mu_- = \mu_+ = U/2$ continuous

$U \ge W$: $\Delta\mu = \mu_+ - \mu_- = U - W \ge 0$ Mott-Hubbard MIT

b) $t-J$ model limit $U = \infty$: (incl. 3-site contributions)

$$\boxed{\hat{H}^{\mathrm{eff}} = \underbrace{\sum_{\varkappa} (-t\varkappa)\, \tilde{n}^{e}_{\varkappa}}_{\text{charge}} - \underbrace{(J/4)\left[\pi^2 - (2\varkappa-\Delta)^2\right] \tilde{n}^{s}_{\varkappa-\Delta,\uparrow}\, \tilde{n}^{s}_{\varkappa,\downarrow}}_{\text{spin}}}$$

charge and spin

separated at all

energy scales

- ground state: for $J \le J_c = 2W/[\pi^2(1-n)]$ the ground state can be identified as the Gutzwiller projected Fermi sea:

$$|\Psi_0\rangle = \prod_{l=1}^{L}(1 - \hat{n}_{l,\uparrow}\hat{n}_{l,\downarrow}) \prod_{\substack{-k_F < k < k_F \\ \sigma}} \hat{c}^{+}_{k,\sigma} |\mathrm{vac}\rangle$$

• Elementary excitations: (14)

$J \leq J_c = 2W/[(U-n)\pi^2]$: – one linear branch for "spinons":

$$v_s = J\pi/2$$

– one linear branch for "holons":

$$v_h = -(1/\pi)[W + J\pi^2(2n-1)/2]$$

charge-spin separation: $v_s \neq v_h$

$J > J_c$: – gap for spin excitations
– two linear branches for "holons":

$$v_h^r = -v_h^l = J\pi n/2$$

V. Summary / Conclusions:

(1) Class of exactly solvable (lattice) models: long-range exchange
$1/r^2$ – spin-models (SU(2), SU(N))
$1/r^2$ – supersymmetric t-J model
$1/r$ – hopping Hubbard model

(2) Common features: • highly degenerate excitation spectra
• Jastrow-Laughlin-type wave functions (link to Quantum-Hall-Effect ?!)

(3) Problems: no complete proofs for lattice-problems, situation poorest for Hubbard model case

(4) Interesting physics present: – Luttinger Liquids
– charge-spin separation
– metal-to-insulator transition
– spin gap develops in t-J model limit of the Hubbard model

More fun and excitement is to be expected ...!

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Exact Results for a Hubbard Chain with Long-Range Hopping

Florian Gebhard[a] and Andrei E. Ruckenstein
Serin Physics Laboratory, Rutgers University, P.O. Box 849, Piscataway, New Jersey 08855-0849
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We give the exact spectrum and thermodynamics for a long-range hopping Hubbard chain with linear dispersion. This model exhibits a Mott-Hubbard metal-insulator transition at half filling when the interaction strength U equals the bandwidth W. The solution for $U \gg W$ also covers the corresponding t-J model, which reduces to the spin model of Haldane and Shastry at half filling. We mention possible extensions of the model in one and higher dimensions.

PACS numbers: 71.30.+h, 05.30.Fk

There has been renewed interest in exactly solvable models of strongly correlated systems, especially following the discovery of the high-T_c oxides in which strong correlations appear to play a key role. A few years ago, Haldane [1] and Shastry [2] independently introduced a spin-$\frac{1}{2}$ Heisenberg model with long-range exchange,

$$\hat{H}_{\text{spin}} = (J/2) \sum_{l \neq m = 1}^{L} [d(l-m)]^{-2} \hat{\mathbf{S}}_l \cdot \hat{\mathbf{S}}_m \,, \tag{1}$$

where $d(l-m) = (L/\pi)\sin[\pi(l-m)/L]$ is the chord distance between two lattice points (l,m) on a ring of length L. The ground state for the antiferromagnetic case $(J>0)$ has been identified [1,2] to be the paramagnetic Gutzwiller-projected Fermi sea (FS) with the number of electrons equal to the number of sites (half filling), $|\Psi_0\rangle = \hat{P}_{D=0}|\text{FS}\rangle = \prod_l [1-\hat{n}_{l,\uparrow}\hat{n}_{l,\downarrow}]|\text{FS}\rangle$, where $\hat{n}_{l,\sigma} = \hat{c}^\dagger_{l,\sigma}\hat{c}_{l,\sigma}$ is the number operator for electrons of spin σ. $|\Psi_0\rangle$ also turns out to be the ground state for a supersymmetric t-J model generalization of the Haldane-Shastry long-range model away from half filling [3]. The exact ground-state correlation functions are known from Ref. [4]. In a remarkable Letter Haldane was also able to derive the full spectrum and thermodynamics of his model [5]. In contrast to the solutions obtained by the Bethe ansatz, the structure of Haldane's spectrum suggests extensions to higher dimensions. A specific scenario—involving a generalization of statistics—was recently outlined in Ref. [6].

In this work we introduce a new Hubbard-type chain with long-range hopping:

$$\hat{H} = \hat{T} + U\hat{D} = \sum_{l \neq m = 1, \sigma}^{L} t_{l,m}\hat{c}^\dagger_{l,\sigma}\hat{c}_{m,\sigma} + U\sum_{l=1}^{L}\hat{n}_{l,\uparrow}\hat{n}_{l,\downarrow} \tag{2}$$

with $t_{l,m} = it(-1)^{l-m}[d(l-m)]^{-1} = t^*_{m,l}$. Since it is most convenient to work with even L, the specific form of the hopping matrix elements forces us to choose antiperiodic boundary conditions. Then the resulting dispersion is *linear*, $\epsilon(k) = tk$, with $k = (2\pi/L)(m+\frac{1}{2})$ $(m = -L/2, \ldots, L/2-1)$. The itinerant model of Eq. (2) is constructed so that at half filling, in the $U \to \infty$ limit, we recover the Haldane-Shastry model. In particular, the corresponding t-J Hamiltonian can be obtained in the usual way [7]:

$$\hat{H}_{tJ} = \hat{P}_{D=0}\left\{\hat{T} + \sum_{l \neq m}\frac{2|t_{l,m}|^2}{U}(\hat{\mathbf{S}}_l \cdot \hat{\mathbf{S}}_m - \tfrac{1}{4}\hat{n}_l\hat{n}_m) - \sum_{l \neq n \neq m \neq l}\frac{t_{l,n}t_{n,m}}{U}\sum_{\sigma\sigma'}(\sigma\sigma')\hat{c}^\dagger_{l\sigma}\hat{c}^\dagger_{n-\sigma}\hat{c}_{n-\sigma'}\hat{c}_{m\sigma'}\right\}\hat{P}_{D=0}\,, \tag{3}$$

where $\hat{n}_l = \sum_\sigma \hat{n}_{l\sigma}$. At half filling this reduces to Eq. (1) with $J = 4t^2/U$ and an energy shift $C \equiv (J/8)\sum_{l\neq m=1}^{L}[d(l-m)]^{-2} = (J/24)(\pi/L)^2 L(L^2-1)$ [8]. Note that $\hat{H}$ [Eq. (2)] has a definite parity, i.e., $\hat{P}\hat{H}\hat{P} = \eta\hat{H}$, only for $U=0$ $(\eta=-1)$, $t=0$ $(\eta=+1)$, or in the limit of half filling and $U \to \infty$ $(\eta=+1)$.

In this paper we use the exact diagonalization results for small systems and analytical results in various limits to conjecture an effective Hamiltonian which recovers the full spectrum and degeneracies of (2) *for arbitrary fillings and interaction strengths*. This effective Hamiltonian is then used to study the ground-state properties and thermodynamics of our model. We conclude by proposing an obvious extension of this type of long-range hopping model which may be solvable even in higher dimensions.

The spectrum of the kinetic-energy operator $\hat{T}$ for a fixed total momentum Q $[\hat{Q} = (\hat{T}/t)\bmod 2\pi]$ consists of *highly degenerate equidistant levels* separated by $2\pi t$. To see this, let the momentum transferred be $q>0$ and consider the two-particle scattering process $(k,p) \to (k+q, p-q)$. The initial and scattered states have the same kinetic energy T if (i) $k+q<\pi$ and $p-q>-\pi$ or (ii) $k+q>\pi$ and $p-q<-\pi$. In the remaining cases T changes by $\pm 2\pi t$.

To gain some intuition we analyze the L^2 states with $S^z = L/2-1$ (single spin flip). There are L states with $T=Q=0$; of these, the $L-1$ states with $S=L/2-1$ have double occupancy $D=1$, and the remaining $S=L/2$ state has $D=0$. For fixed total momentum $Q=(2\pi/L)m_Q \neq 0$ one finds $(L-1)(L-2)$ simultaneous eigenstates of $\hat{T}$ and $\hat{D}$ with $D=1$, of which $(L-1)(L-|m_Q|-1)$ have kinetic energy $T=tQ$, and $(L-1)(|m_Q|-1)$ have $T=t(Q-2\pi\lambda_Q)$ [where $\lambda_Q = \text{sgn}(Q)$]. The $2(L-1)$ remaining $Q\neq 0$ states can be chosen as eigenstates of $\hat{T}$

17

and are superpositions between $D=0$ and $D=1$ charge states. These bonding and antibonding states [$(L-1)$ doublets] for the spin flip and the hole are obtained from the diagonalization of the 2×2 matrix for each Q, with diagonal elements $tQ+U|m_Q|/L$ and $t(Q-2\pi\lambda_Q)+U(1-|m_Q|/L)$, and off-diagonal elements $U[(|m_Q|/L)(1-|m_Q|/L)]^{1/2}$.

In what follows it will be useful to represent the Hilbert space of the above example pictorially: For this purpose we introduce a "quasimomentum" space with "coordinates" $\mathcal{K}$ which, as the physical momenta, are given by $\mathcal{K}=(2\pi/L)(m_{\mathcal{K}}+\frac{1}{2})$ $(m_{\mathcal{K}}=-L/2,\dots,L/2-1)$. A particular state will be depicted by filling the $\mathcal{K}$ states with spin-carrying $(\uparrow,\downarrow)$ and charge-carrying $(\times,\circ)$ bosons representing spin up and down, and doubly occupied and empty $\mathcal{K}$ sites. In addition, a completeness constraint that each $\mathcal{K}$ site is occupied by one and only one boson must be fulfilled. For example, there are L states which we would represent by $\uparrow\cdots\uparrow\downarrow\uparrow\cdots\uparrow$ which involve a single spin flip with $D=0$. However, from our solution above we learn that there is only one *eigenstate* with $D=0$, namely, the state with $S=L/2$ and total momentum $Q=0$. We will arbitrarily select the pictorial representation of this state to be $\downarrow\uparrow\cdots\uparrow$, i.e., the down spin occupies the first position on the $\mathcal{K}$ chain. All configurations which involve an up spin to the immediate left of a down spin—and there are $L-1$ such configurations in the single-spin-flip case—must then belong to the $L-1$ doublets. We depict these configurations by "boxing" the nearest-neighbor up-down pair, $\uparrow\cdots\uparrow(\uparrow\downarrow)_{\mathcal{K}}\uparrow\cdots\uparrow$. Similarly, the $L(L-1)$ states with $D=1$ could be represented as $\uparrow\cdots\uparrow\times\uparrow\cdots\uparrow\circ\uparrow\cdots\uparrow$. Of these, $L-1$ are part of the doublets while the remaining $(L-1)^2$ are simultaneous eigenstates of $\hat{T}$ and $\hat{D}$. We then choose to represent the former by boxing a *neighboring* pair of doubly occupied and empty sites whenever the doubly occupied site occurs to the left of the empty one: $\uparrow\cdots\uparrow(\times\circ)_{\mathcal{K}}\uparrow\cdots\uparrow$. The $L-1$ doublets are then identified with the $2(L-1)$ states involving the superposition $(\uparrow\downarrow)_{\mathcal{K}}$ and $(\times\circ)_{\mathcal{K}'}$ with $\mathcal{K}'=\mathcal{K}$.

We now assign physical momenta and kinetic energies to the various configurations. For configurations without boxes the physical momentum of the singly occupied $\mathcal{K}$ sites is identified with the $\mathcal{K}$ momentum while doubly occupied and empty $\mathcal{K}$ states carry zero momentum. These states can then be described in terms of an effective Hamiltonian diagonal in the occupation-number representation for our set of boson operators for spin-carrying $(\hat{s}_{\mathcal{K},\sigma})$, and charge-carrying $(\hat{e}_{\mathcal{K}},\hat{d}_{\mathcal{K}})$ quasiparticles:

$$\hat{H}_0^{\text{eff}}=\sum_{-\pi<\mathcal{K}<\pi}{}' \{\tfrac{1}{2}t\mathcal{K}[(\hat{s}^\dagger_{\mathcal{K},\uparrow}\hat{s}_{\mathcal{K},\uparrow}+\hat{s}^\dagger_{\mathcal{K},\downarrow}\hat{s}_{\mathcal{K},\downarrow}) -(\hat{d}^\dagger_{\mathcal{K}}\hat{d}_{\mathcal{K}}+\hat{e}^\dagger_{\mathcal{K}}\hat{e}_{\mathcal{K}})]+U\hat{d}^\dagger_{\mathcal{K}}\hat{d}_{\mathcal{K}}\}\,, \tag{4}$$

where we made use of the completeness constraint $\sum_\sigma \hat{s}^\dagger_{\mathcal{K},\sigma}\hat{s}_{\mathcal{K},\sigma}+\hat{d}^\dagger_{\mathcal{K}}\hat{d}_{\mathcal{K}}+\hat{e}^\dagger_{\mathcal{K}}\hat{e}_{\mathcal{K}}=1$ for each $\mathcal{K}$. The prime on the sum is a reminder of the fact that the $L-1$ doublets are not represented by $\hat{H}_0^{\text{eff}}$.

For boxed configurations, $(\uparrow\downarrow)_{\mathcal{K}}$ and $(\times\circ)_{\mathcal{K}}$ are assigned momenta $\pi\lambda_{\mathcal{K}}$ and $-\pi\lambda_{\mathcal{K}}$, respectively. The mixing of the boxed configurations due to the Hubbard interaction can be read off from the solution of our single-spin-flip problem. We use the identity $m_Q/L=[2\pi\lambda_{\mathcal{K}}-(2\mathcal{K}+\Delta)]/4\pi$ with $\Delta=2\pi/L$, so that the second part of the effective Hamiltonian reads

$$\begin{aligned}\hat{H}_1^{\text{eff}}=\sum_{-\pi<\mathcal{K}<\pi-\Delta}\Bigg\{&\left[\frac{U}{2}-\lambda_{\mathcal{K}}\left(\pi t-\frac{U}{4\pi}(2\mathcal{K}+\Delta)\right)\right]\hat{d}^\dagger_{\mathcal{K}}\hat{d}_{\mathcal{K}}\hat{e}^\dagger_{\mathcal{K}+\Delta}\hat{e}_{\mathcal{K}+\Delta}\\ &+\left[\frac{U}{2}+\lambda_{\mathcal{K}}\left(\pi t-\frac{U}{4\pi}(2\mathcal{K}+\Delta)\right)\right]\hat{s}^\dagger_{\mathcal{K},\uparrow}\hat{s}_{\mathcal{K},\uparrow}\hat{s}^\dagger_{\mathcal{K}+\Delta,\downarrow}\hat{s}_{\mathcal{K}+\Delta,\downarrow}\\ &+\frac{U}{4\pi}[(2\pi)^2-(2\mathcal{K}+\Delta)^2]^{1/2}[\hat{d}^\dagger_{\mathcal{K}}e^\dagger_{\mathcal{K}+\Delta}\hat{s}_{\mathcal{K},\uparrow}\hat{s}_{\mathcal{K}+\Delta,\downarrow}+\text{H.c.}]\Bigg\}.\end{aligned} \tag{5}$$

The total effective Hamiltonian is given by $\hat{H}^{\text{eff}}=\hat{H}_0^{\text{eff}}+\hat{H}_1^{\text{eff}}$.

We now *conjecture* that $\hat{H}^{\text{eff}}$ is indeed correct in *all* spin sectors, for *all* fillings and arbitrary interaction strengths. For two particles on a lattice of size L this can be checked by applying a particle-hole transformation and turning the problem into the single-spin-flip case. We have also confirmed our conjecture for lattices of size $L=2$, 4, and 6 at half filling where the dimension of the Hilbert space is 6, 70, and 924, respectively. Further confirmation comes from agreement with (i) the small-U expansion at zero temperature, (ii) the low-density expansion at zero temperature, (iii) the high-temperature expansion, and (iv) all of Haldane's results [5] for the large-U limit at half filling (see below).

The effective Hamiltonian can be easily brought into an occupation-number form by diagonalizing the 2×2 matrices describing the independent doublets for each $\mathcal{K}$. Including an external magnetic field $(\mathcal{H}_0)$ and chemical potentials $\mu_\sigma=\mu-\sigma g\mu_B\mathcal{H}_0/2$ we may write the resulting effective Hamiltonian as

$$\hat{H}^{\text{eff}}=\sum_{-\pi<\mathcal{K}<\pi}\Big(\sum_\sigma h^s_{\mathcal{K},\sigma}\bar{n}^s_{\mathcal{K},\sigma}+h^d_{\mathcal{K}}\bar{n}^d_{\mathcal{K}}+h^e_{\mathcal{K}}\bar{n}^e_{\mathcal{K}} +J_{\mathcal{K}}(\bar{n}^d_{\mathcal{K}-\Delta}\bar{n}^e_{\mathcal{K}}-\bar{n}^s_{\mathcal{K}-\Delta,\uparrow}\bar{n}^s_{\mathcal{K},\downarrow})\Big)\,, \tag{6}$$

with $h^s_{\mathcal{K},\sigma}=t\mathcal{K}/2-\mu_\sigma$, $h^d_{\mathcal{K}}=-t\mathcal{K}/2-2\mu+U$, $h^e_{\mathcal{K}}$

18

$=-t\mathcal{K}/2$, and

$$J_{\mathcal{K}}=\{t(2\mathcal{K}-\Delta)-U+[(2\pi t)^2+U^2-2tU(2\mathcal{K}-\Delta)]^{1/2}\}/2\geq 0\,.$$

Here, the obvious notations $\tilde{n}^s_{\mathcal{K},\sigma}=\tilde{s}^\dagger_{\mathcal{K},\sigma}\tilde{s}_{\mathcal{K},\sigma}$, etc., have been used. To be precise, we also restrict ourselves to $t\geq 0$ and $U\geq -2\pi t$ in which case we may identify $\mathcal{K}_{\min}-\Delta\equiv\mathcal{K}_{\max}$ because $J_{\mathcal{K}}$ vanishes for $\mathcal{K}=\mathcal{K}_{\min}=-[(L-1)/2]\Delta$.

For an even particle number, $N\leq L$, the ground state in zero magnetic field is obtained by only filling $\mathcal{K}$ states with $\tilde{s}^\dagger_{\mathcal{K},\uparrow}\tilde{s}^\dagger_{\mathcal{K}+\Delta,\downarrow}$ pairs from $\mathcal{K}=-\pi$ to $\mathcal{K}=\mathcal{K}^F=\pi(2n-1)$. The ground-state energy per lattice site is found to be

$$e_0(n\leq 1)=\frac{Un-W(1-n)n}{4}-\frac{1}{24WU}\{(W+U)^3-[(W+U)^2-4WUn]^{3/2}\}\,, \tag{7}$$

where $W=2\pi t$ is the bandwidth. Using particle-hole symmetry we obtain $e_0(n\geq 1)=e_0(2-n)+U(n-1)$, and, correspondingly, for the chemical potential at zero temperature, $\mu(n<1)=\partial e_0(n<1)/\partial n$, $\mu(n>1)=U-\mu(2-n)$. At half filling we find for the left derivative of the ground-state energy density $\mu_-(n=1)=(W+U-|W-U|)/4$, while the right derivative is $\mu_+(n=1)=U-\mu_-(n=1)$. For $U\leq W$ the chemical potential is continuous ($\mu=U/2$), while for $U>W$, there is a gap $\Delta\mu(n=1)=U-W$, i.e., our model exhibits a Mott-Hubbard metal insulator transition at half filling for a finite value of the interaction, $U=W$.

We note that, in the limit of small U and *all n*, or small n and *all U*, the ground-state energy (7) agrees with the appropriate limits derived from the Gutzwiller wave function, $|\Psi_g\rangle=\prod_l[1-(1-g)\hat{n}_{l,\uparrow}\hat{n}_{l,\downarrow}]|\mathrm{FS}\rangle$ [9]. It can be checked explicitly from the appropriate perturbation theory that this wave function is indeed the ground-state wave function, but this is true only in these special limits. In addition, given our effective Hamiltonian, it can be seen that the Gutzwiller-projected Fermi sea also becomes the ground state of the t-J model (3) for $J=4t^2/U\ll t$.

It is also interesting to consider the t-J model which is obtained by replacing $4t^2/U$ by J in the Hamiltonian (3) and then treating J as an independent parameter. With the help of the large-U limit of our effective Hamiltonian we obtain the ground-state energy as $e_0^U(J\leq J_c)=-Wn(1-n)/2-J\pi^2n^2(3-2n)/12$ and $e_0^U(J\geq J_c)=-nW^2/2J\pi^2-J\pi^2n(3-n^2)/24$, where $J_c=2W/\pi^2(1-n)$. At half filling we can compare our results with those of Haldane [5]. Since in the large-U limit $\tilde{s}^\dagger_{\mathcal{K},\sigma}\equiv s^\dagger_{\mathcal{K},\sigma}$ the antiferromagnetic ground state is given by $L/2$ $(\uparrow\downarrow)$ pairs, a state which is represented by $L/2+1$ empty "orbitals" in Ref. [5]. For finite L the corresponding energy is given by

$$E_0^{\mathrm{AF}}=C+\sum_{\mathcal{K}}{}'J_{\mathcal{K}}=-(J/24)(\pi/L)^2L(L^2+5)\,,$$

where the prime on the sum indicates that only every second $\mathcal{K}$ had to be included in the sum [10].

The resulting spectrum consists of two fundamental excitations: "spinons" [5,6] as in the Haldane-Shastry model, and "holons," which are already present in the t model ($U=\infty$, i.e., $J\equiv 0$). This charge-spin separation [11] is already evident from the effective Hamiltonian for $U\to\infty$, which can be explicitly written as an itinerant and a spin part

$$\hat{H}^{\mathrm{eff}}_{tJ}=\hat{H}^{\mathrm{eff}}_t+\hat{H}^{\mathrm{eff}}_J$$
$$=\sum_{-\pi<\mathcal{K}<\pi}[(-t\mathcal{K})\tilde{n}^c_{\mathcal{K}}-J_{\mathcal{K}}\tilde{n}^s_{\mathcal{K}-\Delta,\uparrow}\tilde{n}^s_{\mathcal{K},\downarrow}]\,.$$

Here, $J_{\mathcal{K}}=(J/4)(\pi^2-\mathcal{K}^2)$. A "spinon" for $J\leq J_c$ corresponds to a broken spin pair in an otherwise unchanged hole background. It can be represented graphically as

$$(\uparrow\downarrow)\cdots(\uparrow\downarrow)\sigma|_{\mathcal{K}}(\uparrow\downarrow)\cdots(\uparrow\downarrow)\sigma'|_{\mathcal{K}'}(\uparrow\downarrow)\cdots(\uparrow\downarrow)|_{\pi(2n-1)}\mathrm{O}\cdots\mathrm{O}\,.$$

The excitations are always created in pairs, and their separation $\mathcal{K}-\mathcal{K}'$ has to be an even multiple of Δ. It is then convenient to rescale $\mathcal{K}\to 2\mathcal{K}$ to make contact with Haldane's Eq. (17); in the infinite volume limit we obtain for a single spinon $\epsilon_{\mathrm{spinon}}(\mathcal{K})=(J/2)[(\pi/2)^2-\mathcal{K}^2]$ with $-\pi/2\leq\mathcal{K}\leq\pi(2n-1)/2$. The spinon velocity at $\mathcal{K}=-\pi/2$ is $v_{\mathrm{spinon}}=J\pi/2$. For $J>J_c$ the lowest-lying spin excitations are two holes at $\mathcal{K}=\mathcal{K}^F-2\pi n$ and $\mathcal{K}^F-2\pi n+\Delta$ and two spins at $\mathcal{K},\mathcal{K}'$ which can have arbitrary separation. Here, $\mathcal{K}^F=\pi[-J_c/J+n(1+J_c/J)]$. The dispersion for a single spinon is then given by $\epsilon_{\mathrm{spinon}}(\mathcal{K})=t\mathcal{K}-t(\mathcal{K}^F-2\pi n)+J_{\mathcal{K}^F-2\pi n}/2$ with $-\pi\leq\mathcal{K}<\mathcal{K}^F-2\pi n$. The spinon spectrum opens a gap at $J=J_c$, i.e., the spin fluid becomes "incompressible" leading to the vanishing of the spin susceptibility above J_c. At half filling we have $\mathcal{K}^F=\pi$ and the spinon dispersion acquires another linear part around $\mathcal{K}=\pi/2$ with velocity $v^l_{\mathrm{spinon}}=-v^r_{\mathrm{spinon}}=-J\pi/2$. We further notice that we obtain two spinons for every broken $(\uparrow\downarrow)$ pair, so that one has $L/2+1-M$ orbitals for $2M$ spinons ($M=0,1,\ldots,L/2$). Spinons between unbroken $(\uparrow\downarrow)$ pairs occupy the corresponding orbitals. This construction provides the link to Haldane's construction of the spinon states [12].

The other excitation, the holon, involves a hole of momentum $\mathcal{K}$ which is surrounded by unbroken spin pairs. As for the spinon, the allowed $\mathcal{K}$ values are spaced by $4\pi/L$ and we again rescale $\mathcal{K}\to 2\mathcal{K}$. Then the excitation energies are

$$\epsilon_{\mathrm{holon}}(\mathcal{K})=\frac{\mathcal{K}^F-2\mathcal{K}}{2\pi}[W+\tfrac{1}{4}J\pi(\mathcal{K}^F+2\mathcal{K})]\,,\qquad \mathcal{K}^F/2-\pi n\leq\mathcal{K}\leq\mathcal{K}^F/2\,. \tag{8}$$

For $J=0$ the above dispersion agrees with that of a gas of spinless fermions with linear dispersion and Fermi momentum $\mathcal{K}^F=\pi(2n-1)$. For $J\le J_c$ the holon velocity at $\mathcal{K}=\mathcal{K}^F/2$ is $v_{\text{holon}}=(-1/\pi)[W+J\pi^2(2n-1)/2]$. For $J>J_c$ we find *two* linear excitations, the first at $\mathcal{K}=\mathcal{K}^F/2$ with velocity $v^l_{\text{holon}}=-nJ\pi/2$, and the second at $\mathcal{K}=\mathcal{K}^F/2-\pi n$ with velocity $v^r_{\text{holon}}=nJ\pi/2$.

The model (6) belongs to the class of one-dimensional Ashkin-Teller models [13] in external fields. Its free-energy density in the thermodynamic limit can be easily found by the use of the transfer-matrix method. We define $S_{\mathcal{K},\sigma}=\exp(-\beta h^s_{\mathcal{K},\sigma})$, $D_{\mathcal{K}}=\exp(-\beta h^d_{\mathcal{K}})$, $E_{\mathcal{K}}=\exp(-\beta h^e_{\mathcal{K}})$, $X_{\mathcal{K}}=S_{\mathcal{K},\uparrow}+S_{\mathcal{K},\downarrow}+D_{\mathcal{K}}+E_{\mathcal{K}}$, and $P_{\mathcal{K}}=\exp(-\beta J_{\mathcal{K}})$, and obtain

$$f=-\frac{1}{\beta}\int_{-\pi}^{\pi}\frac{d\mathcal{K}}{2\pi}\ln\left(\tfrac{1}{2}X_{\mathcal{K}}+\tfrac{1}{2}\{X_{\mathcal{K}}^2-4[S_{\mathcal{K},\uparrow}S_{\mathcal{K},\downarrow}(1-P_{\mathcal{K}}^{-1})+D_{\mathcal{K}}E_{\mathcal{K}}(1-P_{\mathcal{K}})]\}^{1/2}\right). \tag{9}$$

We checked this expression for high temperatures to first order in β.

At half filling we have $\mu=U/2$ due to particle-hole symmetry. For $U\to\infty$ and finite external magnetic field we can therefore neglect the terms with $D_{\mathcal{K}}$ and $E_{\mathcal{K}}$; the effective Hamiltonian (6) can then be transformed into an Ising model. The corresponding expressions for the entropy density at vanishing external magnetic field, and the static magnetic susceptibility can be easily derived and are found to completely agree with Eqs. (15) and (16) of Ref. [5].

The reason for the integrability of the model is related to the relatively simple algebra satisfied by the kinetic- and potential-energy operators and their highly degenerate spectrum. It is not difficult to see how to systematically construct long-range hopping models which preserve these features in $d\ge 1$. We start from the observation that nearest-neighbor hopping on a cluster of size R gives R values for the kinetic energy, $\epsilon(\mathbf{k}_n)$, with $n=1,\dots,R$. A long-range hopping model on the lattice can then be defined by choosing a dispersion relation $\epsilon(\mathbf{k})$ which is limited piecewise to the constant values $\epsilon(\mathbf{k}_n)$ in regions around $\mathbf{k}_n$. Such a *finite* cluster model should lead to an effective Hamiltonian in an occupation-number basis, describing a *classical* model in the same number of dimensions. The latter, although rarely completely solvable, can be investigated by a variety of techniques of classical statistical mechanics. We expect that, in this new class of models, the metallic ground states are Fermi liquids, like those obtained from variational Gutzwiller wave functions [4], and one might as well call such phases "Gutzwiller liquids."

In this Letter we have studied a new itinerant one-dimensional model with long-range hopping and Hubbard interaction. In spite of its simplicity this model displays a metal-to-insulator transition at half filling for a finite interaction strength. It remains to justify the conjectures of this paper by a direct algebraic approach.

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Note added.—R. Shankar informed us that he has checked the validity of our ground-state energy, Eq. (7), by perturbation theory up to third order in U for all fillings. He also observed that at half filling $e_0(n)$ has a nonanalyticity of the form $(1-n)^2|1-n|$ for all values of the coupling constant.

(a)Address after 1 March 1992: Philipps Universität Marburg, Fachbereich Physik, Renthof 5, W-3550 Marburg/Lahn, Germany.

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$$\cup(\uparrow\downarrow)\cup(\uparrow\downarrow)\overset{\sigma}{\cup}(\uparrow\downarrow)\overset{\sigma'}{\cup}(\uparrow\downarrow)\cup\,,$$

Haldane's orbitals are denoted by $\cup$, and thus, in Haldane's language (the occupation representation for these orbitals), it translates into

$$\cup\cup\overset{\sigma}{\cup}\overset{\sigma'}{\cup}\cup\,.$$

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