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Lectures on Schemes

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These are preliminary lecture notes, intended only for distribution to participants

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Lectures on SCHEMES.

A.J. Scholl

I. Spec R.

In naive algebraic geometry, there is an equivalence of categories:-

$$\left(\begin{array}{c}\text{affine varieties } /k \\ +\text{morphisms}\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{fin. generated integral} \\ k\text{-algebras } + \text{morphisms}\end{array}\right) \quad (\text{contravariant})$$

- quasi-projective varieties can be constructed by gluing affine varieties.

In schemes, take as basic object any ring R (always commutative, with 1) and construct a topological space $\operatorname{Spec} R$, together with a sheaf of rings $\operatorname{Spec} R$. This is an affine scheme - general schemes are defined by gluing.

Def$^{\underline{n}}$. $\operatorname{Spec} R = \{\text{prime ideals } \mathfrak{p} \subsetneq R\}$.

To define the Zariski topology, write for any ideal $\mathfrak{a} \subset R$

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Prop$^{\underline{n}}$ 1. (i) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$

(ii) $V(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} V(\mathfrak{a}_i)$.

(iii) $V(\mathfrak{a}) \subset V(\mathfrak{b}) \iff \sqrt{\mathfrak{a}} \supset \mathfrak{b}$

(iv) $V(R) = \emptyset$; $V((0)) = \operatorname{Spec} R$.

Proof (i) One way is obvious ($\subset$). The other way: let $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$, so that $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$. If $\mathfrak{p} \supseteq \mathfrak{a}$ then $\mathfrak{p} \in V(\mathfrak{a})$. If not, $\exists\, x \in \mathfrak{a}$ s.t. $x \notin \mathfrak{p}$. Let $y \in \mathfrak{b}$; then $xy \in \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, so as $\mathfrak{p}$ is prime $y \in \mathfrak{p}$. Therefore $\mathfrak{p} \supseteq \mathfrak{b}$ i.e. $\mathfrak{p} \in V(\mathfrak{b})$.

(ii), (iv) are easy exercises.

(iii) is easy once one recalls that $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals containing $\mathfrak{a}$.

(i), (ii), (iv) show that the subsets $V(\mathfrak{a})$ satisfy the axioms for the closed subsets of a topology. This is called the <u>Zariski topology</u> on Spec R.

Among the open sets, we single out the <u>distinguished open sets</u>

$$D_f = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\} = \operatorname{Spec} R \setminus V((f))$$

(any $f \in R$). Clearly

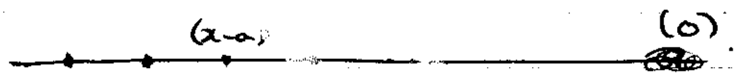
$$D_f \cap D_g = D_{fg} \quad \text{and} \quad \operatorname{Spec} R - V(\mathfrak{a}) = \bigcup_{f \in \mathfrak{a}} D_f .$$

Therefore $\{D_f \mid f \in R\}$ is a basis for the topology on Spec R.

<u>Exs.</u> • k a field. Spec k is a point (picture:- •)

• $k[x]$ where $k = \bar{k}$. Prime ideals are $(x-a)$ for $a \in k$ and (0).

<u>Picture</u>:-



(0) is <u>not a closed point</u> of Spec $k[x]$; in fact $(0) \in V(\mathfrak{a})$ if and only if $\mathfrak{a} = (0)$. So the closure of $\{(0)\}$ is the <u>whole space</u>.

<u>Remark</u>. $\mathfrak{p} \in \operatorname{Spec} R$ is a <u>closed</u> point $\Leftrightarrow$ $\mathfrak{p}$ is a <u>maximal</u> ideal.

<u>Defn</u>. X any topological space, $Y \subset X$ closed. Say $y \in Y$ is a <u>generic point</u> of Y if

$$\overline{\{y\}} = Y.$$

Any such Y is necessarily <u>irreducible</u> (i.e. not the union of 2 proper closed subsets.)

Prop^n 2 $V(\mathfrak{a})$ is irreducible $\iff$ $\sqrt{\mathfrak{a}} = \mathfrak{p}$ is prime. If so, $V(\mathfrak{a})$ has a <u>unique</u> generic point, namely $\mathfrak{p}$.

<u>Proof.</u> First assume $\sqrt{\mathfrak{a}} = \mathfrak{p}$ is prime. Then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = V(\mathfrak{a})$, so $V(\mathfrak{a})$ is irreducible. If it has another generic point $\mathfrak{q}$, then $V(\mathfrak{p}) = V(\mathfrak{q})$, hence $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ and so $\mathfrak{p} = \mathfrak{q}$.

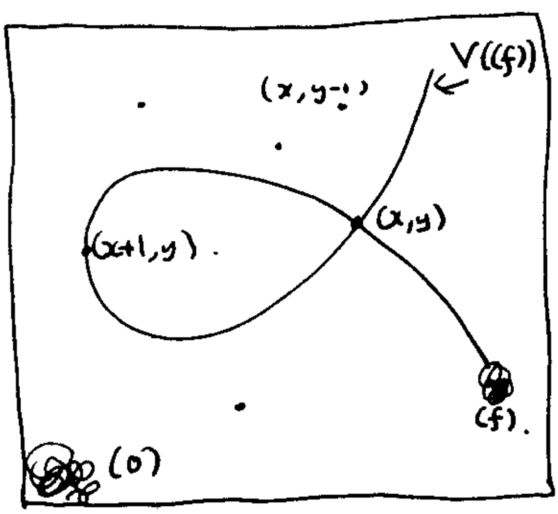
Conversely, suppose $V(\mathfrak{a})$ is irreducible. Assume without loss of generality that $\mathfrak{a} = \sqrt{\mathfrak{a}}$. We show $\mathfrak{a}$ is prime. Let $x, y \in R$ s.t. $xy \in \mathfrak{a}$. Then $V(\mathfrak{a}) \subseteq V((xy)) = V((x)) \cup V((y))$.

$$\therefore \quad V(\mathfrak{a}) = \big[V(\mathfrak{a}) \cap V((x))\big] \cup \big[V(\mathfrak{a}) \cap V((y))\big].$$

As $V(\mathfrak{a})$ is irreducible, $V(\mathfrak{a}) \subseteq V((x))$ or $V((y))$.
$\Rightarrow x \in \mathfrak{a}$ or $y \in \mathfrak{a}$. So $\mathfrak{a}$ <u>prime</u>.

<u>Ex.</u> • $R = k[x,y]$, $k = \bar{k}$. Prime ideals are of 3 types:-

$(x-a, y-b)$ for $a, b \in k$;
$(f(x,y))$ for $f \in k[x,y]$ irreducible; and
(0)



(f) is the generic point of $V((f))$. The other points in $V((f))$ are <u>closed</u>, and are the points
$$\{(x-a, y-b) \mid f(a,b) = 0\}.$$

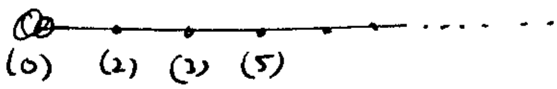
$[f = y^2 - x^2(x+1)]$

In general, if $R = k[X_1, \dots X_N]$ with $k = \bar{k}$, the closed points of $V(\mathfrak{a})$ are just

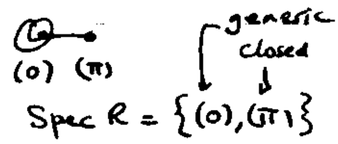
$$\left\{ (X_1 - a_1, \dots, X_N - a_N) \;\middle|\; (a_i) \in k^N \text{ is in the zero-set of } \mathfrak{a} \right\}.$$

(by Hilbert's Nullstellensatz).

- $R = \mathbb{Z}$. Primes are (p), (0).



- R = discrete valuation ring



If $\varphi : R \longrightarrow S$ is a homomorphism (taking 1 to 1!) then $\mathfrak{q} \longmapsto \varphi^{-1}(\mathfrak{q})$ is a map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$, which is <u>continuous</u> for the Zariski topology [exercise]. *

For example: the canonical map $R \longrightarrow R/\mathfrak{a}$ ($\mathfrak{a}$ any ideal) induces a map

$$\operatorname{Spec} R/\mathfrak{a} \longrightarrow \operatorname{Spec} R.$$

<u>Fact (or easy exercise)</u>: this induces a <u>homeomorphism</u> of $\operatorname{Spec} R/\mathfrak{a}$ with $V(\mathfrak{a})$.

Now consider the distinguished open sets D_f $(f \in R)$.

Let R_f be $S^{-1}R$ where $S = \{f^n : n \geq 0\}$.

<u>Prop^n 4</u> $R \longrightarrow R_f$ induces a homeomorphism of $\operatorname{Spec} R_f$ with $D_f \subset \operatorname{Spec} R$.

[* if we had taken Spec R = {<u>maximal ideals</u>} this map would not exist!]

__Proof__. Let $\varphi: R \longrightarrow R_f$ be the canonical mapping:—

$\varphi(x) = x/1 \in R_f$. Let $\mathfrak{p} \in D_f$, so that $f \notin \mathfrak{p}$. Define $\mathfrak{p}_f = \varphi(\mathfrak{p}).R_f \subset R_f$. I claim $\mathfrak{p}_f$ is a prime ideal. For if a/f^n, $b/f^n \in R_f$ with $ab/f^{2n} \in \mathfrak{p}_f$, then for some $r \geq 1$ and $c \in \mathfrak{p}$ we have ~~[illegible]~~. $f^r ab - f^{2n} c = 0$, and $\therefore$ $f^r ab \in \mathfrak{p}$. As $\mathfrak{p}$ is prime and $f \notin \mathfrak{p}$, one of a, b is in $\mathfrak{p}$, hence one of a/f^n, $b/f^n \in \mathfrak{p}_f$. It is now easy to check that $\mathfrak{p} \longmapsto \mathfrak{p}_f$ is a continuous map $D_f \longrightarrow \operatorname{Spec} R_f$, and is an inverse of $\operatorname{Spec}\varphi: \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.

__Propⁿ 4.__ $\operatorname{Spec} R = \bigcup_{i\in I} D_{f_i} \Leftrightarrow (\dots f_i \dots)_{i\in I} = R$.

More generally, $D_f = \bigcup_{i\in I} D_{f_i} \Leftrightarrow \sqrt{(\dots f_i \dots)_{i\in I}} = \sqrt{(f)}$.

__Proof__ By Propⁿ 1 (ii), $D_f = \bigcup_{i\in I} D_{f_i}$ if and only if $V((f)) = V((\dots f_i \dots))$, which holds if and only if $\sqrt{(f)} = \sqrt{(f, \dots f_i \dots)}$ by Propⁿ 1(iii). Taking $f = 1$ we get $(\dots f_i \dots) = R$ since $\sqrt{\mathfrak{a}} = R \Rightarrow \mathfrak{a} = R$.

__Corollary__ Spec R is quasi-compact (every open covering has a finite subcovering).

__Proof__ Since every open subset is a union of D_f's it is enough to show that if $\operatorname{Spec} R = \bigcup_{i\in I} D_{f_i}$, then there is a finite subset $I' \subset I$ such that $\operatorname{Spec} R = \bigcup_{i\in I'} D_{f_i}$. We have $R = (\dots f_i \dots)_{i\in I}$, so for some __finite__ $I' \subset I$ and $c_i \in R$ we can write $1 = \sum_{i\in I'} c_i f_i$. Then $R = (\dots f_i \dots)_{i\in I'}$ and thus $\operatorname{Spec} R = \bigcup_{i\in I'} D_{f_i}$.

<u>Remark</u>: Suppose $f, g \in R$ and $D_f \supset D_g$. Then $g^n = af$ for some $n \geq 1$ and some $a \in R$ [since $V((f)) \subset V((g))$, so $(g) \subset \sqrt{(f)}$ by Propⁿ 1 (iii)]

Then there is defined a homomorphism

$$\rho_{f,g} : R_f \longrightarrow R_g$$

by $\quad x/f^k \longmapsto a^k x/g^{kn}$.

In particular, if $D_f = D_g$ there is an isomorphism $R_f \xrightarrow{\sim} R_g$.

These maps are transitive (if $D_f \supset D_g \supset D_h$ then $\rho_{f,h} = \rho_{g,h} \circ \rho_{f,g}$).

<u>Theorem</u>. There is a unique sheaf of rings $\mathcal{O} = \mathcal{O}_{\operatorname{Spec} R}$, the <u>structure sheaf</u>, such that :-

(i) $\mathcal{O}(D_f) = R_f$, and if $D_f \supset D_g$ the restriction homomorphism $\mathcal{O}(D_f) \to \mathcal{O}(D_g)$ is $\rho_{f,g}$;

(ii) $\mathcal{O}_{\mathfrak{p}} = R_{\mathfrak{p}}$, the localisation of R at $\mathfrak{p}$, for every $\mathfrak{p} \in \operatorname{Spec} R$.

[Actually, (i) already determines $\mathcal{O}$ uniquely by the sheaf axiom, since $\{D_f\}$ generate the topology and $D_f \cap D_g = D_{fg}$.].

The <u>proof</u> is complicated by the fact that not every open set is a D_f. We define $\mathcal{O}(U)$ for any open $U \subseteq \operatorname{Spec} R$:-

$$\mathcal{O}(U) = \Big\{ \text{all maps } s : U \longrightarrow \coprod_{\mathfrak{p} \in U} R_{\mathfrak{p}} \text{ such that for any distinguished } D_f \subset U, \text{ there exists } a/f^n \in R_f \text{ such that for all } \mathfrak{p} \in D_f,\ s(\mathfrak{p}) = a/f^n \in R_{\mathfrak{p}}. \Big\}.$$

Defining $+$ and $\times$ componentwise makes $\mathcal{O}$ into a presheaf of rings on Spec R.

i), ii) are easy to check [the main point is that the product of the localisation maps $R_f \longrightarrow \prod_{\mathfrak{p} \in D_f} R_{\mathfrak{p}}$ is injective].

The tricky part is to show that $\mathcal{O}$ satisfies the sheaf axiom. Arguing formally from the fact that $\{D_f\}$ generate the topology, one is reduced to showing that if

$$D_f = \bigcup_{i \in I} D_{f_i}$$

then for every collection (x_i) with $x_i \in R_{f_i}$ satisfying $\rho_{f_i, f_i f_j}(x_i) = \rho_{f_j, f_i f_j}(x_j) \in D_{f_i f_j}$, there is a unique $x \in R_f$ such that $x_i = \rho_{f, f_i}(x)$ for every i.

This is proved as follows. First, by coroll. to Proposition 4 it is permissible to replace I by a finite subset — so we assume $I = \{1, 2 \ldots N\}$ for some N. Next we remark that replacing f by a positive power f^n does not change D_f or R_f. So by Prop^{n} 4 we can assume

$$f = c_1 f_1 + \ldots + c_N f_N \quad \text{for some} \quad c_i \in R.$$

Now suppose $y \in R_f$ such that $\rho_{f, f_i}(y) = 0 \ \forall i \in I$. Then $f_i^n y = 0$ for all i and some (large) n. So

$$f^m y = (\Sigma c_i f_i)^m y = 0 \quad \text{for some (larger) } m.$$

As f is invertible in R_f, $y = 0$.

This shows that x, if it exists, is unique (the difference between 2 possible x's being taken as y). Finally we assume that $\{x_i\}$ are given such that $\rho_{f_i, f_if_j}(x_i) = \rho_{f_j, f_if_j}(x_j)$ for all i, j. Write

$$x_i = b_i / f_i^n \quad \text{for some large } n \text{ (independent of } i\text{)}$$

The condition on $\{x_i\}$ implies:—

$$(*) \qquad (f_if_j)^m (b_if_j^n - b_jf_i^n) = 0 \quad \text{for large } m.$$

Let $b_i' = b_if_i^m$, $\quad f_i' = f_i^{m+n}$. Then $D_{f_i'} = D_{f_i}$ so $D_f = \bigcup_i D_{f_i'}$. Therefore by Prop^{n} 4 (replacing f by a positive power if necessary) we can find $\{c_i'\}$ in R with

$$f = c_1'f_1' + \dots + c_N'f_N'.$$

$$(*) \quad \Rightarrow \quad b_i'f_j' - b_j'f_i' = 0. \qquad (**)$$

Write $x = f^{-1} \sum_i c_i'b_i' \in R_f$. Then

$$\begin{aligned} f_i'x &= \sum_j f^{-1}c_j'b_j'f_i' \\ &= \sum_j f^{-1}c_j'f_j'b_i' \qquad \text{by } (**) \\ &= f.f^{-1}b_i' = f_i'x_i' \in R_{f_i}. \end{aligned}$$

Hence $\rho_{f,f_i}(x) = x_i \in R_{f_i}$ for all i as required.

II. Schemes.

Defn. A locally ringed space is a pair $(X, \mathcal{O}_X)$, where

- X is a topological space;
- $\mathcal{O}_X$ is a sheaf of rings on X whose stalks $\mathcal{O}_{X,x}$

are local rings for every $x \in X$.

Notation: If $x \in X$, $m_x :=$ maximal ideal of $\mathcal{O}_{X,x}$; $k(x) :=$ residue field $\mathcal{O}_{X,x}/m_x$.

Example. $(X, \mathcal{O}_X) = (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

We use this example to define schemes, which will be locally ringed spaces locally isomorphic to $(\operatorname{Spec} R, \mathcal{O})$.

First some ideas from sheaf theory:-

Let $X \xrightarrow{f} Y$ be a continuous map of topological spaces. If $\mathcal{F}$ is a sheaf on X (of abelian groups or rings) we define the direct image $f_*\mathcal{F}$ on Y by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)), \quad V \overset{\text{open}}{\subseteq} X;$$

it is easy to see that it is a sheaf. [Check axioms!]

A common situation is to have $X \xrightarrow{f} Y$, $\mathcal{F}$ on X and $\mathcal{G}$ on Y, together with a morphism

$$\alpha : \mathcal{G} \longrightarrow f_*\mathcal{F}.$$

Example. Let $\mathcal{F}, \mathcal{G}$ be the sheaves of continuous functions on X, Y respectively. Given $f : X \longrightarrow Y$ there is a canonical morphism $\alpha : \mathcal{G} \longrightarrow f_*\mathcal{F}$, given by:-

$$\begin{array}{ccc} \alpha_V : \mathcal{G}(V) & \longrightarrow & (f_* \mathcal{F})(V) \\ \| & & \| \\ \left\{\begin{array}{c}\text{continuous functions} \\ \text{on } V\end{array}\right\} & \xrightarrow[\text{compose with } f]{} & \left\{\begin{array}{c}\text{continuous functions} \\ \text{on } f^{-1}(V)\end{array}\right\}. \end{array}$$

— this is just "pullback of continuous functions".

Returning to the general situation with $f : X \to Y$, $\mathcal{F}$ and $\mathcal{G}$, there is another way to describe a map $\alpha : \mathcal{G} \to f_* \mathcal{F}$. Construct a sheaf $f^{-1}\mathcal{G}$ on X (the <u>inverse image</u>), to be the sheaf associated to the presheaf:

$$U \rightsquigarrow \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

[the <u>direct</u> <u>limit</u> over all open V containing $f(U)$]. Note the similarity with the definition of stalk: in fact if $i : \{y\} \hookrightarrow Y$ is the inclusion of a point, then $i^{-1}\mathcal{G}$ is just the constant sheaf $\mathcal{G}_y$ on $\{y\}$. It is a basic fact that there is a natural bijection

(*) $\quad \mathrm{Hom}_Y(\mathcal{G}, f_* \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$.

To see how to get this, let $\beta : f^{-1}\mathcal{G} \to \mathcal{F}$ be a morphism. Let $V \subset Y$ be open. Then as $f(f^{-1}(V)) \subset V$, there is a natural map

$$\gamma_V : \mathcal{G}(V) \longrightarrow f^{-1}\mathcal{G}(f^{-1}(V))$$

by the definition of $f^{-1}\mathcal{G}$. Compose with $\beta_{f^{-1}(V)}$ to give

$$\alpha_V = \beta_{f^{-1}(V)} \circ \gamma_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}(V)).$$

Varying $V \subset Y$ gives a morphism $\alpha : \mathcal{G} \to \mathcal{F}$.

Although the definition of $f^{-1}\mathcal{G}$ is not totally simple, it has good properties:–

- The stalk $(f^{-1}\mathcal{G})_x$ is $\mathcal{G}_{f(x)}$;

- if $f : X \hookrightarrow Y$ is the inclusion of an open subset, then $f^{-1}\mathcal{G}$ is simply the restriction of $\mathcal{G}$ to open sets in X.

Property ⊛ is usually expressed by saying "f^{-1} and f_* are adjoint functors".

<u>Consequence</u>: $\alpha : \mathcal{G} \to f_*\mathcal{F}$ induces, for every $x \in X$, a homomorphism of stalks $\alpha_x : \mathcal{G}_{f(x)} \to \mathcal{F}_x$.

<u>Def^{n}</u> A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a pair $(f, f^{\#})$ where:

- $f : X \to Y$ is a continuous map;
- $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of rings, such that $f^{\#}_x : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is a <u>local</u> homomorphism (that means $f^{\#}_x(m_{f(x)}) \subset m_x$).

<u>Remarks</u> (i) One should think of $\mathcal{O}_X$ as being "functions" on X; a morphism from X to Y should contain enough information to pull back functions. This is why $f^{\#}$ is part of the definition.

The last part ($f^{\#}_x$ is a local homomorphism) ensures that if a function on Y vanishes at $f(x)$, then the "pullback" to X vanishes at x.

(ii) The formula $(*)$ on p.10 expresses the fact that f^{-1} and f_* are adjoint functors (f^{-1} is left adjoint to f_*).

If $(X, \mathcal{O}_X)$ is a locally ringed space then if $U \subset X$ is open the pair $(U, \mathcal{O}_X|_U)$ is also a locally ringed space.

<u>Defn.</u> An <u>affine scheme</u> is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some R.

A <u>scheme</u> is a locally ringed space $(X, \mathcal{O}_X)$ which has a covering $X = \bigcup_{i \in I} U_i$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

A <u>morphism</u> of schemes is a morphism of locally ringed spaces.

<u>Ex.</u> $\mathbb{A}^n = (\operatorname{Spec} \mathbb{Z}[x_1, x_2 \dots x_n], \mathcal{O}_{\operatorname{Spec} \mathbb{Z}[x_1 \dots x_n]})$.

– <u>affine n-space</u>.

To save space write now $\operatorname{Spec} R$ for the pair $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

$\mathbb{P}^n$ – <u>projective n-space</u>. Take $n+1$ copies of $\mathbb{A}^n$:-

$$U_i = \operatorname{Spec} \mathbb{Z}[X_0/X_i, \dots\dots, X_n/X_i] \qquad 0 \le i \le n.$$

The distinguished open set

$$U_i \supset U_{ij} = D_{X_j/X_i} = \operatorname{Spec} \mathbb{Z}[X_0/X_i \dots X_n/X_i, X_i/X_j].$$

Now as $X_0/X_i = X_0/X_j \cdot X_j/X_i$ etc. we can identify this with

$$\operatorname{Spec} \mathbb{Z}[X_0/X_j, \dots X_n/X_j, X_j/X_i] = U_{ji}.$$

By gluing the $\{U_i\}$ together along the overlaps we get a topological space. Since $U_{ij} \cong U_{ji}$ as schemes we can glue the sheaves together. This gives a scheme

$$\mathbb{P}^n = \bigcup_i U_i$$

Fact: $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \mathbb{Z}$. So $\mathbb{P}^n$ is definitely not affine for $n \geq 1$ (if so, it would $=$ Spec $\mathbb{Z}$).

Ex. k a field. Let $X = \operatorname{Spec} k[x,y] - \{(x,y)\}$ with the induced structure sheaf. Then one can show that

$$\Gamma(X, \mathcal{O}_X) = k[x,y] = \Gamma(\operatorname{Spec} k[x,y], \mathcal{O}_{\operatorname{Spec} k[x,y]}).$$

In particular X is not affine, as $X \neq \operatorname{Spec} k[x,y]$.
(X = affine plane with origin removed)*

Now let $f: X \to Y$ be a morphism of schemes. Then it determines a ring homomorphism

$$f^\#_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X).$$

If Y is affine, this is enough to determine f:

Theorem (i) Let $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} R'$. Then $f \mapsto f^\#_Y$ gives a bijection

$$\operatorname{Hom}_{\text{Schemes}}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\text{Rings}}(R', R).$$

(ii) Let $Y = \operatorname{Spec} R$. Then $f \mapsto f^\#_Y$ gives a bijection

$$\operatorname{Hom}_{\text{Schemes}}(X, \operatorname{Spec} R) \xrightarrow{\sim} \operatorname{Hom}_{\text{Rings}}(R, \Gamma(X, \mathcal{O}_X)).$$

---

* Exercise: Use the covering $X = D_x \cup D_y \subset \operatorname{Spec} k[x,y]$ to show (a) X is a scheme; and (b) $\Gamma(X, \mathcal{O}_X) = k[x,y]$.

I will not give a complete proof. We have already seen that $\varphi: R' \to R$ (a homomorphism) determines a map of topological spaces
$$f = \mathrm{Spec}(\varphi) : \mathrm{Spec}\, R \to \mathrm{Spec}\, R'.$$
If $\mathfrak{p} \in \mathrm{Spec}\, R$, $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$, and by localisation we get a homomorphism
$$\varphi_{\mathfrak{p}} : R'_{\varphi^{-1}(\mathfrak{p})} \longrightarrow R_{\mathfrak{p}}.$$

This allows us to define $f^{\#}: \mathcal{O}_{\mathrm{Spec} R'} \to f_* \mathcal{O}_{\mathrm{Spec} R}$ as follows: let
$$s = (s_{\mathfrak{q}}) \in \mathcal{O}_{\mathrm{Spec} R'}(V) \subseteq \prod_{\mathfrak{q} \in V} R'_{\mathfrak{q}}.$$
Define $t = f^{\#}_V(s) = (t_{\mathfrak{p}}) \subseteq \prod_{\mathfrak{p} \in f^{-1}(V)} R_{\mathfrak{p}}$ by
$$t_{\mathfrak{p}} = \varphi_{\mathfrak{p}}(s_{\varphi^{-1}(\mathfrak{p})}).$$

One can check this defines a morphism of sheaves. Then one has to check that any $(f, f^{\#})$ is uniquely determined by $f^{\#}_Y$, in case (i).

For (ii) one replaces X by an open covering by affine schemes U_i. The restriction of $f: X \to Y$ to U_i is a morphism $f_i : U_i \to Y$, hence by (i) corresponds to a homomorphism $\varphi_i : R \to \Gamma(U_i, \mathcal{O}_X)$. The sheaf axiom shows that (f_i) come from some $f: X \to Y$ if and only if (φ_i) come from a homomorphism $\varphi: R \to \Gamma(X, \mathcal{O}_X)$.

For details see Hartshorne p. 73 or Mumford p. 111.

Ex. For any scheme X, there is a unique homomorphism

$$\mathbb{Z} \longrightarrow \Gamma(X, \mathcal{O}_X)$$

and so a unique morphism

$$X \longrightarrow \operatorname{Spec} \mathbb{Z}.$$

Ex. If X is a scheme and R is a ring, the set of morphisms $\operatorname{Spec} R \to X$ is called the set of R-valued points of X, denoted $X(R)$. Let's determine it for some special X:-

- $X = \mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \ldots x_n]$.

Then $X(R) = \{\text{homomorphisms } \mathbb{Z}[x_1, \ldots x_n] \xrightarrow{\varphi} R\}$

$$\cong R^n$$

(the last bijection by $\varphi \mapsto (\varphi(x_1), \ldots \varphi(x_n))$. So $\mathbb{A}^n(R) = R^n$.

In particular $\mathbb{A}^n(k) = k^n$, "usual" affine space $/k$.

- $X = \mathbb{P}^n$; we'll restrict to the case $R = k$ (field).

Then as $\operatorname{Spec} k = (\text{point})$, any morphism $\operatorname{Spec} k \to \mathbb{P}^n$ has image contained in one of the U_i's. That is,

$$\mathbb{P}^n(k) = \bigcup U_i(k)$$

and each $U_i(k) \cong k^n$. Examining the gluing data we see that this identifies $\mathbb{P}^n(k)$ ~~as with~~ with the set of lines in k^{n+1}.

~~If X is a scheme then any open subset U of the topological space of X is that has a the sheaf of rings $\mathcal{O}_X|_U$, and is also a scheme* called an open subscheme of X.~~

~~But if $Y \subset X$ is a closed subset, there is no obvious way to get a sheaf of rings on Y. If $i: Y \hookrightarrow X$ is the inclusion map $i^{-1}\mathcal{O}_X$ will generally be "too big".~~

~~* this requires a short proof.~~

A more intrinsic way to define projective space is given by the "Proj" construction, which we now sketch. Let

$$S = \bigoplus_{d \geq 0} S_d$$

be any graded ring. [for example, $S = R[X_0 \dots X_m]$, with S_d = homogeneous polynomials of degree d]. We say a prime ideal $\mathfrak{p} \subset S$ is <u>homogeneous</u> if

$$\mathfrak{p} = \bigoplus_{d \geq 0} (\mathfrak{p} \cap S_d).$$

Let $S_+ = \bigoplus_{d > 0} S_d$, and define

$$\operatorname{Proj} S = \{\text{homogen. prime ideals } \mathfrak{p} \text{ s.t. } \mathfrak{p} \not\supseteq S_+\}$$

Then define $V_+(\mathfrak{a}) = \{\mathfrak{p} \supseteq \mathfrak{a}\}$, $D_f^+ = \{\mathfrak{p} \not\ni f\}$ for $f, \mathfrak{a}$ homogeneous, just as for Spec R.

<u>Thm</u> There is a unique sheaf of rings $\mathcal{O}_{\operatorname{Proj} S}$ on Proj S such that

$$\Gamma(D_f^+, \mathcal{O}_{\operatorname{Proj} S}) = S_{(f)} \overset{\text{def}^n}{=} \left\{ \frac{a}{f^n}, \ a \in S \text{ homog. of degree } n \times \deg f \right\}$$
$$\subseteq S_f.$$

$(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme, and

$$\mathcal{O}_{\operatorname{Proj} S, \mathfrak{p}} = S_{(\mathfrak{p})} \overset{\text{def}^n}{=} \{\text{degree } 0 \text{ elements of the localisation of } S \text{ w.r.t. homog. elts of } S - \mathfrak{p}\}$$

Now we can define $\mathbb{P}^N_R = \operatorname{Proj} R[X_0 \dots X_N]$. See Hartshorne, pp 76-77, for more details, (and EGA for all details!).

A slight generalisation of affine space is given as follows — let E be a vector space over a field k. Consider the <u>Symmetric algebra</u>

$$\mathrm{Sym}\, E = \bigoplus_{n \geq 0} \mathrm{Sym}^n E$$

(recall that $\mathrm{Sym}^0 E = k$, $\mathrm{Sym}^1 E = E$ and $\mathrm{Sym}^n E$ is the quotient of $E \otimes_k \cdots \otimes_k E$ [n copies] by all relations derived from $x \otimes y = y \otimes x$.)

Let $\mathbb{V}[E] = \mathrm{Spec}(\mathrm{Sym}\, E)$. Then $\mathbb{V}[E]$ is an affine k-scheme, and

$$\mathbb{V}[E](k) = \mathrm{Hom}_{k\text{-alg}}(\mathrm{Sym}\, E, k) = \mathrm{Hom}_{k\text{-vect}}(E, k) = E^\vee.$$

This is the <u>natural</u> way to construct an affine space-scheme from a vector space. Choosing a basis $e_1 \ldots e_n$ for E (if it is f. dimensional) gives an isomorphism $\mathrm{Sym}\, E \xrightarrow{\sim} k[e_1, \ldots e_n]$ and so an isomorphism of $\mathbb{V}[E]$ with $\mathbb{A}^n_k$.

Slightly more generally, if E is a free ~~(or projective)~~ module over a ring R, we can form

$$\mathbb{V}[E] = \mathrm{Spec}(\mathrm{Sym}\, E)$$

which is an affine R-scheme. If $E = R^n$ then $\mathbb{V}[E] = \mathbb{A}^n_R$.

There is an analogous construction for projective spaces. Let E/k be a vector space, and consider

$$\mathbb{P}[E] = \mathrm{Proj}(\mathrm{Sym}\, E)$$

where $\mathrm{Sym}\, E$ is graded in the obvious way. It is an easy exercise in the definition of Proj to see that

$$\mathbb{P}[E](k) = (E^\vee - \{0\})/k^* = \{\text{hyperplanes in } E\}$$

the <u>dual</u> projective space to E. This will be generalised later to a relative setting giving vector bundles and projective bundles

III. Schemes (continued)

If $U \subset X$ is an open subset, where $(X, \mathcal{O}_X)$ is a locally ringed space, then $(U, \mathcal{O}_X|_U)$ is also a locally ringed space.

If $(X, \mathcal{O}_X)$ is a scheme then $(U, \mathcal{O}_X|_U)$ is also a scheme [exercise: prove this!]. We call this an <u>open subscheme</u> of X.

To simplify notation we will generally now use $X, Y \ldots$ to denote schemes (<u>ie</u>. spaces + structure sheaves), and $f: X \to Y \ldots$ to denote morphisms of schemes. When we refer to the underlying topological space we will write $sp(X)$, $sp(f)$ . . . . .

The content of the first paragraph is then that if X is a scheme, then ~~any opens~~ there is a bijection between the set of open subspaces of $sp(X)$ and the set of open subschemes of X.

Now look at closed subsets. There is no analogous way to ~~[illegible]~~ restrict a sheaf to rings to a closed subset. To specify a "closed subscheme" we will need to give not only the underlying space, but also the structure sheaf.

We therefore define :-

Def^n A <u>closed subscheme</u> of X is a scheme Y and a morphism $i : Y \longrightarrow X$ such that :-

a) on topological spaces, i is the inclusion of a closed subset of X;

b) the map $i^{\#} : \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$ is a surjection (as a morphism of sheaves).

<u>Examples.</u> $X = \operatorname{Spec} R$. Then if $\mathfrak{a} \subset R$ is any ideal, there is a closed subscheme Y of X isomorphic to $\operatorname{Spec} R/\mathfrak{a}$. In fact, take the canonical map

$$i : \operatorname{Spec} R/\mathfrak{a} \longrightarrow \operatorname{Spec} R$$

and identify the space of $\operatorname{Spec} R/\mathfrak{a}$ with its image $V(\mathfrak{a})$. We need to check (b), but this is essentially obvious:- if

$$x = \mathfrak{p} \in V(\mathfrak{a}) \quad \text{then} \quad i^{\#}_x : \mathcal{O}_{X,x} = R_{\mathfrak{p}} \twoheadrightarrow (i_*\mathcal{O}_Y)_x = R_{\mathfrak{p}} / \mathfrak{a}_{\mathfrak{p}} ;$$

and if $x \notin V(\mathfrak{a})$ then $(i_* \mathcal{O}_Y)_x = 0$ (since $\exists$ open U with $x \in U$, $V(\mathfrak{a}) \cap U = \phi$, and so $(i_* \mathcal{O}_Y)(U) = 0$). Notice that as $sp(Y) = V(\mathfrak{a})$, different $\mathfrak{a}$ with the same radical will give different Y, with the same underlying space. Any closed subscheme of $\operatorname{Spec} R$ is of this type (this is not trivial!)

<u>Ex</u>. $X = \operatorname{Spec} k[x]$., $n \geq 1$. Consider $\mathfrak{a} = (x^n)$. Then $V(\mathfrak{a}) = \{(x)\}$

and the closed subscheme attached to $\mathfrak{a}$ is $Y_n \cong \operatorname{Spec} k[x]/(x^n)$.

Note $\mathcal{O}_Y$ has nilpotent elements! If $s = s(x) \in \Gamma(X, \mathcal{O}_X) = k[X]$

then
$$i^{\#}_{(x)}(s) = s \bmod (x^n)$$
$$= \sum_{j=0}^{n-1} a_j . x^j \bmod (x^n) \quad \text{if } s = \sum a_j x_j.$$

~~mean~~ – so one recovers not also the <u>value</u> $s(0)$, but also the first n terms in the expansion in powers of x, from $i^{\#}$s.

We have an infinite chain of closed subschemes:-

$$Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots \subset X$$

all with the same underlying space $sp(Y_n) = \{(x)\}$. They are called <u>infinitesimal neighbourhoods</u> of (x) in X — these are useful for applying methods of differential geometry (tangent space, connections, ...) to abstract algebraic geometry.

<u>Ex.</u> Let $X = \mathrm{Spec}\, k[x,y]$. We consider the closed subschemes Y', Y of X defined by the ideals

$$\mathfrak{a}' = (xy, y^2), \qquad \mathfrak{a} = (y).$$

Since $\sqrt{\mathfrak{a}'} = \mathfrak{a}$, $sp(Y) = sp(Y') = V((y)) =$ "x-axis". We compare Y and Y' — for simplicity, assume $k = \bar{k}$.

~~If $\mathfrak{p} \in$~~ The closed points of $sp(Y)$ are of the form $\mathfrak{p} = (x-a, y)$ with $a \in k$. If $a \neq 0$ then since $x \equiv a \mod \mathfrak{p}$, x is invertible in $k[x,y]_{\mathfrak{p}}$ and therefore also in

$$\mathcal{O}_{Y',\mathfrak{p}} = \left(k[x,y]/(xy,y^2)\right)_{(x-a,y)}$$

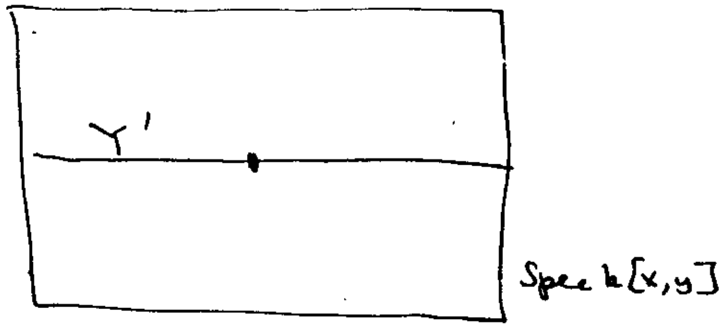
Hence as $xy = 0$ in $\mathcal{O}_{Y',\mathfrak{p}}$, we have $y = 0$ in $\mathcal{O}_{Y',\mathfrak{p}}$ so that

$$\mathcal{O}_{Y',\mathfrak{p}} \cong k[x]_{(x-a)} \cong \mathcal{O}_{Y,\mathfrak{p}}.$$

Now suppose $a = 0$. Then $\mathcal{O}_{Y,\mathfrak{p}} = k[x]_{(x)}$ but

$\mathcal{O}_{Y',\mathfrak{p}} = \left(k[x,y]/(xy,y^2)\right)_{(x,y)} \not\cong k[x]_{(x)}$; in fact the maximal ideal $(x,y)\mathcal{O}_{Y',\mathfrak{p}}$ cannot be generated by one element.

Roughly speaking, Y' is Y with a "thickened" point added at the origin:-



Subschemes like this arise frequently in families in which most members are "nice".

Now we will discuss one of the most important constructions in scheme theory — <u>fibre products</u>.

<u>Defn</u> Let $f : X \to S$, $g : Y \to S$ be morphisms in a category $\mathcal{C}$. A <u>fibre product</u> of f and g is an object W of $\mathcal{C}$, together with morphisms $p_1 : W \to X$, $p_2 : W \to Y$ such that $f \circ p_1 = g \circ p_2$, and satisfying the <u>universal property</u>:-

If $q_1 : Z \to X$, $q_2 : Z \to Y$ are any morphisms of $\mathcal{C}$ such that $f \circ q_1 = g \circ q_1$, then is a <u>unique</u> morphism $h : Z \to W$ such that $q_i = p_i \circ h$ $(i = 1, 2)$.

<u>General fact</u> If a fibre product exists it is unique up to isomorphism (this follows from the universal property).

We denote a fibre product by $X \times_S Y$. The diagram below summarises the definition:-

$$\begin{tikzcd}
 & Z \arrow[dl, bend right, "q_1"'] \arrow[d, dashed, "h"] \arrow[dr, bend left, "q_2"] & \\
 & X \times_S Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & S &
\end{tikzcd}$$

Exs. • In the category of sets, the fibre product of $f: X \to S$, $g: Y \to S$ exists and is

$$X \times_S Y = \{ (x,y) \in X \times Y \mid f(x) = g(y) \}.$$

So $$X \times_S Y = \bigcup_{s \in S} f^{-1}(s) \times g^{-1}(s) \qquad \text{(hence the name).}$$

• If U, V are subsets of a set (space) X, and $f: U \hookrightarrow X$, $g: V \hookrightarrow X$ are the inclusions, then

$$U \times_X V = U \cap V.$$

Theorem. Fibre products of schemes exist.

Idea of proof. (i) The affine case:

$$\begin{tikzcd} X = \operatorname{Spec} A \arrow[dr, "f"'] & & Y = \operatorname{Spec} B \arrow[dl, "g"] \\ & S = \operatorname{Spec} R & \end{tikzcd} \quad \text{corresponds to} \quad \begin{tikzcd} A & & B \\ & R \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] & \end{tikzcd},$$

where φ, ψ are ring homomorphisms. Then ~~...~~ we can form the tensor product $A \otimes_R B$, together with the maps

$$\pi_1 : A \to A \otimes_R B, \quad \pi_2 : B \to A \otimes_R B$$
$$a \mapsto a \otimes 1 \qquad\qquad b \mapsto 1 \otimes b.$$

It has the universal property: for any homomorphisms of rings $\theta_1 : A \to C$, $\theta_2 : B \to C$ such that $\theta_1 \circ \varphi = \theta_2 \circ \psi$ there is a unique $\omega : A \otimes_R B \to C$ such that $\theta_i = \omega \circ \pi_i$.

So if we define $X \times_S Y = \operatorname{Spec} A \otimes_R B$ with the obvious morphisms to X and Y (coming from π_i) then it is immediate that $X \times_S Y$ is a fibre product.

(ii) <u>The general case</u>. One covers X, Y, S by open affines, and by a straightforward but tedious argument defines $X \times_S Y$ by gluing together products of the type (i). In case of necessity consult Hartshorne, p. 88 for (a subset of) the details.

<u>Exs.</u> • As $\mathbb{Z}[x_1 \ldots x_n] \otimes_{\mathbb{Z}} R = R[x_1 \ldots x_n]$ we have

$$\mathbb{A}^n_R = \mathbb{A}^n \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R.$$

Note that as every scheme has a <u>unique</u> morphism to $\operatorname{Spec}\mathbb{Z}$, there is for any pair of schemes X, Y a product

$X \times_{\operatorname{Spec}\mathbb{Z}} Y$, which we abbreviate to $X \times Y$.

• $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$.

• Definition of $\mathbb{P}^n_R := \mathbb{P}^n \times \operatorname{Spec} R$.

[Notations. A product $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is usually abbreviated $X \otimes_R R'$.]

Also we can consider:—

• $Y_1, Y_2 \hookrightarrow X$ closed subschemes. Then their <u>intersection</u> is the fibre product $Y_1 \times_X Y_2$. (This is not always a good notion of intersection, however!)

<u>Ex.</u> $X = \mathbb{A}^2_k$, Y_1 defined by (y), Y_2 by $(y - x^2)$. (char $k \neq 2$ say)

Then $Y_1 \times_X Y_2$ is defined by $(y, y - x^2) = (y, x^2)$: a point with "multiplicity 2 in the x-direction". This is the scheme-theoretic picture of the contact of the conic $y = x^2$ and the line $y = 0$ in the affine plane.

IV.

Important example: fibres of a morphism.

Let $f : X \to Y$ be a morphism, and $y \in Y$ a point, with residue field $k(y)$. We now define a scheme X_y, which for a closed point y will be a closed subscheme of X, playing the role of the "fibre $f^{-1}(y)$".

There is a unique ~~pt~~ morphism $g : T = \operatorname{Spec} k(y) \to Y$ such that:

(i) $g(T)$ is the point $y \in Y$;

(ii) $g^\#_y : \mathcal{O}_{Y,y} \to k(y)$ is the quotient map.

Define X_y to be the fibre product of f and g:–

$$\begin{tikzcd} X \times_Y T = X_y \arrow[r] \arrow[d, "f_y"] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & Y \end{tikzcd}$$

and let f_y be the projection onto the 2nd factor T.

Ex. $X = \mathbb{A}^2_k = \operatorname{Spec} k[u,v] \xrightarrow{f} Y = \mathbb{A}^1 = \operatorname{Spec} k[t]$ by $f^\# : t \mapsto u^2 - v^2$

Then let $y = (t-a) \in Y$, $a \in k$. Then $k(y) = k$ and

$$X_y = X \times_Y \operatorname{Spec} k = \operatorname{Spec} k[u,v] \otimes_{k[t]} k$$ with respect to the ring homomorphisms $t \mapsto u^2 - v^2$, $t \mapsto a$. Therefore

$$X_y = \operatorname{Spec} k[u,v]/(u^2 - v^2 - a),$$ so we have the family of conics $u^2 - v^2 = a$ in $\mathbb{A}^2_k$.

Taking $y = (0) \in Y$ (generic point) gives $k(y) = k(t)$ and so

$$X_{(0)} = \operatorname{Spec} k[u,v] \otimes_{k[t]} k(t)$$
$$= \operatorname{Spec} k(t)[u,v]/(u^2 - v^2 - t) \subset \mathbb{A}^2_{k(t)}.$$

We say $X_{(0)}$ is the <u>generic fibre</u> of $X \xrightarrow{f} Y$.

<u>Ex.</u> $X = \operatorname{Spec} \mathbb{Z}[u,v]/(f)$ where $f \in \mathbb{Z}[u,v]$, $f \neq 0$

$\downarrow$ (an <u>arithmetic surface</u>)

$Y = \operatorname{Spec} \mathbb{Z}$.

Taking $y = (p) \in Y$ for a prime p gives $k(y) = \mathbb{F}_p$ and

$$X_{(p)} = \operatorname{Spec} \mathbb{Z}[u,v]/(f) \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p = \operatorname{Spec} \mathbb{F}_p[u,v]/(\bar{f})$$

where $\bar{f}$ = image of f in $\mathbb{F}_p[u,v]$; taking $y = (0) \in Y$ gives

$$X_{(0)} = \operatorname{Spec} \mathbb{Q}[u,v]/(f) \qquad \text{since } k(y) = \mathbb{Q}.$$

For example, if $f = v^2 - u^3 - 5$ then $X_{(0)}$ is the nonsingular cubic $v^2 = u^3 + 5$ in affine 2-space over $\mathbb{Q}$. $X_{(p)}$ is the cubic $v^2 \equiv u^3 + 5$ over $\mathbb{F}_p$, which is nonsingular if $p > 5$. For $p = 5$, we have

$$X_{(5)} = \operatorname{Spec} \mathbb{F}_5[u,v]/(v^2 - u^3),$$

a <u>cuspidal</u> cubic. (Exercise – determine $X_{(2)}$ and $X_{(3)}$.)

<u>Ex.</u> $X = \operatorname{Spec} \mathcal{O}_K \longrightarrow Y = \operatorname{Spec} \mathbb{Z}$, where K is an algebraic number field. Then :-

- $X_{(0)} = \operatorname{Spec} K$;
- If $(p) = \prod \mathfrak{p}_i$ does not ramify in K, then
$$X_{(p)} = \coprod \operatorname{Spec} \mathcal{O}_K/\mathfrak{p}_i = \coprod \operatorname{Spec} \mathbb{F}_{p^{f_i}} ;$$
- If (p) ramifies in K then $\mathcal{O}_{X_{(p)}}$ has nilpotent elements.

[Prove these assertions using algebraic number theory].

Ex. Let $X = \mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[u]$ be given by

$$\begin{array}{lcl} X = \mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[u] & & f^{\#} : \mathbb{F}_p[v] \longrightarrow \mathbb{F}_p[u] \\ \downarrow f & & \\ Y = \mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[v] & & v \longmapsto u^p. \end{array}$$

Let $y = (g(v)) \in Y$ be a closed point, where $g(v) \in \mathbb{F}_p[v]$ is irreducible. Then

$$k(y) = \mathbb{F}_q \quad \text{where} \quad q = p^{\deg(g)} ;$$

let a = image of v in $\mathbb{F}_q$ and $b \in \mathbb{F}_q$ be the element such that $b^p = a$. Then

$$X_{(g)} = \operatorname{Spec} R, \quad \text{where } R \text{ is the fibre product of}$$

$$\begin{tikzcd} & \mathbb{F}_p[u] \\ \mathbb{F}_p[v] \arrow[ur, "f^{\#}"] \arrow[dr] & \\ & \mathbb{F}_q \end{tikzcd}$$

Therefore $R = \mathbb{F}_q[u]/(u^p - a) = \mathbb{F}_q[t]/(t^p) \qquad (t = u - b).$

So every fibre $X_{(g)} \cong \operatorname{Spec} \mathbb{F}_q[t]/(t^p)$ has nilpotent elements in its structure sheaf. Also $X_{(0)} = \operatorname{Spec} \mathbb{F}_q(u)$ is a (reduced) point. Therefore f is a homeomorphism of topological spaces, but very far from being an isomorphism.

<u>Ex.</u> Let $f : X \longrightarrow S$ be any morphism.
(We often say <u>X is a scheme over S</u> if there is given such a morphism). Consider the fibre product of f with itself:-

$$\begin{tikzcd}
 & X \times_S X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\
 & S &
\end{tikzcd}$$

Then by the univ. property, there is a unique morphism $\Delta : X \to X \times_S X$ such that $p_1 \circ \Delta = p_2 \circ \Delta =$ the <u>identity</u> map $X \to X$.

Δ is called the <u>diagonal</u> – since in the category of <u>sets</u> the analogue is the map $x \mapsto (x, x) \in X \times X$.

If $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$ are <u>affine</u>, then

$X \times_S X = \operatorname{Spec} A \otimes_R A$ and the diagonal is given

by $\Delta^\# : A \otimes_R A \longrightarrow A$, $\Delta^\#(a \otimes b) = ab$.

[since it is the unique map which, when composed with either of the canonical maps $\theta_i : A \to A \otimes_R A$ gives the identity on A).
$a \mapsto a \otimes 1$ or $1 \otimes a$

We say f is <u>separated</u> if $\Delta(X)$ is a <u>closed</u> subset of $X \times_S X$.

If $S = \operatorname{Spec} \mathbb{Z}$, we simply say <u>$X$ is separated</u>.

This is the analogue of the <u>Hausdorff</u> condition in topology. In topology, the image of $X \xrightarrow{\Delta} X \times X$ is closed if and only if, whenever $(p, q) \in X \times X$ is a limit point of the diagonal, we have $p = q$. For a reasonable space X this is the same as saying that a sequence can have at most one limit.

<u>Proposition</u> Every affine scheme is separated.

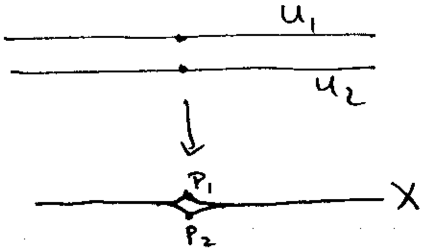
<u>Proof</u> In fact, we show that every morphism $\operatorname{Spec} A \to \operatorname{Spec} R$ of affine schemes is separated (take $R = \mathbb{Z}$ for the proposition).

Let $$J = \ker\left[\Delta^\#: A \otimes_R A \longrightarrow A\right].$$

Clearly $\Delta^\#$ is <u>surjective</u>, so the image of Δ is the closed subscheme of $\operatorname{Spec} A \otimes_R A$ defined by the ideal J; so its topological space is closed.

<u>Example</u> (a non-separated scheme). Let U_1, U_2 be two copies of $\mathbb{A}^n_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x_1 \ldots x_n]$, and join them along the open sets $U_1 - (\text{origin})$, $U_2 - (\text{origin})$: –

(n=1 here)



X is "$\mathbb{A}^n$ with the origin doubled". Consider $\Delta: X \longrightarrow X \times X$.

Then $X \times X$ is the union of 4 copies of $\mathbb{A}^n \times \mathbb{A}^n$ and has 4 origins, namely (P_i, P_j) for $i, j = 1$ or 2. Each of these is in the closure of the diagonal [since the generic point of the diagonal is in the <u>intersection</u> of the 4 copies of $\mathbb{A}^n \times \mathbb{A}^n$]; but $\Delta(X)$ contains only (P_1, P_1) and (P_2, P_2). So $\Delta(X)$ is not closed.

<u>Proposition</u> If X is a separated scheme and $U_1, U_2 \subset X$ are affine open subschemes, then $U_1 \cap U_2$ is also affine.

<u>Proof</u> If X is separated then $\Delta(X)$ is a closed subscheme of $X \times X$, and as $\Delta^\#: \mathcal{O}_{X \times X} \longrightarrow \Delta_* \mathcal{O}_X$ is <u>surjective</u> [it is enough to consider the affine case,

when we already know it is true] the diagonal $\Delta(X)$ is a closed subscheme of $X \times X$, isomorphic to X

Now it is easy to see that $U_1 \cap U_2$ is isomorphic to $\Delta(X) \cap U_1 \times U_2$, which is a closed subscheme of the (affine!) scheme $U_1 \times U_2$. As we mentioned earlier, every closed subscheme of an affine scheme is also affine; hence $U_1 \cap U_2$ is affine.

The previous example for $n \geq 2$ shows how this can fail if X is not separated.

<u>Remark</u> X a scheme, $x \in X$ any point; $m_x \subset \mathcal{O}_{X,x}$ the maximal ideal

If $f \in \Gamma(U, \mathcal{O}_X)$ where $x \in U$, attached to f have:

- f_x = image of f in $\mathcal{O}_{X,x}$ ("germ of f")
- $f(x) = \ldots\ldots\ k(x) = \mathcal{O}_{X,x}/m_x$ – residue field ("value of f")

$\{f_x : x \in U\}$ clearly determines f (as $\mathcal{O}$ is a sheaf).

$\{f(x)\}$ needn't: if $f^n = 0$ then $f(x) = 0$ $\forall x$.

<u>Defn</u> A scheme X is <u>reduced</u> if all $\mathcal{O}_{X,x}$ have no $\neq 0$ nilpotent elements

(equivalently, if same is true for all $\mathcal{O}_X(U)$).

<u>Slightly stronger is</u>: X is <u>integral</u> if each $\mathcal{O}_X(U)$ is an integral domain
(equivalently, iff X is reduced and irreducible)

A (quasi-projective) <u>variety</u> over an alg closed field k is an integral quasi-projective scheme over k.

[The category of such varieties is equivalent to the category of "naïve" varieties over k].

<u>Dimension</u> of a scheme X is the supremum of n s.t. there exists a chain

$$\phi \neq Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \subset X$$

of distinct irreducible subsets of X.

R a ring then the (Krull) dimension of R is the supremum of the lengths of chains of prime ideals:-

$$R \supsetneq \mathfrak{p}_0 \supsetneq \ldots \supsetneq \mathfrak{p}_n$$

Clearly $\dim R = \dim(\operatorname{Spec} R)$.

Recall from local algebra:-

Def^n. R local Noetherian ring, $k = R/\mathfrak{m}$. R is a regular ring if

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$$

(always $\geq$)

Then say a scheme is regular if all its local rings are regular (enough to check closed points).

Regular local rings are UFD's, hence also integrally closed — so regular schemes have very nice local properties* Here are examples in low dimension:-

Exs. of regular schemes of dimension 1: (dim^n = 0 is an exercise!)

- (geometric) $k = \bar{k}$ field. non-singular curves / k
  (i.e. regular schemes of dim^n 1, quasi-projective / k).
- (arithmetic) Spec $\mathcal{O}_K$ (K = number field).

More generally, if R is any Dedekind domain, Spec R is a reg. ~~[illegible]~~ scheme of dim^n 1.

eg. R = discrete valuation ring, uniformiser π.

$X = \operatorname{Spec} R = \{(0), (\pi)\}$ $\circ\!\!-\!\!\bullet$ (0) (π)

generic ↑ ↑ closed

[such an X is called a "trait" in French, for which there exists no English equivalent]

Dimension 2. • (geometric) quasi-proj / $\bar{k}$ non-singular surface

• (arithmetic) The important example for Arakelov theory is an arithmetic surface, which is a regular, irreducible

* eg. every integral closed subscheme of ~~codim~~ dim^n 1 is locally given by one equation.

projective scheme X over $\mathbb{Z}$ of dimension 2, such that $X \to \operatorname{Spec}\mathbb{Z}$ is surjective [this is ~~to~~ ensure X is not a 2-dimensional scheme over some $\mathbb{F}_p$; equivalently, we may require ~~that~~ X be <u>flat</u> over $\operatorname{Spec}\mathbb{Z}$].

Simplest example is $\mathbb{P}^1_{\mathbb{Z}}$.

Note that the top space of an arithmetic surface is quite intricate:— even for $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x]$ have different sorts of points:-

$$\underset{\text{minimal prime}}{(0)} \qquad \underbrace{(p) \qquad \underset{f \in \mathbb{Z}[x] \text{ irreducible}}{(f)}}_{\text{height} = \text{one}} \qquad \underset{\substack{\text{maximal} \\ (f \in \mathbb{Z}[x],\ f \bmod p \text{ irreducible})}}{(p, f)}$$

* * * * *

Also even if one starts with regular schemes, the <u>fibres</u> of a morphism between regular schemes can be quite unpleasant.

<u>eg.</u> $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y] \xrightarrow{f} Y = \operatorname{Spec} k[u,v]$ $\operatorname{char}(k) \neq 2$, $k = \bar{k}$

$f^{\#}(u,v) = (xy, y^2)$.

$P \in Y$ the closed pt. $\{u = a, v = b$

$$f^{-1}(P) = \begin{cases} \operatorname{Spec} k[y]/(y^2 - b) & \text{if } b \neq 0 & \bullet\ \bullet \\ \operatorname{Spec} 0 = \emptyset & \text{if } a \neq 0 = b. & \emptyset \\ \operatorname{Spec} k[x,y]/(xy, y^2) & a = b = 0 & \end{cases}$$

Another striking use of nilpotent elements is in the <u>tangent space</u>.

Let X be a quasi-projective scheme over $k = \bar{k}$. Write $D = k[\varepsilon]/(\varepsilon^2)$ ("dual numbers"). If $x \in X$ is a closed point, let

$$T_{X,x} = \{k\text{-morphisms } \operatorname{Spec} D \xrightarrow{\varphi} X \text{ s.t. } \varphi(\text{point}) = x\}$$

If X is quasi-projective then $k(x) = k$ $^{(*)}$, and then one finds

$$T_{X,x} = \{\text{local } k\text{-algebra homomorphisms } \mathcal{O}_{X,x} \to D\}$$
$$= \{k\text{-linear maps } \mathfrak{m}_x \xrightarrow{\lambda} k.\varepsilon \subset D \text{ such that } \lambda(\mathfrak{m}_x^2) = 0\}$$
$$= \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$$

The space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is called the <u>cotangent space</u> of X at x, and is closely related to <u>differentials</u> (see below). One can think of Spec D therefore as "a point with a tangent vector attached" (but no ambient space!).

(*) because x is a closed point of $\mathbb{P}^n_k$, hence of $\mathbb{A}^n_k$ for some affine $\mathbb{A}^n_k$; and the closed points of $\mathbb{A}^n_k$ are all given by ideals of the form $(x_1 - a_1, \ldots, x_n - a_n)$, $(a_i) \in k^n$ [Hilbert Nullstellensatz]

Sheaves of modules

Let X be a scheme. A <u>sheaf of $\mathcal{O}_X$-modules</u> (or simply an $\mathcal{O}_X$-module) is a sheaf $\mathcal{F}$ on X, and for every open $U \subset X$ an action

$$\mathcal{O}_X(U) \times \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$$

compatible with restriction to open $V \subset U$, which makes each $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$-module. Morphisms of $\mathcal{O}_X$-modules are defined in the obvious way.

A simple example of an $\mathcal{O}_X$-module is the <u>free</u> module:

$$\mathcal{O}_X \oplus \dots \oplus \mathcal{O}_X = \mathcal{O}_X^d$$

defined in an obvious way.

There is a particularly important class of sheaves of modules. First consider the case of an <u>affine</u> scheme $X = \operatorname{Spec} R$. Let M be any R-module. Then the proof of the existence of the structure sheaf $\mathcal{O}_X$ can almost trivially be generalised to give:-

<u>Theorem / Definition</u> There is a unique (up to isomorphism ...) sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ on $X = \operatorname{Spec} R$ such that

$$\Gamma(D_f, \tilde{M}) = M_f = M \otimes D_f$$

(compatible with the restriction maps). Its stalks are the localisations $M_{\mathfrak{p}} = M \otimes R_{\mathfrak{p}}$

This gives a functor $M \mapsto \tilde{M}$

$$(R\text{-modules}) \longrightarrow (\mathcal{O}_{\operatorname{Spec} R}\text{-modules})$$

which is faithful — i.e. $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec} R}}(\tilde{M},\tilde{N})$

However there are $\mathcal{O}_X$-modules not isomorphic to $\tilde{M}$.

Ex. Let R be a discrete valuation ring, with field of fractions ~~uniformizer~~ ~~t,~~ ~~residue field~~ K. Let η, s be the generic and closed points of $X = \mathrm{Spec}\, R$. Then the sheaf $\mathcal{F}$ given by

$$\Gamma(X, \mathcal{F}) = 0$$
$$\Gamma(\{\eta\}, \mathcal{F}) = K$$

is not of the form $\tilde{M}$ [since $\Gamma(X,\tilde{M}) = M$!]

Remark :
This is an example of extension by zero: if $j: U \hookrightarrow X$ is the inclusion of an open subscheme [X now arbitrary] and $\mathcal{G}$ is a sheaf on U, then there is a sheaf $j_!\mathcal{G}$ on X by

$$(j_!\mathcal{G})(V) = \begin{cases} \mathcal{G}(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise}. \end{cases}$$

We have

$$\mathrm{Hom}(j_!\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(\mathcal{G}, j^{-1}\mathcal{F})$$

for sheaves [of abelian groups] $\mathcal{F}, \mathcal{G}$ on X, U ; so $j_!$ is left adjoint to j^{-1} (cf. pp 10 – 11).

Def^{n}. Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is said to be quasi-coherent if for every open affine $U \subset X$, $\mathcal{F}|_U \cong \tilde{M}$ for some $\mathcal{O}(U)$-module M.

A basic fact is that it is enough to check the condition for all U belonging to one affine covering of X so taking $X = \mathrm{Spec}\, R$ we see that the quasi-coherent sheaves on $\mathrm{Spec}\, R$ are precisely those of the form $\widetilde{M}$.

For the most part we deal with quasi-coherent modules, which are closest to the idea of a module over a ring.

<u>Examples</u> • Let $i : Y \hookrightarrow X$ be a closed subscheme. Write $\mathcal{J}_{Y/X} = \ker [i^\# : \mathcal{O}_X \to i_* \mathcal{O}_Y]$. Then $\mathcal{J}_{Y/X}$ is a sheaf of $\mathcal{O}_X$-modules; actually it is a sheaf of $\mathcal{O}_X$-ideals, as

$$\mathcal{J}_{Y/X}(U) = \ker [\mathcal{O}_X(U) \to \mathcal{O}_Y(Y \cap U)]$$

is an $\mathcal{O}_X(U)$-ideal. This construction gives a bijection between the set of closed subschemes of X and the set of quasi-coherent sheaves of ideals in $\mathcal{O}_X$ — this generalises the affine analogue on page 17 (and the proof is the same — cf. Hartshorne §II.5, especially p. 116).

• Let $f : X \to Y$ be any morphism. Then the homomorphism

$$f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$$

gives $f_* \mathcal{O}_X$ the structure of $\mathcal{O}_Y$-module [in fact it is an $\mathcal{O}_Y$-algebra]. It is not hard to see that $f_* \mathcal{O}_X$ is quasi-coherent if (say) f is quasi-projective.

More generally, if $f : X \to Y$ is a morphism and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $f_* \mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module under reasonable conditions on f, X (e.g. if X is quasi-projective over a Noetherian ring)

• (Kähler differentials) Recall that if $\varphi : R \to A$ is a ring homomorphism, the module of Kähler differentials $\Omega_{A/R}$ is defined to be the quotient of the free A-module on symbols $\{da \mid a \in R\}$ by relations -

$$d(a+b) = da + db \qquad\qquad (a,b \in A,\ r \in R)$$
$$d(ab) = a\,db + b\,da$$
$$d(\varphi(r)) = 0$$

<u>Eg</u> $\Omega_{R[x_1 \dots x_n]/R} = \bigoplus_{i=1}^{n} R.dx_i$, free of rank n.

$\Omega_{K/k} = 0$ if K/k is a separable algebraic extension of fields

The formation of $\Omega_{A/R}$ commutes with localisation. So if X/R is a scheme, there is a unique quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/R}$ such that if $U = \operatorname{Spec} A \subset X$ is open affine, then

$$\Omega_{X/R}|_U \cong \widetilde{\Omega_{A/R}}.$$

(in a way compatible with restriction)

Most standard operations on modules extend in an almost obvious way to $\mathcal{O}_X$-modules: for example:

- direct sum $\mathcal{F} \oplus \mathcal{G}$ by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$
- tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ = sheaf associated to presheaf
$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$
- "Sheaf hom" $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by
$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$
- "Dual sheaf" $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.
- Symmetric and exterior powers $\mathrm{Sym}^n \mathcal{F}$, $\Lambda^n \mathcal{F}$

For quasi-coherent modules these operations can described in terms of open affine coverings: let $X = \bigcup U_i$ with $U_i = \mathrm{Spec}\, A_i$. Let $\mathcal{F}|_{U_i} = \widetilde{M}_i$, $\mathcal{G}|_{U_i} = \widetilde{N}_i$ for $\mathcal{O}_X(U_i) = A_i$-modules M_i, N_i. Then

$$\mathcal{F} \otimes \mathcal{G}|_{U_i} = \widetilde{M_i \otimes_{A_i} N_i}, \quad \mathcal{F} \oplus \mathcal{G}|_{U_i} = \widetilde{M_i \oplus N_i},$$

and similarly for Sym^n and Λ^n.

Remark. It is not true in general that $(\mathcal{F} \otimes \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ for every open U (not necessarily affine) or that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$. In fact, for $\mathcal{F}, \mathcal{G}$ quasi-coherent $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ need not even be quasi-coherent.

Inverse image of a sheaf of modules Let $f: X \to Y$ be a morphism of schemes and $\mathcal{G}$ an $\mathcal{O}_Y$-module. Then $f^{-1}\mathcal{G}$ is not usually an $\mathcal{O}_X$-module [consider for example $X = \mathrm{Spec}\, K \to Y = \mathrm{Spec}\, k$ for a field extension K/k!], but it is an $f^{-1}\mathcal{O}_Y$-module. We define

$$f^* \mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

(recall that by adjunction $f^\#$ can be viewed as a homomorphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$). If $\mathcal{Y}$ is quasi-coherent one can be more explicit: if $U = \operatorname{Spec} A \subset X$, $V = \operatorname{Spec} B \subset Y$ are open affines with $f(U) \subset V$ and $\mathcal{Y}|_V = \tilde{M}$ for an B-module M then $f^*\mathcal{Y}|_U = \widetilde{M \otimes_B A}$. In the category of $\mathcal{O}_X$-modules (f^*, f_*) are adjoint:–

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{Y}, \mathcal{G}) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{Y}, f_*\mathcal{G}).$$

<u>Basic finiteness condition for $\mathcal{O}_X$-modules</u>. In what follows we will assume X is <u>quasi-projective</u> over a <u>Noetherian</u> ring R.

<u>Def</u>: An $\mathcal{O}_X$-module $\mathcal{Y}$ is <u>coherent</u> if for all open affines $U = \operatorname{Spec} A \subset X$, $\mathcal{Y}|_U \cong \tilde{M}$ for a <u>finitely generated</u> A-module M

(remark – it is enough to consider all U belonging to a fixed open covering of X).

The category of finitely-generated R-modules (R Noetherian) is stable under the usual operations $\oplus$, $\otimes_R$, $\operatorname{Hom}_R(\ ,\ )$, Λ^p, Sym^p, and dual. Therefore ~~the same is true for~~ the category of coherent sheaves is stable under $\oplus$, $\otimes_{\mathcal{O}_X}$, $\mathcal{H}om_{\mathcal{O}_X}$, Λ^p, Sym^p and $^\vee$. Also $f^*\mathcal{Y}$ is a coherent $\mathcal{O}_X$-module if $\mathcal{Y}$ is a coherent $\mathcal{O}_Y$-module [since $M \otimes_B A$ is finitely generated over A if M is finitely generated over B].

However $f_*\mathcal{Y}$ need not be coherent if $\mathcal{Y}$ is. For example, if $f : \mathbb{A}^1_R \to \operatorname{Spec} R$ and $\mathcal{Y} = \mathcal{O}_{\mathbb{A}^1_R}$, then $\mathcal{Y}$ is coherent but $f_*\mathcal{Y} = \widetilde{R[x]}$, and $R[x]$ is of course not finitely generated over R.

<u>Theorem</u> If $f : X \to Y$ is <u>projective</u>* and $\mathcal{Y}$ is a coherent $\mathcal{O}_X$-module, then $f_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_Y$-module.

[(*) <u>i.e.</u> f factors as $X \underset{\text{closed subscheme}}{\hookrightarrow} \mathbb{P}^N \times Y \to Y$]

For example, let X be projective over a field k. Then $f_* \mathcal{O}_X = \widetilde{M}$ where M is the k-vector space $\Gamma(X, \mathcal{O}_X)$. So the theorem implies that the space of global sections of X is finite-dimensional.

The earliest proof of this theorem uses cohomology [see the lectures of Srinivas, or Hartshorne III.5.2].

<u>More examples.</u> (X always q-proj. over Noetherian R)

- $Y \overset{i}{\hookrightarrow} X$ a closed subscheme, then $\mathcal{J}_{Y/X}$ is coherent (essentially by Hilbert's basis theorem)

- <u>locally free sheaves.</u> If X can be covered by open sets U_i such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module, then $\mathcal{F}$ is said to be <u>locally free</u>. A locally free sheaf is coherent $\Leftrightarrow$ the free modules $\mathcal{F}|_{U_i}$ are $\cong \mathcal{O}_{U_i}^{d_i}$ for $d_i < \infty$.

Locally free sheaves are essentially the same as <u>algebraic vector bundles</u>. Suppose $\cancel{\mathcal{F}}$ $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, and pick a covering $\{U_i\}$ such that $\mathcal{E}|_{U_i}$ is free. We may as well assume each U_i is affine ($=\operatorname{Spec} A_i$ say) so that

$$\mathcal{E}|_{U_i} \cong \widetilde{M}_i \qquad \text{(canonically!)}$$

where $M_i = \Gamma(U_i, \mathcal{E})$ is a free A_i-module. For each U_i we can then form (see p. 15.2 above)

$$\mathbb{V}[\mathcal{E}|_{U_i}] = \operatorname{Spec}\left[\operatorname{Sym}_{A_i}(M_i)\right]$$

These glue together to give a scheme $\mathbb{V}[\mathcal{E}]$ together with a morphism $\mathbb{V}[\mathcal{E}] \xrightarrow{\pi} X$. Notice that

$$\mathbb{V}[\mathcal{E}|_{U_i}] \cong \mathbb{A}^{d_i} \times U_i, \qquad d_i = \operatorname{rank}_{A_i}(M_i)$$

and in particular the fibres of π are all affine spaces. $\mathbb{V}[\mathcal{E}]$ is the <u>algebraic vector bundle</u> attached to $\mathcal{E}$. To recover $\mathcal{E}$ from $\mathbb{V}[\mathcal{E}]$ observe that if $U \subset X$ is open, then

$\{\text{sections of } \pi \text{ over } U\} \xrightarrow{\sim} \mathcal{E}^\vee(U)$

$\quad\ \ \|$

$\{\text{morphisms } s : U \to \mathbb{V}[\mathcal{E}] \text{ such that } \pi\circ s = \text{inclusion } U \hookrightarrow X\}$

(compare the construction on page 15.2)

Similar, the schemes $\mathrm{Proj}\,(\mathrm{Sym}_{A_i} M_i)$ glue together to give a scheme $\mathbb{P}[\mathcal{E}]$, with a morphism to X, whose fibres are projective spaces. One cannot recover $\mathcal{E}$ from $\mathbb{P}[\mathcal{E}]$ in general; if fact it is not hard to see that if $\mathcal{L}$ is a rank 1 locally free sheaf then

$$\mathbb{P}[\mathcal{E}\otimes\mathcal{L}] \cong \mathbb{P}[\mathcal{E}].$$

- Invertible sheaves. A locally free sheaf $\mathcal{L}$ of rank 1 is called an invertible sheaf (since $\mathcal{L}\otimes\mathcal{L}^\vee \cong \mathcal{O}_X$). The associated vector bundles are line bundles. Invertible sheaves are particular important for projective schemes. In fact, projective space $\mathbb{P}^n_R$ carries a special invertible sheaf, denoted $\mathcal{O}_{\mathbb{P}^n}(1)$. One way to define it is as follows. Consider the open subsets

$$U_i = \mathrm{Spec}\ A_i, \qquad A_i = R\left[\frac{X_0}{X_i}, \dots \frac{X_n}{X_i}\right]$$

which cover $\mathbb{P}^n_R$. Let M_i be the free A_i-module $A_i \cdot X_i = \left\{ \frac{f}{X_i^m} : f \text{ a homogeneous poly. of degree } m+1 \right\}$. Then

$\tilde{M}_i|_{U_i\cap U_j}$, $\tilde{M}_j|_{U_i\cap U_j}$ are canonically isomorphic (since X_i/X_j is invertible on $U_i\cap U_j$) and in fact there is a sheaf $\mathcal{O}(1)$ such that $\mathcal{O}(1)|_{U_i} = \tilde{M}_i$, compatible with these isomorphisms.

We define
$$\mathcal{O}(d) = \begin{cases} \bigotimes^d_{\mathcal{O}_X} \mathcal{O}(1) & \text{for } d>0 \\ \mathcal{O}_X & \text{for } d=0 \\ \mathcal{O}(-d)^\vee & \text{for } d<0. \end{cases}$$

The vector bundle $\mathbb{V}[\mathcal{O}(1)]$ is just the tautological line bundle on $\mathbb{P}^n$.

Then $\mathcal{O}(m) \otimes \mathcal{O}(n) = \mathcal{O}(m+n)$ for all m, n, and

$$\Gamma(\mathbb{P}^n_R, \mathcal{O}(d)) = \{\text{homogeneous } f \in R[X_0 \ldots X_n] \text{ of degree } d\}.$$

In particular this gives $\mathcal{O}(1) \otimes \mathcal{O}(-1) \cong \mathcal{O}$, but

$$0 = \Gamma(\mathbb{P}^n_R, \mathcal{O}(1)) \otimes \Gamma(\mathbb{P}^n_R, \mathcal{O}(-1)) \neq \Gamma(\mathbb{P}^n_R, \mathcal{O}) = R$$

(cf. p36 above)

If $X \overset{i}{\hookrightarrow} \mathbb{P}^n_R$ is any quasi-projective scheme over R, then we define

$$\mathcal{O}_X(1) = i^*\left[\mathcal{O}_{\mathbb{P}^n_R}(1)\right]$$

A crucial property of the sheaves $\mathcal{O}_X(d)$ is that, even though a coherent sheaf $\mathcal{F}$ on X need not have any global sections, for d sufficiently large $\mathcal{F} \otimes \mathcal{O}_X(d)$ has plenty of them: more precisely

<u>Theorem</u> Let $X \overset{i}{\hookrightarrow} \mathbb{P}^n_R$ be a quasi-projective scheme over R (a Noetherian ring) and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then there exists $d_0 \in \mathbb{Z}$ such that:— for each $d \geq d_0$, the images of the global sections $\Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(d))$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x \otimes \mathcal{O}_{X,x}(d)$ for all $x \in X$.

Another way to state this is that for every $d \geq d_0$ there exists a <u>surjection</u>

$$\mathcal{O}_X^r \twoheadrightarrow \mathcal{F} \otimes \mathcal{O}_X(d)$$

for some r (depending on d)

<u>Remark</u> In algebraic number theory a module M over a Dedekind domain R is said to be <u>locally free</u> if the <u>localisations</u> $M_\mathfrak{p}$ are free $R_\mathfrak{p}$-modules. This is not quite analogous to the sheaf-theoretical definition. Here is an example of Srinivas:—

Let $M = \{ \frac{m}{n} \in \mathbb{Q} \mid n \text{ is square-free} \}$.

Then for every prime number p

$$M_{(p)} = \mathbb{Z}_{(p)} \cdot \tfrac{1}{p} \quad \text{is free (of rank 1 !)}$$

and $M_{(0)} = \mathbb{Q}$. But M is clearly not a free $\mathbb{Z}$-module, and moreover $\tilde{M}$ is not locally free on $\operatorname{Spec} \mathbb{Z}$.

For finitely generated modules over a Noetherian ring, it is however true that

$$M_{\mathfrak{p}} \text{ is free over } R_{\mathfrak{p}} \text{ for all } \mathfrak{p} \iff \tilde{M} \text{ is a locally free } \mathcal{O}_{\operatorname{Spec} R}\text{-module}$$
$$\iff M \text{ is a projective } R\text{-module}.$$

<u>Line bundles and divisors</u> Assume that X is quasi-projective over a Noetherian ring R, and that X is <u>regular</u>. (weaker conditions will in fact suffice).

By a <u>divisor</u> on X we mean a finite formal sum $D = \sum m_i Y_i$, where $Y_i \subset X$ is an integral closed subscheme of codimension 1, and $m_i \in \mathbb{Z}$.

We associate to D the invertible sheaf $\mathcal{O}_X(D)$ as follows:- consider the ideal sheaves $\mathcal{J}_{Y_i/X}$. If $x \in X \setminus Y_i$ then $\mathcal{J}_{Y_i/X,x} = \mathcal{O}_{X,x}$, and if $x \in Y_i$ then $\mathcal{J}_{Y_i/X,x} \subset \mathcal{O}_{X,x}$ is an <u>ideal of height 1</u>. Since $\mathcal{O}_{X,x}$ is a UFD this implies that $\mathcal{J}_{Y_i/X,x}$ is a <u>principal</u> ideal, and so since $\mathcal{J}_{Y_i/X}$ is coherent, it is locally free of rank 1. Define

$$\mathcal{O}_X(Y_i) = \mathcal{J}_{Y_i/X}^{\vee} = \mathcal{J}_{Y_i/X}^{\otimes -1}$$

$$\mathcal{O}_X(D) = \bigotimes_i \mathcal{O}_X(Y_i)^{\otimes m_i}.$$

To go the other way, let $\mathcal{L}$ be an invertible sheaf on X. Let η be the generic point of X; $\mathcal{O}_{X,\eta} = k(\eta)$ is a field, the field of <u>rational functions</u> on X. Pick a non-zero $s \in \mathcal{L}_\eta$. If $y \in X$ is a point whose closure $\overline{\{y\}}$ has codimension 1, then $\mathcal{O}_{X,y}$ is a regular local ring of dimension 1, hence a discrete valuation ring, with uniformiser π_y say. Let $\mathrm{ord}_y(s) \in \mathbb{Z}$ be the unique integer such that

$$s.\mathcal{O}_{X,y} = \pi^{\mathrm{ord}_y(s)}.\mathcal{L}_y \subset \mathcal{L}_\eta .$$

We associate to $(\mathcal{L}, s)$ the divisor $\mathrm{div}(s) = \sum_y \mathrm{ord}_y(s).\overline{\{y\}}$, where we give $\overline{\{y\}}$ the reduced subscheme structure.

A principal divisor is one of the form $\mathrm{div}(s)$ where $s \in \mathcal{O}_{X,\eta}^*$ is a rational function on X. This construction gives a bijection

$$\{\text{isom}^{=} \text{ classes of invertible sheaves on } X\} \longleftrightarrow \frac{\{\text{divisors on } X\}}{\{\text{principal divisors}\}}$$

and the sum of divisors corresponds to the $\otimes$ of invertible sheaves. The group of isomorphism classes of invertible sheaves is denoted $\mathrm{Pic}\, X$, and called the Picard group.

For example, $\Gamma(\mathbb{P}^n_k, \mathcal{O}(1)) = \{\text{degree 1 homogeneous polynomials}\}$, so the divisor of any non-zero section of $\mathcal{O}_{\mathbb{P}^n}(1)$ is simply a hyperplane section H, so $\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_{\mathbb{P}^n}(H)$. However $\mathcal{O}_{\mathbb{P}^n}(1)$ is better than any of the sheaves $\mathcal{O}_{\mathbb{P}^n}(H)$, since it is canonical. In fact, if $\mathcal{E}$ is a locally free sheaf on X, and if $\pi : \mathbb{P}[\mathcal{E}] \to X$ is the associated projective bundle, then for each open $U \subset X$ over which $\mathcal{E}$ is free, we get a sheaf $\mathcal{O}(1)$ on $\pi^{-1}(U)$. These patch together to give an invertible sheaf $\mathcal{O}_{\mathbb{P}[\mathcal{E}]}(1)$ on $\mathbb{P}[\mathcal{E}]$. However there need not an algebraic family of hyperplane sections $\{H_x \subset \pi^{-1}(x) \mid x \in X\}$.

Now take $X = \mathbb{P}^1_k$, and consider $\Omega^1_{X/k}$, which as we saw earlier is locally free of rank $\dim(X) = 1$. The differential dx/x at the generic point has divisor

$\mathrm{div}(dx/x) = -(0) - (\infty)$, so $\Omega^1_{\mathbb{P}^1_R} \cong \mathcal{O}(-2)$. For $\mathbb{P}^n$, $n > 1$ the situation is slightly more complicated. There is an exact sequence for $X = \mathbb{P}^n_R$.

$$0 \longrightarrow \Omega_{X/R} \xrightarrow[\alpha]{} \mathcal{O}_X(-1)^{\oplus n+1} \xrightarrow[\beta]{} \mathcal{O}_X \longrightarrow 0$$

where α, β are defined as follows: let $e_0, \dots e_n$ be the "basis elements" of the middle $\mathcal{O}_X$-module, so

$$\Gamma(U, \mathcal{O}_X(-1)^{\oplus n+1}) = \{ \sum_0^n f_j e_j \mid f_j \in \Gamma(U, \mathcal{O}_X(-1)) \}.$$

On the distinguished open subset $U_i = \mathrm{Spec}\, R[\frac{x_0}{x_i} \dots \frac{x_n}{x_i}]$ define

$$\alpha : d(x_j/x_i) \longmapsto x_i^{-2}(x_i e_j - x_j e_i) \in \Gamma(U_i, \mathcal{O}(-1)^{\oplus n+1})$$
$$\beta : \sum f_j e_j \longmapsto \sum_j f_j x_j \in \Gamma(U, \mathcal{O}_X).$$

The exterior powers then give $\Lambda^n \Omega_{X/R} \cong \mathcal{O}_X(-n-2)$ for the <u>canonical sheaf</u> $\omega_{X/R} = \Lambda^n \Omega_{X/R}$. Any divisor K such that $\mathcal{O}(K) \cong \omega_{X/R}$ is called a <u>canonical divisor</u>.

In general, if $\mathcal{L} \in \mathrm{Pic}\, X$ the associated divisor class is denoted $c_1(\mathcal{L})$. One of the aims of <u>intersection theory</u> is to define groups of (suitable equivalence classes) of codimension r subschemes, $CH^r(X)$ (<u>Chow groups</u>) and <u>Chern classes</u>

$$c_r(\mathcal{E}) \in CH^r(X)$$

for any vector bundle $\mathcal{E}$; $CH^1(X)$ is the divisor class group introduced above.