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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Topology and Differential Geometry - Lecture 2

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Topology and Differential Geometry

Lecture II

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Chern-Weil theory of characteristic classes

1 Review of lecture I

In lecture I the notion of vector bundles and connections were explained as well as curvature and related notions.
Let me add a few comments.
We are often dealing with differential operators on vector bundles and switch between local and global consideration.
Thus I think it is usefull to apply sheaves as an appropriate language. If $E \to X$ is a vector bundle we get an associate sheaf of germs of sections, denoted by $\mathcal{C}^\infty_X(E)$, i.e. $\mathcal{C}^\infty_X(E)(U) = \Gamma(E|U)$.

(smooth sections over the open set U)
It is a sheaf of $\mathcal{C}^\infty_X$-modules (where $\mathcal{C}^\infty_X$ denotes the sheaf of germs of C^∞-functions with real or complex values according to the case of real or complex bundles)
Each map of ~~bundles~~ vector bundles $E \xrightarrow{\varphi} F$ induces a $\mathcal{C}^\infty_X$-linear morphism of sheaves, also denoted by φ. In this way we get an equivalence of categories

$$\left\{\begin{matrix}\text{Category of vector bundles}\\ \text{and maps of vector bundles}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Category of}\\ \text{locally free sheaves}\\ \text{of } \mathcal{C}^\infty_X\text{-modules and}\\ \mathcal{C}^\infty_X\text{-linear morphisms}\end{matrix}\right\}$$

A connection on E was defined as a $\mathbb{R}$-linear map $\nabla : \mathcal{C}^\infty_X(E) \to \mathcal{C}^\infty_X(T^*X \otimes E)$ satisfying Leibnitz rule.
It can be considered a a generalization of the differential of a function and also extends to exterior differentiation
To be precise, denoting

$$\mathcal{C}^\infty_X(\Lambda^p T^*X \otimes E) = \mathcal{A}^p(E)$$

we get a sequence of operators

$$\mathcal{A}^0(E) \xrightarrow{\nabla} \mathcal{A}^1(E) \xrightarrow{\nabla} \mathcal{A}^2(E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{A}^p(E) \to \cdots$$

$\mathcal{A}^p(E) \xrightarrow{\nabla} \mathcal{A}^{p+1}(E)$ satisfies

$$\nabla(\alpha\otimes e) = d\alpha\otimes e + (-1)^p\,\alpha\wedge\nabla e$$

which could be used for defining ∇, or we can define ∇ on $\mathcal{A}^p(E)$ more explicitely by considering

$$\mathcal{A}^p(E) = \mathcal{C}^\infty(\Lambda^p T^*X\otimes E) = \mathcal{C}^\infty_X(HOM(\Lambda^p TX, E))$$

$$\underset{def}{=} \mathcal{H}om(\Lambda^p TX, E)$$

(here $HOM(F,E)$ denotes the <u>bundle</u> of homomorphismes of ~~the~~ vector bundle F to E, as explained in lecture I)

If $\varphi \in \mathcal{H}om(\Lambda^p TX, E)$ and $v_0,\dots,v_p$ are germs of vector fields on X then $\nabla\varphi$ can be defined as

$$(\nabla\varphi)(v_0\wedge v_1\wedge\dots\wedge v_p) = \sum_{j=0}^{p}(-1)^j\nabla_{v_j}(\varphi(v_0\wedge\dots\wedge\check{v}_j\wedge\dots\wedge v_p)) + \sum_{j<k}(-1)^{j+k}\varphi([v_j,v_k]\wedge v_0\wedge\dots\wedge\check{v}_j\wedge\dots\wedge\check{v}_k\wedge\dots\wedge v_p)$$

Exercise: Check that the right hand side of this formula defines a $(p+1)$-linear alternating map of $(v_0,\dots,v_p)$, "linear" meaning "$\mathcal{C}^\infty_X$-linear".

Contrary to case of $d\circ d = 0$ we don't have this equality for ∇ in general. But at least we get that $\nabla\circ\nabla$ is $\mathcal{C}^\infty_X$-linear, i.e. defines a bundle map $F: E \to \Lambda^2 T^*X\otimes E$, called the curvature $F = F^\nabla$ (and not a 2nd order differential operator as one would expect).
I leave this as an exercise
Note also that for $\alpha\otimes e \in \mathcal{A}^p(E)$

$$\nabla\nabla(\alpha\otimes e) = \alpha\otimes Fe$$

Remarks: 1) The explicite formula for $\nabla\varphi$ yields the expression:
For vector fields v_0, v_1 and $e \in E$ we get

$$F(v_0\wedge v_1)e = [\nabla_{v_0},\nabla_{v_1}]e - \nabla_{[v_0,v_1]}e.$$

2) The curvature tensore $F = F^{\nabla}$ was introduced by B. Riemann under the name "4-indicees-symbol" as integrability condition to find r $(= rk(E))$ linearly independant solutions (locally) of the differential equation $\nabla e = 0$, which exist if and only if $F = F^{\nabla} = 0$

2. Chern-Weil theory

It relates curvature properties with characteristic classes of vector bundles via the isomorphisme

$$H^*_{DR}(X) \longrightarrow H^*(X, \mathbb{R})$$

between de Rham and singular cohomology.
I will explain this for complex vector bundles first (structure group $Gl_r(\mathbb{C})$ or $U(r)$)
If ϕ is a polynomial function on $gl_r(\mathbb{C})$ (space of $r \times r$-matrices over $\mathbb{C}$), invariant under conjugation by $Gl_r(\mathbb{C})$

we can define $\phi(F^{\nabla})$ which is a differential form on X.
We will prove the following two facts
(1) $\phi(F^{\nabla})$ is a closed form
(2) $\phi(F^{\nabla})$ depends on ∇ only up to an exact form.
Thus it defines a class in the de Rham cohomology, and therefore in $H^*(X, \mathbb{R})$, which only depends on the bundle E, not on the choice of the connection.
The coefficients of the characteristic polynomials form a basis of the space of invariant polynomials.
If we write

$$\det(Id + \frac{i}{2\pi} F^{\nabla}) = 1 + \gamma_1(E, \nabla) + \gamma_2(E, \nabla) + \cdots$$

we get closed forms $\gamma_j(E, \nabla)$ of degree $2j$ and via the de Rham isomorphisme

$$[\gamma_j(E, \nabla)] = c_j(E)$$

where $c_j(E)$ are the topologically defined Chern classes (Jones lecture)

First we prove (1) and (2).
We can assume ϕ ~~to~~ be homogeneous, of degree m say. Remember that homogeneous polynomials of degree m are in 1-1-correspondence to symmetric m-linear functions, which I denote by the same letter, so

$\phi(F)$ is a short hand notion for $\phi(F, F, \dots, F)$

If $A_1, \dots, A_m, B \in gl_r(\mathbb{C})$, conjugation by $\exp(tB)$, invariance and applying $\frac{d}{dt}|_0$ gives

$$\phi([B,A_1], A_2, \dots, A_m) + \phi(A_1, [B,A_2], A_3, \dots, A_m) + \dots + \phi(A_1, A_2, \dots, A_{m-1}, [B, A_m]) = 0$$

as consequence of invariance (which is indeed equivalent to invariance for connected Lie groups)
Also remember the 2nd Bianchi identity, F considered as section of $\Lambda^2 T^*X \otimes END(E)$ has covariant derivative 0

$\nabla F = 0$

[illegible]

Locally, choosing a gauge of E, the connection is expressed by a $r \times r$-matrix of 1-forms as

$$\nabla e = de + \omega \otimes e$$

and F corresponds to $\Omega = d\omega + \omega \wedge \omega$
Then ∇F is expressed as

$$d\Omega + [\omega, \Omega]$$

Now, if $F_1, \dots, F_m$ are sections of $\Lambda^2 T^*X \otimes END(E)$, given locally by matrices Ω_j

$$d\phi(F_1, \dots, F_m) = \phi(d\Omega_1, \Omega_2, \dots, \Omega_m) + \dots + \phi(\Omega_1, \Omega_2, \dots, d\Omega_m)$$

$$(\text{by invariance}) = \phi(d\Omega_1 + [\omega, \Omega_1], \Omega_2, \dots, \Omega_m) + \dots + \phi(\Omega_1, \dots, \Omega_{m-1}, d\Omega_m + [\omega, \Omega_m])$$

$$= \phi(\nabla F_1, F_2, \dots, F_m) + \dots + \phi(F_1, \dots, F_{m-1}, \nabla F_m)$$

This formula together with $\nabla F = 0$ gives formula (1).

As for (2) we observe that the space of connections on E is an affine space under the vector space $\mathrm{Hom}(E, T^*X \otimes E)$.
Thus any two connections ∇_1, ∇_2 on E

can be joint by a smooth path,

$t \mapsto \nabla(t)$, $\nabla(0)=\nabla_1$, $\nabla(1)=\nabla_2$

(for example $(1-t)\nabla_1 + t\nabla_2$, the straight line connecting ∇_1, ∇_2)

Now we prove

Lemma If $\nabla(t)_{t\in\mathbb{R}}$ is a family of connections on E depending smoothly on t. Then

$$\frac{d}{dt}\phi(F(t)) = m\, d\,\phi\left(\frac{d\nabla}{dt}, F(t),\dots,F(t)\right)$$

(where $F(t) = F^{\nabla(t)}$)

There are 2 ways to prove this, either to use again invariance, or reducing it to the relation (1). It goes as follows:

Consider $Y = X\times\mathbb{R}$ and the pull back $\tilde{E} = E\times\mathbb{R} \to X\times\mathbb{R}$ of E.

Define a connection $\tilde{\nabla}$ on E by

$$(\tilde{\nabla} s)(-,t) = \nabla(t)(s(-,t)) + dt\otimes \frac{\partial s}{\partial t}(-,t)$$

It has curvature

$$\tilde{F} s = Fs + dt\wedge \frac{\partial\nabla}{\partial t} s$$

So $\phi(\tilde{F},\dots,\tilde{F}) = \phi(F,\dots,F) + m\, dt\wedge\phi\left(\frac{\partial\nabla}{\partial t}, F,\dots,F\right)$

(because of $dt\wedge dt = 0$)

$$0 = d\phi(\tilde{F},\dots,\tilde{F}) = dt\wedge\frac{\partial}{\partial t}\phi(F,\dots,F) - m\, dt\wedge d\phi\left(\frac{\partial\nabla}{\partial t}, F,\dots,F\right)$$

which proves the lemma.

Now integrating along the path we get

$$\phi(F(1)) - \phi(F(0)) = d\left(\int_0^1 m\,\phi\left(\frac{d\nabla}{dt}, F(t),\dots,F(t)\right)dt\right)$$

which proves (2)

It is not difficult to prove functoriality and additivity of the classes $[\gamma_j(E,\nabla)]$ or rather $[1+\gamma_1+\dots+\gamma_r] = [\gamma(E,\nabla)]$, $\gamma(E,\nabla) = \det(Id + \frac{i}{2\pi}F^{\nabla})$.

If $Y \xrightarrow{f} X$ is a differentiable map, (E,∇) a vector bundle with a connection then

$$\gamma(f^*E, f^*\nabla) = f^*\gamma(E,\nabla)$$

which proves functoriality already on the level of forms for suitable choice of the connection.

For multiplicativity, consider (E', ∇'), (E'', ∇'') and $(E' \oplus E'', \nabla' \oplus \nabla'') = (E, \nabla)$ then $F^{\nabla' \oplus \nabla''} = F^{\nabla'} \oplus F^{\nabla''}$, hence $\gamma(E, \nabla) = \gamma(E', \nabla')\gamma(E, \nabla)'$.

Therefore we can use splitting principle to reduce the proove of relations between ~~these~~ these classes to the case of line bundles. The proof that the $[\gamma_j]$ yield the topological Chern classes is therefore reduced to the case, that they coincide for the Hopf bundle over $\mathbb{P}^n(\mathbb{C})$ (it suffices even for $\mathbb{P}^1(\mathbb{C})$), and to specify the de Rham isomorphisme. This will be explained in Dold's lecture, so I will only compute $[\gamma_1(E, \nabla)]$ for the ~~Hopf~~ Hopf bundle $E \to \mathbb{P}^n(\mathbb{C})$.

It can be described as central projection $E = \mathbb{P}^{n+1}(\mathbb{C}) \setminus \{P_0\} \xrightarrow{\pi} \mathbb{P}^n(\mathbb{C})$

We take $P_0 = (0 : \cdots : 0 : 1)$, then $\pi(z_0 : \cdots : z_n : z) = (z_0 : \cdots : z_n)$

We have to choose a connection on E. The bundle is holomorphic and has a hermitian metric by

$$\langle (z_0 : \cdots : z_n : z), (z_0 : \cdots : z_n : z') \rangle = \frac{\bar{z} z'}{|z_0|^2 + \cdots + |z_n|^2}$$

We take the unique Hermitian connection with the property that ∇s is ~~always~~ of type $(1,0)$ for <u>holomorphic</u> section which always exist on holomorphic Hermitian bundles.

With respect to a (local) holomorphic frame where $\langle\, ,\, \rangle$ has the matrix h the connection has the matrix $h^{-1}\partial h$. Then in case of rank 1 we get

$$F = \bar{\partial}\partial \log h$$

(since $h = \|s\|^2$ is a scalar in this case).

We take s on $U_0 = [z_0 \neq 0] \subset \mathbb{P}^n(\mathbb{C})$ as $s(1 : z_1 : \cdots : z_n) = (1 : z_1 : \cdots : z_n : 1)$

then $\|s\|^2 = \dfrac{1}{1 + \|z\|^2}$, $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$

and

(this is the Kähler form of the Fubini Study metric, see lectures of H. Azad)

Remarks

1) It is easy to see why the $[\gamma_j(E,\nabla)]$ are real classes.
We can choose a Hermitian metric on E and a Hermitian connection ∇. Then the curvature $F = F^\nabla$ is skew-Hermitian and iF Hermitian. Therefore the coefficients of the characteristic polynomials are real, i.e. $\gamma_j(E,\nabla)$ are real forms

2) We can get more general versions by taking a formal power series $f(T)$ and functions of $\mathrm{Tr}\left(f\left(\frac{i}{2\pi}F\right)\right)$
Example:
a) $\mathrm{Tr}\left(\exp\left(\frac{i}{2\pi}F\right)\right) = ch(E,\nabla)$
represents the Chern character
b) $\exp\left(\mathrm{Tr}\left(\frac{\frac{i}{2\pi}F}{\exp\left(\frac{i}{2\pi}F\right)-1}\right)\right) = td(E,\nabla)$
represents the Todd class
(see Jones lecture)

3) For real bundles the same approach, considering $\det\left(I + \frac{F^\nabla}{2\pi}\right)$, gives the Pontrjagin classes.
Since we can assume E to be a metric bundle and ∇ to respect the metric, we get skew-symmetric F^∇. In this case only forms of degree $0 \bmod 4$ appear, thus Pontrjagin classes are numbered by $p_1(E) \in H^4$, $p_2(E) \in H^8$ etc.

If $rk(E) = 2r$ even and E is oriented and metric, we have another invariant polynomial on $\mathfrak{so}(2r,\mathbb{R})$, namely the Pfaffian $Pf(A) = \det^{\frac{1}{2}}(A)$.
Then $Pf\left(-\frac{F^\nabla}{2\pi}\right) = e(E,\nabla)$ is a form which represents the Euler class.

In case $E = TX$ (dim $X = 2r$), ~~X~~ compact oriented it gives the topological Euler characteristic $\chi(X)$
($[e(E,\nabla)] = \chi(X)\cdot$ orientation class of X)