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# Algebraic Surfaces and Mordell's Conjecture over Function Fields

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# Algebraic surfaces and Mordell's conjecture over function fields.

## I

**ampleness:**

$X$ proper scheme over $\operatorname{Spec} A$, $A$ noetherian ring.
$L$ invertible sheaf ($=$ vector bundle of rank 1) on $X$

Then

$L$ ample $\iff$ $L|_{X_{red}}$ ample $\iff$ $L|_{X_i}$ ample $\forall$ irreducible component $X_i$ of $X_{red}$ $\iff$ $\exists\, n>0$, $X \hookrightarrow \mathbb{P}(H^0(L^n))$ $\iff$ if $L = f^*M$, $f: X \to Y$ finite morphism, then $M$ ample $\iff$ $\forall\, \mathcal{F}$ coherent sheaf, $\mathcal{F}\otimes L^n$ generated by global sections for $n \gg 0$ $\iff$ $\forall\, \mathcal{F}$ coherent sheaf, $H^i(\mathcal{F}\otimes L^n) = 0$ $i>0$, $n \gg 0$.

In particular, this implies: $L$ ample, $\mathcal{F}\otimes L^n$ ample $\forall\, \mathcal{F}$ invertible, for $n \gg 0$ (even very ample)

["Proof": may assume: $L$ very ample and $\mathcal{F}$ generated
i.e. $\exists \oplus \mathcal{O}_X \twoheadrightarrow \mathcal{F} \Rightarrow \exists \oplus L \to \mathcal{F}\otimes L$. As
$\exists \oplus \mathcal{O}_X \twoheadrightarrow \oplus L$, one sees that $\mathcal{F}\otimes L$ is generated and those sections separate the points... ]

In the sequel, we consider a *surface* $X$, i.e.: a 2 dimensional smooth projective variety over a field $k$, $k = \bar{k}$ (algebraically closed).

Recall that one considers the intersection pairing $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$, which is by definition non degenerate on $NS(X) := \dfrac{\operatorname{Pic} X}{\operatorname{Pic}^n X}$, where $\operatorname{Pic}^n X = \{ L \in \operatorname{Pic} X,$ $L.M = 0$ for all $M \in \operatorname{Pic} X \}$.

Hodge Index Thm. Let $H$ be an ample divisor on a surface $X$. Let $D$ be a divisor such that $D.H = 0$ and $D^2 \geq 0$. Then $D$ is numerically equivalent to zero.

Proof. Assume $D.H = 0$, $D$ not num. eq. 0.
Take $E$ such that $D.E \neq 0$.
If $D^2 = 0$, take $E' = aE + bH$ $(a, b \in \mathbb{Z})$, s.t. $E'.H = 0$, $E'.D \neq 0$; then $D' = nD + E'$ has $D'^2 = 2nDE' + E'^2$ can be made $> 0$.
So may assume: $D^2 > 0$. Then $H' = D + nH$
ample for $n \gg 0$ and $H'.D = D^2 > 0 \Longrightarrow$
$h^2(mD) = h^0(K - mD) = 0$ for $m \gg 0$ as if
$K - mD$ is effective, then $K - mD . H' \geq 0 \Longrightarrow$
$\chi(mD) \geq h^0(mD)$ grows as $m^2 D^2$. ($> 0$ constant)
$m \gg 0$
$\Longrightarrow mD$ effective for $m \gg 0$ which contradicts
$mD.H = D.H = 0$.

(Rmk. $NS(X)$ is in fact a free $\mathbb{Z}$-module of finite rank. The quadratic form on it has then an orthogonal basis in which the matrix has the shape $\operatorname{diag}(+, -, -, \ldots, -)$. Hence the name of the Thm.)

Nakai - Moishezon criterion. Let $D$ be a divisor on a surface such that $D^2 > 0$ and $D.C > 0$ for all irreducible curves $C$ on a surface $X$. Then $D$ is ample. (and conversely).

Corollary. If $D$ is such that $D.C \geq 0$ $\forall$ irreducible curves $C$, then $D+H$ is ample for all $H$ ample, and $D^2 \geq 0$.

Proof. The Nakai - Moishezon criterion implies that $mD+H$ is ample for all $m > 0$. Therefore $(mD+H)^2 > 0$ for all $m >> 0$.

Proof of Nakai - Moishezon.
Again $h^2(mD) = h^0(K - mD) = 0$ for $m >> 0$ as if $K - mD$ effective, then $H$ is very ample and $H.(K - mD) \geq 0$ and $D.H > 0$.
So $\chi(mD) \geq h^0(mD)$

for $m >> 0$, grows like $m^2 D^2$. (positive constant),
and $mD$ is effective for $m >> 0$.
So may assume $D$ effective.
But $\mathcal{O}_X(D)|_D$ ample as $\mathcal{O}_X(D)|_{D_{red}}$ ample for all components $D_{red}$ of $D_{red}$. Therefore
$H^1(\mathcal{O}_X(nD)|_D) = 0$ for $n >> 0$, and
$H^1(X, \mathcal{O}_X(nD)) \xrightarrow{\sim} H^1(X, \mathcal{O}_X((n+1)D))$ for $n >> 0$ (as $H^1(X, \mathcal{O}_X(nD))$ is a finite dimensional $k$-vector space). In particular
$$H^0(X, \mathcal{O}_X(nD)) \twoheadrightarrow H^0(X, \mathcal{O}_X(nD)|_D)$$
for $n >> 0$. As $\mathcal{O}_X(nD)|_D$ is generated for

$n \gg 0$, $\mathcal{O}_X(nD)$ is generated as well. Consider now $\varphi: X \to Y \subset \mathbb{P}(H^0(\mathcal{O}_X(nD)))$ the linear map to $\mathcal{O}(nD)$, such that $\mathcal{O}_X(nD) = \varphi^*\mathcal{O}_Y(1)$. As $\mathcal{O}(D).C > 0$ for all curves $C$, no such curve can map to a point because of the projection formula $\mathcal{O}_X(nD).C = \mathcal{O}_Y(1).\varphi(C)$.

<u>Kodaira-Vanishing Thm</u>. <u>Assume</u> char $k = 0$, <u>and let $X$ a smooth projective variety</u> (<u>we may here assume</u> $\dim X \geq 2$). Then $H^i(X, L^{-1}) = 0$ <u>for all</u> $i < \dim X$, $L$ <u>ample</u>, <u>or</u> <u>dually</u> $H^j(X, L \otimes \omega_X) = 0$ <u>for all</u> $j > 0$ <u>and</u> $L$ <u>ample</u>.

<u>Proof</u>. As $L^n$ is generated, for $n \gg 0$, one may write $L^n = \mathcal{O}(D)$, $D$ smooth divisor. Then one has:

(see p. 4-bis) $H^i(L^{-1}(-D) = L^{-1-n}) \xrightarrow[D]{} H^i(L^{-1})$

is surjective, or equivalently the restriction map

$$H^i(L^{-1}) \xrightarrow[|_D]{} H^i(L^{-1}|_D)$$

is zero in cohomology. To see this, one can observe that $|_D$ factorizes as:

$$\begin{tikzcd} H^i(L^{-1}) \arrow[r, "H^i(\nabla)"] \arrow[dr, "\frac{1}{n}|_D"'] & H^i(L^{-1} \otimes \Omega^1_X(\log D)) \arrow[d] \\ & H^i(L^{-1}|_D) \end{tikzcd}$$

where $\nabla$ is a connection, and that $H^i(\nabla) = 0$ as a consequence of the Hodge theoretical statement: $d\, H^i(\mathcal{O}_Y) = 0$ in

(1) We use throughout these notes the classical notations:

$$L(D) := L \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$$

(2) In fact the (algebraic) Kähler differential $d: \mathcal{O}_Y \to \Omega^1_Y$ has as image

$\pi_* d: \pi_* \mathcal{O}_Y \to \pi_* \Omega^1_Y$ where one direct factor of it is $\nabla: L^{-1} \to \Omega^1_X(\log D) \otimes L^{-1}$.

Now, the fact that $d\, H^i(\mathcal{O}_Y) = 0$ is

proven (algebraically) in [DI]. Over $k = \mathbb{C}$, one can use GAGA thms to say that $H^i(\mathcal{O}_Y) = H^i(Y_{an}, \mathcal{O}_{Y_{an}})$ and the classical Hodge theory to prove $d\, H^i(Y_{an}, \mathcal{O}_{Y_{an}}) = 0$.

$H^i(\Omega^i_Y)$, where $Y = \mathrm{Spec}_{\mathcal{O}_X} \bigoplus_{\bullet}^{N-1} L^{i}$ (see [EV LN])

Now as $L$ is ample, $H^i(L^{-m}) = H^{\dim X - i}(L^m \otimes \omega) = 0$

for $m >> 0$, $i < \dim X$. (see p. 4 bis)

Rmks: ① If one does not assume $L$ ample, but only $L^N = \mathcal{O}_X(D)$ for $D$ smooth, then one obtains

$$H^i(L^{-1}) \xrightarrow[0]{} H^i(L^{-1}|_D) \quad \text{for all } i \geq 0,$$

or equivalently $H^j(L \otimes \omega_X(D)) \to H^j(L \otimes \omega_D)$ for all $j$. This is

Kollár's vanishing thm.

② The assumption on the char $k$ is necessary because of the use of $dH^i(\mathcal{O}_Y) = 0$. Deligne-Illusie [DI] proved this (and more) in char $p > 0$ under some very mild assumption, and gave thereby a direct proof of Kodaira vanishing in char $p > 0$ (under this assumption).

We assume now that we have a family $f : X \to B$ of curves, $f^{-1}(b) = X_b$, where $X$ is again a surface, and $B$ is a smooth curve. We will denote by $g$ the genus of the generic fiber of $f$, which is the curve $\Gamma = X_{k(B)}$, which is smooth as $X$ is smooth and we assume that char $k = 0$.

Conversely, starting with a curve $\Gamma$ defined over the field $K$ of functions of a curve $B$, we can produce a family $f : X \to B$ as follows. Write $k(B)$ as the field of fractions of the regular functions of some open set $U \subset B$. The equations of $\Gamma$ with coefficients in $K$ give, by reducing to the same denominator in $\mathcal{O}(U)$, the equations of a family $X_U \to U$ extending $\Gamma$. Take the Zariski closure and desingularize to obtain $f$.

We will be interested in the set $\Gamma(K)$ of $K$-rational points of $\Gamma$ (i.e: diagrams

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\sigma"] \arrow[dr, "\sim"'] & \Gamma \arrow[d] \\ & \operatorname{Spec} K \end{tikzcd}$$

). A $K$ rational point $\sigma$ is extendable to a section of $f : X \to B$ produced above, a priori over some open set in $B$, and, as $B$ is a smooth curve, everywhere. Conversely a section $\sigma$ of $f$ determines $\sigma \in \Gamma(K)$ by restricting $\sigma$ to $\operatorname{Spec} K$. So

$$\Gamma(K) = \{ \sigma : B \to X, \text{ s.t } f \circ \sigma = \mathrm{Id}_B \}.$$

The aim of the rest of these notes is to give two different proofs of the Mordell conjecture over function fields:

If $g \geq 2$, and $f$ is not isotrivial, then $\Gamma(K)$ is finite.

(We explain later "isotrivial").

This is Manin's thm [M], and Manin introduced the so called "Gauß-Manin" connection to prove it, a very powerful tool in algebraic geometry. (Grauert proved it later independently, and then Parshin [P] understood the relation between Shafarevich and Mordell conjectures, and thereby gave a third proof of Mordell).

We give here a later proof of Parshin [PF] relying on Bogomolov-Miyaoka-Yau inequality (see later in these notes) and motivated by what one would hope in the arithmetic case (i.e. $K$ is a number field). We also give a proof [EV M] based on the vanishing thm (Kollár) explained above. Both proofs are effective (see later what it means).

## Minimal model and semi-stable reduction.

thm: ① Let $f: X \to B$ be as above. Then there is a finite $\tau: B' \to B$, $B'$ smooth, and a desingularization $X'$ of $X \times_B B'$, such that

$f' : X' \to B'$ has only singularities of the type $\times$ with reduced fibers (i.e. $t' = \text{unit} \cdot x' y'$, where $x', y'$ are local parameters of $X'$ and $t'$ of $B'$).

② Let $\omega_{X|B} := \omega_X \otimes f^* \omega_B^{-1}$ where $\omega_X = \Omega^2_X$, $\omega_B = \Omega^1_B$.

Then there is a contraction $\kappa$ such that

$$\begin{array}{ccc} X' & \xrightarrow{\kappa} & X'' \\ & \searrow^{f'} & \downarrow f'' \\ & & B' \end{array}$$

$\omega_{X''|B'} \cdot C \geq 0$ for all curves $C$ s.t. $f''(C) = \text{point}$, when $g \geq 1$.

(This is called
① semi-stable reduction
② minimal (relative) model of $f'$).

On Proof: ① first blow up the bad fibers of $f$ such as to obtain bad fibers which are normal crossing divisors), that is $f$ is locally of the shape $t = x^a y^b \cdot \text{unit}$. Then take a finite cover $\tau$ of $B$ such that in the bad points of $f$, $t'^n = t$, the equation (locally) of $\tau$ is where $n$ is a multiple of all $a, b$ appearing. The normalization of the new fiber singularities have now the shape $t'^n = x^a y^b \cdot \text{unit}$, whose normalization is of type $A_N$ for some $N$. The minimal desingularization of those does it.

② Apply Castelnuovo criterion to the vertical

(-1) rational curves E. Then $\omega_{X/B'} \cdot E = 1$, and therefore E can only cut (Fiber $- E$) in one point: So the contraction of the fiber is still a normal crossing divisor.

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<u>Isotriviality</u> $f: X \to B$ <u>is isotrivial</u> if and only if $\exists\ \tau: B' \to B$ <u>finite</u>, $B'$ <u>smooth</u>, <u>such that</u> $f: X \times_B B' \to B'$ <u>is birational to the projection</u> $F \times B' \to B'$ <u>where</u> $F$ <u>is a generic fiber of</u> $f$.

So, as different sections of $f$ give different sections of the (relative) minimal model $f''$, <u>we may assume</u>, (and will assume), to prove Mordell, that <u>$f$ is a relative minimal model.</u>

---

<u>heights</u>. Let $f: X \to B$ be a relative minimal (semi-stable) family of curves. Let $H$ be an ample divisor on $X$. Then the <u>height</u> of the section $\sigma$ of $f$ is

$$\text{height}\ \sigma = \sigma(B) \cdot H.$$

<u>fact</u>: <u>in order to prove</u> <u>Mordell</u>, <u>it is enough to prove that</u>

$$\exists\ M > 0, \quad \sigma(B) \cdot H \leq M \ \text{for all}\ \sigma \in \Gamma(K).$$

<u>on proof</u>: As one considers a numerical criterion,

one may assume that H is very ample. Then one looks at the Hilbert scheme

$\mathcal{H} = \{$ curves in $X$, curve $\cdot H \leq M\}$, which has finitely many components, $T_i$ say, with $\mathcal{C}_i \subset T_i \times X$ parametrizing a universal family

those curves.

Assume now that one has infinitely many of them are lying above a curve sections. Then $\infty$ many of them (with family $\mathcal{C} \subset T \times X$).

T in one component of $\mathcal{H}$ (with family

Then $\varphi: \mathcal{C} \to T \times B$ is such that $\varphi^{-1}(t \times b) = X_b \cap \mathcal{C}_t$ consists of one point for $t$ being in $T'$ and $b \in B$.

Therefore $\varphi^{-1}(t \times b)$ consists of one point everywhere and $\mathcal{C} = T \times B$. One has a generically finite map $\mathcal{C} = T \times B \to X$. Then $T$ maps to $X_b$ for all $b \in B$.

This implies that almost all $X_b$ are isomorphic (as there are only finitely many such finite maps $T \to$ curve of degree $\geq 2$), which implies isotriviality (see previous proof in [L], using p.o. bians).

second fact: in order to prove Mordell, it is enough to prove that

$$\exists M > 0, \quad \sigma(B) \cdot \omega_{X/B} \leq M \quad \text{for all } \sigma \in \Gamma/K$$

($f$ is semi-stable, minimal).

Proof: It will be shown below that in fact, not only $\omega_{X/B} \cdot C \geq 0$ for

all vertical curves $C$, but also for all curves $C$.
(one says: $\omega_{X/S}$ is numerically effective, or nef). Then $(\omega_{X/S} + F)$, where $F$ is a fiber, verifies $(\omega_{X/S} + F)^2 = \omega_{X/S}^2 + 2F.\omega_{X/S} = \omega_{X/S}^2 + 2(2g-2) > 0$
and it will be proved later again that a power $(\omega_{X/S}(F))^N$ contains some ample sheaf $\mathcal{H}$: $(\omega_{X/S}(F))^N = \mathcal{H}(R)$ where $R$ is effective, $R = \sum \nu_i E_i + R'$, $R'$ is in fibers and $E_i \to B$ is finite. Then for any section $\sigma(B)$, $\sigma(B).R' \geq 0$, and $\sigma(B).E_i \geq 0$ unless possibly if $\sigma(B) = E_i$.
Therefore $\mathcal{H}.\sigma(B) \leq NM + N$ except possibly if $\sigma(B) =$ one of the $E_i$. So $\mathcal{H}.\sigma(B)$ is bounded as well.

This is what is going to be proved in the sequel: $\sigma(B).\omega_{X/B}$ is bounded.

## $\boxed{\mathrm{II}}$ Kodaira-Spencer classes

Let $f: X \to B$ be a semi stable family of curves. One considers the sheaf of global differentials of degree 1 with logarithmic singularities along the bad locus $D := f^{-1}(S)$ ($S$ is the bad locus, $D$ is a normal crossing divisor.), denoted by $\Omega_X^1(\log D)$. It is an extension

(*) $0 \to f^*\Omega_B^1(\log S) \to \Omega_X^1(\log Y) \to \Omega_{X/B}^1(\log Y) \to 0$.

It is easy to compute that, because of

semi-stability, one has $\omega_{X/B} = \Omega^1_{X/B}(\log Y)$, the latter being just defined as the quotient $\Omega^1_X(\log Y)/f^*\Omega^1_B(\log S)$.

The connecting morphism

$$f_*\omega_{X/B} \xrightarrow{KS} \Omega^1_B(\log S)\otimes R^1f_*\mathcal{O}_X$$

of $f$, is called the Kodaira–Spencer class.

Thm (see [K]): $f$ is isotrivial if and only if $KS = 0$. if $k = \mathbb{C}$

On proof: $KS = 0 \Rightarrow KS|_{B-S=U} = 0$. If $f$ is isotrivial, then $KS = 0$ on some cover of $U$, and as $KS$ is compatible on some open set with base change, $KS$ itself is zero.

Conversely, $R^1f_*\mathbb{Q}|_{B-S}$ is a variation of Hodge structures of weight $1$, with $F^1 = f_*\omega_{X/S}|_{B-S}$, such that $\nabla F^1 \subset \Omega^1_U\otimes F^1$ if $KS = 0$, where $\nabla$ is the Gauß–Manin connection. This implies that the monodromy representation of $\pi_1(B-S)$ on $H^1(X_b,\mathbb{Q})$ is finite ($b\in B-S$), and by Torelli for curves, this implies that on the cover of $B-S$ trivializing the variation of Hodge structures, the family of curves is trivial.

Corollary: Let $C := \sigma(B)$ be a section of $f$. If $f$ is not isotrivial, then there is a nontrivial map $f_*\omega_{X/B}^{\otimes 2}(C) \longrightarrow \omega_B(\log S)$.

Proof: One has a commutative diagram

$$\begin{tikzcd}
f_*\omega_{X/S}\otimes f_*\omega_{X/S} \arrow[r, "KS\otimes 1"] \arrow[d, "\text{multiplication}"'] \arrow[dr, "\lambda"] & \omega_B(\log S)\otimes R^1f_*\Theta\otimes f_*\omega_{X/S} \arrow[d, "1\otimes\text{multiplication}"] \\
f_*\omega_{X/S}^{\otimes 2} \arrow[r, "\mu"'] & \omega_B(\log S)\otimes R^1f_*\omega_{X/S}=\omega_B(\log S)
\end{tikzcd}$$

As $R^1f_*\Theta_X\otimes f_*\omega_{X/S}\to\mathcal{O}_B$ is a perfect pairing (Serre duality) and $KS\neq 0$, then $\lambda\neq 0$, and a fortiori $\mu\neq 0$.

Furthermore, from the factorization of $(*)$, to

$$(*')\quad 0\to f^*\Omega^1_B(\log S)\to\Omega^1_X(\log Y+C)\to\omega_{X/B}(C)\to 0$$

for any section $C=\sigma(B)$ of $\Gamma(K)$, one obtains the factorization of $\mu$ through

$$f_*\omega_{X/S}^{\otimes 2}(C)\longrightarrow\omega_B(\log S).$$

## II] A proof of Manin's theorem [M] (see [EVM])

Let $f: X \to B$ be, as before, a morphism from a smooth surface $X$ to a curve $B$, both projective and defined over an algebraically closed field $k$ of characteristic zero. As we have seen in part I, in order to prove the Mordell conjecture over the function field $k(B)$ we can assume that:

The fibres of $f$ are semi-stable,
$X$ is relatively minimal over $B$ and $f$ is not isotrivial.

Let $S \subseteq B$ denote the set of points $b \in B$ with $f^{-1}(b)$ singular, $s = \# S$, $g = g(F)$ the genus of a general fibre $F$ of $f$ and $q = g(B)$ the genus of $B$.

The Mordell conjecture (= Manin's theorem) follows from

<u>Theorem</u>: Keeping the assumptions made above, let $\sigma: B \to X$ be a section of $f$ and $C = \sigma(B)$. Then

$$h(C) := c_1(\omega_{X/B}) \cdot C \leq 2 \cdot (2g-1)^2 (2q-2+s).$$

---

The first half of the proof has been done already, using the Kodaira-Spencer map:

A] There exists an invertible sheaf $\mathcal{N}$ and a surjective morphism
$$f_*(\omega^2_{X/B} \otimes \mathcal{O}_X(C)) \twoheadrightarrow \mathcal{N}$$
such that $\underline{\text{degree}(\mathcal{N}) \leq 2 \cdot g - 2 + s}$.

Obviously the Theorem follows from:

B] If $\mathcal{N}$ is invertible and $f_*(\omega^2_{X/B} \otimes \mathcal{O}_X(C)) \longrightarrow \mathcal{N}$ surjective, then
$$\underline{\text{degree}(\mathcal{N}) \geq \eta := \tfrac{1}{2} h(C) \cdot \frac{1}{(2 \cdot g - 1)^2}}.$$

To prove B] we are allowed to replace $B$ by a finite cover $\tau : B' \longrightarrow B$, say of degree $m$, as long as $\tau$ is étale (unramified) in a neighbourhood of $S$. In fact, $X' = X \times_B B'$ is non singular and, if

$$\begin{tikzcd} C' \arrow[r] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ & B' \arrow[ul, "\sigma'", "\simeq"'] \arrow[r, "\tau"] & B \end{tikzcd}$$

is the induced diagram, flat base change ([H I]) gives
$$\tau^* f_*(\omega^2_{X/B} \otimes \mathcal{O}_X(C)) = f'_*(\omega^2_{X'/B'} \otimes \mathcal{O}_{X'}(C')).$$
Moreover, for $\mathcal{N}' = \tau^* \mathcal{N}$ one has
$$\deg(\mathcal{N}') = m \cdot \deg(\mathcal{N})$$
and, by projection formulae,
$$h(C') = C' \cdot c_1(\omega_{X'/B'}) = m \cdot C \cdot c_1(\omega_{X/B}) = m \cdot h(C).$$

In particular we may assume that the number $\eta$ in B) is an integer. Of course, it is enough to consider the case $h(c) > 0$.

<u>Definition & Lemma:</u> Let $\mathcal{F}$ be a locally free sheaf on $B$. $\mathcal{F}$ is called semi-positive, if one of the following equivalent conditions holds true.

i) For all $\nu > 0$ the sheaf $S^{\nu}(\mathcal{F}) \otimes \mathcal{O}_B(\mathrm{pt.})$ is ample.
($S^{\nu}$ = symmetric product. pt = any point in $B$).

ii) If $\mathbb{P} = \mathbb{P}(\mathcal{F})$ is the projective bundle and $\mathcal{O}_{\mathbb{P}}(1)$ the tautological sheaf, then $\mathcal{O}_{\mathbb{P}}(1)$ is numerically effective (i.e.: $c_1(\mathcal{O}_{\mathbb{P}}(1)) \cdot D \geq 0$ for all curves $D \subseteq \mathbb{P}$).

iii) For any finite covering $\tau : B' \longrightarrow B$ and for all invertible sheaves $\mathcal{N}'$ on $B'$ with allow a surjection $\tau^* \mathcal{F} \longrightarrow\!\!\!\!\to \mathcal{N}'$, one has $\deg(\mathcal{N}') \geq 0$.

B) follows from:

B') For $\eta$ as in B) the sheaf

$$f_*(\omega^2_{X/B} \otimes \mathcal{O}_X(c)) \otimes \mathcal{O}_B(-\eta \cdot \mathrm{pt.})$$

is semi-positive.

<u>Properties:</u>

i) $\overset{m}{\bigoplus} \mathcal{O}_B$ is semi-positive

ii) If $\mathcal{G}$ is semi-positive and if $\mathcal{G} \longrightarrow \mathcal{F}$ is a morphism, surjective over some non empty open $U \subseteq B$, then $\mathcal{F}$ is semi-positive.

iii) In particular, if $\mathcal{F}$ is generated by global sections $s_1, \ldots, s_\nu \in H^0(B, \mathcal{F})$ over some $\emptyset \neq U \subseteq B$, then $\mathcal{F}$ is semi-positive. (We will say: "$\mathcal{F}$ is globally generated in the general point of $B$).

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The main example (due to Fujita) is:

<u>Thm:</u> $\quad f_* \omega_{X/B}$ is semi-positive

(In fact, here $\dim(X)$ may be larger than 2).

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<u>Proof</u> (Kollár). Let $F = f^{-1}(pt)$ be a fibre, $\mathcal{L} = \mathcal{O}_X(F)$. Kollár's vanishing theorem (in the easiest case, see I J) shows that the upper sequence in the following diagram is exact:

$$\begin{array}{ccccccc} 0 \to H^0(X, \omega_X \otimes \mathcal{L}) & \longrightarrow & H^0(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{O}(F)) & \longrightarrow & H^0(F, \omega_F \otimes \mathcal{L}) \to 0 \\ & & \| & & \| \\ & & H^0(B, f_*(\omega_X) \otimes \mathcal{O}(2 \cdot pt)) & \longrightarrow & \mathbb{C}_{pt}^{h^0(F, \omega_F \otimes \mathcal{L})} \end{array}$$

Hence $\quad f_*(\omega_X) \otimes \mathcal{O}(2 \cdot pt) = f_*(\omega_{X/B}) \otimes \mathcal{O}(2 \cdot pt) \otimes \omega_B$

is globally generated in the general point of $B$.

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For $\nu > 1$, consider the $\nu$-fold product

$\tilde{X} = X \times_B \cdots \times_B X \xrightarrow{\tilde{f}} B$ and a desingularization

$f' : X' \longrightarrow B$ of $\tilde{X}$. It is easy to show that

$$f'_* \omega_{X'/B} \subset \bigotimes^{\nu} f_* \omega_{X/B}$$

$\cong$ over a non empty open set. Using "$f$ semistable" one even obtains equality.

Applying the first part to $X'$ instead of $X$ we find that $(\bigotimes^{\nu} f_* \omega_{X/B}) \otimes \mathcal{O}_B(2 \cdot pt) \otimes \omega_B$ is globally generated in the general point of $B$ for all $\nu > 0$.

This implies, of course, that $f_* \omega_{X/B}$ can not have an invertible quotient sheaf $\mathcal{N}$ of negative degree, not even over some finite covering $\tau : B' \to B$.

qed

---

The next tool needed is "integral parts of $\mathbb{Q}$-divisors" and "cyclic coverings". (see [EVLN] for example).

Let $D = \sum n_j \cdot D_j$ be an effective normal crossing divisor on $X$ (i.e.: the components $D_i$ are non-singular and the local intersection numbers $(D_i \cdot D_j)_p$ are zero or one for $i \neq j$). If, for $a \in \mathbb{R}$, $[a]$ denotes the "integral part (i.e: $[a] \in \mathbb{Z}$ and $[a] \leq a < [a]+1$), then write $\left[\frac{D}{N}\right] = \sum \left[\frac{n_j}{N}\right] \cdot D_j$.

<u>Lemma</u>: Let $\mathcal{L}$ be an invertible sheaf such that, for $N > 1$, $\mathcal{L}^N = \mathcal{O}_X(D)$. Then there exists a surface $Y$ and a surjective morphism $\delta : Y \to X$ such that $\omega_{X/B} \otimes \mathcal{L} \otimes \mathcal{O}_X(-\left[\frac{D}{N}\right])$ is a direct summand

of $\delta_* \omega_{Y/B}$.

Idea of proof. Write $\mathcal{L} = \mathcal{O}_X(\Gamma)$. Then the section $s$ of $\mathcal{L}^N$ with zero locus $D$ corresponds to a rational function $\sigma \in k(X)$ with divisor $(\sigma) = -N \cdot \Gamma + D$.
Let $\tilde{\delta}: \tilde{Y} \longrightarrow X$ be the covering obtained by taking $\tilde{Y}$ to be the normalization of $X$ in the field $k(X)(\sqrt[N]{\sigma})$. $\tilde{Y}$ has at most rational singularities (over the singularities of $D_{red} = \sum D_j$). One takes $Y \xrightarrow[\varsigma]{} \tilde{Y}$ to be a desingularization. An elementary (tedious) calculation shows that

$$\tilde{\delta}_* \omega_{\tilde{Y}} = \bigoplus_{i=0}^{N-1} \omega_X \otimes \mathcal{L}^i \otimes \mathcal{O}_X(-[\tfrac{i \cdot D}{N}])$$

and that $\varsigma_* \omega_Y = \omega_{\tilde{Y}}$.

---

Corollary. Under the assumptions made above, the sheaf

$$f_*(\omega_{X/B} \otimes \mathcal{L} \otimes \mathcal{O}_X(-[\tfrac{D}{N}]))$$

is semi-positive.

Proof. Apply Fujita's theorem to $Y \longrightarrow B$ (This is possible, since "semi-stable" was not used in it's proof).

---

Now we have the technical tools to prove B'].

Let us start with:

**Step 1:** $\omega_{X/B}$ is numerically effective.

Proof. Let $D \subseteq X$ be an irreducible curve. If $f(D)$ is a point $c_1(\omega_{X/B}) \cdot D \geq 0$ since $X$ is relatively minimal over $B$. If $f|_D = \tau : D \to B$ is surjective, since $\omega_F$ is generated by global sections for a general fibre $F$ of $f$, one obtains a non-trivial map

$$\tau^* f_* \omega_{X/B} = f^* f_* \omega_{X/B}\big|_D \longrightarrow \omega_{X/B}|_D .$$

and, the semi-positivity of $f_* \omega_{X/B}$ implies

$$c_1(\omega_{X/B}) \cdot D = \deg(\omega_{X/B}|_D) \geq 0 .$$

**Corollary:** $c_1(\omega_{X/B})^2 \geq 0$.

**Remark:** Since $f$ is non-isotrivial, it is even true that $c_1(\omega_{X/B}) > 0$. We will not need this in the sequel. One even obtains, by similar methods as used in the sequel:

**Corollary:**

1) For all $n > 0$ the sheaves $f_* \omega^n_{X/B}$ are semi-positive.

2) (Since $f$ is non isotrivial:) For $n > 1$ the sheaves $f_* \omega^n_{X/B}$ are ample.

Step 2.: Let us write $\mathcal{L} = \omega_{X/B} \otimes \mathcal{O}_X(C)$. By adjunction formula one has $c_1(\mathcal{L}) \cdot C = \deg(\mathcal{L}|_C) = 0$. Since $\omega_{X/B}$ is numerically effective, $\mathcal{L}$ is numerically effective.

Moreover $\quad c_1(\mathcal{L})^2 = (c_1(\omega_{X/B}) + C)^2 = c_1(\omega_{X/B})^2 + h(C) \geq h(C) > 0.$

Lemma: $\mathcal{L}$ numerically effective and $c_1(\mathcal{L})^2 > 0$. Then there exists an effective divisor $E$ and $\nu_0 > 0$ such that $\quad \mathcal{L}^\nu \otimes \mathcal{O}_X(-E)$ is ample for all $\nu \geq \nu_0$.

proof.

Let $\mathcal{H}$ be an ample invertible sheaf on $X$. By Riemann-Roch, up to terms, linear in $\nu$

$h^0(X, \mathcal{L}^\nu \otimes \mathcal{H}^{-1}) + h^2(X, \mathcal{L}^\nu \otimes \mathcal{H}^{-1})$ rises like $\frac{1}{2} c_1(\mathcal{L})^2 \cdot \nu^2$.

By Serre-duality $h^2(X, \mathcal{L}^\nu \otimes \mathcal{H}^{-1}) = h^0(X, \mathcal{H} \otimes \omega_X \otimes \mathcal{L}^{-\nu})$, and the second term can not rise like $\frac{1}{2} c_1(\mathcal{L})^2 \cdot \nu^2$.

Hence for some $\nu_0 \gg 0$ one gets an inclusion $\mathcal{H} \hookrightarrow \mathcal{L}^{\nu_0}$. By the Nakai-Moižeson criterium for ampleness, $\mathcal{H} \otimes \mathcal{L}^{\nu - \nu_0}$ is ample for $\nu \geq \nu_0$.

---

Step 3.:

Claim: For some $N > 1$ and $\eta = \dfrac{h(C)}{2 \cdot (2g-1)^2}$ there is a non-trivial section

$$\sigma \in H^0(X, (\mathcal{L} \otimes \mathcal{O}_X(-\eta \cdot F))^N)$$

whose zero-divisor $D$ satisfies: ($F$ general fibre)

$D|_F = \sum \mu_j \cdot P_j$ with $P_j$ pairwise different points of $F$

and $\mu_j < N$ for all $j$. (Denote $m(D) = \mathrm{Max}\{\mu_j\}$).

Let us assume the Claim. Then, blowing up $X$, one can assume $D$ to be a normal crossing divisor.

By the corollary of Fujita's theorem $f_*(\omega_{X/B} \otimes \mathcal{L} \otimes \mathcal{O}_X(-[\frac{D}{N}]))$ is semi-positive. On the other hand, the assumption "$\mu_j < N$" implies that $[\frac{D}{N}]|_F$ is zero and the natural inclusion

$$f_*(\omega_{X/B} \otimes \mathcal{L} \otimes \mathcal{O}_X(-[\tfrac{D}{N}])) \longrightarrow f_*(\omega_{X/B} \otimes \mathcal{L})$$

is an isomorphism over an open set. One obtains B') and hence the theorem.

<u>Step 4.</u> Consequences of Riemann Roch

a) Since $\mathcal{L}$ is numerically effective, $h^i(X, \mathcal{L}^\nu)$ for $i = 1, 2$ is bounded by a polynomial of degree $i$.

<u>Proof.</u> By Kodaira vanishing for some very ample sheaf $\mathcal{A} = \mathcal{O}_X(A)$ $H^i(X, \mathcal{L}^\nu \otimes \mathcal{O}_X(A)) = 0$ for $i = 1, 2$, and all $\nu \geq 0$. Now use the exact sequence

$$0 \to \mathcal{L}^\nu \to \mathcal{L}^\nu \otimes \mathcal{O}_X(A) \to \mathcal{L}^\nu|_A \to 0 .$$

b) $\mathrm{rank}(f_* \mathcal{L}^\nu) = (2g-1) \cdot \nu - g + 1$

<u>Proof:</u> Riemann-Roch for $\mathcal{L}^\nu|_F$.

c) $h^1(B, f_* \mathcal{L}^\nu)$ is bounded by a linear polynomial in $\nu$.

<u>Proof.</u> By the Leray spectral sequence $H^1(B, f_* \mathcal{L}^\nu)$ is

contained in $H^1(X, \mathcal{L}^\nu)$.

Let us write $\tilde{\geq}$ for " $\geq$ up to a linear polynomial in $\nu$ "
and $\cong$ for " $=$ up to a linear polynomial in $\nu$ ".

d) $\deg(f_* \mathcal{L}^\nu) \tilde{\geq} \frac{1}{2} h(C) \cdot \nu^2$

Proof. $h^0(X, \mathcal{L}^\nu) = h^0(B, f_* \mathcal{L}^\nu)$. By R.R. for locally free sheaves on curves and by c) one gets

$$h^0(X, \mathcal{L}^\nu) \cong \deg(f_* \mathcal{L}^\nu) - \operatorname{rank}(f_* \mathcal{L}^\nu) \cdot (q-1) \cong \deg(f_* \mathcal{L}^\nu).$$

By R.R. for surfaces and by a)

$$h^0(X, \mathcal{L}^\nu) \cong \frac{1}{2} c_1(\mathcal{L})^2 \cdot \nu^2 \geq \frac{1}{2} h(C) \cdot \nu^2.$$

e) For all $\mu > 0$ and $\nu \gg 0$ one has

$$h^0((\mathcal{L}^{\mu+1} \otimes \mathcal{O}_X(-(2g-1) \cdot \eta \cdot \mu \cdot F))^\nu) \tilde{\geq} \frac{1}{2} h(C) \cdot (\mu+1) \cdot \nu^2.$$

Proof. Using R.R. on curves again one gets for the left hand side the lower bound

$$\deg(f_* \mathcal{L}^{\nu(\mu+1)}) - \operatorname{rank}(f_* \mathcal{L}^{\nu \cdot (\mu+1)})(q-1+(2g-1) \cdot \eta \cdot \nu \cdot \mu)$$

$$\tilde{\geq} \frac{1}{2} h(C) (\mu+1)^2 \cdot \nu^2 - (2g-1)^2 \cdot \nu (\mu+1) \cdot \eta \cdot \nu \cdot \mu$$

---

Step 5: Proof of the claim:

By the lemma in Step 2, for $\mu \gg 0$ the sheaf

$$\mathcal{L}^{(\mu+1)(2g-2)} \otimes \mathcal{O}_X(-E) \text{ is ample.}$$

Replacing $(\mu+1)$ and $E$ by some common multiple we can as well assume that

$\mathcal{L}^{(\mu+1)(2g-2)} \otimes \mathcal{O}_X(-E - (2g-1)\cdot\eta\cdot F)$ is ample.

In fact, having found some $\mu_0$ (and $E$) with this property, the same will hold true for all $\mu \geq \mu_0$.

By Step 4, e), if $E'$ is any given effective divisor on $X$, we can choose $\mu$ big enough, such that $h^0((\mathcal{L}^{\mu+1} \otimes \mathcal{O}_X(-(2g-1)\cdot\eta\cdot\mu\cdot F - E'))^\nu) \geqq a\cdot\nu^2$ for some $a>0$.

For a general divisor $D_0 \geq 0$ with

$$\mathcal{O}_X(D_0) = (\mathcal{L}^{\mu+1} \otimes \mathcal{O}_X(-(2g-1)\cdot\eta\cdot\mu\cdot F - E'))^\nu ,$$

the maximal multiplicity of components of $D_0|_F$, let us call it again $m(D_0)$, will satisfy

$$m(D_0) \leq \nu\cdot(\mu+1)\cdot(2g-1) - \deg(E'|_F)\cdot\nu .$$

Choosing $E'$ sufficiently large and such that $E'|_F$ is disjoint from $E|_F$, we can assume that for $\mu$ sufficiently large,

$$m(D_0 + \nu E' + \nu E) \leq \nu\cdot(\mu+1)(2g-1) .$$

Hence, if $D_1$ is the zero divisor of a general section of

$$(\mathcal{L}^{(\mu+1)(2g-2)} \otimes \mathcal{O}_X(-E - (2g-1)\cdot\eta\cdot F))^\nu$$

and $D = D_0 + D_1 + \nu \cdot E' + \nu \cdot E$, then

$$\mathcal{O}_X(D) = \mathcal{L}^{(\mu+1)(2g-1)\cdot\nu} \otimes \mathcal{O}_X(-(\mu+1)(2g-1)\cdot\eta\cdot\nu\cdot F)$$

and $m(D) \leq \nu\cdot(\mu+1)(2g-1)$. Hence, for $N = \nu\cdot(\mu+1)(2g-1)$ we obtain the claim.

---

Excersises:

1) Use R.R. for vector bundles on curves to show that $c_1(\omega_{X/B})^2 > 0$ if and only if $\deg(f_*\omega_{X/B}) > 0$.

2) Show (as in Step 1) that the ampleness of $f_*\omega_{X/B} \otimes \mathcal{O}_B(pt)$ implies that one has:
There exists some effective divisor $E$ on $X$, concentrated in a finite number of fibres of $f$, and $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ the sheaf

$$(\omega_{X/B} \otimes \mathcal{O}_X(F))^\nu \otimes \mathcal{O}_X(-E)$$

is ample.

3) Try to verify, whether in Steps 4 and 5 we got the numbers right.

---

Remark: The use of the Kodaira-Spencer map in this proof makes it hard to imagine, how to do something similar in the arithmetic case.

One does not have a global $\Omega^1_X$ for $f: X \to \operatorname{Spec} \mathbb{Z}$.
For the same reason, the second proof we want to present now, does not carry over. Again, the only know proofs of the Bogomolov - Miyaoka - Yau - inequality use the existence of $\Omega^1_X$.

## III Parshin's proof of Manin's theorem, using the Bogomolov - Miyaoka - Yau inequality.

Let $X$ be a surface of general type.
If $f: X \to B$ is a surjective morphism, then $q = g(B) \geq 2$ and $g = g(F) \geq 2$ implies that $X$ is of general type. Moreover, assume that $X$ is minimal. (for $f: X \to B$ with $g(B) \geq 1$ this follows if $X$ is relatively minimal over $B$).

---

**Theorem** (van de Ven, Bogomolov, Miyaoka, Yau)

$$c_1(\omega_X)^2 \leq 3 \cdot c_2(X).$$

---

Here $c_2(X) = c_2(\Omega^1_X) = c_2((\Omega^1_X)^*)$ is the second Chern class.

Remark: $X$ "minimal and of general type" implies that $c_1(\omega_X)^2 > 0$.

**Sketch of Miyaoka's proof (see [B.P.V], VI, 4):**

The inequality follows from a careful study of subbundles of $S^n(\Omega^1_X)$:

Assume that $\alpha = c_2(X)\cdot(c_1(\omega_X)^2)^{-1} < \frac{1}{3}$.

Let us write $\gamma = \frac{1}{4}(1+\alpha)$.

---

R.R. for vector bundles on surfaces, or on the projective bundles $\mathbb{P} = \mathbb{P}(\Omega^1_X) \xrightarrow{p} X$ allows to determine the leading term (in $n$) of

$$\chi(n) := \chi(S^n\Omega^1_X \otimes \omega_X^{-n\cdot\gamma}) = \chi(\mathcal{O}_{\mathbb{P}}(n)\otimes p^*\omega_X^{-n\cdot\gamma}).$$

The splitting principle for projective bundles (see [H]) gives

$$c_1(\mathcal{O}_{\mathbb{P}}(1))^2 = -p^*c_1(\omega_X)\cdot c_1(\mathcal{O}_{\mathbb{P}}(1)) - p^*c_2(X).$$

Moreover, $c_1(\mathcal{O}_{\mathbb{P}}(1))\cdot p^*(c_2(X)) = c_2(X)$ (as number) and

$$c_1(\mathcal{O}_{\mathbb{P}}(1))\cdot p^*(c_1(\omega_X)^2) = c_1(\omega_X)^2.$$

One obtains for the leading term of $\chi(n)$:

$$\frac{1}{6}\cdot c_1(\mathcal{O}_{\mathbb{P}}(n)\otimes p^*\omega_X^{-n\cdot\gamma})^3 = \frac{1}{6}\cdot n^3\cdot\left(c_1(\mathcal{O}_{\mathbb{P}}(1)) - \gamma\cdot p^*c_1(\omega_X)\right)^3 =$$

$$\frac{1}{6}\cdot n^3(1-3\gamma+3\gamma^2-\alpha) = \frac{1}{6\cdot 16}\cdot n^3(3\alpha^2-22\alpha+7).$$

Since the roots of the polynomial in $\alpha$ are $\frac{1}{3}$ and $7$, one obtains:

A) **Lemma.** $\alpha < \frac{1}{3}$ implies that $\chi(n) > 0$ for $n \gg 0$.

One has $(\Omega^1_X)^* = \Omega^1_X \otimes \omega_X^{-1}$ and, by Serre duality

$$h^2(X, S^n\Omega^1_X \otimes \omega_X^{-n\cdot\gamma}) = h^0(X, S^n\Omega^1_X \otimes \omega_X^{n(\gamma-1)+1}).$$

Hence A] implies that:

B] Lemma: $\alpha < \frac{1}{3}$ implies that for $n >> 0$

i] $H^0(X, S^n(\Omega^1_X) \otimes \omega_X^{-n\cdot\gamma}) \neq 0$

or ii] $H^0(X, S^n(\Omega^1_X) \otimes \omega_X^{n(\gamma-1)+1}) \neq 0$.

---

C] Proposition: Let $\mathcal{L}$ be a line bundle and $\mathcal{F} \subset \Omega^1_X$ a locally free subsheaf of rank 2. Assume that

a) $c_1(\mathcal{F})$ is numerically effective.

b] $H^0(X, S^n(\mathcal{F}) \otimes \mathcal{L}^{-1}) \neq 0$.

Then $c_1(\mathcal{F}) \cdot c_1(\mathcal{L}) \leq \mathrm{Max}\{n \cdot c_2(\mathcal{F}), 0\}$.

---

The proposition, applied to case i] gives (for $\mathcal{F} = \Omega^1_X$)

$\gamma \cdot n \cdot c_1(\omega_X)^2 \leq n \cdot c_2(X)$ or $\frac{1}{3} \leq \alpha$.

In case ii] it implies

$(-n(\gamma-1)-1) \cdot c_1(\omega_X)^2 \leq n \cdot c_1(X)$, hence (for $n>>0$)

$(1-\gamma) \leq \alpha$ or $\frac{3}{5} \leq \alpha$. In both cases one has a contradiction to the assumption $\alpha < \frac{1}{3}$.

---

Sketch of the proof of C]:

Step 1: Reduction to the case $n = 1$:

If $\mathbb{P} = \mathbb{P}(\mathcal{F}) \xrightarrow{p} X$ is the projective bundle, then a] gives the existence of a section of $\mathcal{O}_{\mathbb{P}}(n) \otimes p^*\mathcal{L}^{-1}$, let us say with divisor $\Sigma$. One shows the existence of a surface $Y$ and a surjective morphism $\tau$ (of high degree) such that $\tau'^{*} \mathfrak{F} = \sum^{n} \mathfrak{F}_i$ for

$$\begin{array}{ccc} \mathbb{P}(\tau^{*}\mathcal{F}) & \xrightarrow{\tau'} & \mathbb{P}(\mathcal{F}) \\ \downarrow p' & & \downarrow \\ Y & \xrightarrow{\tau} & X \end{array}$$

such that $\qquad \mathcal{O}_{\mathbb{P}(\tau^{*}\mathcal{F})}(\mathfrak{F}_i) = \mathcal{O}_{\mathbb{P}(\tau^{*}\mathcal{F})}(1) \otimes p'^{*}\mathcal{L}_i^{-1}$

and $\bigotimes^{n} \mathcal{L}_i = \tau^{*}\mathcal{L}$. Since $\tau^{*}\mathcal{F} \subset \tau^{*}\Omega^1_X \subset \Omega^1_Y$,

Proposition CJ for $n=1$ on $Y$ implies Proposition CJ on $X$.

---

Step 2: Assume that $n=1$. Then one has

**Lemma:** If $H^0(X, \mathcal{F} \otimes \mathcal{L}^{-1}) \neq 0$, then $H^0(X, \mathcal{L}^{\nu})$ is bounded by a linear polynomial in $\nu$.

Proof: If not, $H^0(X, \mathcal{L}^{\nu})$ has 3 sections $s_0, s_1, s_2$ for $\nu \gg 0$ with $f_1 = \frac{s_1}{s_0}$ and $f_2 = \frac{s_2}{s_0}$ algebraic independent. Replacing $X$ by a suitably choosen finite covering, one may assume that $\nu = 1$. Then $s_0, s_1, s_2 \in H^0(X, \mathcal{L}) \subset H^0(X, \Omega^1_X)$.

Hence $ds_i = 0$ and $0 = ds_1 = f_1 \cdot ds_0 + df_1 \wedge s_0 = df_1 \wedge s_0$

One obtains $df_1 \wedge df_2 = s_0 \wedge s_0 = 0$, contradicting the algebraic independence of $f_1$ and $f_2$.

---

We assumed in Prop. CJ that $(n=1)$ $H^0(X, \mathcal{F} \otimes \mathcal{L}^{-1}) \neq 0$. For some effective divisor $B$ one obtains an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-B) \to \mathcal{M} \otimes I_Z \to 0$$

where $\mathcal{M}$ is invertible and $Z$ the ideal sheaf of a zero dimensional subscheme of $X$. Of course,

$c_2(\mathcal{F} \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-B)) = \dim_k \mathcal{O}_X / I_Z \geq 0$, or

$$c_1(\mathcal{F}) \cdot c_1(\mathcal{L}) \leq (c_1(\mathcal{L}) + B)^2 - c_1(\mathcal{F}) \cdot B + c_2(\mathcal{F}).$$
$$\leq (c_1(\mathcal{L}) + B)^2 + c_2(\mathcal{F}).$$

If $(c_1(\mathcal{L}) + B)^2 \leq 0$, then we are done.

If $c_1(\mathcal{L} + B)^2 > 0$, then by R.R. for surfaces and the Lemma $h^2(X, (\mathcal{L} \otimes \mathcal{O}_X(B))^\nu) = h^0(X, \omega_X \otimes \mathcal{L}^{-\nu} \otimes \mathcal{O}_X(-\nu \cdot B))$ rises like $\frac{1}{2} d \cdot \nu^2$ for some $d > 0$. Then, for infinitly many $\nu$ one has $\nu \cdot c_1(\mathcal{F}) \cdot (c_1(\mathcal{L}) + B) \leq c_1(\mathcal{F}) \cdot c_1(\omega_X)$, hence $c_1(\mathcal{F}) \cdot c_1(\mathcal{L}) \leq c_1(\mathcal{F}) \cdot B + c_1(\mathcal{F}) \cdot c_1(\mathcal{L}) \leq 0$.

---

## An improvement of the BMY-inequality:

The existence of $(-2)$-curves on $X$ allows to improve the inequality. Recall that a $(-2)$-curve is a rational curve $\Gamma$, non singular, with $\Gamma^2 = -2$ (and hence $\Gamma \cdot c_1(\omega_X) = 0$). Let

$\Delta$ be the union of all $(-2)$ curves and

$\Delta_1, \dots, \Delta_r$ the connected components of $\Delta$.

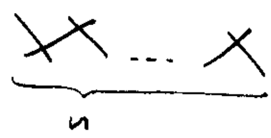
Then one has

**Thm:** $$c_1(\omega_X)^2 \leq 3 c_2(X) - \sum_{i=1}^{r} (\chi(\Delta_i) - \beta_i).$$

(See [PF] or [BPV] for the reference)

$\chi(\Delta_i)$ and $\beta_i$ are defined as follows. On the surface $X$ (of general type) one can contract the $\Delta_i$ to points $p_i$. $p_i$ is a quotient singularity, say for a group $G_i$. Then $\beta_i = \frac{1}{\# G_i}$.
$\chi(\Delta_i)$ denotes the Euler-characteristic of $\Delta_i$.

<u>Example</u>: If $p_i$ is a $A_n$-singularity, then $\chi(\Delta_i) = n+1$ and $\beta_i = \frac{1}{n+1}$. $\Delta_i$ looks like




## <u>Application to "families of curves"</u>

Let $f : X \longrightarrow B$ be a surjective morphism, $X$ relatively minimal over $B$, $f$ semi-stable, $g = g(F) \geq 2$ and (for simplicity $q = q(B) \geq 2$).
Blowing down the $A_n$-singularities (which must lie in fibres) one obtains a model

$$f' : X' \longrightarrow B .$$

Let us define for $b \in B$

$$\delta_b = \# \text{ singular points of } f^{-1}(b)$$

$$\delta'_b = \# \text{ singular points of } f'^{-1}(b) .$$

Following Parshin (see [PF]) one obtains:

Theorem (using BMY)

$$c_1(\omega_{X/B})^2 \leq 3 \sum_{b\in B} \delta_b + (2g-2)(2g(B)-2)$$

---

Proof: Since $c_1(\omega_X)^2 = c_1(\omega_{X/B})^2 + 2\cdot c_1(\omega_{X/B})\cdot c_1(f^*\omega_B)$
$= c_1(\omega_{X/B})^2 + 2(2g-2)(2q-2)$,

we just have to show that

$$c_2(X) = \sum \delta_b + (2g-2)(2q-2).$$

This follows easily from the exact sequence

$$0 \to f^*\omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0$$

and a (local) calculation, showing that

$$\Omega^1_{X/B} = \omega_{X/B} \otimes I_Z$$

where $Z = \bigcup_j P_j$ where the union is taken over all singular points of fibres.

---

Using the "Improved BMY inequality" one can improve the theorem. In fact, $X$ can only have $A_n$ singularities and,

$$c_1(\omega_X)^2 \leq 3\, c_2(X) - \sum_{\substack{\text{sing. of } X':\\ x_i \text{ of type } A_{n_i}}} \left(n_i + 1 - \frac{1}{n_i+1}\right)$$

gives:

Theorem'

$$c_1(\omega_{X/B})^2 \leq 3\cdot \sum_{b\in B} \delta'_b + (2g-2)(2q-2).$$

Corollary:

$$c_1(\omega_{X/B})^2 \leq (2g-2)(2q-2+s) \quad \text{where } s = \#\,\text{singular fibres}.$$

Proof.

Let $\tau : B_1 \to B$ be a covering of degree $\mathfrak{s}$, and $X_1$ a minimal desingularization of $X \times_B B_1$, $f_1 : X_1 \to B_1$. If $\tau$ is étale, the formula given in the Corollary is equivalent for $B$ and $B_1$.

Hence we may assume that $s \geq 2$. Choose $B_1$ to be totally ramified in all $b \in B$ with $f^{-1}(b)$ singular and nowhere else. Then

$c_1(\omega_{X_1/B_1})^2 = \mathfrak{s} \cdot c_1(\omega_{X/B})$, $g_1 = g(f_1^{-1}(a)) = g$, and for $q_1 = g(B_1)$

$2q_1 - 2 = \mathfrak{s}(2q - 2) + (\mathfrak{s} - 1) \cdot s$. Since $\sum \delta_b'$ does not change, one obtains for all $\mathfrak{s} \gg 0$:

$$c_1(\omega_{X/B})^2 \leqslant \frac{3}{\mathfrak{s}} \sum \delta_b' + (2g-2)(2q-2+s).$$

---

<u>Remark</u>: Using the methods from part II, one can show the weaker inequality

$$c_1(\omega_{X/B})^2 \leqslant (2g-2)^2(2q-2+s)$$

using the Kodaira-Spencer map, instead of BHY inequalities.

---

<u>Remark</u>: The upper bound for $c_1(\omega_{X/B})^2$ can be applied to prove the Shafarevich-conjecture over function fields. As Parshin did in [P], one can apply it to prove Mordell's conjecture over function fields, using what is called now "the Parshin trick". Let us fix from now on a section $\sigma : B \to X$ and write $C = \sigma(B)$.

## Parshin's covering construction:

**Thm [P]:** $C$ given, one can find a finite covering $B_1 \to B$, unramified over $B - S$ (for $S = \{ b ; f^{-1}(b) \text{ singular} \}$) whose degree depends only on $g$, such that one has for the fibred product

$$\begin{array}{ccc} X_1 & \xrightarrow{\sigma'} & X \\ {\scriptstyle f_1}\downarrow & & \downarrow \\ B_1 & \xrightarrow[\sigma]{} & B \end{array} \quad \text{and} \quad C_1 = \sigma'^{-1}(C) :$$

There is a semi-stable family $\varphi : Y \to B_1$, relatively minimal and smooth over $B_1 - \sigma^{-1}(S)$ and there are morphisms

$$\begin{tikzcd} Y \arrow[dr, "\varphi"'] & Y_1 \arrow[l, "\eta"'] \arrow[r, "\tau"] \arrow[d, "\varphi_1"] & X_1 \arrow[dl, "f_1"] \\ & B_1 & \end{tikzcd}$$

such that:

a) degree $(\tau)$ depends only on $g$.

b) $\eta$ is a sequence of blowing ups.

c) $\tau$ is étale over $X_1 - C_1 - f_1^{-1}(\sigma^{-1}(S))$, $\tau^* C_1$ consists of sections and fibre components and one component $\Delta_1$ $\tau^* C_1$, with $\varphi_1(\Delta_1) = B_1$, is verified.

1. __Corollary.__ $\deg(\varsigma)\cdot C\cdot c_1(\omega_{X/B}) = C_1\cdot c_1(\omega_{X_1/B_1}) \leq c_1(\omega_{Y/B_1})^2$

__2. Corollary:__ $h(C) = C\cdot c_1(\omega_{X/B})$ is bounded by a constant depending on $g$, $q$ and $s = \# S$.

__"proof" of 2:__ By the corollary to the BMY-inequality

$c_1(\omega_{Y/B_1})^2 \leq (2\cdot g_1 - 2)(2\cdot q_1 - 2 + s_1)$ where

$g_1$ is the genus of the general fibre of $Y \xrightarrow{\varphi} B_1$, $q_1 = q(B_1)$ and $s_1 = \# \varsigma^{-1}(S)$. By construction and the Hurwitz formula $q_1$ is bounded by a constant depending on $g, q$ as well as $s_1 \leq s\cdot\deg(\varsigma)$ and $g_1$.

__"proof" of 1:__ The first equality is nothing but the projection formula, since $\omega_{X_1/B_1} = \varsigma'^*\omega_{X/B}$.

For the second one, let us remark first, that

$$\omega_{Y_1/B_1} = \eta^*\omega_{Y/B_1}\otimes\mathcal{O}_{Y_1}(E)$$

for some effective divisor $E$, with $\eta(E)$ a finite number of points. Moreover,

$$\omega_{Y_1/B_1} = \tau^*\omega_{X_1/B_1}\otimes\mathcal{O}_Y(\Sigma(e_i - 1)\cdot\Delta_i + \vartheta),$$

where $\vartheta$ is an effective divisor contained in fibres.

$E$ and $\vartheta$ are both effective, contained in fibres and none of them can contain a whole fibre. Hence $\eta^*\omega_{Y/B_1} = \tau^*\omega_{X_1/B_1} \otimes \mathcal{O}_Y(\Sigma(e_i-1)\Delta_i + \vartheta')$ where again $\vartheta'$ is effective. One obtains, since $\tau^*\omega_{X_1/B_1}$ and $\eta^*\omega_{Y/B_1}$ are numerically effective:

$$c_1(\omega_{Y/B_1})^2 = c_1(\omega_{Y_1/B_1}) \cdot c_1(\eta^*\omega_{Y/B_1}) \geq$$

$$c_1(\tau^*\omega_{X_1/B_1}) \cdot c_1(\eta^*\omega_{Y/B_1}) \geq$$

$$c_1(\tau^*\omega_{X_1/B_1})^2 + c_1(\tau^*\omega_{X_1/B_1}) \cdot (e_1-1)\Delta_1$$

$$\geq c_1(\tau^*\omega_{X_1/B_1}) \cdot \Delta_1 = c_1(\omega_{X_1/B_1}) \cdot C_1 .$$

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Altogether, the second proof of Manin's theorem is finished, if we prove the Theorem, i.e. the existence of the Parshin-construction. To illustrate what has to be done, let us first consider the generic fibre $F = X \times_B \operatorname{Spec} \overline{k(B)} \; \left(= X_1 \times_{B_1} \operatorname{Spec} \overline{k(B)}\right)$.

Since $q(F) \geq 2$, $F$ has unramified cover.

Step 1: Choose a non trivial unramified cover

$$F_2 \xrightarrow[\tau_1]{} F.$$

Step 2.: Choose two points $Q_1, Q_2 \in F_2$ lying over the point $P$ induced by $C$.

$\mathcal{O}_{F_2}(Q_1 - Q_2)$ is in $\mathrm{Pic}^0(F_2)$, hence we find some $\mathcal{N} \in \mathrm{Pic}^0(F_2)$ with $\mathcal{N}^2 = \mathcal{O}_{F_2}(Q_1 - Q_2)$.

Take $\mathcal{L} = \mathcal{N} \otimes \mathcal{O}_{F_2}(Q_2)$. Then

$\mathcal{L}^2 = \mathcal{O}_{F_2}(Q_1 + Q_2)$ and, as in section II, one finds a cover $F_1 \xrightarrow[\tau_2]{} F_2$ of degree 2 with $\tau_2^*(Q_1 + Q_2) = 2 \cdot Q_1' + 2 \cdot Q_2'$ and étale over $F_2 - Q_1 - Q_2$.

$F_1$ will be defined over some finite extension $L$ of $k(B)$. Choose $B_1$ to be the normalization of $B$ in $L$, i.e. $L = k(B_1)$, and choose $Y_1$ to be a desingularization of the induced family of curves over $B_1$ induced by $F_1$ (as explained in part 1).

The only problem is, to do this construction such that $B_1 \to B$ is unramified over $B - S$ and of bounded degree. To this aim, one has to modify the construction in the following way:

<u>In Step 1</u>: Let $U = B - S$ and $V = f^{-1}(U) \to U$. $V \to U$ is smooth and hence one can consider the relative Jacobian variety $J = J(V/U) \to U$.

Let
$$\begin{tikzcd} J \arrow[rr, "\cdot 2"] \arrow[dr] & & J \arrow[dl] \\ & U & \end{tikzcd}$$
be the multiplication by 2

and
$$\begin{tikzcd} W_2 \arrow[rr, "\tau_1"] \arrow[dr] & & V \arrow[dl] \\ & U & \end{tikzcd}$$
the étale cover induced.

Over some etale cover $U_2 \to U$ of bounded degree ($\leq 2^g$) $\tau_1^{-1}(C)$ will become the sum of sections.

<u>In Step 2</u>: If $W_3 \to U_2$ is the pullback family and $\Delta'_1$, $\Delta'_2$ two sections lying over $C$, then $\mathcal{O}_{W_3}(\Delta'_1 - \Delta'_2)$

will not nescessarily be a square.

However, using the multiplication with 2 in $\mathrm{Pic}^\circ(W_3/U_2)$ one finds $U_3 \to U_2$, étale of order $\leq 2^9$, such that for $W_4 = W_3 \times_{U_2} U_3$ all 2-division points in $\mathrm{Pic}^\circ(W_4/U_3)$ are sections. Over $U_3$ our construction will work. Hence, $B_1$ should be a completion of $U_3$ and $X_1$ a completion of $W_4$.

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Of course, one still has to replace $B_1$ by a finite cover to obtain semistable reduction for $Y_1 \to B_1$. However, here again the degree can be bounded.

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