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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Complex Geometry - Lecture II

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1. Hodge Decomposition

Let M be a compact oriented C^∞ manifold of dimension m. We have seen that, by introducing a Riemannian metric on M, we get a unique harmonic representative in each de Rham class of M:

$$A^p(M) = \ker\Delta \perp \operatorname{Im}\Delta$$

$$\omega = H\omega \perp d\alpha \perp \delta\beta$$

(where $\delta\beta = 0$ if $d\omega = 0$). Here, the coefficients can be either real or complex, since Δ is a real operator.

Observe that ω is harmonic $\Leftrightarrow$ $*\omega$ is harmonic $\Longleftrightarrow$ $d\omega = 0 = d(*\omega)$. These conditions are forced by the requirement that we look for the element of minimal L^2-norm in a given cohomology class. Since $A^p(M)$ is not complete in the L^2-norm, regularity theorems are needed to make this intuitive argument into a proof.

<u>Corollary</u>. $H^p(M)$ and $H^{m-p}(M)$ are canonically dual to each other, and in particular have the same dimension.

<u>Proof</u>. By Stokes' theorem, the bilinear form

$$H^p \times H^{m-p} \longrightarrow \mathbb{R}$$
$$([\omega_1], [\omega_2]) \longrightarrow \int_M \omega_1 \wedge \omega_2$$

is well-defined; it is non-degenerate because $\int \omega \wedge * \omega > 0$ if $\omega (\neq 0)$ is harmonic. $\square$

We have also seen that the Dolbeault cohomology groups can be studied similarly on a compact complex manifold (with its canonical orientation) via a Hermitian metric.

<u>Main Theorem</u>. If M is compact complex manifold with a <u>Kähler</u> metric, then

$$\Delta (= d\delta + \delta d) = 2\,\Box\, (= 2\,(\bar\partial \bar\partial^{*} + \bar\partial^{*}\bar\partial))$$

We shall assume this and give some applications.

1. Since $\Box$ preserves bidegrees, and $\Delta\omega = 0 \iff \Box\omega = 0$, it is clear that the space of Δ-harmonic forms of degree p (complex-valued) is the direct sum of the space of harmonic forms of bidegree $(\cancel{p,q})$

(r,s), $r+s=p$. In particular, we have an isomorphism

$$H^p_{dR}(M,\mathbb{C}) \simeq \bigoplus_{r+s=p} H^{r,s}_{\bar\partial}(M),$$

This isomorphism seems to depend on the choice of a Kähler metric. But it is important to observe that there is a canonical decomposition of $H^p_{dR}(M,\mathbb{C})$ into the direct sum of subspaces $H^{r,s}$ isomorphic to $H^{r,s}(M)$.

To see ~~that~~ this, ~~let~~ us call an $[\omega] \in H^p_{dR}(M,\mathbb{C})$ of type (p,q) if it contains a (~~$\bar\partial$-closed~~) form ω_0 of bidegree ($\cancel{p,q}$ r,s), which is necessarily $\bar\partial$-closed. In this case the $\bar\partial$-class of ω_0 in $H^{p,q}$ is well-defined:

$$\omega_0 - \omega_0' = \Delta x = \Box\beta = \bar\partial\varphi + \bar\partial^*\psi,$$

and $\bar{\partial}(\omega_0 - \omega_0') = 0 \iff \bar{\partial}^*\psi = 0$.

It is now clear that the subspace of H^p consisting of classes of type (r,s) $(r+s=p)$ is isomorphic to $H^{r,s}$ and that H^p is the direct sum of these subspaces.

<u>Corollary</u> $h^{r,s} = h^{s,r}$
$= h^{m-r,m-s}$

<u>Proof</u> Follows from the fact that if ω is harmonic, so are $\bar{\omega}$ and $*\omega$ (since Δ, and hence $\Box = \frac{1}{2}\Delta$, are real operators). In particular, $h^{0,q} = h^{m,m-q}$, which is a special case of the Serre duality between $H^q(M,\mathcal{O})$ and $H^{m-q}(M,\Omega^m)$.

<u>Corollary</u> h^p is even if p is odd.
Each class in $H^1_{dR}(M,\mathbb{C})$ has a unique representative $\omega_1 + \overline{\omega_2}$, ω_i holomorphic 1-forms. $H^{0,1} \approx \{\bar{\omega} : \omega \text{ a holomorphic 1-form}\}$.

2. Applications

An important consequence of the Kähler condition is the following:

Proposition Every holomorphic form on a compact Kähler manifold is d-closed.

Remark This conclusion is false if either the compactness or the Kähler condition is omitted. (Example: $\mathbb{C}^2$ with the usual euclidean metric, and certain compact homogeneous threefolds).

Proof Observe that, on any compact complex manifold, $\bar{\partial}^*\omega = 0$ for any $(p,0)$ form with respect to any Hermitian metric. If now $\bar{\partial}\omega = 0$, then ω is $\Box$-harmonic, hence Δ-harmonic if the metric is Kähler, hence d-closed. $\Box$

This result is important for defining the Albanese map of compact Kähler (e.g. projective) manifolds, generalising the classical Abel-Jacobi mapping of a compact Riemann surface to its Jacobian.

We recall quickly how this is done. Let M be compact Kähler. For any path γ on M, we have $I(\gamma) \in (H^{1,0}(M))^*$ given by

$$I(\gamma)(\omega) = \int_\gamma \omega.$$

Because each $\omega \in H^{1,0}$ is closed, $I(\gamma)$ depends only on the homology class of γ when γ is a loop, or more generally a 1-cycle. The image of $H_1(M, \mathbb{Z})$ in $(H^{1,0}(M))^*$ is of course an additive subgroup, but it is in fact a lattice (i.e. is generated by an $\mathbb{R}$-basis of $(H^{1,0}(M))^*$). So we get a complex torus $A(M) = (H^{1,0}(M))^* / \operatorname{Im} H_1(M,\mathbb{Z})$, called the Albanese torus of M

We also get a holomorphic map $M \to A(M)$ as follows. Fix a point 0 in M. Then, for any variable $\xi \in M$, $I(\gamma_\xi)$ (with γ any path from 0 to ξ) is uniquely determined by ξ as a point of $A(M)$. This map is easily checked to be holomorphic, and has the obvious universal property for holomorphic maps into complex tori. The Albanese map is a useful tool in the study of M; if e.g.

$\dim_{\mathbb{C}} H^{1,0} = 1$, ~~M, ...~~ or more generally if $\dim(\operatorname{Im} M) = 1$, M "fibres over a curve", and this has interesting geometric consequences.

More generally, given a "zero-cycle of degree 0" on M, that is a formal linear combination $\sum n_i \xi_i$ $\xi_i \in M$, $n_i \in \mathbb{Z}$, $\sum n_i \xi_i = 0$, we can choose a 1-chain γ (= integral linear combination of paths) such that $\partial\gamma = \sum n_i \xi_i$, and then $I(\gamma) \in A(M)$ is uniquely determined by $\sum n_i \xi_i$. In particular, fixing a point $0 \in M$, we get for any k a map $\underbrace{M \times \cdots M}_{k \text{ times}} \longrightarrow A(V)$ by considering the zero-cycle $\xi_1 + \dots + \xi_k - k\,\xi_0$

We can state in this context two classical theorems on compact Riemann surfaces of (genus ≥ 1).

<u>Abel's Theorem</u> Let M be a compact Riemann surface. Then a zero-cycle (of degree zero) $D = \sum n_i \xi_i$ is the divisor of a meromorphic function on M iff it goes to zero in $A(M)$, i.e if γ is a 1-chain of paths such that $\partial\gamma = D$, then there is a <u>loop</u> γ_0 in M such that

$$I(\gamma) = I(\gamma_0).$$

<u>Jacobi's Inversion Theorem</u>. If g is the genus of M, then the map $\underbrace{M \times \cdots \times M}_{g\text{-times}} \to A(M)$ defined above is surjective.

To prove these theorems, we observe that there is a canonical isomorphism $(H^{1,0})^* \longrightarrow H^{0,1}$ given by the non-degenerate pairing

$$H^{1,0} \times H^{0,1} \longrightarrow \mathbb{C}$$
$$(\omega_1 , \omega_2) \longrightarrow \int \omega_1 \wedge \omega_2$$

(non-degenerate because $\int \omega \wedge \bar{\omega} > 0$ for $\omega (\neq 0)$ in $H^{1,0}$). As we have observed earlier without proof, $H^{0,1}(M)$ is the sheaf cohomology $H^1(M, \mathcal{O})$, and the exact cohomology sequence of the sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^* \longrightarrow 0$$

shows that $H^1(M, \mathcal{O}) / H^1(M, \mathbb{Z}) = \mathrm{Pic}^0(M)$, the group of isomorphism classes of holomorphic line bundles of degree (= first Chern class) zero on M.

Since the natural injection $H^1(M, \mathbb{R}) \longrightarrow H^{0,1}(M) = H^1(M, \mathcal{O})$ is also surjective for dimension reasons, and $H^1(X, \mathbb{Z})$ is a lattice in $H^1(M, \mathbb{R})$, $\mathrm{Pic}^0(M)$ is also a complex torus. By Poincaré duality, the isomorphism $(H^{1,0})^* \to H^1(M, \mathcal{O})$ mentioned above carries the lattice in $(H^{1,0})^*$ onto $H^1(X, \mathbb{Z})$. Thus we ~~s~~have a natural isomorphism ~~A(X)~~ $A(M) \longrightarrow \mathrm{Pic}^0(M)$. One verifies that, under this isomorphism the point of $A(M)$ corresponding to a degree zero zero cycle D goes to the line bundle $\mathcal{O}(D)$. Abel's theorem now becomes a tautology. Jacobi's inversion theorem is equivalent to the assertion that every $L \in \mathrm{Pic}^0(M)$ is of the form $\mathcal{O}(D)$, $D = \xi_1 + \dots + \xi_g - g\,\xi_0$. This assertion is equivalent to the fact that the bundle $L \otimes \mathcal{O}(g\xi_c)$ of degree g has a non-trivial section, and this is a consequence of the Riemann-Roch inequality

$$H^0(M, L) \geqslant \deg L - g + 1.$$

We conclude by proving the following important lemma of Kodaira, which has many applications (e.g. to Arakelov theory).

<u>Lemma</u>. Let M be a compact Kähler manifold, and ω a real d-~~closed~~ exact form of type $(1,1)$. Then there exists $f \in A^0(M, \mathbb{R})$ such that $\omega = i\partial\bar\partial f$.

<u>Proof</u>. We are given that $\omega = d\varphi$, $\varphi \in A^1(M, \mathbb{R})$. Now $\varphi = \alpha + \bar\alpha$, $\alpha \in A^{1,0}(M, \mathbb{C})$. Since $d\varphi = \omega \in A^{1,1}(M)$, we must have $\bar\partial\bar\alpha = \partial\alpha = 0$. Hence $\bar\alpha = \bar\psi + \bar\partial\tilde f$, with ψ a <u>holomorphic</u> $(1,0)$-form.

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\begin{aligned}
\text{Hence } \omega &= d(\psi + \partial\bar{\tilde f} + \bar\psi + \bar\partial\tilde f) \\
&= \bar\partial\partial\bar{\tilde f} + \partial\bar\partial\tilde f \\
&= \partial\bar\partial(\tilde f - \bar{\tilde f}) \\
&= i\partial\bar\partial(2\,\mathrm{Im}\,\tilde f). \qquad \square
\end{aligned}
$$

This lemma shows that, if L is a holomorphic line bundle on a compact Kähler manifold, and ω is any real $(1,1)$-form in the de Rham class of $c_1(L)$, then there is a metric on L whose curvature form is ω.