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The Abel-Jacobi Theorem

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In modern language this famous theorem says that for a compact Riemann surface X, its Picard variety $\mathrm{Pic}_0(X)$ and its Jacobian $J(X)$ are canonically isomorphic. Of course, the theorem was originally formulated and proved by Abel and Jacobi around 1825 in much more elementary terms (recall that at this time, even the notion of a Riemann surface itself had to wait for 25 more years). The aim of these notes is to present a proof of this theorem as elementary as possible, but indicating in many places the relation to more advanced topics and further developments.

The whole story began with the attempt of making any sense out of an integral of the form

$$\int \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}$$

or, more generally,

$$\int R(x,y)\,dx$$

with a rational function R of two complex variables x and y which are related by a polynomial equation $f(x,y) = 0$ of degree ≥ 3 (such integrals are still called Abelian).

The first step is now to consider

$$\omega = R(x,y)\,dx$$

as a meromorphic differential on the Riemann surface X defined by f. Let us assume that ω is even holomorphic on X (this is the first case to consider: abelian differentials of the first kind). Recall from the Riemann-Roch theorem that the holomorphic differentials on X form a g-dimensional complex vector space $\Omega(X) = H^0(X,\Omega)$, where g is the genus of X.

Now the definite integral

$$\int_P^Q \omega$$

on X depends on the chosen path from P to Q, or more precisely on its homology class. Hence $\int_P^Q \omega$ is only determined up to adding $\int_\alpha \omega$ for a closed path α on X. The homology classes of these cycles form a free abelian group

$$H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$$

of rank $2g$, hence the complex numbers $\{\int_\alpha \omega : \alpha \in H_1(X,\mathbb{Z})\}$ form a subgoup of $\mathbb{C}$ which, if $g \geq 2$, in almost all cases is a dense subset. So it is impossible to attribute any reasonable meaning to $\int_P^Q \omega$, except on an elliptic curve.

The situation definitely improves if instead of looking at a single holomorphic differential ω, we consider a basis $\omega_1, \ldots, \omega_g$ of $\Omega(X)$ (which shall be fixed throughout these notes): Let

$$\Lambda := \{(\int_\alpha \omega_1, \ldots, \int_\alpha \omega_g) \in \mathbb{C}^g : \alpha \in H_1(X,\mathbb{Z})\}$$

Let us fix moreover a canonical (or symplectic) basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ of $H_1(X,\mathbb{Z})$. Then obviously Λ is an additive subgroup of $\mathbb{C}^g$ generated by the $2g$ vectors

$$A_i := (\int_{\alpha_i} \omega_1, \ldots, \int_{\alpha_i} \omega_g), \qquad i = 1, \ldots, 2g$$

where from now on we make the convention

$$\alpha_{i+g} := \beta_i, \qquad i = 1, \ldots, g$$

Proposition: *Λ is a lattice*
(i.e. discrete and not contained in any hyperplane).
equivalently: $A_1, \ldots, A_{2g}$ are $\mathbb{R}$-linearly independent.

The proof will be an easy consequence of Abel's theorem and is therefore postponed. We note the following important consequence:

Corollary: *The map*

$$\mu : X \times X \longrightarrow J(X) := \mathbb{C}^g/\Lambda$$
$$(P \quad) \longmapsto (\int_P^Q \omega_1, \ldots, \int_P^Q \omega_g) \bmod \Lambda$$

is well defined.

We will exploit this fact and study this map in detail, but first note that

$$J(X) := \mathbb{C}^g/\Lambda$$

is not only an abelian group, but also a complex manifold: topologically $J(X)$ is obtained from the (fundamental) parallelotop spanned by the $2g$ vectors $A_1, \ldots, A_{2g}$ in $\mathbb{C}^g \cong \mathbb{R}^{2g}$ by identifying opposite sides, so $J(X)$ is a $2g$-dimensional real torus. Moreover the quotient map $\mathbb{C}^g \to J(X)$ is a local homeomorphism (even the universal covering map), hence can be used to put an analytic structure on this torus. Note that with this complex structure on $J(X)$, the map μ of the Corollary is analytic. Using Riemann's bilinear relations and theta functions one shows that $J(X)$ can even be embedded into a projective space, i.e. $J(X)$ is an abelian variety. It is called the ***Jacobian*** of X.

Finally note that the analytic torus $J(X)$ does not (up to analytic isomorphism) depend on the choice of a basis of $\Omega(X)$ since the change from one basis to another is given by a matrix in $\mathrm{GL}_g(\mathbb{C})$, hence biholomorphic. The analytic structure of $J(X)$ depends however on X, although as groups and even as real analytic manifolds all tori of the same dimension are isomorphic.

We now come to the formulation of the Abel–Jacobi theorem: Fix a point $P_0 \in X$ and consider the map $P \mapsto \mu(P_0, P)$; this is an analytic map $X \to J(X)$, which we shall again denote by μ. One important consequence of the Abel–Jacobi theorem is that this μ is injective and even an analytic embedding.

We need to extend our map μ to divisors: recall that the group of (Weil) divisors $\mathrm{Div}(X)$ on X is defined to be the free abelian group on the points of X, i.e.

$$\mathrm{Div}(X) = \{\sum_{P \in X} n_P P : n_P \in \mathbb{Z},\ n_P = 0 \text{ for all but finitely many } P\}$$

with formal addition as group law. For $D = \sum_{P\in X} n_P P \in \mathrm{Div}(X)$, the degree of D is $\deg D = \sum_{P \in X} n_P$.

Theorem (Abel–Jacobi): *The map*

$$\mu : \mathrm{Div}_0(X) \longrightarrow J(X)$$
$$D = \sum_{P\in X} n_P P \longmapsto (\sum_{P\in X} n_P \int_{P_0}^P \omega_1, \ldots, \sum_{P\in X} n_P \int_{P_0}^P \omega_g)$$

is a surjective group homomorphism with kernel $\mathrm{Div}_h(X)$.

Moreover μ is independent of the choice of P_0.

($\mathrm{Div}_0(X)$ is the group of divisors of degree 0; $\mathrm{Div}_h(X)$ is the group of principal divisors of X, i.e. divisors of the form $D = \mathrm{div}(f)$ for a meromorphic function f on X; recall that for $f \in \mathcal{M}(X)$, $\mathrm{div}(f) = \sum_{P\in X} \mathrm{ord}_P(f) \cdot P$ is a divisor of degree 0.)

At several places in the proof we shall need the following

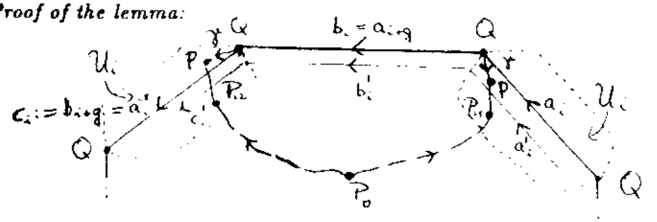
Lemma 1: *Let $\mathcal{P} \subset \mathbb{H}$ be a fundamental polygon for X (obtained from a canonical dissection of X, say). Choose $P_0 \in \mathring{\mathcal{P}}$ and let η be a meromorphic differential on X without poles on $\partial\mathcal{P}$. For a holomorphic differential $\omega \in \Omega(X)$, consider the holomorphic function f on $\mathring{\mathcal{P}}$ given by $f(P) := \int_{P_0}^P \omega$ (which is well defined since $\mathring{\mathcal{P}}$ is simply connected).*
__Then:__

$$2\pi i \sum_{P\in X} \mathrm{Res}_P(f\eta) = -\sum_{i=1}^{2g} \left(\int_{a_i} \eta\right)\left(\int_{b_i} \omega\right)$$

where a_i and b_i are the edges of $\mathcal{P}$ as in the figure on the next page, and where we use the conventions $a_{i+g} := b_i$ and $b_{i+g} := a_i^{-1}$.

(This lemma is sometimes called the first reciprocity law; it also plays a key rôle in the proof of Riemann's period relations.)

Proof of the lemma:



Let $U_i (i = 1, \dots, g)$ be a simply connected neighbourhood of $a_i - \{Q\}$ in X, and denote, as will always be done, by the same symbol its inverse image in a neighbourhood of $\mathcal{P}$ in $\mathbb{H}$. Choose points P_{i1} and P_{i2} on either side of a_i in U_i. For a point $P \in U_i$ define

$$f_{i1}(P) := \int_{P_0}^{P_{i1}} \omega + \int_{P_{i1}}^{P} \omega$$
$$f_{i2}(P) := \int_{P_0}^{P_{i2}} \omega + \int_{P_{i2}}^{P} \omega$$

where in both formulas the first integral is computed along a path in $\mathring{\mathcal{P}}$, and the second one in U_i.

Finally choose a path γ in $U_i \cup \{Q\}$ joining P to Q as indicated. Then clearly the path $\gamma P P_{i2} P_0 P_{i1} P \gamma^{-1} b_i$ is nullhomotopic in X, therefore we have

$$f_{i1}(P) - f_{i2}(P) = -\int_{b_i} \omega_i$$

Now for a polygon $\mathcal{P}'$ inside $\mathcal{P}$ which is close enough to $\mathcal{P}$ such that all the poles of η lie inside $\mathcal{P}'$, we can apply the residue theorem for domains in $\mathbb{C}$ to obtain

$$\int_{\partial \mathcal{P}'} f\eta = 2\pi i \sum_{P \in \mathcal{P}'} \mathrm{Res}_P(f\eta) = 2\pi i \sum_{P \in X} \mathrm{Res}_P(f\eta)$$

The left hand side is $\sum_{i=1}^{2g} (\int_{a_i'} f\eta + \int_{c_i'} f\eta)$. Letting $\mathcal{P}'$ tend to $\mathcal{P}$ we finally obtain

$$2\pi i \sum_{P \in X} \mathrm{Res}_P(f\eta) = \sum_{i=1}^{2g} \int_{a_i} (f_{i1} - f_{i2})\eta = -\sum_{i=1}^{2g} \left(\int_{a_i} \eta\right)\left(\int_{b_i} \omega\right)$$

Now we come to the proof of the theorem:
It is clear from the definition that μ is a group homomorphism.
If we replace P_0 by a point $P_0' \in X$, μ is changed by adding $\sum_{P \in X} n_P \int_{P_0}^{P_0'} \omega_i$ to the i-th component. Since $\sum_{P \in X} n_P = \deg D = 0$, we see that μ is independent of the choice of P_0.

To show that μ factors through $\mathrm{Pic}_0(X)$, i.e. $\mathrm{Div}_h(X) \subset \ker \mu$, let $f \in \mathcal{M}(X)$ and let $D = \mathrm{div}(f) = \sum_{P \in X} \mathrm{ord}_P(f) \cdot P$ be the corresponding principal divisor. Then according to lemma 1 we have

$$\begin{aligned}
\mu(D) &= \left(\sum_{P \in X} \mathrm{ord}_P(f) \int_{P_0}^{P} \omega_1, \dots, \sum_{P \in X} \mathrm{ord}_P(f) \int_{P_0}^{P} \omega_g\right) \\
&= \left(\sum_{P \in X} (\int_{P_0}^{P} \omega_1) \mathrm{Res}_P \frac{df}{f}, \dots, \sum_{P \in X} (\int_{P_0}^{P} \omega_g) \mathrm{Res}_P \frac{df}{f}\right) \\
&= \left(\sum_{P \in X} \mathrm{Res}_P (\int_{P_0}^{P} \omega_1) \frac{df}{f}, \dots, \sum_{P \in X} \mathrm{Res}_P (\int_{P_0}^{P} \omega_g) \frac{df}{f}\right) \\
&= \left(-2\pi i \sum_{i=1}^{2g} \left(\int_{a_i} \frac{df}{f}\right)\left(\int_{b_i} \omega_1\right), \dots, -2\pi i \sum_{i=1}^{2g} \left(\int_{a_i} \frac{df}{f}\right)\left(\int_{b_i} \omega_g\right)\right)
\end{aligned}$$

and this is an element of Λ because $\frac{1}{2\pi i} \int_{a_i} \frac{df}{f}$ is an integer (since a primitive of $\frac{df}{f}$ is $\log f$, which is well defined up to $2\pi i \mathbb{Z}$).

(The third equality in the above computation holds because all poles of $\frac{df}{f}$ are simple.)

The next step in the proof is ***Abel's theorem:*** the injectivity of the map $\mu : \mathrm{Pic}_0(X) \to J(X)$. Thus we have to show that $\ker \mu \subset \mathrm{Div}_h(X)$.

Let $D = \sum_{P \in X} n_P P \in \ker \mu$. We have to find $f \in \mathcal{M}(X)$ with $D = \mathrm{div} f$. We shall show that there exists a meromorphic differential η on X with the following properties:

(i) η is holomorphic on $X - \mathrm{Supp}\, D$
(ii) $\mathrm{ord}_P(\eta) = -1$ for all $P \in \mathrm{Supp}\, D$
(iii) $\mathrm{Res}_P(\eta) = n_P$ for all $P \in \mathrm{Supp}\, D$
(iv) $\int_{a_i} \omega \in 2\pi i \mathbb{Z}$ for $i = 1, \dots, 2g$

Once we have found such η we set

$$f(P) := \exp(\int_{P_0}^{P} \eta)$$

This f is well defined because of (iv), holomorphic outside $\mathrm{Supp}(D)$ because of (i); near $P \in \mathrm{Supp}\, D$ it behaves like z^{n_P} because of (ii) and (iii), hence is meromorphic. Finally $\mathrm{ord}_P(f) = n_P$.

To prove the existence of η with properties (i) – (iii) we use Riemann-Roch: Let $\operatorname{Supp} D =: \{P_1, \ldots, P_n\}$, $\bar{D} := P_1 + \ldots + P_n$ and $\Omega(\bar{D})(X)$ the set of all meromorphic differentials η on X with $\operatorname{div} \eta \geq -\bar{D}$. If K denotes a canonical divisor on X, we get from the Riemann-Roch theorem

$$\dim \Omega(\bar{D})(X) = h_0(K + \bar{D}) = h_0(-\bar{D}) - \deg(-\bar{D}) - 1 + g = n - 1 + g$$

Now consider the $\mathbb{C}$-linear map

$$\begin{array}{rcl} \Omega(\bar{D})(X) & \xrightarrow{\rho} & \mathbb{C}^n \\ \eta & \longmapsto & (\operatorname{Res}_{P_1}\eta, \ldots, \operatorname{Res}_{P_n}\eta) \end{array}$$

Since the sum of the residues of a meromorphic differential on a Riemann surface is always 0, the image of ρ is contained in the hyperplane $H := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} z_i = 0\}$. Obviously $\ker \rho = \Omega(X)$, which is g-dimensional. So the image of ρ is of dimension $n + 1 + g - g = n + 1 = \dim H$. Thus ρ is surjective, and we find $\eta \in \rho^{-1}(n_{P_1}, \ldots, n_{P_n})$, i.e. satisfying properties (i) – (iii). Note that this argument works for all divisors, not only those of degree 0.

It remains to show that we can choose $\eta \in \rho^{-1}(n_{P_1}, \ldots, n_{P_n})$ satisfying property (iv). We introduce the notations

$$f_k(P) := \int_{P_0}^{P} \omega_k, \qquad k = 1, \ldots, g$$

$$A_{ik} := \int_{a_i} \omega_k, \qquad k = 1, \ldots, g;\ i = 1, \ldots, 2g$$

$$B_{ik} := \int_{b_i} \omega_k, \qquad k = 1, \ldots, g;\ i = 1, \ldots, 2g$$

we can compute the k-th component of $\mu(D)$, using lemma 1:

$$(\mu(D))_k = \sum_{P \in X} n_P f_k(P) = \sum_{P \in X} \operatorname{Res}_P(f_k \eta)$$

$$= -\frac{1}{2\pi i} \sum_{i=1}^{2g} \left(\int_{a_i} \eta \right) B_{ik}$$

On the other hand we have assumed that $\mu(D)$ is in Λ; hence we find $m_i \in \mathbb{Z}$, $i = 1, \ldots, 2g$ such that

$$(\mu(D))_k = \sum_{i=1}^{2g} m_i B_{ik}$$

These two computations imply

$$\sum_{i=1}^{2g} B_{ik}(\int_{a_i} \eta + 2\pi i m_i) = 0$$

Now we use the following

Lemma 2: *For $i = 1, \ldots, 2g$, let $A_i := (A_{i1}, \ldots, A_{ig})$ and $B_i := (B_{i1}, \ldots, B_{ig})$. Then for any vector $x = (x_1, \ldots, x_{2g}) \in \mathbb{C}^{2g}$ we have*

$\sum_{i=1}^{2g} x_i \cdot B_i = 0$ if and only if there is $b \in \mathbb{C}^g$ such that $x_i = b \cdot A_i$ for all i.

In our situation the hypothesis of this lemma is satisfied with $x_i = \int_{a_i} \eta + 2\pi i m_i$. Then taking $b = (b_1, \ldots, b_g)$ as in the conclusion of the lemma, and putting

$$\tilde{\eta} := \eta - \sum_{j=1}^{g} b_j \omega_j$$

we find

$$\int_{a_i} \tilde{\eta} = \int_{a_i} \eta - \sum_{j=1}^{g} b_j \int_{a_i} \omega_j = \int_{a_i} \eta - b \cdot A_i = \int_{a_i} \eta - x_i = 2\pi i m_i$$

Hence $\tilde{\eta}$ satisfies all four desired properties.

Proof of lemma 2:

" $\Leftarrow$ ": Let $b = (b_1, \ldots, b_g)$ and $\omega := \sum_{j=1}^{g} b_j \omega_j$. Then $b \cdot A_i = \sum_{j=1}^{g} b_j \int_{a_i} \omega_j = \int_{a_i} \omega$, and by lemma 1 we have, since ω is holomorphic: $\sum_{i=1}^{2g} (\int_{a_i} \omega) \cdot B_i = 0$.

" $\Rightarrow$ ": Let us show first that $A_1, \ldots, A_{2g}$ (and hence also $B_1, \ldots, B_{2g}$) generate $\mathbb{C}^g$: assume that this is not true, and let $y = (y_1, \ldots, y_g) \in \mathbb{C}^g$ be orthogonal to the subspace spanned by $A_1, \ldots, A_{2g}$, and let $\omega := \sum_{j=1}^{g} y_j \omega_j \in \Omega(X)$.

Then $\int_{a_i} \omega = \sum_{j=1}^{g} y_j A_{ij} = y \cdot A_i = 0$ for all i, hence $P \mapsto \int_{P_0}^{P} \omega$ is a well defined holomorphic function on X. As X is compact, this function is constant, and thus $\omega = 0$. But then $y = 0$, because the ω_j are linearly independent.

Consequences of this fact are

1) $\rho : \mathbb{C}^g \to \mathbb{C}^{2g}$, $y \mapsto (y \cdot A_1, \ldots, y \cdot A_{2g})$ is injective

2) $\varphi : \mathbb{C}^{2g} \to \mathbb{C}^g$, $x \mapsto \sum_{i=1}^{2g} x_i B_i$ is surjective

From the first part of the proof we already know that the kernel of φ contains the image of ρ. On the other hand, 1) and 2) show that $\dim(\ker \varphi) = \dim(\operatorname{im} \rho) = g$, hence $\ker \varphi = \operatorname{im} \rho$, as desired.

This concludes the proof of Abel's theorem, and we now turn to Jacobi's part of the theorem: the surjectivity of μ.

Since μ is a group homomorphism it is sufficient to show that μ is surjective onto a neighbourhood of 0. For arbitrary points $a_1, \ldots, a_g$ in X choose local charts (U_i, t_i), $i = 1 \ldots, g$, and consider the map

$$v = v_{a_1,\ldots,a_g} : \prod_{i=1}^{g} U_i \longrightarrow \mathbb{C}^g$$

$$(P_1, \ldots, P_g) \longmapsto \mu\left(\sum_{i=1}^{g}(P_i - a_i)\right) = \left(\sum_{i=1}^{g}\int_{a_i}^{P_i}\omega_1, \ldots, \sum_{i=1}^{g}\int_{a_i}^{P_i}\omega_g\right)$$

It suffices to find $a_1, \ldots, a_g$ such that the map $v_{a_1,\ldots,a_g}$ is locally surjective. By the implicit function theorem this happens as soon as the determinant Δ of the Jacobi matrix of v is nonzero. If in local coordinates $\omega_j = f_{ij}(t_i)\,dt_i$, we have $\Delta = \det(f_{ij}(0))$. Hence Δ can also be interpreted as the determinant of the linear map

$$\Omega(X) \to \mathbb{C}^g,\ \omega \mapsto (f_1(0), \ldots, f_g(0))$$

where $\omega = f_i(t_i)\,dt_i$ in local coordinates. Writing $\omega(a_i) := f_i(0)$ we are done if we find $a_1, \ldots, a_g$ in such a way that there is no nonzero $\omega \in \Omega(X)$ with $\omega(a_i) = 0$ for $i = 1, \ldots, g$.

Now for any $a \in X$ consider the vector space $N(a) := \{\omega \in \Omega(X) : \omega(a) = 0\}$. It is either equal to $\Omega(X)$, or it is of codimension 1. But of course $\bigcap_{a \in X} N(a) = \{0\}$, hence we can find $g = \dim \Omega(X)$ of theses vector spaces such that their intersection becomes trivial.

This finally ends the proof of the Abel-Jacobi theorem.

It still remains to prove the proposition, i.e. that Λ, the additive subgroup of $\mathbb{C}^g$ generated by $A_1, \ldots, A_{2g}$, is a lattice.

This will follow from the surjectivity of μ by purely topological considerations: Consider the map

$$\widetilde{\varphi} : X^g \to \mathrm{Pic}_0(X),\ (P_1, \ldots, P_g) \mapsto \text{class of } \sum_{i=1}^{g}(P_i - P_0)$$

Let us show that this map is surjective:
let $D \in \mathrm{Div}_0(X)$; by Riemann-Roch $l(D + gP_0) \geq g + 1 - g = 1$; so there exists $f \in L(D + gP_0)$, hence also an effective divisor $\widetilde{D}$ of degree g such that

$$\operatorname{div} f = -D - gP_0 + \widetilde{D}$$

Let $\widetilde{D} = P_1 + \ldots + P_g$; then $\sum_{i=1}^{g}(P_i - P_0)$ is linearly equivalent with D, which shows the surjectivity of $\widetilde{\varphi}$.

From the theorem it follows that the composite map

$$\Phi := \mu \circ \widetilde{\varphi} : X^g \to J(X),\ (P_1, \ldots, P_g) \mapsto (\sum_{i=1}^{g}\int_{P_0}^{P_i}\omega_1, \ldots, \sum_{i=1}^{g}\int_{P_0}^{P_i}\omega_g)$$

is also surjective. Since Φ is obviously continuous, the image $J(X) = \Phi(X^g)$ of the compact space X^g is also compact.

Now assume Λ were not a lattice; then Λ were contained in some real hyperplane $H \subset \mathbb{R}^{2g}$, and $\mathbb{C}^g/\Lambda \cong (H \times \mathbb{R})/\Lambda \cong H/\Lambda \times \mathbb{R}$ could not be compact.

We end these notes with a few remarks:

1) *Algebraic construction of the Jacobian*

In view of the Abel-Jacobi theorem an algebraic construction of $J(X)$ comes down to endow the group $\mathrm{Pic}_0(X)$ with the structure of an algebraic variety (and hence an algebraic group).

First note that $\widetilde{\varphi}_g(P_1, \ldots, P_g)$ does not depend on the order of the g points, hence $\widetilde{\varphi}_g$ induces a map

$$\varphi_g : X^{(g)} \longrightarrow \mathrm{Pic}_0(X)$$

where $X^{(g)} := X^g/S_g$ is the g-fold symmetric product of X. By the theorem of elementary symmetric functions, $X^{(g)}$ is a complex manifold of dimension g. It is convenient to write the elements of $X^{(g)}$ as effective divisors of degree g.

The fibre $\varphi_g^{-1}(\varphi_g(D))$ for $D = P_1 + \ldots + P_g \in X^{(g)}$ consits of all effective divisors $D' \in X^{(g)}$ linearly equivalent with D, thus

$$\varphi_g^{-1}(\varphi_g(D)) = \{D + \operatorname{div} f : f \in L(D)\} \cong \mathbb{P}(L(D))$$

In particular the fibres of φ_g, being projective spaces, are connected. Identifying $\mathrm{Pic}_0(X)$ and $J(X)$ by the Abel-Jacobi theorem we may view $\mathrm{Pic}_0(X)$ as a compact g-dimensional complex manifold. The same holds for $X^{(g)}$, and since φ_g is continuous, it is generically finite. The above observation then shows that φ_g is generically injective (but definitely not globally, except for $g = 1$!).

In the abstract situation of a smooth projective curve X over an algebraically closed field k, we still have the map φ_g from the projective variety $X^{(g)}$ to the abelian group $\mathrm{Pic}_0(X)$ of divisor classes of degree 0, and it is still generically injective (although, of course, the proof is completely different: essentially it requires a refinement of the Riemann-Roch theorem). So on an open (Zariski-) dense subset of $X^{(g)}$, φ_g induces "almost" a group law; A. Weil developped a general technique to construct an algebraic group out of such a "group chunk". This abelian variety is then defined to be the Jacobian of X and turns out to be, as a group, isomorphic to $\mathrm{Pic}_0(X)$.

2) ***Elliptic curves***

In the particular case of genus 1, the map φ_1 is not only generically injective, but also globally, hence it is an isomorphism. This is because for an effective divisor of degree 1, i.e. a point $P \in X$, we have $L(P) = \mathbb{C}$, because there is no meromorphic function on X having only a single simple pole.

Note that this (analytic) isomorphism $X \to J(X) = \mathbb{C}/\Lambda$ is one way to define a group law on an elliptic curve. It is not hard to deduce from this isomorphism (i.e. from Abel's theorem) the addition theorem for the Weierstraß $\wp$-function.

3) ***Exponential sequence***

A more abstract approach to the Abel–Jacobi theorem is through the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 0$$

which is a short exact sequence on any complex manifold. For compact connected X, the associated long exact cohomology sequence begins

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(X, \mathbb{Z}) \to \dots$$

Of course, the map $\mathbb{C} \to \mathbb{C}^*$ is the usual exponential map, hence surjective. So we obtain an exact sequence of abelian groups*)

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \dots$$

Now if X is a compact Riemann surface of genus g, $H^1(X, \mathbb{Z})$, being dual to $H_1(X, \mathbb{Z})$, is a free abelian group of rank $2g$, $H^1(X, \mathcal{O}_X)$ is by Serre duality isomorphic to $H^0(X, \Omega)$, hence a complex vector space of dimension g, $H^1(X, \mathcal{O}_X^*)$ is isomorphic to $\mathrm{Pic}(X)$ (this holds for any locally ringed space), $H^2(X, \mathbb{Z})$ is dual to $H_0(X, \mathbb{Z})$, hence isomorphic to $\mathbb{Z}$, and $H^2(X, \mathcal{O}_X) = 0$ by the general theory (Grothendieck's vanishing theorem).
Thus we obtain an exact sequence

$$0 \to \mathbb{Z}^{2g} \to \mathbb{C}^g \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0$$

The last map in this sequence is the Chern class or in this case simply the degree of the divisor (or the line bundle), so that we get from this sequence immediately the isomorphism of groups

$$\mathrm{Pic}_0(X) \cong \mathbb{C}^g / \mathbb{Z}^{2g}$$

*) Note that the exponential sequence is only defined in the analytic category; by GAGA theorems we may identify the analytic and the algebraic cohomology groups of $\mathcal{O}_X$, and also the analytic and the algebraic Picard group.

References:

Our proof of the Abel–Jacobi theorem follows mainly

E. Reyssat: Quelques aspects des surfaces de Riemann, Ch.XII,§1.
Progress in Mathematics, vol.77; Birkhäuser 1989

For slightly different proofs see

P. Griffiths, J. Harris: Principles of Algebraic Geometry, Ch.2,§2;
Wiley Interscience 1978

O. Forster: Lectures on Riemann surfaces.
Graduate Texts in Mathematics 81, Springer 1980.

The approach indicated in Remark 3) is carried out in

R. Gunning: Lectures on Riemann surfaces, §8.
Princeton Mathematical Notes 1966.