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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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ALGEBRAIC GEOMETRY:
Cohomology;
The Chow Ring and the Riemann-Roch Theorem.

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These are preliminary lecture notes, intended only for distribution to participants

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Cohomology

If X is a topological space, a sequence of sheaves of abelian groups

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \cdots \longrightarrow \mathcal{F}_n$$

is <u>exact</u> if for each $x \in X$, the sequence of stalks

$$\mathcal{F}_{1,x} \longrightarrow \mathcal{F}_{2,x} \longrightarrow \cdots \longrightarrow \mathcal{F}_{n,x}$$

is an exact sequence of abelian groups. If

$$0 \to \mathcal{F}' \xrightarrow{a} \mathcal{F} \xrightarrow{b} \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of sheaves, then we have an exact sequence

$$0 \to \Gamma(X, \mathcal{F}') \xrightarrow{a_*} \Gamma(X, \mathcal{F}) \xrightarrow{b_*} \Gamma(X, \mathcal{F}''),$$

where b_* is in general <u>not</u> surjective.

<u>Example</u>. On $X = \mathbb{P}^1_k$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{2P} \longrightarrow 0$$

where $P = (1:0)$, and $2P$ is the subscheme whose ideal sheaf $\mathcal{O}_X(-2P)$ is the square of the maximal ideal sheaf of P.

Then we have

$$H^0(X, \mathcal{O}_X) = k, \quad \text{and} \quad H^0(X, \mathcal{O}_{2P}) \cong k[t]/(t^2).$$

Hence $H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_{2P})$ is not surjective.

The cohomology groups $H^i(X, \mathcal{F})$ are the defined as the values of) the <u>right derived functors</u> of the left exact functor $\Gamma(X, -) = H^0(X, -)$. We recall how these are defined:

a sheaf $\mathcal{I}$ is called <u>injective</u> if any monomorphism $0 \to \mathcal{I} \longrightarrow \mathcal{F}$ has a splitting $s: \mathcal{F} \twoheadrightarrow \mathcal{I}$. Equivalently, if

$0 \to \mathcal{F} \to \mathcal{G}$ is a monomorphism, then any homomorphism $\mathcal{F} \to \mathcal{I}$ extends to a homomorphism $\mathcal{G} \to \mathcal{I}$.

emma 1 – Let X be a topological space.

1) For any sheaf $\mathcal{F}$ of abelian groups on X, there is a resolution
$$0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots$$
where the $\mathcal{I}_j$ are injective.

2) If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a homomorphism of sheaves, $0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots$ a resolution, $0 \to \mathcal{G} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$ a ~~injective resolution~~ complex where $\mathcal{I}_j$ ~~are~~ is injective $\forall i$, then $\exists$ a map of complexes $\mathcal{F}_\bullet \to \mathcal{I}_\bullet$ compatible with f. If $0 \to \mathcal{G} \to \mathcal{I}_\bullet$ is exact, this map is unique upto homotopy.

3) Any two injective resolutions of a sheaf $\mathcal{F}$ are chain homotopic.

roof :– 2) ~~follows from the~~ is proved by a standard argument using the defining property of an injective (for example, the composite $\mathcal{F} \xrightarrow{f} \mathcal{G} \to \mathcal{I}_0$ extends to $\mathcal{F}_0 \to \mathcal{I}_0$; if $\mathcal{G} = \ker(\mathcal{F}_1 \to \mathcal{F}_2) = \operatorname{coker}(\mathcal{F} \to \mathcal{F}_0)$, then ~~$\mathcal{F}_1 \to$~~ the composite $\mathcal{F}_0 \to \mathcal{I}_0 \to \mathcal{I}_1$ factors through $\mathcal{G}$, and the map $\mathcal{G} \to \mathcal{I}_1$ extends to $\mathcal{F}_1$ since $\mathcal{I}_1$ is injective, etc.). Clearly 2) $\Rightarrow$ 3).

To prove 1), it suffices to prove that for any sheaf $\mathcal{F}$, there is a monomorphism $0 \to \mathcal{F} \to \mathcal{I}$ with $\mathcal{I}$ an injective sheaf. For any $x \in X$ and any abelian group A, ~~if $i_x : \{x\} \to X$ is the inclusion,~~ let ~~$A_{\{x\}}$~~ $A_{\{x\}}$ denote the constant sheaf corresponding to A on $\{x\}$; if A is an injective (= divisible) abelian group, then $(i_x)_* A_{\{x\}}$ (where $i_x : \{x\} \hookrightarrow X$) is an injective sheaf, such that for any $\mathcal{F}$, $\operatorname{Hom}_X(\mathcal{F}, (i_x)_* A_{\{x\}}) = \operatorname{Hom}(\mathcal{F}_x, A)$. Now for each $x \in X$, there is a monomorphism of abelian groups $0 \to \mathcal{F}_x \to I_x$, where I_x is divisible: if
$$0 \to F_1 \to F_0 \to \mathcal{F}_x \to 0$$

is a presentation of $\mathcal{F}_x$, where F_0, F_1 are free abelian, then $F_0 \hookrightarrow F_1 \hookrightarrow F_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ induces a monomorphism

$$\mathcal{F}_x \cong F_1/F_0 \hookrightarrow (F_1 \otimes_{\mathbb{Z}} \mathbb{Q})/F_0 = I_x$$

where the latter is divisible.

Now $\mathcal{F} \longrightarrow \prod_{x \in X} (i_x)_*(I_x)_{\{x\}} = \mathcal{I}$

is the desired monomorphism to an injective sheaf.

<u>Definition</u>: $H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}_\cdot))$ where $0 \to \mathcal{F} \to \mathcal{I}_\cdot$ is an injective resolution (By 3) of the lemma, this is independent of the resolution).

It is also useful to compute cohomology using other types of resolutions.

<u>Lemma 2</u> – 1) For any sheaf $\mathcal{F}$ and any resolution

$$0 \to \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \cdots$$

such that $H^i(X, \mathcal{F}_j) = 0 \ \forall\, i > 0, \forall\, j \geq 0$, we have isomorphisms

$$H^i(X, \mathcal{F}) \cong H^i(\Gamma(X, \mathcal{F}_\cdot)) = i^{\text{th}} \text{ cohomology of the complex of}$$

abelian groups $\quad 0 \to \Gamma(X, \mathcal{F}_0) \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \cdots$

2) Suppose $\mathcal{F}$ is <u>flasque</u> i.e. for any open $U \subset X$,

$$\rho_{XU} : \mathcal{F}(X) \longrightarrow \mathcal{F}(U)$$

is surjective. Then $H^i(X, \mathcal{F}) = 0 \ \forall\, i > 0$.

~~) Let $i : Y \hookrightarrow X$ be the inclusion of a closed subset of X~~,

3) Let $\mathcal{I}$ be an injective sheaf. Then $\mathcal{I}$ is flasque.

4) Let $i : Y \hookrightarrow X$ be the inclusion of a closed subset. Then for any sheaf $\mathcal{F}$ on Y, there is a natural isomorphism

$$H^i(Y, \mathcal{F}) \xrightarrow{\cong} H^i(X, i_* \mathcal{F}), \quad \forall\, i \geq 0.$$

<u>Sketch of proof</u> :– 1) By induction on i; for $i = 0$, this follows from the left exactness of $\Gamma(X, -)$. Given any $\mathcal{F}$, and a resolution by

sheaves with vanishing cohomology,

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots,$$

we may split up this resolution into a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{G} \to 0$$

and a ~~too~~ resolution

$$0 \to \mathcal{G} \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots$$

Then $H^1(X,\mathcal{F}) \cong \operatorname{coker}\left(H^0(X,\mathcal{F}_0) \to H^0(X,\mathcal{G})\right)$

$= \text{cohomology of } \left(H^0(X,\mathcal{F}_0) \to H^0(X,\mathcal{F}_1) \to H^0(X,\mathcal{F}_2)\right),$

and $H^i(X,\mathcal{F}) \cong H^{i-1}(X,\mathcal{G})$ for $i \geq 2$.

The formula for $H^1(X,\mathcal{F})$ is the desired one, so the result is proved for $i=1$. If the result holds for ~~i-1~~ H^{i-1}, ~~then~~ where $i \geq 2$, then

$H^i(X,\mathcal{F}) \cong H^{i-1}(X,\mathcal{G}) \cong H^{i-1}$ of the complex $\left(H^0(X,\mathcal{F}_1) \to H^0(X,\mathcal{F}_2) \to \cdots\right)$

$= H^i$ of ~~of~~ the complex $\left(H^0(X,\mathcal{F}_0) \to H^0(X,\mathcal{F}_1) \to H^0(X,\mathcal{F}_2) \to \cdots\right)$.

This proves the result for H^i.

2) This follows from 3) by induction, using the following ~~two~~ (easily proved) statements: (i) if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact and $\mathcal{F}'$ is flasque, then $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact for each open $U \subset X$ (ii) if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact and $\mathcal{F}', \mathcal{F}$ are flasque, so is $\mathcal{F}''$.

3) Let $U \overset{j}{\hookrightarrow} X$ be the inclusion of an open set, $\mathcal{I}$ an injective sheaf. Then $\mathcal{I}(U) = \operatorname{Hom}_X(j_!\mathbb{Z}_U, \mathcal{I})$; since $\exists$ a natural monomorphism $0 \to j_!\mathbb{Z}_U \to \mathbb{Z}_X$, and $\mathcal{I}$ is injective, any map $j_!\mathbb{Z}_U \to \mathcal{I}$ (i.e. section of $\mathcal{I}$ on U) extends to a map $\mathbb{Z}_X \to \mathcal{I}$ (i.e. a global section of $\mathcal{I}$).

4) Let $0 \to \mathcal{F} \to \mathcal{I}_\bullet$ be an injective resolution on Y. Then $0 \to i_*\mathcal{F} \to i_*\mathcal{I}_\bullet$ is exact, since Y is closed; it is a flasque resolution

n X with $\Gamma(X, i_* \mathcal{I}_\cdot) = \Gamma(Y, \mathcal{I}_\cdot)$, so that

$$H^j(X, i_* \mathcal{F}) \cong H^j(\Gamma(X, i_* \mathcal{I}_\cdot)) = H^j(\Gamma(Y, \mathcal{I}_\cdot)) \cong H^j(Y, \mathcal{F}) \quad \forall j.$$

some properties of cohomology.

) $\mathcal{F} \mapsto H^j(X, \mathcal{F})$ is a functor from sheaves of abelian groups on X to the category of abelian groups.

2) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, then there is a long exact sequence of cohomology groups

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}'') \to H^2(X, \mathcal{F}') \to \cdots$$

To see this, note that if $0 \to \mathcal{F}' \to \mathcal{I}'_\cdot$, $0 \to \mathcal{F}'' \to \mathcal{I}''_\cdot$ are injective resolutions, then one can find an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}_0' \oplus \mathcal{I}_0'' \to \mathcal{I}_1' \oplus \mathcal{I}_1'' \to \mathcal{I}_2' \oplus \mathcal{I}_2'' \to \cdots$$

such that there is a commutative diagram of resolutions

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{I}_0' \arrow[r] \arrow[d] & \mathcal{I}_1' \arrow[r] \arrow[d] & \mathcal{I}_2' \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{I}_0' \oplus \mathcal{I}_0'' \arrow[r] \arrow[d] & \mathcal{I}_1' \oplus \mathcal{I}_1'' \arrow[r] \arrow[d] & \mathcal{I}_2' \oplus \mathcal{I}_2'' \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F}'' \arrow[r] \arrow[d] & \mathcal{I}_0'' \arrow[r] \arrow[d] & \mathcal{I}_1'' \arrow[r] \arrow[d] & \mathcal{I}_2'' \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where the vertical sequences $0 \to \mathcal{I}_j' \to \mathcal{I}_j' \oplus \mathcal{I}_j'' \to \mathcal{I}_j'' \to 0$ are the obvious split exact sequences). Thus we obtain a short exact sequence of complexes of abelian groups

$$0 \to \Gamma(X, \mathcal{I}'_\cdot) \to \Gamma(X, \mathcal{I}'_\cdot \oplus \mathcal{I}''_\cdot) \to \Gamma(X, \mathcal{I}''_\cdot) \to 0$$

whose associated long exact sequence of cohomology groups is the one we wanted to construct.

) If we have a commutative diagram, whose rows are exact,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0
\end{tikzcd}$$

then the maps on cohomology given by the vertical maps yield a commutative diagram

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \Gamma(X,\mathcal{F}') \arrow[r] \arrow[d] & \Gamma(X,\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(X,\mathcal{F}'') \arrow[r] \arrow[d] & \cdots \arrow[r] & H^i(X,\mathcal{F}'') \arrow[r] \arrow[d] & H^{i+1}(X,\mathcal{F}') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Gamma(X,\mathcal{G}') \arrow[r] & \Gamma(X,\mathcal{G}) \arrow[r] & \Gamma(X,\mathcal{G}'') \arrow[r] & \cdots \arrow[r] & H^i(X,\mathcal{G}'') \arrow[r] & H^{i+1}(X,\mathcal{G}') \arrow[r] & \cdots
\end{tikzcd}$$

) Since $\mathcal{F} \mapsto H^i(X,\mathcal{F})$ is a functor, $H^i(X,\mathcal{F})$ is naturally a module over the ring $\mathrm{End}(\mathcal{F}) = \mathrm{Hom}(\mathcal{F},\mathcal{F})$. In particular, if $\mathcal{F}$ is a sheaf of R-modules, then so is $H^i(X,\mathcal{F})$ for each i.

In fact, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves of R-modules, then the long exact cohomology sequence in 2) above is an exact sequence of R-modules. This is because ⓐ if $\mathcal{I}$ is an injective object in the category of sheaves of R-modules, then $\mathcal{I}$ is flasque (this is similar to the proof that injective sheaves are flasque) ⓑ if $\mathcal{I}$ is an injective sheaf of abelian groups, R_X the constant sheaf on X associated to R, then the sheaf $\mathcal{H}om(R_X, \mathcal{I})$ is an injective object in the category of sheaves of R-modules. From ⓑ, and sheaf of R-modules has a resolution by injectives in the category of sheaves of R-modules, and by ⓐ, this resolution may be used to compute cohomology. Now the long exact sequence, ~~may be~~ constructed as in 2), is an exact sequence of R-modules.

<u>Čech Cohomology</u> If $U \overset{j}{\hookrightarrow} X$ is an open set, $\mathcal{F}$ a sheaf on X, let $\mathcal{F}_U = j_*(\mathcal{F}|_U) = j_* j^{-1}\mathcal{F}$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X, where I is a well ordered set. For each $i_0, \dots, i_p \in I$, let $U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Define the Čech sheaves and Čech cochain groups as follows:

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{\substack{i_0<\cdots<i_p \\ i_\nu \in I}} \mathcal{F}_{U_{i_0\cdots i_p}}, \quad C^p(\mathcal{U},\mathcal{F}) = \Gamma(X, \mathcal{C}^p(\mathcal{U},\mathcal{F})) = \prod_{\substack{i_0<\cdots<i_p \\ i_\nu \in I}} \mathcal{F}(U_{i_0\cdots i_p}).$$

There are differentials $d: \mathcal{C}^p(\mathcal{U},\mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F})$, and $C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$, given on $\alpha \in \mathcal{C}^p(\mathcal{U},\mathcal{F})(U)$ by

$$(d\alpha)_{i_0\cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\cdots \hat{i}_k \cdots i_{p+1}}\big|_{U\cap U_{i_0\cdots i_{p+1}}},$$

where $\hat{i}_k$ means i_k is omitted. Define the <u>Čech cohomology groups</u> $\check{H}^i(\mathcal{U},\mathcal{F}) = H^i(C^\bullet(\mathcal{U},\mathcal{F}))$.

From 2) of lemma 1, there are natural maps $\check{H}^i(\mathcal{U},\mathcal{F}) \to H^i(X,\mathcal{F})$.
For $i=0$, these are isomorphisms, and monomorphisms for $i=1$.

<u>Theorem of Leray</u>:- In the above situation, if $H^i(U_{i_0\cdots i_p}, \mathcal{F}_{U_{i_0\cdots i_p}})=0$ $\forall i>0$, ($\forall p\geq 0$), $\forall i_0<\cdots<i_p$, $i_\nu\in I$, Then $\check{H}^i(\mathcal{U},\mathcal{F}) \to H^i(X,\mathcal{F})$ is an isomorphism for each $i\geq 0$.

Cohomology of quasi-coherent and coherent sheaves

We will work only with schemes X which are quasi-projective over a Noetherian ring R.

<u>Affine schemes</u>:- Let $X=\operatorname{Spec} R$, where R is a Noetherian ring.

Then one has

<u>Lemma 3</u>: 1) If I is an injective R-module, and $f\in R$, the localisation map $I\longrightarrow I_f$ is surjective.

2) $\widetilde{I}$ is flasque on $\operatorname{Spec} R$.

<u>Cor</u>: Let $X=\operatorname{Spec} R$, where R is Noetherian, and let M be any R-module. Then $H^i(X,\widetilde{M})=0$ $\forall i>0$.

<u>Proof</u>:- Let $0\to M\to I_0\to I_1\to\cdots$ be a ~~injecti~~ resolution by injective R-modules. Then $0\to\widetilde{M}\to\widetilde{I}_0\to\widetilde{I}_1\to\cdots$ is a resolution of $\widetilde{M}$ by flasque sheaves, so that by lemma 2, $H^i(X,\widetilde{M})\cong H^i(\Gamma(X,\widetilde{I}_\bullet)) = H^i(I_\bullet)=0$ for $i>0$.

~~Thus~~ If X is quasi-projective over a Noetherian ring R, then X has an open covering by affine open sets $\{U_i\}_{i\in I}$ (in fact, by a finite number) such that $U_{i_0\cdots i_p}$ is affine for each $i_0,\dots,i_p\in I$ (if X is quasi-projective, the intersection of any two affine open subsets is affine, since X is <u>separated</u>). Hence by the above Corollary, we see that $H^i(X,\mathcal{F})\cong\check{H}^i(\mathcal{U},\mathcal{F})$, the Cech cohomology groups, from the theorem of Leray, provided $\mathcal{F}$ is quasi-coherent. In practice, this is often the most convenient way to compute cohomology.

In particular, we may use this to compute the cohomology of the invertible sheaves $\mathcal{O}(r)$ on $\mathbb{P}^n_R$. ~~This is~~ The results are ~~most~~ conveniently stated by introducing the graded, quasi-coherent $\mathcal{O}_{\mathbb{P}^n_R}$-module $\mathcal{S} = \bigoplus_{r\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n_R}(r)$.

Theorem 1 (1) There is an isomorphism of graded rings

$$H^0(\mathbb{P}^n_R, \mathcal{S}) \cong R[x_0, \ldots, x_n],$$

where the polynomial ring has the usual grading ($\deg a = 0$ $\forall\ a \in R$, $\deg x_i = 1\ \forall\ i$). In particular,

$$H^0(\mathbb{P}^n_R, \mathcal{O}(r)) \cong \begin{cases} 0 & \text{if } r<0 \\ R^{\oplus m} \text{ with } m = \binom{n+r}{r} & \text{if } r \geq 0. \end{cases}$$

(2) $H^i(\mathbb{P}^n_R, \mathcal{S}) = 0$ for $0<i<n$. Thus $H^i(\mathbb{P}^n_R, \mathcal{O}(r)) = 0\ \forall\, r\in\mathbb{Z}$ $\forall\ 0<i<n$.

(3) $$H^n(\mathbb{P}^n_R, \mathcal{S}) = R\left[x_0, \ldots, x_n, \frac{1}{x_0\cdots x_n}\right] \Big/ \sum_{\substack{a_j \geq 0 \\ \text{for some } j}} R.\, x_0^{a_0}\cdots x_n^{a_n}.$$

$$\cong \bigoplus_{\substack{b_j > 0 \\ \text{for all } j}} R.\, \frac{1}{x_0^{b_0}\cdots x_n^{b_n}}$$

We note that the multiplication on $\mathcal{S}$ makes $H^n(\mathbb{P}^n_R, \mathcal{S})$ a graded $H^0(\mathbb{P}^n_R, \mathcal{S}) = R[x_0, \ldots, x_n]$-module; the isomorphism in 3) is as $R[x_0,\ldots,x_n]$-modules. Thus

(i) $H^n(\mathbb{P}^n_R, \mathcal{O}(r)) = 0$ if $r \geq -n$

(ii) $H^n(\mathbb{P}^n_R, \mathcal{O}(-n-1)) \cong R$

(iii) there is a perfect pairing between free R-modules of finite rank $H^0(\mathbb{P}^n_R, \mathcal{O}(m)) \otimes_R H^n(\mathbb{P}^n_R, \mathcal{O}(-m-n-1)) \to R$, which in terms of the above descriptions is given by

$$f\otimes g \mapsto fg \pmod{\sum_{a_j\geq 0} R.\, x_0^{a_0}\cdots x_n^{a_n}}$$
$$= \text{coefficient of } \frac{1}{x_0\cdots x_n} \text{ in } fg.$$

The computation of the above cohomology groups for $\mathbb{P}^n_R$ implies a basic finite generation theorem for ~~any~~ the cohomology of a coherent sheaf on an arbitrary scheme X with is projective over a Noetherian ring.

Theorem 2: Let X be projective over a Noetherian ring R, say $X \hookrightarrow \mathbb{P}^n_R$. Let $\mathcal{O}_X(1)$ be the restriction of $\mathcal{O}_{\mathbb{P}^n}(1)$ to X.
~~Then~~ (1) for any coherent sheaf $\mathcal{F}$ on X, $H^i(X,\mathcal{F})$ is a finitely generated R-module. ~~There~~
(2) There is an integer n_0 such that $H^i(X,\mathcal{F}(n))=0$ for all $n\geq n_0$, for all $i>0$ (here $\mathcal{F}(n)=\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_X(n)$).
(3) If $f\in R$, $X_f = X\times_{\operatorname{Spec} R} D_f$, then
$$H^i(X_f, \mathcal{F}|_{X_f}) \cong H^i(X,\mathcal{F})\otimes_R R_f \text{ for any quasi-coherent sheaf } \mathcal{F}.$$

Proof:- Let $i: X\hookrightarrow \mathbb{P}^n_R$. Then $\underbrace{H^i(X,\mathcal{F}) \cong H^i(\mathbb{P}^n_R, i_*\mathcal{F})}_{\text{(by lemma 2)}}$. Hence we reduce to the case when $X=\mathbb{P}^n_R$. Since $\mathbb{P}^n_R$ has an affine open cover by $n+1$ open subsets, $\mathcal{U}=\{U_i\}_{i=0}^n$,
$$H^i(\mathbb{P}^n_R,\mathcal{F}) \cong \check{H}^i(\mathcal{U},\mathcal{F}) = 0 \text{ for } i>n,$$
for any (quasi-)coherent sheaf $\mathcal{F}$. Hence the theorem holds for H^i with $i>n$. Next, for any coherent sheaf $\mathcal{F}$, there is a surjection $\mathcal{O}(-m)^{\oplus N} \twoheadrightarrow \mathcal{F}$, with kernel $\mathcal{G}$, say; then $\mathcal{G}$ is also coherent, and we have a long exact cohomology sequence

$$0\to H^0(\mathbb{P}^n_R,\mathcal{G})\to H^0(\mathbb{P}^n_R, \mathcal{O}(-m))^{\oplus N} \to H^0(\mathbb{P}^n_R,\mathcal{F})\to\cdots$$
$$\cdots\to H^n(\mathbb{P}^n_R,\mathcal{O}(-m))^{\oplus N}\to H^n(\mathbb{P}^n_R,\mathcal{F})\to 0.$$

From the computation of the cohomology of the sheaves $\mathcal{O}(r)$, we see that the theorem for H^i follows by descending induction on i (for example, $H^n(\mathbb{P}^n_R, \mathcal{F})$ is a quotient of $H^n(\mathbb{P}^n_R, \mathcal{O}(-m))^{\oplus N}$, and is hence finitely generated; since $\mathcal{F}$ was arbitrary, so is $H^n(\mathbb{P}^n_R, \mathcal{G})$, which implies from the long exact sequence that $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ is finitely generated, and so on). The vanishing result is proved similarly.

To prove (3), note that $\mathcal{U}_f = \{U_i \cap X_f\}$ is the standard affine open cover of X_f, and $C^\bullet(\mathcal{U}_f, \mathcal{F}|_{X_f}) = C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_R R_f = C^\bullet(\mathcal{U}, \mathcal{F})_f$. Hence $\check{H}^i(\mathcal{U}, \mathcal{F})_f \cong \check{H}^i(\mathcal{U}_f, \mathcal{F}|_{X_f})$.

Corollary :- Let X be a projective scheme over a field k. Then for ~~$H^i(X,\mathcal{F})$~~ any coherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.

Thus, we can define $\chi(X, \mathcal{F}) = \chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F})$, the Euler characteristic of $\mathcal{F}$. The long exact cohomology sequence implies that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Definition : Let X, Y be quasi-projective over a Noetherian ring R. A morphism $f : X \to Y$ is called projective if it has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y = \mathbb{P}^n_R \times_{\operatorname{Spec} R} Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

for some n, where i is a closed embedding (an isomorphism of X with a closed subscheme of $\mathbb{P}^n_R$), and p is the projection.

The above theorem can be generalized as follows:

Theorem 3 : Let $f : X \to Y$ be a projective morphism, and $\mathcal{F}$ a oherent sheaf on X. Then for each $i \geq 0$, there is a unique coherent heaf $R^i f_* \mathcal{F}$ on Y such that for each affine open set $U \subset Y$,

$$R^i f_* \mathcal{F}(U) = H^i(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}).$$

The sheaves $R^i f_* \mathcal{F}$, $i \geq 0$, have the following properties:

i) if $d = \sup_{y \in Y} \{\dim f^{-1}(y)\}$,

then $R^i f_* \mathcal{F} = 0$ for $i > d$

) if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules, then there is a long exact sequence of coherent $\mathcal{O}_Y$-modules

$$0 \to f_* \mathcal{F}' \to f_* \mathcal{F} \to f_* \mathcal{F}'' \to R^1 f_* \mathcal{F}' \to R^1 f_* \mathcal{F} \to R^1 f_* \mathcal{F}'' \to \cdots$$

$$\cdots \to R^n f_* \mathcal{F}'' \to R^{n+1} f_* \mathcal{F}' \to \cdots.$$

Given a diagram ~~wi~~ ~~of~~ with short exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0
\end{tikzcd}$$

there is a commutative diagram ~~of long ex~~ of coherent $\mathcal{O}_Y$-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{F}' \arrow[r] \arrow[d] & f_*\mathcal{F} \arrow[r] \arrow[d] & f_*\mathcal{F}'' \arrow[r] \arrow[d] & R^1 f_*\mathcal{F}' \arrow[r] \arrow[d] & \cdots \arrow[r] & R^n f_*\mathcal{F}'' \arrow[r] \arrow[d] & R^{n+1} f_*\mathcal{F}' \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & f_*\mathcal{G}' \arrow[r] & f_*\mathcal{G} \arrow[r] & f_*\mathcal{G}'' \arrow[r] & R^1 f_*\mathcal{G}' \arrow[r] & \cdots \arrow[r] & R^n f_*\mathcal{G}'' \arrow[r] & R^{n+1} f_*\mathcal{G}' \arrow[r] & \cdots
\end{tikzcd}$$

iii) if $g: Y' \longrightarrow Y$ is any morphism of quasi-projective R-schemes, then $f': X' = X \times_Y Y' \longrightarrow Y'$ is projective, and for any coherent sheaf $\mathcal{F}$ on X, there are natural maps of $\mathcal{O}_{Y'}$-modules

$$g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* \mathcal{F}', \qquad (\#)$$

where if $g': X' \to X$, then $\mathcal{F}' = g'^* \mathcal{F}$. In particular, for any $y \in Y$, there are natural maps

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \longrightarrow H^i(X_y, \mathcal{F}|_{X_y})$$

where X_y is the fibre over y of f.

iv) in iii), suppose g is <u>flat</u> i.e. for each pair $U = \operatorname{Spec} A \subset Y'$, $V = \operatorname{Spec} B \subset Y$ with $g(U) \subset V$, the ring homomorphism ~~&~~ $g^\#: B \to A$ is flat (i.e. $- \otimes_A B$ is an exact functor); then the maps $(\#)$ are all isomorphisms.

<u>Remark</u>:- For any continuous map $f: X \to Y$ of topological spaces, ~~an~~ one can define $R^i f_*$, $i \geq 0$, as the right derived functors of f_*; then one checks that $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$.
If ~~&~~ $f: X \to Y$ is a projective morphism, this agrees with the $R^i f_* \mathcal{F}$ in the theorem, if $\mathcal{F}$ is coherent. But now we know $R^i f_* \mathcal{F}(V)$ whenever V is affine. To prove iii) and iv), it suffices to do it when Y, Y' are affine, say $Y = \operatorname{Spec} B$, $Y' = \operatorname{Spec} A$; then $C^\cdot(\mathcal{U}', \mathcal{F}') \cong C^\cdot(\mathcal{U}, \mathcal{F}) \otimes_B A$, so there are maps $\check{H}^i(\mathcal{U}, \mathcal{F}) \otimes_B A \to \check{H}^i(\mathcal{U}', \mathcal{F}')$, which are isomorphisms if B is flat over A.

Example: Let X be a non-singular projective curve over an algebraically closed field k.

Riemann-Roch theorem (I^{st} form): Let D be a divisor on X. Then $\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X)$.

Proof:- Let $P \in X$ be a closed point. Then there is an exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+P) \to i_{P*} k_P \to 0,$$

where $i_P : \{P\} \hookrightarrow X$, and k_P is ~~regarded as a const~~ the field k regarded as a constant sheaf on $\{P\}$. Then

$$H^j(X, i_{P*} k_P) \cong H^j(\{P\}, k_P) = \begin{cases} k & \text{if } j=0, \\ 0 & \text{if } j>0, \end{cases}$$

~~Whence the vanishing of~~ so that $\chi(i_{P*} k_P) = 1$. Thus

$\chi(D+P) - \chi(D) = 1 = \deg(D+P) - \deg D$ (where $\chi(D) = \chi(\mathcal{O}_X(D))$)

i.e. $\chi(D) - \deg D$ is unchanged if we add a point to D.

From this, it follows easily that $\chi(D) - \deg D$ is independent of D, and hence equal to its value for $D=0$. Further, $\chi(D_1) = \chi(D_2)$ if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$; hence $\deg D_1 = \deg D_2$ in that case. Now $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \Leftrightarrow \mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X$. If $\varphi : \mathcal{O}_X \to \mathcal{O}_X(D)$ is an isomorphism, for some D, ~~then~~ let $\varphi(1) = s \in \Gamma(X, \mathcal{O}_X(D))$. If $D = \sum_{i=1}^{r} n_i P_i$, let $U = X - \{P_1, \dots, P_r\}$. Then $\mathcal{O}_X(D)|_U = \mathcal{O}_U$, so $s|_U$ is a nowhere-zero regular function on U. From the definitions, one sees that φ is an isomorphism $\Leftrightarrow$ s is a rational function on X which is regular on U, with $\operatorname{ord}_{P_i}(s) = -n_i$ for each i. ~~Thus D~~ Conversely, if f is any non-zero rational

function on X, with a divisor $(f) = \sum_{P \in X} \mathrm{ord}_P(f) \cdot P$, then f determines an isomorphism $\mathcal{O}_X \to \mathcal{O}_X(-(f))$.
$\therefore$ $\deg (f) = 0$.

We see from this example that we need a better understanding of the higher cohomology groups. One important tool for this is the Serre duality theorem.

Let X be a regular, projective integral scheme over a field k. Then $H^0(X, \mathcal{O}_X)$ is an integral domain which is a finite dimensional k-vector space, hence a finite algebraic extension of k. We can then regard X as a projective scheme over $H^0(X, \mathcal{O}_X)$. So we will assume $H^0(X, \mathcal{O}_X) = k$.

Now suppose $\dim X = d$. Then the sheaf $\Omega_{X/k}$ of Kähler differentials is locally free of rank d (this may not hold if $H^0(X, \mathcal{O}_X) \neq k$). Let $\Omega^i_{X/k} = \Lambda^i_{\mathcal{O}_X} \Omega_{X/k}$. Then $\Omega^0_{X/k} = \mathcal{O}_X$, $\Omega^1_{X/k} = \Omega_{X/k}$, and $\Omega^i_{X/k}$ is locally free of rank $\binom{d}{i}$. In particular, $\Omega^d_{X/k}$ is an invertible $\mathcal{O}_X$-module.

Theorem 4 (Serre duality) In the above situation,

1) $H^d(X, \Omega^d_{X/k}) \cong k$

2) if $\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$-module, there is a natural pairing (the cup product):

$$H^i(X, \mathcal{E}) \otimes_k H^{d-i}(X, \mathcal{E}^\vee \otimes \Omega^d_{X/k}) \longrightarrow H^d(X, \Omega^d_{X/k}) \cong k$$

(here $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual sheaf), which is a perfect pairing ~~of~~ between finite dimensional k-vector spaces.

Thus, $H^i(X, \mathcal{E})^\vee \cong H^{d-i}(X, \mathcal{E}^\vee \otimes \Omega^d_{X/k})$, so that these two vector spaces have the same dimension. Recall that a canonical divisor K_X on X is a divisor such that $\mathcal{O}_X(K_X) \cong \Omega^d_{X/k}$.

Suppose ~~...~~ $d=1$ i.e. X is a curve. For any divisor $D = \sum n_i P_i$, where P_i are closed points, define ~~deg~~ the degree $\deg D = \sum_i n_i \dim_k k(P_i)$, where $k(P_i)$ is the residue field of $\mathcal{O}_{X,P_i}$. The earlier argument is easily modified to prove that $\chi(D) = \chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X)$ even in this case. Using Serre duality, this may be reformulated as follows:

Riemann-Roch theorem (2^{nd} form): Let k be a field, and X an integral regular projective k-scheme of dimension 1 with $H^0(X, \mathcal{O}_X) = k$. Then for any divisor D on X,

$$\begin{aligned} \ell(D) - \ell(K_X - D) &= \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)) \\ &= \deg D + 1 - g, \end{aligned}$$

where $g = \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \Omega_{X/k})$.

Proof of Serre duality (sketch):

Step 1 We prove the following stronger assertion on $\mathbb{P}^n_k$ — if $\mathcal{F}$ is any coherent sheaf on $\mathbb{P}^n_k$, there is a natural ~~pairing~~ perfect pairing

$$(*) \quad H^i(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{Ext}^{n-i}_{\mathbb{P}^n}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \longrightarrow H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong k.$$

There is a natural isomorphism $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}(r), \mathcal{G}) \cong \Gamma(\mathbb{P}^n, \mathcal{G}(-r))$ for any $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{G}$, so that the derived functors of $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}(r), -)$ and $\Gamma(\mathbb{P}^n, - \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(-r))$ are isomorphic (if $\mathcal{G}$ is injective, so is $\mathcal{G} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{E}$ for any coherent locally free sheaf $\mathcal{E}$, since

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{A}, \mathcal{G} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{E}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{A} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{E}^\vee, \mathcal{G})).$$

Thus

$$\mathrm{Ext}^i_{\mathbb{P}^n}(\mathcal{O}(r), \mathcal{G}) \cong H^i(\mathbb{P}^n, \mathcal{G}(-r)) \ \forall\, i \geq 0.$$

In particular, $\mathrm{Ext}^i_{\mathbb{P}^n}(\mathcal{O}(r), \mathcal{O}(s)) \cong H^i(\mathbb{P}^n, \mathcal{O}(s-r))$. Now (*) is a restatement of parts of Theorem 1.

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, there is a presentation

$$\mathcal{O}(-r)^{\oplus n_1} \longrightarrow \mathcal{O}(-s)^{\oplus n_2} \longrightarrow \mathcal{F} \longrightarrow 0,$$

and hence (since $H^n(\mathbb{P}^n, -)$ is right exact)

$$H^n(\mathbb{P}^n, \mathcal{O}(-r)^{\oplus n_1}) \longrightarrow H^n(\mathbb{P}^n, \mathcal{O}(-s)^{\oplus n_2}) \longrightarrow H^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0.$$

Thus the dual vector spaces satisfy

$$\begin{aligned} H^n(\mathbb{P}^n, \mathcal{F})^\vee &\cong \ker\left(H^n(\mathbb{P}^n, \mathcal{O}(-s)^{\oplus n_2})^\vee \longrightarrow H^n(\mathbb{P}^n, \mathcal{O}(-r)^{\oplus n_1})^\vee\right) \\ &\cong \ker\left(\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}(-s)^{\oplus n_2}, \mathcal{O}(-n-1)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}(-r)^{\oplus n_1}, \mathcal{O}(-n-1))\right) \\ &= \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}(-n-1)), \end{aligned}$$

and the isomorphism is easily checked to be natural in $\mathcal{F}$. Now given an exact sequence $0 \to \mathcal{G} \to \mathcal{O}(-s)^{\oplus n_2} \to \mathcal{F} \to 0$, we have long exact sequences

$$0 \to H^n(\mathbb{P}^n, \mathcal{F})^\vee \to H^n(\mathbb{P}^n, \mathcal{O}(-s)^{\oplus n_2})^\vee \to H^n(\mathbb{P}^n, \mathcal{G})^\vee \to \cdots \to H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee \to$$
$$H^i(\mathbb{P}^n, \mathcal{O}(-s)^{\oplus n_2})^\vee \to H^i(\mathbb{P}^n, \mathcal{G})^\vee \to \cdots \to H^0(\mathbb{P}^n, \mathcal{G})^\vee \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}(-n-1)) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}(-s)^{\oplus n_2}, \mathcal{O}(-n-1)) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G}, \mathcal{O}(-n-1)) \to$$

$$\cdots \to \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}(-n-1)) \to \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}(-s)^{\oplus n_2}, \mathcal{O}(-n-1)) \to \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G}, \mathcal{O}(-n-1))$$

$$\to \cdots \to \mathrm{Ext}^n_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G}, \mathcal{O}(-n-1)) \to 0.$$

Since $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}(-s)^{\oplus n_2}, \mathcal{O}(-n-1)) \cong H^i(\mathbb{P}^n, \mathcal{O}(s-n-1)^{\oplus n_2}) = 0$ if $0 < i < n$, we ~~see~~ that there is ~~an isom~~ a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}(-n-1)) \cong H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee,$$

by induction on i (the result for $i-1$, for $\mathcal{G}$, implies the result for i, for $\mathcal{F}$; but $\mathcal{F}$ is arbitrary, so it holds for $\mathcal{G}$ as well).

Step 2 :- Now let $X \overset{f}{\hookrightarrow} \mathbb{P}^n_k$ be a closed subscheme such that X is regular and integral. Then there is a natural isomorphism $H^i(X, \mathcal{F})^\vee = H^i(\mathbb{P}^n, f_*\mathcal{F})^\vee \cong \mathrm{Ext}^{n-i}_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{F}, \mathcal{O}(-n-1))$. So the Theorem will follow if we construct a natural isomorphism, for locally free $\mathcal{O}_X$-modules $\mathcal{F}$,

$$\mathrm{Ext}^m_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{F}, \mathcal{O}(-n-1)) \cong H^{m-n+d}(X, \mathcal{F}^\vee \otimes \Omega^d_{X/k})$$

We will in fact give a natural isomorphism

$$\mathrm{Ext}^m_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{F}, \mathcal{O}(-n-1)) \cong \mathrm{Ext}^{m-n+d}_{\mathcal{O}_X}(\mathcal{F}, \Omega^d_{X/k}),$$

which yields the earlier isomorphism if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module.

If $m=0$, we see that for any $\mathcal{O}_X$-module $\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{G}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X, \mathcal{G})).$$

If $\mathcal{G}$ is an injective $\mathcal{O}_{\mathbb{P}^n}$-module, then $f^* \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X, \mathcal{G})$ is an injective $\mathcal{O}_X$-module (this generalizes the easy fact that if $R \to S$ is a map of rings, I an injective R-module, then $\mathrm{Hom}_R(S, I)$ is an injective S-module). Hence there is a Grothendieck spectral sequence (for composite functors)

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{F}, \mathcal{G}).$$

The important point is that ~~Ext~~ for $x \in X$,

$$\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X, \mathcal{G})_x = \mathrm{Ext}^q_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{O}_{X,x}, \mathcal{G}_x),$$

and the ~~[illegible]~~ $\mathcal{E}xt$-sheaf is supported only on X (has vanishing stalks outside X). In fact $\mathcal{O}_{\mathbb{P}^n,x} \twoheadrightarrow \mathcal{O}_{X,x}$ is a surjection between regular local rings of dimensions n and d respectively. Hence the maximal ideal in $\mathcal{O}_{\mathbb{P}^n,x}$ is generated by n elements $x_1, \dots, x_n$ such that

(i) $x_1, \dots, x_{n-d}$ generate $I_x = \ker(\mathcal{O}_{\mathbb{P}^n,x} \twoheadrightarrow \mathcal{O}_{X,x})$

(ii) $x_{n-d+1}, \dots, x_n$ map to a regular system of parameters in $\mathcal{O}_{X,x}$

(iii) $x_1, \dots, x_n$ form a <u>regular sequence</u> in $\mathcal{O}_{\mathbb{P}^n,x}$.

In particular, from (i) and (iii), the <u>Koszul complex</u> over $\mathcal{O}_{\mathbb{P}^n,x}$ with respect to $x_1, \dots, x_{n-d}$ gives a free resolution for $\mathcal{O}_{X,x}$ as an $\mathcal{O}_{\mathbb{P}^n,x}$-module,

$$0 \to \mathcal{O}_{\mathbb{P}^n,x} \to \mathcal{O}_{\mathbb{P}^n,x}^{\oplus n-d} \to \cdots \to \mathcal{O}_{\mathbb{P}^n,x}^{\oplus \binom{n-d}{j}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n,x}^{\oplus n-d} \to \mathcal{O}_{\mathbb{P}^n,x} \to \mathcal{O}_{X,x} \to 0.$$

Further, the Koszul complex for a regular sequence has a "self duality" property i.e. its dual ~~is~~ is isomorphic to itself, with a shift in degree ("dual" means $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n,x}}(-,\mathcal{O}_{\mathbb{P}^n,x})$). Thus

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{O}_{X,x},\mathcal{O}_{\mathbb{P}^n,x}) \cong \begin{cases} 0 & \text{if } i \neq n-d \\ \mathcal{O}_{X,x} & \text{if } i = n-d. \end{cases}$$

Taking into account the dependence of the second isomorphism on the choice of generators for the ideal I_x, one sees that

$$\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{O}_{X,x},\mathcal{E}) \begin{cases} \cong \Lambda^{n-d}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2)^\vee \otimes_{\mathcal{O}_X} f^*\mathcal{E}, & \text{if } i = n-d \\ = 0 & \text{otherwise} \end{cases}$$

for any locally free coherent $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{E}$, where $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^n}$ is the ideal sheaf of X (then $\mathcal{J}/\mathcal{J}^2$ is locally free on X of rank $n-d$).

Thus ~~in~~ the spectral sequence ~~de~~ mentioned above yields an isomorphism

$$\mathrm{Ext}^m_{\mathcal{O}_{\mathbb{P}^m}}(f_*\mathcal{F},\mathcal{O}(-n-1)) \cong \mathrm{Ext}^{m-n+d}_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{E}xt^{n-d}_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X,\mathcal{O}(-n-1)))$$

$$\cong \mathrm{Ext}^{m-n+d}_{\mathcal{O}_X}(\mathcal{F}, \Lambda^{n-d}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2)^\vee \otimes_{\mathcal{O}_X} f^*\mathcal{O}(-n-1)).$$

Thus we have proved the duality theorem with

$$\Lambda^{n-d}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2)^\vee \otimes_{\mathcal{O}_X} f^*\mathcal{O}(-n-1)$$

instead of $\Omega^d_{X/k}$. So it remains to identify these two $\mathcal{O}_X$-modules.

There is an exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{\mathbb{P}^n/k} \otimes \mathcal{O}_X \to \Omega_{X/k} \to 0,$$

where $(\mathcal{J}/\mathcal{J}^2) \otimes k(x)$ is identified with the co-normal space to $X \hookrightarrow \mathbb{P}^n$ at x (its dual is the normal space, the cokernel of the map $T_{x,X} \to T_{x,\mathbb{P}^n}$ on Zariski tangent spaces). This is an exact sequence of locally free sheaves of ranks $n-d$, n and d respectively, and yields a "determinant" isomorphism of invertible sheaves

$$\Lambda^n(\Omega_{\mathbb{P}^n/k} \otimes \mathcal{O}_X) \cong \Lambda^d \Omega_{X/k} \otimes \Lambda^{n-d} \mathcal{J}/\mathcal{J}^2.$$

Further, the exact sheaf sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

similarly yields an isomorphism

$$\Lambda^n \Omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Hence there is an isomorphism $\Lambda^d \Omega_{X/k} \cong \Lambda^{n-d}(\mathcal{J}/\mathcal{J}^2)^\vee \otimes \mathcal{O}_X(-n-1)$.

This proves the Serre duality theorem.

<u>Remark</u> :- The above proof proves the Serre duality theorem

$$H^i(X,\mathcal{F})^\vee \cong \mathrm{Ext}^{d-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X),$$

with $\omega_X \cong f^*\mathcal{E}xt^{n-d}_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X, \mathcal{O}(-n-1)) = f^*\mathcal{E}xt^{n-d}_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X, \Omega^n_{\mathbb{P}^n/k})$,

provided $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^n}}(f_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})_x = \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P}^n,x}) = 0 \ \forall\ i \neq n-d$.

From theorems in homological algebra, this holds precisely when $\mathcal{O}_{X,x}$ is <u>Cohen-Macaulay</u> of dimension d. The proof also gives the duality theorem with the <u>invertible</u> sheaf

$$\omega_X = \Lambda^{n-d}(\mathcal{J}/\mathcal{J}^2)^\vee \otimes_{\mathcal{O}_X} f^*\Omega^n_{\mathbb{P}^n/k},$$

provided $I_x = \mathcal{J}_x \subset \mathcal{O}_{\mathbb{P}^n,x}$ has a resolution by the Koszul complex associated to a regular sequence i.e. provided I_x is generated by a regular sequence of length $n-d$ in $\mathcal{O}_{\mathbb{P}^n,x}$. If this holds, we say $X \hookrightarrow \mathbb{P}^n_k$ is a <u>local complete intersection</u>. This is in fact an intrinsic condition on the local rings of X.

The Chow Ring and the Riemann-Roch theorem.

Hirzebruch-Riemann-Roch theorem: If X is a smooth, projective variety over the complex number field $\mathbb{C}$, and $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, then one has an associated complex manifold X_{an}, and an analytic vector bundle $\mathcal{E}_{an}$, such that $H^i(X, \mathcal{E}) \cong H^i(X_{an}, \mathcal{E}_{an})$ (this isomorphism on cohomology is a particular case of Serre's GAGA). Hence $\chi(\mathcal{E}) = \chi(\mathcal{E}_{an})$ may be computed in terms of the Chern classes $c_i(\mathcal{E}_{an}) \in H^{2i}(X_{an}, \mathbb{Z})$, by the Hirzebruch-Riemann-Roch theorem.

Our goal is to give a suitable algebraic definition of Chern classes, valid for non-singular varieties over more general fields, so that this theorem may be appropriately generalized.

The Chow ring Let X be a non-singular quasi-projective variety over an algebraically closed field k. The group of algebraic cycles of codimension p is the free abelian group $Z^p(X)$ on integral subschemes $T \subset X$ of codimension p. For $p=1$, $Z^1(X) = \mathrm{Div}(X)$, the group of divisors on X.

The notion of linear equivalence of divisors is generalized as follows. The group $R^p(X)$ of codimension p cycles rationally equivalent to 0 on X is defined to be the subgroup generated by cycles of the form $(f)_W$, where $W \subset X$ is an integral subscheme of codimension $p-1$, and $f \in k(W)^*$ is a non-zero rational function on W. A little care is needed in defining $(f)_W$, the divisor of f on W, since W may have singularities. If $T \subset W$ is irreducible of codimension 1 (so that T has codimension p in X), and $x \in T$ is the generic point, then $\mathcal{O}_{W,x}$ is a local integral domain of Krull dimension 1, such that f lies in its quotient field. Define $\mathrm{ord}_T(f) = \ell(\mathcal{O}_{W,x}/a.\mathcal{O}_{W,x}) - \ell(\mathcal{O}_{W,x}/b\mathcal{O}_{W,x})$, where $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{W,x} - \{0\}$, and where "$\ell$" denotes the length of an Artinian module. Then in fact $f \notin \mathcal{O}_{W,x}^*$ for only a finite set of such T, so the sum

$$(f)_W = \sum_T{}' \mathrm{ord}_T(f).[T] \in Z^1(W) \subset Z^p(X).$$

An equivalent definition of $(f)_W$ is the following: let $\pi: \widetilde{W} \to W$ be the <u>normalization</u> of W (if $U = \mathrm{Spec}\, A \subset W$ is an affine open subset, $\pi^{-1}(U) = \mathrm{Spec}\, \widetilde{A}$, where $\widetilde{A}$ is the integral closure of A in its quotient field). Then $k(W) = k(\widetilde{W})$, and for any irreducible $\widetilde{T} \subset \widetilde{W}$ of codimension 1, if $\tilde{x} \in \widetilde{T}$ is the generic point, then $\mathcal{O}_{\widetilde{W},\tilde{x}}$ is an integrally closed, Noetherian domain of dimension 1 i.e. a discrete valuation ring. Thus $\mathrm{ord}_{\widetilde{T}}(f)$ may be defined as the (normalized) valuation of f with respect to the discrete valuation corresponding to $\mathcal{O}_{\widetilde{W},\tilde{x}}$. Hence we may define

$$(f)_{\widetilde{W}} = \sum \mathrm{ord}_{\widetilde{T}}(f).[\widetilde{T}] \in Z^1(\widetilde{W}).$$

Now define $(f)_W = \pi_*(f)_{\widetilde{W}}$, where $\pi_*: Z^1(\widetilde{W}) \to Z^1(W)$ is defined by $\pi_*[\widetilde{T}] = [k(\widetilde{T}) : k(\pi(\widetilde{T}))].[\pi(\widetilde{T})]$.

One can check this agrees with the earlier definition of $(f)_W$.

Finally, we may define the Chow group

$CH^p(X) = Z^p(X)/R^p(X)$.

Thus, $CH^1(X) \cong \mathrm{Pic}(X)$, the group of divisors modulo linear equivalence.

If $\dim X = n$, then in particular,

$Z^n(X)$ = free abelian group on closed points of X.

Hence there is a natural homomorphism

$\deg : Z^n(X) \longrightarrow \mathbb{Z}$, $\deg\left(\sum_i n_i[P_i]\right) = \sum n_i$.

Now $R^n(X) \subset Z^n(X)$ is the subgroup of $Z^n(X)$ generated by cycles $(f)_C$, where $C \subset X$ is an irreducible curve, and $f \in k(C)^*$. If $\pi : \tilde{C} \to C$ is the normalization, then $(f)_C = \pi_*(f)_{\tilde{C}}$. If X is projective, then $\tilde{C}$ is an irreducible, projective curve whose local rings are integrally closed of dimension ≤ 1 i.e. are regular. We have seen that for any non-zero rational function f on a non-singular projective curve, its divisor has degree 0. This implies $(f)_{\tilde{C}}$ has degree 0. One checks that $\pi_* : Z^1(\tilde{C}) \to Z^1(C)$ preserves degrees, so that $(f)_C$ also has degree 0. Hence there is a well defined homomorphism

$$\deg : CH^n(X) \longrightarrow \mathbb{Z}.$$

This is of importance in defining numerical invariants in terms of Chern classes, etc.

Next, one may define an <u>intersection product</u>

$CH^p(X) \otimes_{\mathbb{Z}} CH^q(X) \longrightarrow CH^{p+q}(X)$.

The idea is that if Y, Z are irreducible, smooth subvarieties of X, of codimensions p, q respectively, such that Y and Z "meet transversally", then the intersection ~~of~~ product of the classes $[Y] \in CH^p(X)$ and $[Z] \in CH^q(X)$ should equal $[Y \cap Z]$.* More generally, one uses the theory of <u>intersection multiplicities</u>.* If Y, Z are irreducible in X of codimensions p, q respectively, we say Y, Z <u>meet properly</u> if $Y \cap Z = W_1 \cup \cdots \cup W_r$ where ~~so~~ each W_i is irreducible of codimension $p+q$. In this case, one defines

$$[Y] \cdot [Z] = \sum_{i=1}^{r} I(Y, Z; W_i)[W_i]$$

for certain intersection multiplicities $I(Y, Z; W_i) \in \mathbb{Z}$.

~~The~~ If $x \in W_i$ is the generic point, then Y, Z determine (prime) ideals $I_Y, I_Z \subset \mathcal{O}_{X,x}$ such that $I_Y + I_Z$ is primary to the maximal ideal ($\mathcal{O}_{X,x}$ is a regular local ring of dimension $p+q$, and I_Y, I_Z have heights p, q respectively).

The "naive" definition of intersection multiplicities is motivated by the definitions ~~of~~ in intersection theory on a surface: namely $\ell(\mathcal{O}_{X,x}/I_Y + I_Z)$. An example suggesting the inadequacy of this is the following ~~problem arises because $\mathcal{O}_{X,x}/I$~~

Example: — Let $Y \subset \mathbb{A}^4$, $Z \subset \mathbb{A}^4$ be the subschemes with ideals $I_Y = (x,y) \cap (z,w) = (xz, xw, yz, yw)$ and $I_Z = (x-z, y-w)$. Then $Y = Y_1 + Y_2$, where Y_1 has ideal (x,y), and Y_2 has ideal (z,w). Thus Y_1, Y_2, Z are planes in $\mathbb{A}^4$, such that $Y_1 \cap Z = Y_2 \cap Z = \{P\}$, where $P \in \mathbb{A}^4$ is the origin. Thus one would want $I(Y,Z;P) = 2$. But

$$\ell(\mathcal{O}_{\mathbb{A}^4,P} / I_Y \cdot \mathcal{O}_{\mathbb{A}^4,P} + I_Z \cdot \mathcal{O}_{\mathbb{A}^4,P}) \cong$$
$$\ell(k[x,y]/(x^2, xy, y^2)) = 3.$$

Instead, one uses the idea that intersecting Y and Z in X is equivalent to intersecting $Y \times_k Z$ with the diagonal in $X \times_k X$; now the diagonal has an ideal which is locally defined by a <u>regular sequence</u> (i.e. the diagonal in $X \times_k X$ is a local complete intersection). This led Serre to propose that

$$I(Y,Z;W_i) = \sum_{j \geq 0} (-1)^j \ell\left(\mathrm{Tor}_j^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/I_Y, \mathcal{O}_{X,x}/I_Z)\right)$$

He showed that this definition has reasonable properties (it is non-negative, for example) in the above context, and led him to conjecture similar properties ~~in the context ...~~ for the "intersection multiplicity" function for pairs of suitable ideals in any regular local ring.
(Partial progress towards the proof of this conjecture has been made ~~indep~~ independently by P. Roberts, and by H. Gillet and C. Soulé).

However, this only suggests a definition for $[Y].[Z]$ when Y, Z meet properly. Traditionally, a second ingredient used to construct the Chow ring is a <u>moving lemma</u>, which should state that

(i) if $\alpha \in CH^p(X)$, $\beta \in CH^q(X)$ are arbitrary, $\beta = \sum' m_i [Z_i]$, then the rational equivalence class ~~of~~ α contains a cycle $\sum n_j [Y_j]$ such that each Y_j meets every Z_i properly

(ii) if we choose 2 cycles $\sum n_j [Y_j]$ and $\sum n'_\ell [Y'_\ell]$ ~~with the~~ which "meet each Z_i properly" in the above sense, then the cycles

$$\sum m_i n_j \, [Y_j] \cdot [Z_i] , \qquad \sum m_i n'_\ell \, [Y'_\ell] \cdot [Z_i],$$

defined using intersection multiplicities, represent the same element in $CH^{p+q}(X)$.

Unfortunately, the traditional moving lemmas seem to deal with (ii) inadequately. However, a direct definition of an intersection product

$$CH^p(X) \otimes_{\mathbb{Z}} CH^q(X) \longrightarrow CH^{p+q}(X)$$

has been given by Fulton and MacPherson, such that if Y, Z meet properly, then $[Y].[Z]$ is the class of the cycle defined using intersection multiplicities. For details, see :

W. Fulton : "Intersection Theory", Ergebnisse Math., 3 Folge, Band 2, Springer-Verlag, 1984.

We now have:

<u>Theorem</u>:- (1) $CH^*(X) = \bigoplus_{p \geq 0} CH^p(X)$ is an associative, commutative ring, if X is any non-singular quasi-projective variety over k.

(2) If $f : X \longrightarrow X'$ is any morphism of non-singular quasi-projective varieties, then there is a ring homomorphism (preserving gradings) $f^* : CH^*(X') \longrightarrow CH^*(X)$, such that if $Y' \subset X'$ is irreducible of codimension p, and $f^{-1}(Y') = Y_1 \cup \cdots \cup Y_r$ where each $Y_j \subset X$ has codimension p, then

$$f^*[Y] = \sum n_j [Y_j],$$

where Y_j appears with multiplicity n_j in the subscheme $f^{-1}(Y')$ (i.e. if $y_j \in Y_j$ is the generic point, $\ell(\mathcal{O}_{X,y_j}/I_{f^{-1}(Y)}) = n_j$, where $I_{f^{-1}(Y)}$ is the ideal of $f^{-1}(Y)$). If $g : X' \to X''$ is another such morphism, then $(g \circ f)^* = f^* \circ g^*$.

(3) For any proper morphism $f : X \longrightarrow X'$ of non-singular quasi-projective varieties, there is a homomorphism

$$f_* : CH^p(X) \longrightarrow CH^{p + \dim X' - \dim X}(X'),$$

defined by

$$f_*([Y]) = \begin{cases} 0 & \text{if } \dim f(Y) < \dim Y \\ [k(Y) : k(Y')] \cdot [Y'] & \text{if } \dim \underset{f(Y)}{\underset{\parallel}{Y'}} < \dim Y \end{cases}$$

(4) If $f : X \to X'$ is as in (3), the <u>projection formula</u>

$$f_*(x \cdot f^* y) = f_*(x) \cdot y$$

holds for $x \in CH^*(X)$, $y \in CH^*(X')$.

(5) If $\mathcal{E}$ is locally free of finite rank on X, then $p^* : CH^*(X) \longrightarrow CH^*(\mathbb{V}(\mathcal{E}))$ is an isomorphism.

6) If $\mathcal{E}$ is locally free on X of rank r, and $\xi \in CH^1(\mathbb{P}(\mathcal{E})) = \mathrm{Pic}(\mathbb{P}(\mathcal{E}))$ is the class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, ~~then~~ $f: \mathbb{P}(\mathcal{E}) \longrightarrow X$ the structure morphism, then $f^*: CH^*(X) \longrightarrow CH^*(\mathbb{P}(\mathcal{E}))$ makes $CH^*(\mathbb{P}(\mathcal{E}))$ a free $CH^*(X)$-module with generators $1, \xi, \xi^2, \dots, \xi^{r-1}$.

As a corollary of the last property, one has that

$$\xi^r - f^*(c_1)\xi^{r-1} + f^*(c_2)\xi^{r-2} + \cdots + (-1)^r f^*(c_r) = 0$$

in $CH^r(\mathbb{P}(\mathcal{E}))$, for unique classes $c_i = c_i(\mathcal{E}) \in CH^i(X)$.

Definition: The i^{th} Chern class of $\mathcal{E}$ is defined to be $c_i(\mathcal{E}) \in CH^i(X)$. The total Chern class

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E}) \in CH^*(X).$$

Define $c_i(\mathcal{E}) = 0$ for $i > r$.

Theorem: (1) If $\mathcal{E} = \mathcal{O}_X(D)$ for a divisor D, then $c_1(\mathcal{E}) = [D] \in CH^1(X)$, and $c_i(\mathcal{E}) = 0$ for $i > 0$.

(2) If $f: X \to X'$ is a morphism of smooth, ~~proje~~ quasi-projective varieties, then for any coherent locally free $\mathcal{O}_{X'}$-module $\mathcal{E}'$,

$$c(f^*\mathcal{E}') = f^* c(\mathcal{E}').$$

(3) If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of coherent locally free $\mathcal{O}_X$-modules, then

$$c(\mathcal{E}) = c(\mathcal{E}') \cdot c(\mathcal{E}'') \text{ in } CH^*(X).$$

These properties are similar to properties of the Chern classes defined in topology. Another important result useful in omputations ~~and~~ is the <u>splitting principle</u>: if $\mathcal{E}$ is a locally free sheaf of rank r on X, then there is a projective morphism $f: X' \to X$ such that

(i) $f^*: CH^*(X) \longrightarrow CH^*(X')$ is injective

(ii) $f^*\mathcal{E}$ has a filtration $0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_r = f^*\mathcal{E}$, where each $\mathcal{E}_j$ is locally free, and $\mathcal{E}_j/\mathcal{E}_{j-1}$ is an invertible $\mathcal{O}_{X'}$-module (where $\mathcal{E}_0 = 0$).

If $\mathcal{E}_j/\mathcal{E}_{j-1} \cong \mathcal{L}_j$, then

$$f^*c(\mathcal{E}) = \prod_{j=1}^{r} (1 + c_1(\mathcal{L}_j)),$$

so that $f^* c_i(\mathcal{E}) = i^{\text{th}}$ elementary symmetric polynomial in $c_1(\mathcal{L}_1), \dots, c_1(\mathcal{L}_r)$.

As in topology, this allows us to compute the Chern classes of locally free sheaves obtained from X using tensor perations: one formally writes

$$c(\mathcal{E}) = \prod_{j=1}^{r} (1 + x_j), \quad c(\mathcal{F}) = \prod_{l=1}^{s} (1 + y_l), \ \dots$$

hen, for example,

$$c(\Lambda^m \mathcal{E}) = \prod_{1 \le i_1 < \cdots < i_m \le r} (1 + x_{i_1} + \cdots + x_{i_m}),$$

$$c(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \prod_{j,l} (1 + x_j + y_l), \quad \text{etc.}$$

where the right sides are interpreted as follows — they are ~~sym~~ expressible as polynomials with integer coefficients in the elementary symmetric polynomials of $\{x_1, \dots, x_r\}, \{y_1, \dots, y_s\}, \dots$

One then replaces the elementary symmetric polynomials by the corresponding Chern classes. In particular, $c_1(\mathcal{E}) \cong c_1(\Lambda^r \mathcal{E})$, where $\mathcal{E}$ is locally free of rank r.

As in topology, one may define the Chern character of a locally free sheaf $\mathcal{E}$ of rank r as an element of $CH^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, as

$$ch(\mathcal{E}) = \sum_{j=1}^{r} e^{x_j},$$

where $c(\mathcal{E}) = \prod_{j=1}^{r} (1+x_j)$. Again, this is interpreted by rewriting the right side as a sum of polynomials in the elementary symmetric polynomials in x_j. One also defines the Todd class of $\mathcal{E}$ by

$$td(\mathcal{E}) = \prod_{j=1}^{r} \frac{x_j}{1-e^{-x_j}}.$$

The expressions for $ch(\mathcal{E})$, $td(\mathcal{E})$ appear to be infinite series in the Chern classes of $\mathcal{E}$, but are in fact polynomials, since $CH^i(X)=0$ for $i>n$.

Hirzebruch-Riemann-Roch theorem: Let X be a smooth, projective variety of dimension n over an algebraically closed field k, and let $\mathcal{E}$ be a coherent ~~locally free~~ $\mathcal{O}_X$-module. ~~of rank~~ Then

$$\chi(\mathcal{E}) = \deg\,(ch(\mathcal{E}) \cdot td(\Omega^\vee_{X/k}))_n,$$

where for $y \in CH^*(X)\otimes\mathbb{Q}$, y_n is the component in $CH^n(X)\otimes\mathbb{Q}$, and 'deg : $CH^n(X)\otimes\mathbb{Q} \to \mathbb{Q}$ is the degree homomorphism).

Examples :-

1) If X is a non-singular projective curve, D a divisor on X, then $c(\mathcal{O}_X(D)) = 1 + [D]$, so that $ch(\mathcal{O}_X(D)) = 1 + [D]$ also. Next, $\Omega^\vee_{X/k} \cong \mathcal{O}_X(-K_X)$, where K_X is a canonical divisor, so that

$$td(\Omega^\vee_{X/k}) = 1 - \tfrac{1}{2}[K_X].$$

Hence $ch(\mathcal{O}_X(D)) . td(\Omega^\vee_{X/k}) = 1 + [D] - \frac{1}{2}[K_X]$, and so

$$\chi(\mathcal{O}_X(D)) = \deg D - \tfrac{1}{2}\deg(K_X).$$

Taking $D=0$, we have

$$\chi(\mathcal{O}_X) = 1 - \dim H^1(X, \mathcal{O}_X) = -\tfrac{1}{2}\deg(K_X)$$

i.e. $\deg K_X = 2\dim H^1(X, \mathcal{O}_X) - 2$

$$= 2g - 2.$$

2) If X is a non-singular projective surface, and D is a divisor on X, then

$$ch(\mathcal{O}_X(D)) = 1 + [D] + \tfrac{1}{2}[D]^2.$$

Let $c(\Omega^\vee_{X/k}) = 1 + c_1 + c_2$. Then

$$td(\Omega^\vee_{X/k}) = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2).$$

$$\therefore \chi(\mathcal{O}_X(D)) = \overbrace{\tfrac{1}{2}[D].([D] + c_1) + \tfrac{1}{12}(c_1^2 + c_2)}^{\text{(degree of)}}.$$

Now $c_1 = c_1(\Omega^\vee_{X/k}) \cong c_1(\Lambda^2 \Omega^\vee_{X/k}) = c_1(\mathcal{O}_X(K_X)^\vee)$
$= c_1(\mathcal{O}_X(-K_X))$. Hence $\chi(\mathcal{O}_X(D)) = \underbrace{\tfrac{1}{2}([D].[D] - K_X) + \tfrac{1}{12}([K_X]^2 + c_2)}_{\text{(degree of}}$.

This combines the two formulas (in terms of intersection numbers)

$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D \cdot D - K_X) + \chi(\mathcal{O}_X)$ (Riemann – Roch on a surface)

and

$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + c_2)$ (Noether's formula).

Grothendieck – Riemann – Roch theorem

Let X be a non-singular quasi-projective variety over an algebraically closed field k. One defines the Grothendieck group $K_0(X)$ of locally free $\mathcal{O}_X$-modules by

$K_0(X) = \dfrac{\text{Free abelian group on isomorphism classes of locally free coherent } \mathcal{O}_X\text{-modules}}{\text{relations from exact sequences}}$,

where if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is ~~exact,~~ any exact sequence of locally free coherent sheaves, we impose the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$.

There is an analogous group $G_0(X)$, defined as the free abelian group on isomorphism classes of coherent $\mathcal{O}_X$-modules modulo similar relations obtained from short exact sequences.

Then the following properties hold.

1) The natural homomorphism $K_0(X) \to G_0(X)$ is an isomorphism (so henceforth, we may identify the two groups).

This follows from the fact that on a non-singular quasi-projective variety X, any coherent sheaf $\mathcal{F}$ has a ~~finite~~ resolution
$0 \to \mathcal{E}_d \to \mathcal{E}_{d-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$, with $\mathcal{E}_i$ locally free coherent

sheaves, and $d = \dim X$; further, any two such resolutions can be compared with a third one which maps to each of them, so that the class $\sum_{i\geq 0} (-1)^i [\mathcal{E}_i] \in K_0(X)$ depends only on $\mathcal{F}$. This gives the inverse isomorphism $G_0(X) \to K_0(X)$.

2) The tensor product of coherent, locally free $\mathcal{O}_X$-modules makes $K_0(X)$ into an associative, commutative ring. Then $[\mathcal{E}] \mapsto ch(\mathcal{E})$ extends to a <u>ring</u> homomorphism $K_0(X) \to CH^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

3) If $f: X \to X'$ is a morphism between smooth, quasi-projective varieties, then $\mathcal{E} \mapsto f^*\mathcal{E}$ induces a ring homomorphism

$f^*: K_0(X') \to K_0(X)$.

4) If $f: X \to X'$ is a projective morphism between smooth, quasi-projective varieties, then there is a homomorphism of abelian groups

$$f_! : K_0(X) \to K_0(X'),$$

defined as the composite

$$K_0(X) \longrightarrow G_0(X') \xrightarrow{\cong} K_0(X').$$

$$\mathcal{E} \longmapsto \sum_{i\geq 0} (-1)^i [R^i f_* \mathcal{E}]$$

The projection formula $f_!(x \cdot f^* y) = (f_! x) \cdot y$ holds.

The Grothendieck-Riemann-Roch theorem is:

<u>Theorem</u>: Let $f: X \to X'$ be a projective morphism between smooth, quasi-projective varieties. Then

$$ch(f_! x) \cdot td(\Omega^\vee_{X'/k}) = f_*(ch(x) \cdot td(\Omega^\vee_{X/k}))$$

in $CH^*(X') \otimes_{\mathbb{Z}} \mathbb{Q}$.

The case when $X' =$ point is the Hirzebruch-Riemann-Roch theorem stated earlier. One point about the theorem is that it ~~is a~~ gives a formula in $CH^*(X') \otimes_{\mathbb{Z}} \mathbb{Q}$. There are some contexts where one may hope to get a formula "without denominators".

Define the <u>topological filtration</u> on $K_0(X)$ via the isomorphism $K_0(X) \xrightarrow{\cong} G_0(X)$, and the filtration

$$F^p G_0(X) = \text{subgroup generated by classes } [\mathcal{O}_Y] \text{ where } Y \subset X \text{ is integral of codimension} \geq p.$$

This is also equal to the subgroup generated by classes of coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{F}|_{X-Y} = 0$ for some (perhaps reducible) $Y \subset X$ of codimension p. The "Riemann-Roch theorem without denominators" of Jouanolou states:

<u>Theorem</u>:- 1) The assignment $[Y] \mapsto [\mathcal{O}_Y] \pmod{F^{p+1} K_0(X)}$ gives a surjection

$$\psi_p : CH^p(X) \longrightarrow F^p K_0(X) / F^{p+1} K_0(X).$$

2) If $c_p : K_0(X) \longrightarrow CH^p(X)$ denotes the p^{th} Chern class mapping, then c_p is a homomorphism when restricted to $F^p K_0(X)$, and vanishes on $F^{p+1} K_0(X)$.

3) The composites $c_p \circ \psi_p$ and $\psi_p \circ c_p$ equal multiplication (on the groups $CH^p(X)$ and $F^p K_0(X)/F^{p+1} K_0(X)$, respectively) by $(-1)^{p-1}(p-1)!$

<u>Some further properties of the Chow groups</u> Let X be a smooth, projective variety over $k = \bar{k}$.

Let $N^i(X) = CH^i(X) / (\text{numerical equivalence})$, where $x \in CH^i(X)$ is numerically equivalent to 0 if $\deg(x \cdot y) = 0$

for all $y \in CH^{n-i}(X)$ $(n = \dim X)$. We would like to have a good description of $N^i(X)$ and $\ker CH^i(X) \twoheadrightarrow N^i(X)$.

If $i=1$, $N^1(X)$ is a free abelian group of finite rank, and $\ker (CH^1(X) \to N^1(X)) = Pic^0(X) \times G$, where $Pic^0(X)$ is the group of k-rational points of an abelian variety, the Picard variety of X, and G is a finite abelian group. If $n=1$ i.e. X is a non-singular projective curve, then $N^1(X) = \mathbb{Z}$, $CH^1(X) \twoheadrightarrow N^1(X)$ is the degree map, and $Pic^0(X) = \ker (CH^1(X) \to N^1(X))$ is the group of k-rational points of the Jacobian variety of X.

If $n=2$, and $\Gamma(X, \Omega^2_{X/k}) \neq 0$, then Mumford showed that

$$\ker\left(CH^2(X) \xrightarrow{\deg} \mathbb{Z}\right)$$

is "infinite dimensional" in a suitable sense, and in particular, cannot be the group of k-rational points of an abelian variety in any reasonable way.

Finally, we define a cycle $x \in CH^p(X)$ to be algebraically equivalent to 0 if it is a sum of cycles of the form $p_{2*}(p_1^*(y).z)$, where $y \in Pic^0(C)$ for some non-singular projective curve, $z \in CH^p(C \times X)$, and $p_1 : C \times X \to C$, $p_2 : C \times X \to X$ are the projections. Then the cycles algebraically equivalent to 0 are also numerically equivalent to 0. For divisors, $x \in Pic(X)$ is algebraically equivalent to 0 $\Leftrightarrow$ $x \in Pic^0(X)$ corresponds to a point of the Picard variety.

In $CH^n(X)$, any zero cycle of degree 0 is algebraically equivalent to 0. However, Griffiths showed that there exist examples of smooth, 3 dimensional projective varieties X over $\mathbb{C}$ for which

$$\left(\frac{\ker\left(CH^2(X) \twoheadrightarrow N^2(X)\right)}{\text{algebraic equivalence}} \right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is non-zero; Clemens showed that this $\mathbb{Q}$-vector space can even have infinite dimension; see H. Clemens, "Homological equivalence, modulo algebraic equivalence, is not finitely generated", Publ. Math. I.H.E.S. 58 (1983) 19-38. There are examples of 3 dimensional projective smooth varieties X over $\mathbb{Q}$, and elements $z \in CH^2(X_{\mathbb{Q}})$ such that $z_{\mathbb{C}} \in CH^2(X_{\mathbb{C}})$ ~~lies in~~ has non-zero image

$$\frac{\ker\left(CH^2(X_{\mathbb{C}}) \to N^2(X_{\mathbb{C}})\right)}{\text{algebraic equivalence}} \otimes_{\mathbb{Z}} \mathbb{Q} \; ; \text{ see :}$$

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