INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

UNITED NATIONS INDUSTRIAL DEVELOPMENT ORGANIZATION
INTERNATIONAL CENTRE FOR SCIENCE AND HIGH TECHNOLOGY

SMR: 630/2

MINIWORKSHOP ON NONLINEARITY:
Dynamics of Surfaces in Nonlinear Physics

(13 - 24 July 1992)

"Dynamic Scaling in Surface Growth Phenomena"

presented by:

F. Family
Department of Physics
Emory University
Atlanta, GA 30322
U.S.A..

To appear in *Growth Patterns in Physical Sciences and Biology*, NATO ASI Series, edited by E. Louis, L. M. Sander, P. Meakin, and J. M. Garcia-Ruiz (Plenum, NY, 1992).

DYNAMIC SCALING IN SURFACE GROWTH PHENOMENA

Fereydoon Family

Department of Physics
Emory University
Atlanta, GA 30322, U. S. A.

INTRODUCTION

The most characteristic feature of many of the growth patterns that are formed in physical, chemical and biological processes is the existence of an evolving interface. Thus, in order to develop a better understanding of pattern formation in growth phenomena we need to study the structure and the dynamics of growing surfaces and interfaces. Recently considerable advances have been made [1] in this direction using the concept of dynamic scaling [2] which is based on the fact that growing surfaces exhibit non-trivial scaling behavior and naturally evolve to a steady-state without a characteristic time or spatial scale. Due to its generality, the dynamic scaling approach has become a standard language in the study of growing surfaces and interfaces [1]. In this review I will discuss some recent advances in theoretical, simulational and experimental studies of surface and interface growth phenomena using this approach.

DYNAMIC SCALING

If growth conditions are such that an evolving interface is neither stable nor unstable against perturbations, the resulting interface is a self-affine fractal which can be characterized by a single-valued functions $h(\mathbf{r}, t)$. The function $h(\mathbf{r}, t)$ gives the height of the interface at position $\mathbf{r}$ at time t measured from an initially flat $d-1$ dimensional surface at time $t = 0$. For multi-valued surfaces, $h(\mathbf{r}, t)$ is the maximum height of the surface at $\mathbf{r}$. We concentrate on a section of the surface having an extent L in $d-1$ dimensions perpendicular to the growth direction. As the surface evolves, the roughness or the width of the surface increases with time. The root mean-square of the height fluctuations $w(L, t)$ gives a quantitative measure of the surface width and is defined by

$$w(L, t) = [\langle h^2(\mathbf{r}, t)\rangle_r - \langle h(\mathbf{r}, t)\rangle_r^2]^{1/2}, \tag{1}$$

where $\langle \ldots \rangle_r$ denotes an average over $\mathbf{r}$. In the absence of a characteristic time in the growth process $w(L, t)$ increases with some power of time [2],

$$w(L, t) \sim t^{\beta}. \tag{2}$$

Within a region of length L the surface fluctuations cannot increase indefinitely but reach a steady-state with a constant value of the width which depends on L. These steady-state values, $w(L, t \rightarrow \infty)$, have a power law dependence on L and can be characterized by the exponent α [2],

$$w(L,t \rightarrow \infty) \sim L^{\alpha}. \tag{3}$$

The exponents α and β defined in (2) and (3) characterize the structure and time evolution of a growing surface, respectively. The dependence of $w(L,t)$ on t and L can be combined into the dynamic scaling form [2],

$$w(L,t) = L^{\alpha} f(t/L^{z}). \tag{4}$$

where the dynamic exponent z is defined by $z = \alpha/\beta$, and the scaling function $f(x)$ is a constant for $x >> 1$, and for $x << 1$ it has to be of the form $f(x) \sim x^{\beta}$. Similar scaling behavior can be observed in the height difference correlation function $c(r,t)$, defined by,

$$c(r,t) = \langle [\tilde{h}(r',t') - \tilde{h}(r+r',t'+t)]^2 \rangle_{r',t'}, \tag{5}$$

where $\tilde{h} = h - \langle h \rangle$ and $t' >> L^{z}$. Dynamic scaling implies that,

$$c(r,0) \sim r^{2\alpha}, \qquad \text{for } r << L, \tag{6}$$

and for fixed r and short times, $c(0,t)$ scales as,

$$c(0,t) \sim t^{2\beta}, \qquad \text{for } t << L^{z}. \tag{7}$$

The correlation functions $c(r,0)$ and $c(0,t)$ reach a constant steady-state value in the limits $r >> L$ and $t >> L^{z}$, respectively.

SURFACE GROWTH MODELS

In recent years a number of models have been proposed for describing growing surfaces [1]. Here, I describe two computer simulation models, the ballistic deposition [2, 4, 5] and Eden model [6-10], and the Kardar, Parisi and Zhang (KPZ) model [11,12] which is a nonlinear Langevin equation.

The ballistic deposition model has been used extensively to simulate surface growth by random deposition of molecules on cold substrates, which is a common experimental arrangement particularly in thin film growth. In this model particles follow a straight-line trajectory until they encounter a particle on the surface, or a particle in one of the nearest-neighbor columns. The surface of the ballistic deposition is a self-affine fractal and its evolution has been shown to be described by the dynamic scaling approach [2,4,5].

The Eden model was originally proposed [6] as a model of cell growth in biological systems. In the Eden model [6] every surface site, which are the nearest-neighbor perimeter sites of the cluster, can grow with equal probability. Thus, in each step of the simulation a randomly chosen perimeter site is occupied. In this way a compact pattern is formed which has a fluctuating interface. Surface properties of the Eden model have been extensively investigated using the dynamic scaling approach [7-10].

A phenomenological model based on a nonlinear Langevin type equation has been proposed by Kardar, Parisi and Zhang [11] for describing the evolution of growing surfaces and interfaces. Kardar *et al* have argued that the evolution of an interface can be described by the equation

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + (\lambda/2)|\nabla h|^2 + \eta(\mathbf{r},t) \tag{8}$$

where ν is the surface tension, the parameter λ is the growth velocity perpendicular to the interface, and the noise term $\eta(\mathbf{r},t)$ is assumed to be Gaussian with delta-function correlation:

$$\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = 2D\delta(\mathbf{r}-\mathbf{r}',t-t'). \tag{9}$$

The KPZ equation has been analyzed [11,12] by renormalization group methods in $d=2$, i.e. for a one dimensional interface, and the results are $\alpha = 1/2$, $\beta = 1/3$, and $z = 3/2$. In addition, on the basis of general arguments [4] it has been shown that,

$$\alpha + \alpha/\beta = 2. \tag{10}$$

Simulations of various surface growth models, such as the ballistic deposition [2,4,5] and the Eden model [7-10], are in excellent agreement with the exact results $\alpha = 1/2$, $\beta = 1/3$, and $z = 3/2$ in two dimensions. Similar exact results are not available for $d > 2$ from the KPZ equation, and the numerical simulation results for various models are somewhat controversial [1,5,13,14] . On the basis of numerical results several conjectures have been made for α and β in $d > 2$ [9,13,14]. In particular, the most recent calculations [10,15] appear to be in much better agreement with the Kim-Kosterlitz conjecture [14], $\alpha = 2/(d+2)$ and $\beta = 1/(d+1)$, than with the other conjectures [9,13].

NUMERICAL SOLUTION OF THE KPZ EQUATION

Since the values of α and β for $d \geq 3$ are controversial [1] we have carried out [15] a direct solution of the KPZ equation in $d=3$. We first perform a change of scale $h = y\sqrt{2D/\nu}$ and $t = \tau/\nu$ and write (8) in the form,

$$\frac{\partial y}{\partial \tau} = \nabla^2 y + \epsilon|\nabla y|^2 + \xi(\mathbf{r},\tau) \tag{11}$$

where the non-linearity parameter has been written as $\epsilon = \lambda^2 D/2\nu^3$, and the noise correlations as, $\langle \xi(\mathbf{r},\tau)\xi(\mathbf{r}',\tau')\rangle = \delta(\mathbf{r}-\mathbf{r}',\tau-\tau')$. We have integrated (11) on a discrete grid in $d = 2$, and 3 for $\epsilon = 1$, 2, 5, 10, and 25 using finite-difference methods. As a test of our simulation method, we duplicated the known exact results, $\alpha = 1/2$ and $\beta = 1/3$ in $d=2$. In addition, for the linear model ($\epsilon = 0$) we recovered the exact solution in $d=2$ and 3.

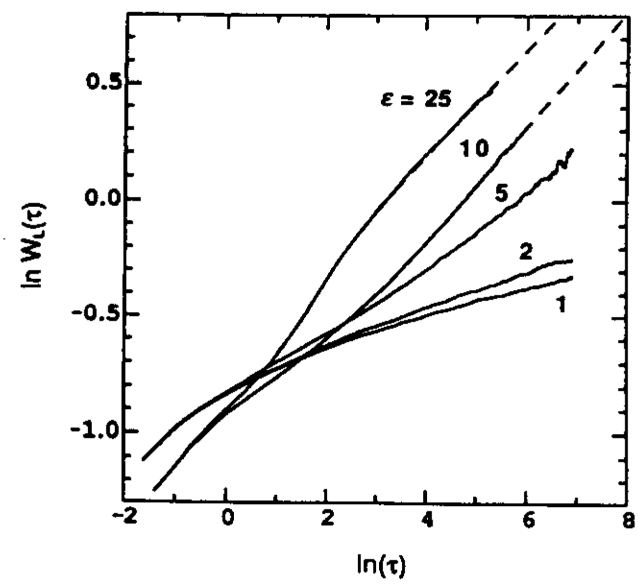

Fig. 1: Log-log plots of width $w_L(\tau)$ for the KPZ equation in $d=3$ for $\epsilon = 1$, 2, 5, 10 and 25.

Figure 1 shows our results [15] for the early-time growth behavior from our numerical solution of the non-linear equation in $d = 3$. For large ϵ ($\epsilon = 10$ and 25), β appears to be close to the value of $\beta = 1/4$ conjectured by Kim and Kosterlitz [14]. Similarly for the surface roughness exponent α we find 0.39 ± 0.01 for $\epsilon = 25$, which is also close to the Kim-Kosterlitz value $\alpha = 0.4$. For $\epsilon = 10$, the slope 0.37 ± 0.02 is somewhat below this value but appears to be increasing with increasing L.

Our results for the KPZ exponents in $d = 3$ are significantly closer to the conjecture of Kim and Kosterlitz [14] than to any of the other conjectures [9,13]. In this connection, recent simulations of the Eden model [10], ballistic deposition model [5] and directed polymers [16] in $d = 3$ appear to show an increase towards the Kim-Kosterlitz values as well. In addition, we see no evidence of a phase transition in $d = 3$ to logarithmic behavior for $\epsilon \geq 1$. While not completely ruling out the possibility of a transition [17, 18] , our results seem to support Tang, Nattermann and Forrest's idea [19] of a very long crossover region for small ϵ.

A NEW APPROACH TO SCALING IN SURFACE GROWTH

As indicated above, one of the most challenging questions in surface growth phenomena is an analytic determination of the exponents for the KPZ equation in arbitrary dimensions. Here we present one approach [20] which is not exact, but does enable us to determine the surface exponents for the KPZ equation [11]. This approach is quite general and we have shown that in addition to surface growth phenomena it can be applied to a wide variety of other nonequilibrium systems.

The key element in our approach [20] is the analogy between the Langevin-type equations in growth phenomena and the forced Navier-Stokes equation. Let us begin by noting that if all the transport coefficients such as ν, λ, D, in the KPZ equation depend on purely microscopic length-scales a; then on scales $l \gg a$ this equation describes the macroscopic behavior in the same manner as the Navier-Stokes equation describes turbulent flow. This suggests that modified and generalized versions of the type of scaling arguments introduced by Kolmogorov [21] in the context of turbulence might be useful as a method to identify the different scaling regimes observable in surface growth phenomena and other extended open dissipative systems.

Basically for any Langevin-type equation such as the KPZ equation to show scaling each separate term (including the noise), when coarse-grained over length scales l, must be of the same order of magnitude or negligible. Only under these circumstances can scaling behavior arise. To apply this concept to the KPZ equation, we assume that at long times $t \gg t_l$, the typical magnitude of the fluctuations in the interfacial height averaged over a length scale l scale as $\langle (h(\mathbf{r}+\mathbf{l},t) - h(\mathbf{r},t))^{1/2} \rangle_l \sim h_l$. This implies that $h_l \sim l^\alpha$. Then, apart from the noise, for times $t \gg t_l$ and averaged over scales l, the various terms in the KPZ equation may be estimated as $\langle |\partial h/\partial t| \rangle_l \sim h_l/t_l$, $\nu \langle |\nabla^2 h| \rangle_l \sim \nu h_l/l^2$, and $\lambda/2 \langle (\nabla h)^2 \rangle_l \sim \lambda h_l^2/l^2$.

To proceed further we need to estimate the average noise on these length and time scales. For white noise we estimate [20] its mean square fluctuations on length scales l and time scales t_l as $\eta_l \sim \sqrt{D/(S_l t_l)}$ where S_l is the average surface area of the interface on length scales l. This is a simple consequence of adding uncorrelated random variables. We estimate the surface area of the growth on length scales l as $S_l \sim (h_l^2 + l^2)^{(d-1)/2}$. In the limit $l >> h_l$ we have $\eta_l \sim \sqrt{D/l^{(d-1)} t_l}$. We have analyzed a variety of growth models using our scaling arguments and this form of the coarse-grained noise and have found the exact results in all cases.

In order to obtain the Kim-Kosterlitz conjecture for the KPZ exponents [14], however, we must assume [20] that the coarse-grained noise has the form $\eta_l \sim (D/h_l^{(d-1)} t_l)^{1/2}$. In addition, we assume that at sufficiently large length scales the nonlinear term in the KPZ equation will dominate the surface diffusion. Equating the $\partial h/\partial t$ term with the nonlinear term implies that a typical fluctuation lasts for time $t_l \sim l^2/\lambda h_l$. The scaling behavior of these two terms implies $\alpha + z = 2$. Equating our estimate for the noise fluctuation to the inertial term then yields

$$[illegible] \quad ([illegible]/\lambda)^{1/(d+2)} l^{2/(d+2)}, \tag{12}$$

and consequently $\alpha = 2/(d+2)$. The result $\beta = 1/(d+1)$ immediately follows from the scaling relation $\alpha + \alpha/\beta = 2$.

Thus we have derived theoretically the expressions conjectured by Kim and Kosterlitz for α and β. However, this result is clearly based on the specific form we used for the noise term [20]. As already mentioned, we have also used this approach [20] to study a number of other models and obtained the exact results in all cases. This approach is quite similar to those used in Flory theory for equilibrium systems and it should be useful in the study of a wide variety of non-equilibrium problems.

INTERFACE DYNAMICS IN FLUID FLOW IN POROUS MEDIA

We have experimentally studied [22] the evolution of the interface in quasi-two-dimensional displacement of viscous fluids in inhomogeneous media in order to test the dynamic scaling idea and to determine the exponents. The experimental setup [22] was a linear Hele-Shaw cell made of parallel plexiglass plates of size 24 cm×100 cm. To produce a porous medium we packed 220 μm diameter glass beads between the plates. The beads were spread randomly and homogeneously in one layer and glued to the lower plate.

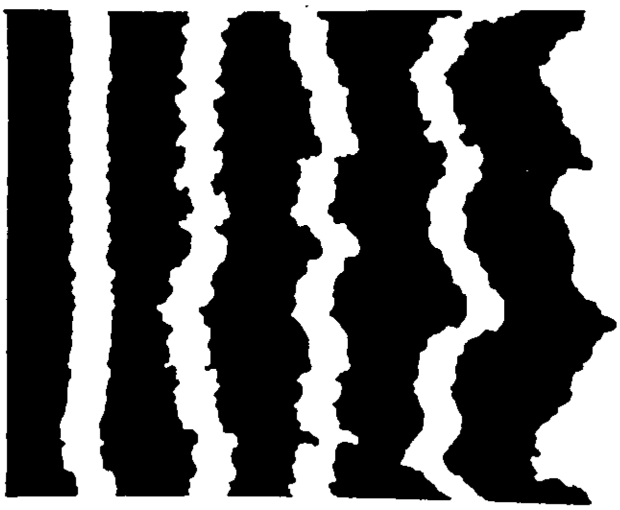

Fig. 2: The evolution of the fluid-air interface, moving from left to right, is shown by plotting the digitized image of the meniscus at different times.

The upper plate was placed directly on the beads and iron rods and clamps were used to prevent the lifting of the plates. Colored glycerine with 4 vol% of water was injected at a fixed flow rate into air between the plates along a line at a shorter sidewall. The viscosity of the glycerine was ~180 cP and the air-liquid surface tension was about ~65 dyn/cm.

The evolution of the interface was recorded on a videotape and digitized with 768×620 spatial resolution. Each digitized image was saved as an array of height values and this process was continued until all images in a run were recorded. We used the average height

$\bar{h} = \langle h(x,t)\rangle_t$ as the time scale in the calculations, because $\bar{h}$ is proportional to the time due to the constant flow velocity and the absence of holes in the region occupied by the more viscous fluid. Fig. 2 shows the development of the fluid-air interface at different times. Clearly the initially smooth interface roughens as it moves into the porous medium.

The dynamic scaling of the growing interface is demonstrated by the data [22] displayed in Fig. 3, where the logarithm of the square root of the height-height correlation function for $x = 0$ is plotted versus $\ln t$. The slope gives $\beta \simeq 0.65$. We have also studied [22] the spatial scaling of the correlations by plotting the logarithm of the square root of the correlation function against $\ln x$. The slope of the straight line fitting the initial part of the data gave [22] $\alpha \simeq 0.81$. These values are different from $\alpha = 1/2$ and $\beta = 1/3$ obtained in various models of interface growth [11]. Our values of α and β are consistent with the scaling relation $\alpha + \alpha/\beta = 2$.

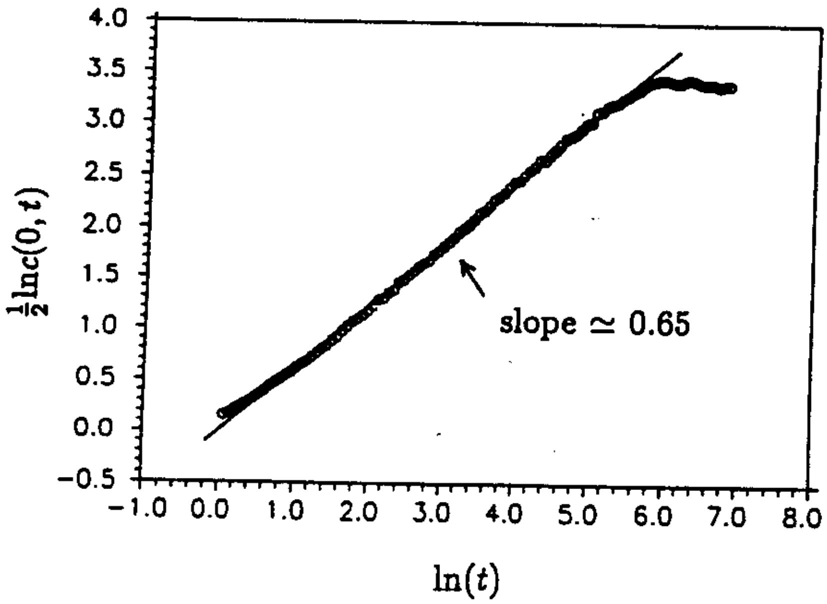

Fig. 3: The logarithm of the square root of the height-height correlation function for $x = 0$ is plotted versus $\ln t$. The slope gives $\beta \simeq 0.65$.

The non-universality of our results is most likely due to the fact that the fluctuations of the interface are not governed by a Gaussian noise [12]. One possibility is that there exists temporal or spatial correlation in the noise. This has been shown to lead to non-universality by Medina et al [12] and Amar, Lam and Family [23], but the results indicate that $\alpha > 2/3$ and $\beta > 1/2$ is impossible for translationally invariant spatially correlated noise. In order to account for the anomalous exponents, Zhang [24] has proposed that the amplitude of the noise is distributed according to a power law instead of a Gaussian distribution. In the next section we discuss the results of the Zhang model and we will compare them with the experimental findings.

SURFACE GROWTH WITH POWER-LAW NOISE

Zhang [24] has suggested that the anomalous exponents arise from the fact that the amplitude of the random noise in the experiments [22] has a non-Gaussian, power law distribution of the form

$$P(\eta) \sim \frac{1}{\eta^{1+\mu}} \quad for \quad \eta > 1 \;\; ; P(\eta) = 0 \;\; otherwise, \tag{13}$$

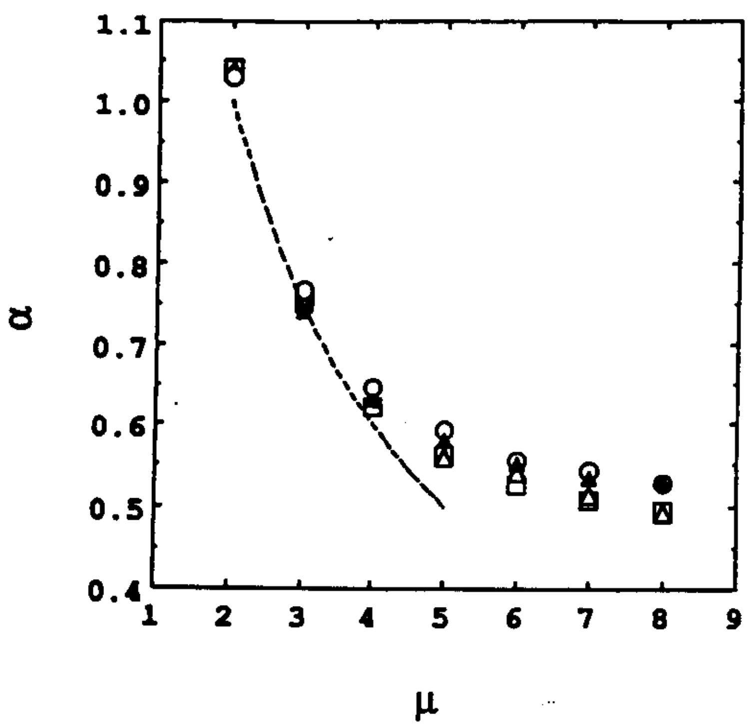

Fig. 4: Variation of the exponent α with μ in the Zhang model and variants in two dimension. Dashed line is prediction of the Flory formula.

where η is the delta-correlated noise. We have carried out [25,26] extensive simulations of two different variants of the Zhang model as well as a ballistic deposition model with power-law noise in order to study the effects of power law noise in surface growth.

Figure 4 summarizes our results [25] for the exponent α for all three models, for $2 \leq \mu < 7$. We also find [25] that the relation $\alpha + z = 2$ is approximately satisfied by all three models and therefore it is reasonable to assume that all 3 models belong to the KPZ universality class. These results indicate that for $\mu < 7$, the exponents α and β for all three models remain clearly above the Gaussian values $\alpha = 1/2$, $\beta = 1/3$. However at $\mu = 8$, for both versions of the Zhang model [24], we obtained results [25,26] which are essentially indistinguishable within error bars from the Gaussian values. We note that our results for α and β for all three models are somewhat higher than the Flory-type formula [27] $\alpha = (1 + d)/(\mu + 1)$ which predicts $\mu_c = 5$ in $d = 2$. We note that when a cutoff is introduced in the power law distribution, the exponents are observed to crossover to the known Gaussian values [26]. We also find [25] power-law distribution of the height fluctuations which should be useful in testing the existence of power-law noise in experiments.

NOISE DISTRIBUTION IN FLUID FLOW IN POROUS MEDIA

We have directly determined [28] the distribution of the noise in experiment on two-phase flow in porous media in order to determine whether the Zhang model [24] does or does not apply to the experiments. In all our calculations we used the digitized images and additional averaging was made over independent experiments with the same system. We applied several different definitions of noise [28] in order to extract the noise distribution from the digitized interfaces. The basic approach was to determine the distribution of amplitudes between two successive profiles separated by a short time interval.

The resulting noise distribution [28] averaged over the experiments is shown in Fig. 5 with circles. Our experimental data points [28] can be fitted by a straight line on log-log

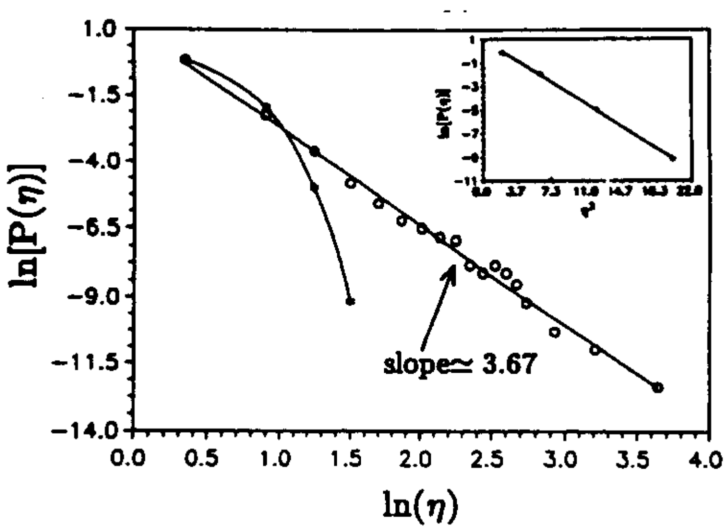

Fig. 5: The logarithm of the noise distribution $P(\eta)$ in the experiment is plotted against the logarithm of the noise amplitude η. In the inset the logarithm of $P(\eta)$ is plotted against η^2 for a computer model, demonstrating that the noise distribution in the simulations is Gaussian.

plot indicating the algebraic decay of noise amplitudes. The exponent of the corresponding power-law behavior $P(\eta) = c\eta^{-(1+\mu)}$ is $\mu = 2.67 \pm 0.19$. From figure 4 we see that for $\mu = 2.67 \pm 0.19$ the value of $\alpha = 0.81 \pm 0.02$ in close agreement with the experimental result [22] of $\alpha \simeq 0.81$. This agreement suggests that in fact the anomalously large values of the exponents found in various experiments could be due to the presence of a power-law noise distribution of the form suggested by Žhang [24]. Clearly this possibility should be tested for other experiments.

UNIVERSAL SCALING IN SURFACE GROWTH

Despite the great activity [1] in determining the exponents from discrete models and continuum equations, there has been no rigorous demonstration of the relation between the KPZ equation and these models. Thus, a more detailed study of the scaling behavior of the KPZ equation would be helpful in establishing a connection between the discrete models and the continuum description.

We have derived [29] expressions for the scaling behavior of the asymptotic coefficients C_t ($C_t = w(\infty, t)/t^{1/3}$) and C_L ($C_L = w(L, \infty)/L^{1/2}$) as a function of the hydrodynamic parameters λ, D, and ν from the scaling properties of the KPZ equation in $d = 2$. Our scaling analysis predicts the existence of a universal scaling function as well as universal amplitude ratios in $d = 2$. We have tested these predictions by simulations of three different surface growth models and by a mode-coupling calculation [29, 30]. Similar results have been derived for the correlation function $c(r, t)$ in the steady-state limit [30].

The scaling behavior of C_t and C_L as a function of D, ν, and λ may be derived as follows. We perform a scale change to a parameterless equation with *dimensionless* variables, $h' = (\lambda/2\nu)\, h$, $\tau = (\lambda^4 D^2/4\nu^5)t$, $x' = (\lambda^2 D/2\nu^3)x$, and $\xi = (2\nu^4/\lambda^3 D^2)\eta$, so that (8) may be rewritten in $d = 2$ as,

$$\partial h'/\partial \tau = \nabla'^2 h' + |\nabla' h'|^2 + \xi(x', \tau) \tag{14}$$

where $< \xi(x_1', \tau_1)\xi(x_2', \tau_2) >= \delta(x_1' - x_2')\delta(\tau_1 - \tau_2)$. If we write $w'(L', \tau) = g(L', \tau)$, then transforming back to x, h and t we obtain a universal scaling form for the surface width,

$$w(L,t) = L^{1/2} C_L \; F(|\lambda| C_L \; t/L^{3/2}) \tag{15}$$

where $F(x)$ is a universal scaling function. Similarly, we can define [29] a universal amplitude ratio R as,

$$R = C_t/(|\lambda| C_L^4)^{1/3} \tag{16}$$

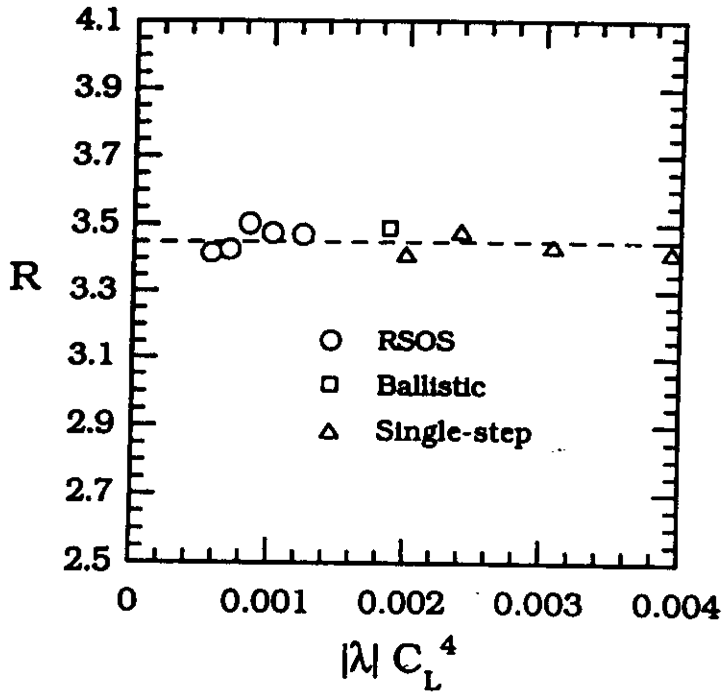

Fig. 6: Universal amplitude ratio R as a function of the scaling parameter λC_L^4 is shown for three different growth models. Average value (dashed line) is 3.45 ± 0.05.

In order to verify our predictions we simulated three different growth models in the KPZ universality class in $d = 2$. Fig. 6 shows a summary of our results [29, 30] for the amplitude ratio R plotted as a function of the scaling parameter λC_L^4 as well as the driving force f for each model. Average value (dashed line) is 3.45 ± 0.05. We also numerically solved the KPZ equation in $d = 2$ from which we obtained $R \simeq 3.2$ in approximate agreement with this value.

In order to test the universality of the scaling function in (15) we also determined $w(L,t)$ for two models. Fig. 7 shows a scaling plot of our results [29, 30]. The asymptotic scaling functions for both models are essentially identical when scaled in the form of (15) as predicted.

The universal amplitude ratio R and the scaling function $F(x)$ were also calculated using a mode-coupling approximation [30]. The solid line in figure 7 shows the mode-coupling results obtained for the scaling function, which is in good agreement with our simulation results.

We note that the same scaling analysis that we have applied to the KPZ equation may be applied to a variety of other models of surface growth at and below their critical dimension d_c, including models with power-law noise. We are currently carrying out simulations on different models with power-law noise in order to test our predictions.

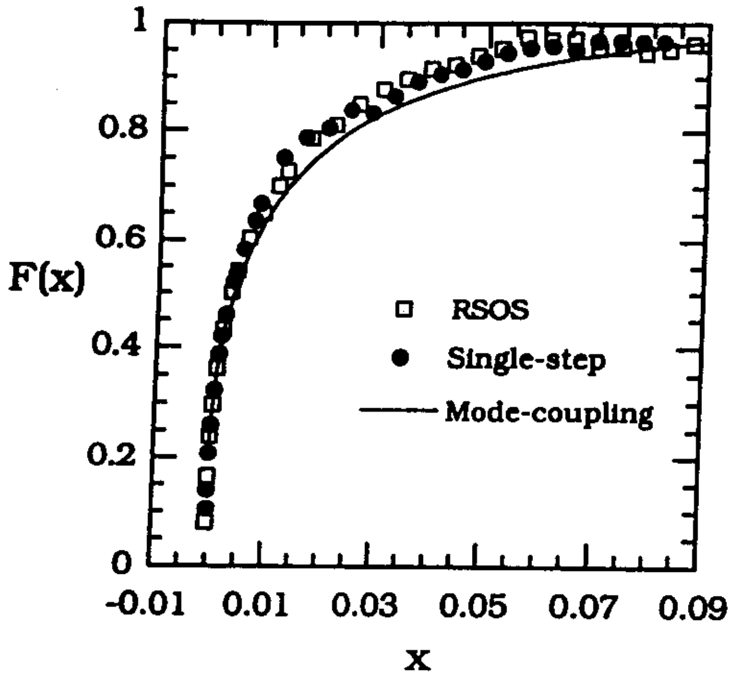

Fig. 7: Scaling function $F(x)$ defined in (16) for two different discrete models. The solid line is result of mode-coupling calculation.

CONCLUSIONS

Studies of the scaling properties of rough surfaces and interfaces provide an effective method for describing the evolution and the structure of growth patterns in physical, chemical and biological processes. The dynamic scaling approach has become a standard tool in analyzing the data from simulations, analytical theories and experiments on growing surfaces. Although considerable progress has been made in recent years in understanding surface growth phenomena, many unresolved questions still remain. For example, only a single experiment has been carried out that has directly investigated the dynamic scaling behavior of a two-dimensional interface. The idea of universal scaling has only recently been developed and could provide an important tool fin understanding the relation between experiments and microscopic models. Finally, both the results of the experiments and discrete models indicate that in order to fully understand surface growth phenomena a much clearer understanding of the nature of the noise must be developed.

ACKNOWLEDGEMENTS

I would like to thank Jacques Amar, Viktor Horváth, Tamás Vicsek and Pui-Man Lam for their collaborations on some of the works discussed here. I would also like to thank Jacques Amar for his helpful comments on the manuscript. This research was supported by the Office of Naval Research and the Petroleum Research Fund administered by the American Chemical Society.

REFERENCES

1. F. Family and T. Vicsek, eds., *Dynamics of Fractal Surfaces* (World-Scientific, Singapore, 1990).
2. F. Family and T. Vicsek, *J. Phys. A* **18**, L75 (1985).
3. B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
4. P. Meakin, P. Ramanlal, L. M. Sander and R. C. Ball, *Phys. Rev. A* **34**, 5091 (1986).
5. F. Family, *Physica A* **168**, 561 (1990).
6. M. Eden, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume IV: Biology and Problems of Health*, edited by J. Neyman (Univ. of Calif. Press, Berkeley, 1961), pp. 223-239.
7. R. Jullien and R. Botet, *J. Phys. A* **18**, 2279-2287 (1985).
8. D. Stauffer, and J. G. Zabolitzky, *Phys. Rev. Lett.* **57**, 1809 (1986); *Phys. Rev. A* **34**, 1523 (1986).
9. J. Kertész and D. E. Wolf, *J. Phys. A* **21**, 747 (1988); *Europhys. Lett.* **4**, 651 (1989).
10. P. Devillard and H. E. Stanley, *Physica A* **160**, 298 (1989)
11. M. Kardar, G. Parisi and Y.-C. Zhang, *Phys. Rev. Lett.* **56**, 889 (1986).
12. E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang, *Phys. Rev. A* **39**, 3053 (1989).
13. T. Halpin-Healy, *Phys. Rev. A* **42**, 711 (1990).
14. J. M. Kim and J. M. Kosterlitz, *Phys. Rev. Lett.* **62**, 2289 (1989).
15. J. G. Amar and F. Family, *Phys. Rev. A* **41**, 3399 (1990).
16. J. M. Kim, M. A. Moore and A. J. Bray, *Phys. Rev. A* **44**, 2345 (1991).
17. H. Yan, D. Kessler and L. M. Sander (1990), *Phys. Rev. Lett.* **64**, 926 (1990).
18. Y. P. Pellegrini and R. Jullien, *Phys. Rev. Lett.* **64**, 1745 (1990).
19. L.-H. Tang, T. Nattermann and B. M. Forrest, *Phys. Rev. Lett.* **65**, 2422 (1990); J. M. Kim, A. J. Bray and M. A. Moore, , *Phys. Rev. A* **44** R4782 (1991).
20. H. G. E. Hentschel and F. Family, *Phys. Rev. Lett.* **66**, 1982 (1991).
21. A. N. Kolmogorov, A. N., *Dokl. Akad. Nauk. SSSR* **30**, 299 (1941).
22. V. K. Horváth, F. Family and T. Vicsek *J. Phys. A* **24**, L25 (1991).
23. J. G. Amar, P.-M. Lam and F. Family, *Phys. Rev. A* 43, 4548 (1991).
24. Y. C. Zhang, *J. de Physique* 51, 2129 (1990).
25. J. G. Amar, and F. Family, *J. Phys. A* **24**, L79 (1991).
26. J. G. Amar, J. G. and F. Family, *J. de Physique I* **1**, 175 (1991).
27. Y. C. Zhang, Y.-C., *Physica A* **170**, 1 (1990).
28. V. K. Horváth, F. Family and T. Vicsek *Phys. Rev. Lett.* **67**, 3207 (1991).
29. J. G. Amar and F. Family, *Phys. Rev. A* **45** (Rapid Communications), *in press* (1992).
30. J. G. Amar and F. Family, *Phys. Rev. A* **45** , *in press* (1992).