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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Arakelov Theory - the basic ideas

U. Jannsen
Mathematisches Institut
Universität zu Köln
Weyertal 86-90
5000 Köln 41
Germany

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 224559 TELEX 460392

Arakelov theory - the basic ideas

(Trieste, August 31, 1992)

Uwe Jannsen

In this talk we concentrate on the theory of arithmetic surfaces and don't discuss the higher-dimensional theory. In fact first consider the one-dimensional situation.

I Curves

A) Geometric curves

Let X be a smooth projective curve over an algebraically closed field k. Recall the following basic concepts and results from the theory of curves.

divisors: formal linear combinations $D = \sum\limits_{\substack{P \in X \\ \text{closed}}} n_P P$,

$n_P \in \mathbb{Z}$, almost all zero

degree: $\deg D = \sum n_P \in \mathbb{Z}$

principal divisors: $f \in K(X)^*$ meromorphic function, then

$\operatorname{div}(f) = \sum v_P(f) P$, $v_P(f)$ = order of zero of f at P

Since X is projective ("compact"), one has the fundamental relation

$$(*) \qquad \sum_P v_p(f) = 0 ,$$

i.e., principal divisors have degree zero.

The Riemann-Roch problem is: compute the dimension of

$$\{ f \in K(X)^\times \mid \operatorname{div}(f) \geq -D \}$$

$$= H^0(X, \mathcal{O}(D)) .$$

<u>Recall</u>

$$\frac{\operatorname{Div}(X)}{\text{principal divisors}} \cong \operatorname{Pic}(X) = \left\{ \begin{array}{l} \text{isomorphy} \\ \text{classes of} \\ \text{line bundles} \end{array} \right\}$$

$$D \longmapsto \mathcal{O}(D)$$

$$\operatorname{div}(s),\ s \text{ meromorphic section} \longleftarrow \mathcal{L}$$

(Hence have $\deg : \operatorname{Pic}(X) \to \mathbb{Z}$).

<u>Riemann-Roch</u>:

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^1(X, \mathcal{O}(D)) = \deg D + 1 - g$$

(g = genus of X)

<u>Serre duality</u>: $\dim H^1(X, \mathcal{O}(D)) = \dim H^0(X, \mathcal{O}(K-D))$

K = canonical divisor $\Longleftrightarrow$ class of $\Omega^1_{X/k}$

The final result is

$$\dim H^0(X, \mathcal{O}(D)) = \begin{cases} 0 & \deg D < 0 \quad \underline{\text{vanishing theorem}} \\ 0 & \text{unless } \mathcal{O}(D) \cong \mathcal{O}_X \\ \deg D + 1 - g & \deg D \gg 0 \end{cases}$$

B) Arithmetic curves

$\mathcal{O}_K$ = ring of integers in a number field K.

(e.g., $\mathbb{Z} \subset \mathbb{Q}$)

For $f \in K^\times$ (a "meromorphic function on $\mathcal{O}_K$") the analogue of the relation $(*)$ is the product formula of number theory

$$(*)' \qquad \prod_{\mathfrak{p}} |f|_{\mathfrak{p}} \prod_{v|\infty} \|f\|_v = 1$$

where $\mathfrak{p}$ runs through the (maximal) prime ideals of $\mathcal{O}_K$ and $v|\infty$ through the archimedean places

(e.g., $\prod_p |f|_p \cdot |f|_\infty = 1$ for $f \in \mathbb{Q}$).

Additively, $(*)'$ reads as

$$\sum_{\mathfrak{p}} v_{\mathfrak{p}}(f) \log N\mathfrak{p} - \sum_{v|\infty} \log \|f\|_v = 0 .$$

Although this is well-known and this analogy was pursued quite far by Weil, the following notions were in this form only introduced by Arakelov (following some ideas of Parshin).

<u>Arakelov divisor</u>: $$D = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \, \mathfrak{p} + \sum_{v | \infty} \lambda_v \, v \, ,$$

$$n_{\mathfrak{p}} \in \mathbb{Z} \, , \text{ a.a. zero} \, , \quad \lambda_v \in \mathbb{R}$$

<u>Arakelov degree</u>: $$\overline{\deg} \, D = \sum n_{\mathfrak{p}} \log N\mathfrak{p} + \sum \lambda_v \in \mathbb{R}$$

<u>A.-principal divisor</u>: $$\overline{\operatorname{div}} (f) = \sum v_{\mathfrak{p}}(f) \, \mathfrak{p} - \sum \log \|f\|_v \, v$$

$$\frac{\operatorname{Div}(\overline{\mathcal{O}}_K)}{\text{principal divisors}} \cong \operatorname{Pic}(\overline{\mathcal{O}}_K) = \left\{ \begin{array}{l} \text{isometry} \\ \text{classes of} \\ \text{metrized} \\ \text{line bundles} \end{array} \right\}$$

Analogues of Riemann-Roch etc. exist and gets a reformulation and a new look at classical results of number theory, especially of Minkowski's geometry of numbers.

II Surfaces

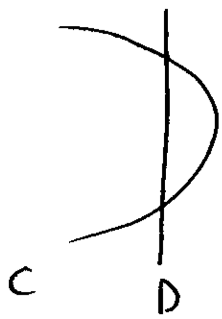
A) Geometric surfaces

X smooth projective surface over an algebraically closed field k.

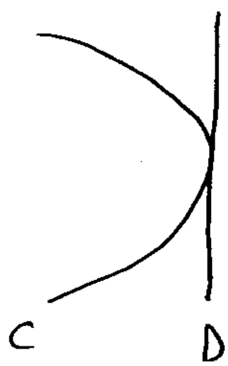
divisor : $$D = \sum_{\substack{C \subset X \\ \text{closed, integral curve}}} n_C \, C \quad , \; n_C \in \mathbb{Z} \, , \text{ a.a. zero}$$

The new feature now is : one can intersect curves on surfaces, i.e., define an intersection number

$$C_1 . C_2 = \sum_{\substack{P \in X \\ \text{closed points}}} (C_1 . C_2)_P \in \mathbb{Z} .$$



$C . D = 2$



$C . D = 2$.

Hence an intersection product for divisors, by linear extension.

Essential fact $\quad f \in K(X)^* \Rightarrow \operatorname{div}(f) . D = 0 \quad \forall D .$

Hence get a bilinear pairing

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z} .$$

<u>Global formula</u>

$C \subset X$ irreducible curve. Then

$$C.D = \deg_C \left(\mathcal{O}(D)\big|_C \right) .$$

<u>Riemann-Roch</u> Let $\chi(D) = \sum_{i=0}^{2} (-1)^i \dim_k H^i(X, \mathcal{O}(D))$.

$$\chi(D) = \tfrac{1}{2} D.(D - K) + \chi(\mathcal{O}) ,$$

$$\chi(\mathcal{O}) = \tfrac{1}{12} \left(K.K - c_2(-K) \right) = \tfrac{1}{12} (c_1^2 - c_2) ,$$

K = canonical divisor $\longleftrightarrow$ class of $\Omega^2_{X/k}$

$c_i = c_i(-K)$ = i-th Chern class

<u>vanishing theorem</u> $H^0(X, \mathcal{O}(D)) = 0$ if $D.H < 0$ for some ample divisor H

<u>Hodge index theorem</u> $D.H = 0$, $\exists$ E with $D.E \neq 0$ $\Rightarrow$ $D^2 < 0$

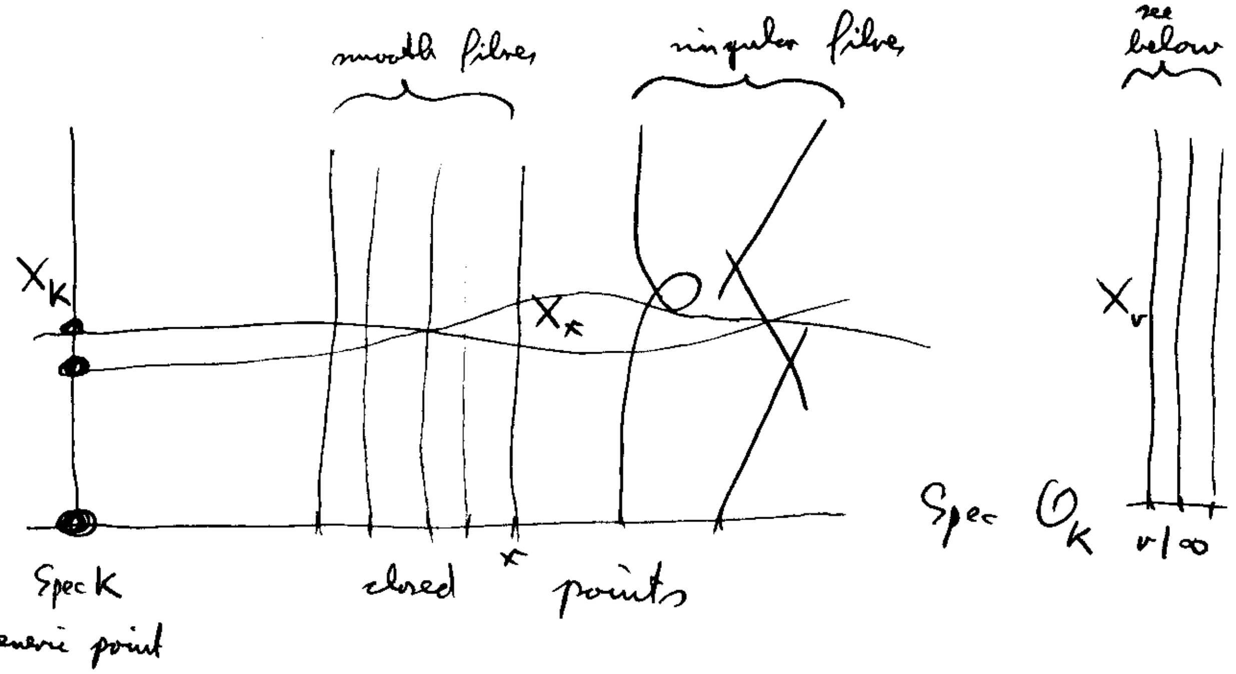
<u>ampleness criterion</u> D ample $\Leftrightarrow$ $D^2 > 0$ and $D.C > 0$ for all irreducible curves $C \subset X$.

If char $k = 0$, then one has the Bogomolov-Miyaoka-Yau inequality: if X is of general type (in the sense of classification of surfaces), then

$$c_1^2 \leq 3 c_2 .$$

B) Arithmetic surfaces

If one considers a curve X_K over a number field K and rational points on this curve, then it is very useful to introduce some notion of integrality and ~~to~~ treat the points more geometrically. This is both achieved by spreading out X_K over the ring of integers $\mathcal{O}_K$ ("find equations over $\mathcal{O}_K$"), ~~both~~ and by considering the closures of the rational points.



To do this correctly, one needs the general (= Grothendieck's) language of algebraic geometry (schemes rather than varieties).

<u>Definition</u> An arithmetic surface is a regular, projective, flat scheme X over $\mathcal{O}_K$ such that the generic fibre $X_K = X \otimes_{\mathcal{O}_K} K$ is a smooth curve over K.

X has dimension 2, hence the term surface. X will not in general be smooth over $\mathcal{O}_K$, there may be finitely many singular fibres.
To get the analogue of a projective ("compact") geometric surface, one has to "do something at infinity".
Historically, this theory was developed as follows:

Arakelov (Parshin)	metrised line bundles intersection theory
Hriljac, Faltings	Riemann-Roch
Bismut, Gillet, Soulé Faltings	generalisations to higher dimensions
Vojta, Faltings	applications to diophantine questions (Mordell conjecture, points on abelian varieties/K, ...)

To describe the definition, assume that $K = \mathbb{Q}$, for simplicity (this is no restriction for the following, since we can always regard X_K as a curve over $\mathbb{Q}$ (not necessarily connected)). Then $\{v|\infty\} = \{\infty\}$.

Let $X_\infty = X \times_{\mathbb{Z}} \mathbb{C} = X_{\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{C}$. This is a smooth projective curve over $\mathbb{C}$, i.e., corresponds to a Riemann surface. Choose a Kähler form μ_∞ (e.g., the "canonical" one) on X_∞, invariant under complex conjugation (= "real")

<u>Arakelov divisor</u>

$$D = \sum_{\substack{C \subset X \\ \text{closed, integral,} \\ \text{dimension } 1}} n_C \, C + \lambda_\infty X_\infty ,$$

$n_C \in \mathbb{Z}$, a.a. zero, $\lambda_\infty \in \mathbb{R}$, where

X_∞ is used as a formal symbol

<u>principal divisor</u> of $f \in K(X)^\times = K(X_K)^\times$:

$$\overline{\mathrm{div}}(f) = \underbrace{\mathrm{div}(f)}_{\text{classical}} + \lambda_\infty \infty ,$$

$$\lambda_\infty = - \int_{X_\infty} \log |f| \, \mu_\infty$$

<u>metrized line bundle</u> on X : line bundle $\mathcal{L}$ on X

(= locally free $\mathcal{O}_X$-module of rank 1) with a "real" metric $\| \ \|_\infty$ on $\mathcal{L}|X_\infty$.

<u>admissible metric</u> : if $\underbrace{\text{curve}_{\mathcal{L}, \| \ \|_\infty}}_{\text{curvature}} = \lambda \mu_\infty$, $\lambda \in \mathbb{R}$

<u>Remark</u> Hodge theory $\Rightarrow$ admissible metrics only differ by a scalar

<u>Proposition</u> $\dfrac{\text{Arakelov divisors}}{\text{principal divisors}} \cong \mathrm{Pic}(\overline{X}) = \left\{ \begin{array}{l} \text{isometry classes} \\ \text{of line bdls} \\ \text{with admiss.} \\ \text{metric} \end{array} \right\}$

$D = D_f + D_\infty \longmapsto \mathcal{O}(D_f)$ with metric from D_∞

Essential point : given D_f construct a canonical admissible metric $\| \ \|_{D_f, \mathrm{can}}$ on $\mathcal{O}(D_f)$: by Hodge theory

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<u>intersection product</u> can define $D_1 \cdot D_2 \in \mathbb{R}$

1) $D_1 \subset X$ irreducible ~~...~~ "curve" (closed integral subscheme of dimension 1), i.e.

vertical = irred. component of closed fibre
or = curve over a finite field
horizontal = closure of closed point of $X_{\mathbb{Q}}$
= order (not necessarily = maximal order) in $\mathcal{O}_L$, $L/\mathbb{Q}$ finite extension

Then

$$D_1 . D_2 = \deg_{D_1} \left(\mathcal{O}(D_2)\big|_{D_1} \right) .$$

(for vertical D_1, $\deg_{D_1} = \log \# \kappa(x) \cdot \deg^{\text{geometric}}_{D_1}$, if $D_1 \subset X_x$; for horizontal D_1, $\mathcal{O}(D_2)\big|_{D_1}$ is a metrized line bundle and $\deg_{D_1}$ is the Arakelov degree)

2) $\overline{\mathrm{div}}(f) \; D_2 = 0$

3) $$D_1 . D_2 = \sum_{\substack{P \in X \\ \text{closed}}} \underbrace{(D_1 . D_2)_P}_{\text{classical}} + (D_1 . D_2)_\infty$$

$(D_1 . D_2)_\infty = - \log G(P_1, P_2)$, if D_1, D_2 both horizontal, corresponding to closed points P_1, P_2 of $X_{\mathbb{Q}}$, $G(P, Q)$ = Green's function on X_∞

<u>Riemann - Roch</u>

1) Put a canonical metric on $\omega = \omega_{X/\mathcal{O}_K}$ = canonical sheaf, by putting a metric on $\Omega^1_{X_K/K}$:

$\delta : X_K \hookrightarrow X_K \times X_K$ diagonal

$$\Omega^1_{X_K} = \delta^*(\underbrace{\mathcal{O}_{X_K\times X_K}(-X_K)}_{\text{canonical metric by :}})$$

Green's functions / flat metric on Jacobian

2) <u>relative Riemann - Roch</u> : define

$$\chi(X/\mathcal{O}_K, \underset{\substack{|\\ \text{admissible metrics}}}{\mathcal{L}}) = \text{"}\chi(\cancel{H^0(X,\mathcal{L}) -} H^1(X,\mathcal{L}))\text{"}$$

$$= \chi\left(\frac{\det H^0(X,\mathcal{L})}{\det H^1(X,\mathcal{L})}\right)$$

$f: X \to \operatorname{Spec} \mathcal{O}_K$

$$= \chi(\underbrace{\det Rf_*\mathcal{L}}_{\text{Knudsen-Mumford determinant}})$$

Faltings / Quillen metric on it

again theory of Jacobians

<u>R.R.</u>

$$\chi(X/\mathcal{O}_K, \mathcal{L}) = \tfrac{1}{2}\mathcal{L}.(\mathcal{L} - \omega) + \chi(X/\mathcal{O}_K, \mathcal{O}_X)$$

$$\chi(X/\mathcal{O}_K, \mathcal{O}_X) = \frac{1}{12}(\omega.\omega - \Sigma\,\delta_v)$$

$$\delta_v = \begin{cases} \text{measure of singularity} & v \nmid \infty \\ \underbrace{\text{Faltings invariant}}_{\Updownarrow\ \text{moduli space of curves}} & v \mid \infty \end{cases}$$

vanishing theorems
Hodge index theorem
ampleness criterion
} analogues exist

III Applications to diophantine problems

Principal aim: use Arakelov theory to get finiteness results

For example:

a) Faltings used the ideas and methods of Arakelov theory to prove the
Mordell conjecture: if C is a smooth projective curve of genus $g \geq 2$ over a number field K, then C has only finitely many rational points

b) Parshin observed: if one had a suitable analogue of the Bogomolov-Miyaoka-Yau inequality, for arithmetic surfaces, one could derive highly interesting consequences, e.g.,
an effective proof of the Mordell conjecture (e.g., an effective bound for $C(K)$ in a)),

a proof of the <u>Fermat conjecture</u>.

Always, an essential tool is the relation between metrised line bundles and heights.

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Appendix: The dictionary between geometric and arithmetic curves

Geometric curves		Arithmetic curves
X smooth, projective curve over an algebraically closed field k	objects	$X = \mathrm{Spec}\, \mathcal{O}_K$, $\mathcal{O}_K$ = ring of integers in an algebraic number field K (e.g., $\mathbb{Z} \subset \mathbb{Q}$)
$f \in K(X)^\times$	meromorphic function	$f \in K^\times$
$\sum_{\substack{P \in X \\ \text{closed}}} v_P(f) = 0$ ($v_P(f)$: zero order of f at P)	closedness "compactness" ($\deg \mathrm{div}(f) = 0$)	$\prod_{\mathfrak{p}} \Vert f\Vert_{\mathfrak{p}} \cdot \prod_{v \mid \infty} \Vert f\Vert_v = 1$ ($\mathfrak{p}$ max'l ideal) (e.g., $f \in \mathbb{Q}^\times$: $\prod_p \vert f\vert_p \cdot \vert f\vert_\infty = 1$) $\sum_{\mathfrak{p}} v_{\mathfrak{p}}(f) \log N\mathfrak{p} - \sum_{v \mid \infty} \log \Vert f\Vert_v = 0$
$\sum_P n_P P$, $n_P \in \mathbb{Z}$, a.a. zero $\sum n_P \in \mathbb{Z}$	divisor and its degree	Arakelov divisor: $\sum_{\mathfrak{p}} n_{\mathfrak{p}} \mathfrak{p} + \sum_{v \mid \infty} \lambda_v v$ $n_{\mathfrak{p}} \in \mathbb{Z}$, a.a. zero, $\lambda_v \in \mathbb{R}$ $\sum n_{\mathfrak{p}} + \sum \lambda_v \in \mathbb{R}$
$\sum_P v_P(f) P = \mathrm{div}(f)$	principal divisor	$\sum_{\mathfrak{p}} v_{\mathfrak{p}}(f) \mathfrak{p} - \sum_{v \mid \infty} \log \Vert f\Vert_v \, v = \overline{\mathrm{div}}(f)$

$\dfrac{\mathrm{Div}(X)}{\text{princ. divisors}} \cong \mathrm{Pic}(X)$ isomorphy classes of line bundles $D \longmapsto \mathcal{O}(D)$ $\mathrm{div}(s) \longleftarrow \mathcal{L}$ merom. section	divisors and line bundles	$\dfrac{\mathrm{Div}(\overline{O}_K)}{\text{princ. divisors}} \cong \mathrm{Pic}(\overline{O}_K)$ isometry classes of metrized line bundles (line bundle $\mathcal{L} \Leftrightarrow L$ projecti[ve] O_K-module of rank 1 metric $\Leftrightarrow$ norm on $L \otimes \mathbb{R}$)
$\{ f \in K(X)^* \mid \mathrm{div}(f) \geq -D \}$ $= H^0(X, \mathcal{O}(D))$	global sections	analogous def'n with $\overline{\mathrm{div}}$
$\chi(X, \mathcal{L}) = \dim H^0(X, \mathcal{L}) - \dim H^1(X, \mathcal{L})$	Euler-Poincaré characteristic	$\chi(X, \mathcal{L}) = -\log \mathrm{vol}\left(L \otimes \mathbb{R} / L\right)$ metrized, $\Leftrightarrow L$
$\chi(X, \mathcal{L}) = \deg \mathcal{L} + \chi(X, \mathcal{O}_X)$ $\chi(X, \mathcal{O}_X) = 1 - g$ genus of X	Riemann-Roch	analogues exist and give reformulation of classical algebraic number theory
$H^1(X, \mathcal{L})^\vee \cong H^0(X, \Omega^1_{X/k} \otimes \mathcal{L}^{-1})$	Serre duality	
$H^0(X, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$ $\left(\Rightarrow H^1(X, \mathcal{L}) = 0 \text{ if } \deg \mathcal{L} > \deg \Omega^1_{X/k} = 2g-2 \right)$	vanishing theorems	