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The Hodge Decomposition Theorems (II)

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The Hodge decomposition theorems II

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In this writeup we give a proof of the identity $\Delta = 2\bar{\Box}$ where $\Delta = dd^* + d^*d$ is the ordinary Laplacian and $\bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is the $\bar{\partial}$-Laplacian on a Kähler manifold M. From the harmonic forms representation of cohomology classes it follows from this that if M is compact:

$$H^n_{dR}(M, \mathbb{C}) = \bigoplus_{p+q=n} H^{pq} \quad , \quad \overline{H^{pq}} = H^{qp}$$

where $H^{pq} \cong H^q(M, \Omega^p)$. This is the Hodge decomposition of cohomology.

The proof of $\Delta = 2\bar{\Box}$ on a Kähler manifold is surprisingly involved. We will largely follow [We]. Other approaches can be found in [G-H] and [Weil]. Let us begin with some representation theory of the Lie algebra $sl_2 = sl_2(\mathbb{C})$ of 2×2 complex matrices with trace zero and bracket $[A,B] = AB - BA$. Recall that a representation of a Lie algebra $\mathfrak{g}$ in a $\mathbb{C}$-vector space V is a homomorphism of Lie algebras

of $\xrightarrow{\rho}$ End V where End V is the Lie algebra of endomorphisms of V with bracket $[\varphi,\psi] = \varphi\circ\psi - \psi\circ\varphi$.
Note that sl_2 is 3-dimensional with basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $[X,Y] = H$, $[H,X] = 2X$, $[H,Y] = -2Y$.

Recall that since $SL_2 = SL_2(\mathbb{C}) = \{ g \in GL_2(\mathbb{C}) \mid \det g = 1 \}$ is simply connected we have a 1-1 correspondence

(1.1) $$\left\{ \begin{matrix} \text{Lie alg. repr.} \\ sl_2 \xrightarrow{\rho} \operatorname{End} V \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Lie group repr.} \\ SL_2 \xrightarrow{\pi} GL(V) \end{matrix} \right\} \quad \text{if } \dim V < \infty$$

given by: $$\rho \longmapsto \pi(\exp A)\cdot v := (\exp \rho(A))\cdot v$$
$$d\pi \longleftarrow \pi .$$

In particular for every Lie algebra representation ρ we can consider the action on V of the element

(1.2) $$w = \exp\left(\tfrac{i\pi}{2}(X+Y)\right) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in SL_2 .$$

It will be very important for us to have a formula for this action in a particular instance. To deduce this formula let us prove it first for a very concrete representation: The group SL_2 acts on polynomials $P \in \mathbb{C}[T_1,T_2]$ by the formula

$$(g\cdot P)(T_1,T_2) := P((T_1,T_2)\,g)$$

This defines a representation π of SL_2 on $\mathbb{C}[T_1, T_2]$.
Note that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$(1.3) \quad (\pi(g) P)(T_1, T_2) = P(a T_1 + c T_2, \; b T_1 + d T_2).$$

The representation π decomposes into the representations π_m of SL_2 on homogeneous polynomials of degree m.
Note that $\pi_m(g)(T_1^{m-k} T_2^k) = (\pi_1(g) T_1)^{m-k} (\pi_1(g) T_2)^k$
because π acts by $\mathbb{C}$-algebra automorphisms.
If $\rho_m = d\pi_m$ is the corresponding Lie algebra representation of $\mathfrak{sl}_2$ on homogeneous polynomials of degree m then

$$(1.4) \quad (\rho_1 P)(T_1, T_2) = P(a T_1 + c T_2, \; b T_1 + d T_2)$$

for P homogeneous of degree 1 and ρ_m is obtained by extending ρ_1 as a derivation. In particular we have:

$$\rho_1(H) T_1 = T_1, \quad \rho_1(X) T_1 = 0, \quad \rho_1(Y) T_1 = T_2$$
$$\rho_1(H) T_2 = T_2, \quad \rho_1(X) T_2 = T_1, \quad \rho_1(Y) T_2 = 0.$$

By induction it follows from this that if we set

$$u_{-1} = 0, \quad u_0 = T_1^m, \quad u_k = \binom{m}{k} T_1^{m-k} T_2^k$$

then we have the following relations:

$$u_k = \frac{1}{k!}\, \rho_m(Y)^k u_0 \qquad \text{for } k = 0, 1, \ldots, m$$

$$\rho_m(H)\, u_k = (m-2k)\, u_k$$

(1.5)

$$\rho_m(Y)\, u_k = (k+1)\, u_{k+1}$$

$$\rho_m(X)\, u_k = (m-k+1)\, u_{k-1}\,.$$

It is clear that the representation ρ_m is irreducible of dimension $m+1$. It is known that every irreducible representation of dimension $m+1$ of sl_2 is equivalent to ρ_m but we will not need this fact. Now let us compute the action of the element w in Sl_2 on homogeneous polynomials of degree m i.e. the action of $\pi_m(w)$. Since $w = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ we have

by (1.3)

$$\pi_m(w)\, T_1^{m-k} T_2^{k} = (i\, T_2)^{m-k} (i\, T_1)^k$$

$$= i^m\, T_1^k\, T_2^{m-k}\,.$$

Hence

(1.6) $$\pi_m(w)\left(\rho_m(Y)^k u_0\right) = i^m \frac{k!}{(m-k)!}\left(\rho_m(Y)^{m-k} u_0\right).$$

This is the formula we have been refering to above. We will require it in a more abstract setting:

Let $\rho : sl_2 \to \operatorname{End} V$ be any finite dimensional representation of sl_2. A vector $v_0 \in V$ is said to be primitive of weight λ if

$$v_0 \neq 0 \quad , \quad \rho(H)\, v_0 = \lambda\, v_0 \quad \text{and} \quad \rho(X)\, v_0 = 0 .$$

Then letting $v_{-1} = 0$ and setting $v_k = \frac{1}{k!}\rho(Y)^k v_0$ for $k = 0, 1, 2, \ldots$ one obtains:

<u>Lemma</u>: (a) $\rho(H)\, v_k = (\lambda - 2k)\, v_k$

(b) $\rho(Y)\, v_k = (k+1)\, v_{k+1}$

(c) $\rho(X)\, v_k = (\lambda - k + 1)\, v_{k-1}$.

Moreover $\lambda = m$ where $m+1 = \dim \langle v_0, v_1, \ldots \rangle$ and the subrepresentation of V generated by the v_i is irreducible.

The <u>proof</u> follows essentially from the commutation relations in sl_2, e.g. for (a): Let $\rho(H)v = \lambda v$, then:

$$\begin{aligned} \rho(H)(\rho(Y)v) &= (\rho(H)\rho(Y) - \rho(Y)\rho(H))v + \rho(Y)\rho(H)v \\ &= \rho([H,Y])v + \lambda\, \rho(Y)v \\ &= -2\rho(Y)v + \lambda\, \rho(Y)v \\ &= (\lambda - 2)\, \rho(Y)v . \end{aligned}$$

This implies (a) by induction. The rest of the proof is left to the reader as an exercise c.f. [We] V 3.7 ff. (5)

It is clear from the lemma and from (1.5) that the mapping u_k to v_k induces an equivalence of the sl_2-representations ρ_m and $\rho|_{\langle v_0, v_1, \dots \rangle}$. Hence we obtain from (1.6) the formula:

$$(1.7) \qquad \pi(w)\left(\rho(Y)^k v_0\right) = i^m \frac{k!}{(m-k)!}\left(\rho(Y)^{m-k} v_0\right)$$

where $d\pi = \rho$, v_0 as above. ($\lambda =: m \in \mathbb{N}$ automatic by the lemma).

After these preparations let us do some hermitian exterior algebra which will eventually lead to a particular sl_2-representation.

Let E be a $\mathbb{C}$-vector space of dimension n. Set $E' = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ i.e. the real dual space of the underlying real vector space of E and set

$$F = E' \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C}).$$

Clearly $\dim_{\mathbb{C}} F = 2n$ and F is equipped with a complex conjugation. One checks that

$$F = F^{1,0} \oplus F^{0,1}$$

where $F^{1,0} = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ ($\mathbb{C}$ - lin. Homs)

$$F^{0,1} = \mathrm{Hom}_{\mathbb{C}\text{-anti}}(E, \mathbb{C}) \quad (\mathbb{C}\text{-antilin. Homs}).$$

Clearly $\overline{F^{1,0}} = F^{0,1}$. On the complex exterior algebra ΛF of F we get an induced decomposition:

$$\Lambda F = \bigoplus_{p=0}^{2n} \Lambda^p F$$

where $\Lambda^p F = \bigoplus_{r+s=p} \Lambda^{r,s} F$, $\Lambda^{r,s} F = \Lambda^r F^{1,0} \otimes_{\mathbb{C}} \Lambda^s F^{0,1}$.

Clearly $\overline{\Lambda^{r,s} F} = \Lambda^{s,r} F$ with respect to the natural complex conjugation on $\Lambda^p F = (\Lambda^p_{\mathbb{R}} E') \otimes_{\mathbb{R}} \mathbb{C}$. The elements $\omega \in \Lambda^p F$ are called p-forms. We say that ω is real if $\overline{\omega} = \omega$. Let $(\Lambda^p F)_{\mathbb{R}} \cong \Lambda^p_{\mathbb{R}} E'$ denote the subspace of real p-forms.

Now suppose that E carries a Hermitian inner product $\langle\, , \rangle$. We may view $\langle\, , \rangle$ as an element of $\Lambda^2 F$. As such it lies actually in $\Lambda^{1,1} F$. There is a basis $z_1, \ldots, z_n$ of $\Lambda^{1,0} F = F^{1,0} = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ such that

$$\langle\, , \rangle = \sum_{\mu=1}^{n} z_\mu \otimes \overline{z}_\mu . \qquad \text{(Exercise !)}$$

The fundamental 2-form associated to $\langle , \rangle$ is by definition

$$(2.1) \qquad \Omega := -\tfrac{1}{2} \operatorname{Im} \langle , \rangle = \tfrac{i}{2} \sum_{\mu=1}^{p} z_\mu \wedge \bar{z}_\mu = \sum_{\mu=1}^{p} x_\mu \wedge y_\mu$$

if we set $x_\mu = \frac{1}{2}(z_\mu + \bar{z}_\mu)$, $y_\mu = \frac{1}{2i}(z_\mu - \bar{z}_\mu)$. Note here that $z_\mu \wedge \bar{z}_\mu = \frac{1}{2}(z_\mu \otimes \bar{z}_\mu - \bar{z}_\mu \otimes z_\mu)$. Obviously Ω is a real form of type $(1,1)$. By (2.1) we have:

$$\Omega^n = n! \; x_1 \wedge y_1 \wedge \dots \wedge x_n \wedge y_n \quad \text{in } \Lambda^{2n} E' = (\Lambda^{2n} F)_{\mathbb{R}}.$$

The bilinear form $S = \operatorname{Re} \langle , \rangle$ defines a real scalar product on the $\mathbb{R}$-vector space underlying E. We have

$$S = \sum_\mu (x_\mu \otimes x_\mu + y_\mu \otimes y_\mu).$$

Let S' be the scalar product induced on the real dual E'. By definition the x_μ, y_μ form an orthonormal basis of E' with respect to S' and hence

$$(2.2) \qquad \mathrm{vol} = \frac{1}{n!} \Omega^n$$

is an orientation of E' i.e. an ON basis of $\Lambda^{2n} E'$ with respect to the scalar product induced by S'. Note that vol depends only on the Hermitian inner product $\langle , \rangle$ on E.

With respect to S' on E' and vol there is a Hodge star operator

$$* : \Lambda^p E' \longrightarrow \Lambda^{2n-p} E' \qquad \text{for every } 0 \le p \le 2n.$$

We will extend it $\mathbb{C}$-linearly to a $*$-operator

$$* : \Lambda^p F \longrightarrow \Lambda^{2n-p} F.$$

Let $\pi_p : \Lambda F \to \Lambda^p F$, $\pi_{p,q} : \Lambda F \to \Lambda^{pq} F$ be the natural projections and set

$$w = \sum_p (-1)^p \pi_p \quad : \Lambda F \to \Lambda F$$

$$J = \sum_{p,q} i^{p-q} \pi_{pq} \quad : \Lambda F \to \Lambda F.$$

Note that since $J = i$ on $\Lambda^{10} F = F^{10}$, $J = -i$ on $\Lambda^{01} F = F^{01}$ the operator J is the $\mathbb{C}$-algebra extension to ΛF of the operator J on F given by $(J\varphi)(e) = \varphi(ie)$.

Let

$$L : \Lambda F \to \Lambda F \quad \text{be defined by } L(v) = \Omega \wedge v.$$

Clearly $L(\Lambda^{pq} F) \subset \Lambda^{p+1,q+1} F$ and L is a real operator i.e. commutes with complex conj. on ΛF.

Recall that $\Lambda^p F$ has a natural Hermitian inner product given by:

$$\langle \alpha, \beta \rangle \,\mathrm{vol} = \alpha \wedge * \bar{\beta} \quad \text{for} \quad \alpha, \beta \in \Lambda^p F .$$

Let L^* be the adjoint of L with respect to this inner product. We have the following relations:

(2.3) Thm.: 1) $* * = w$, $J^2 = w$

2) $L^* = w * L *$ in particular L^* is real , $L^*(\Lambda^p F) \subset \Lambda^{p-2} F$.

3) $* \Pi_{pq} = \Pi_{n-q, n-p} *$

4) $[L, w] = [L, J] = [L^*, w] = [L^*, J] = 0$

5) $[L^*, L] = \sum_{p=0}^{2n} (n-p) \Pi_p$.

Proof: 1), 2) follow by an immediate computation
3) implies that $L^*(\Lambda^{r,s} F) \subset \Lambda^{r-1, s-1} F$ and hence 4)
follows from 3). Finally 3) and 5) are consequences of the next lemma which is proved by tedious but straightforward calculation:

(2.4) Lemma: Suppose that A, B, M are mutually disjoint increasing multiindeces (e.g. $A = \{\mu_1 < \mu_2 < \dots < \mu_a\}$ etc.) Set $z_A = z_{\mu_1} \wedge \dots \wedge z_{\mu_a}$ and similarly $\bar{z}_B$. Also set $w_M = \prod_{\mu \in M} z_\mu \wedge \bar{z}_\mu$. Then we have:

$$*(z_A \wedge \bar{z}_B \wedge w_M) = i^{a-b} (-1)^{\frac{p(p+1)}{2} + m} (-2i)^{p-m} z_A \wedge \bar{z}_B \wedge w_{M'}$$

where $a = |A|$, $b = |B|$, $m = |M|$, $p = a+b+2m$ and $M' = \{1,\dots,n\} \setminus (A \cup B \cup M)$. (Our multiindeces A, B, M are supposed to be subsets of $\{1,\dots,n\}$.)

Let us set $\mathcal{B} = \sum_{p=0}^{2n} (n-p)\, \Pi_p$ then we have the commutation relations which follow from 4), 5):

$$[L^*, L] = \mathcal{B} \quad , \quad [\mathcal{B}, L^*] = 2L^* \quad , \quad [\mathcal{B}, L] = -2L .$$

Hence we obtain a sl_2-representation:

$$\rho : sl_2 \longrightarrow \mathrm{End}(\Lambda F)$$

by setting $\rho(X) = L^*$, $\rho(Y) = L$ and $\rho(H) = \mathcal{B}$. Let π_ρ be the associated representation of SL_2.

Our main application of the sl_2-theory will be to relate the $*$-operator to the ~~the~~ automorphism of ΛF $\# := \pi_\rho(w)$ for w as in (1.2). We will prepare this result by a lemma. If $\eta \in \Lambda^p F$ we set

$$e(\eta) \in \mathrm{Hom}(\Lambda F, \Lambda F) \quad , \quad e(\eta)\varphi = \eta \wedge \varphi$$

e.g. $L = e(\Omega)$. Note that if η is a real 1-form then we have

$$(2.5) \qquad e^*(\eta) := e(\eta)^* = * \, e(\eta) \, * .$$

Using this one checks that

$$(2.6) \qquad [L^*, e(\eta)] = -J\, e^*(\eta)\, J^{-1} .$$

<u>(2.7) Lemma:</u> Let η be a real 1-form i.e. $\eta \in (\Lambda^1 F)_{\mathbb{R}}$.

Then

$$\# \, e(\eta) \, \#^{-1} = -i \, J \, e^*(\eta) \, J^{-1}$$

where $\# = \pi_\rho(w) = \exp\left(\frac{i\pi}{2} \rho(X+Y)\right) = \exp\left(\frac{i\pi}{2}(L^* + L)\right)$.

<u>Proof</u>: For $t \in \mathbb{C}$ set

$$e_t(\eta) = \exp(it(L^*+L)) \cdot e(\eta) \cdot \exp(-it(L^*+L))$$

$$= \mathrm{Ad}\,(\exp it(L^*+L)) \cdot e(\eta)$$

$$= \exp(it \; \mathrm{ad}(L^*+L)) \cdot e(\eta)$$

since $d\,\mathrm{Ad} = \mathrm{ad}$ and $\exp$ commutes with taking the derivative. Recall that $\mathrm{ad}(u)\, v = [u, v]$.

Note that

$$e_0(\eta) = e(\eta)$$

$$e_{\frac{\pi}{2}}(\eta) = \# \, e(\eta) \, \#^{-1} .$$

$$\frac{d}{dt} e_t(\eta) = i\, (\mathrm{ad}(L^*) + \mathrm{ad}\, L)\, e_t(\eta) .$$

The relations

$$(\mathrm{ad}\, L^*)^2 = 0 \quad , (\mathrm{ad}\, L)\cdot e(\eta) = 0 \quad (\text{clear since } \deg \Omega = 2)$$

$$\mathrm{ad}(-B)\circ e(\eta) = e(\eta) \quad (\text{dto.}) \quad \text{and} \quad [L, L^*] = -B$$

show that

$$\tilde{e}_t(\eta) := \cos t \;\; e(\eta) \; + i \sin t \;\; \mathrm{ad}(L^*)\, e(\eta)$$

is a solution of

$$\frac{d}{dt}\, \tilde{e}_t(\eta) = i\, (\mathrm{ad}(L^*) + \mathrm{ad}(L))\; \tilde{e}_t(\eta) .$$

Since $\tilde{e}_0(\eta) = e(\eta)$ we must have $e_t(\eta) = \tilde{e}_t(\eta)$ i.e.

$$e_t(\eta) = \cos t \;\; e(\eta) + i \sin t \;\; \mathrm{ad}(L^*)\, e(\eta) .$$

In particular

$$e_{\frac{\pi}{2}}(\eta) = i\, \mathrm{ad}(L^*)\, e(\eta) = i\, [L^*, e(\eta)]$$

$$= -i\, J\, e^*(\eta)\, J^{-1} \qquad \text{by (2.6)} .$$

This implies the lemma.

Now we come to the main technical result of this section.

<u>(2.8) Main-Lemma</u>: For $\varphi \in \Lambda^p F$ we have:

$$* \varphi = i^{p^2 - m}\; J^{-1} \# \varphi .$$

Proof: The $*$-operator on ΛF satisfies:

(1) $$* 1 = \mathrm{vol} = \frac{1}{n!} \Omega^n = \frac{1}{n!} L^n(1)$$

(2) For all real 1-forms η we have on $\Lambda^p F$:

$$* e(\eta) = (-1)^p e^*(\eta) * \quad (\text{use } (2.5)) \quad , \quad p \geq 0 .$$

Now $*$ is the only $\mathbb{C}$-linear operator $\Lambda F \to \Lambda F$ satisfying (1) and (2) since the forms obtained from 1 by repeated application of $e(\eta)$, η as above span ΛF.

Let $$\tilde{*} = i^{p^2-n} J^{-1} \# \quad : \Lambda^p F \to \Lambda F$$

and extend $\tilde{*}$ to an endomorphism of ΛF.

In the notation of (1.7) the form $v_0 = 1$ is primitive of weight n (since $B v_0 = n v_0$, $L^* v_0 = 0$) hence by (1.7) for $k=0$ we obtain:

$$\# 1 = \frac{i^n}{n!} L^n(1) .$$

Thus
$$\tilde{*} 1 = i^{-n} \left(\frac{i^n}{n!} \right) L^n(1) = \mathrm{vol} .$$

On the other hand for η as above and $\varphi \in \Lambda^p F$ we have:

$$\begin{aligned} \tilde{*} e(\eta) \varphi &= i^{(p+1)^2-n} J^{-1} \# e(\eta) \varphi \\ &= i^{p^2-n} (-1)^p i J^{-1} \# e(\eta) \#^{-1} \# \varphi \\ &\overset{(2.7)}{=} i^{p^2-n} (-1)^p e^*(\eta) J^{-1} \# \varphi \\ &= (-1)^p e^*(\eta) \tilde{*} \varphi \end{aligned}$$

. Thus (1), (2) hold for $\tilde{*}$. Hence $* = \tilde{*}$.

We now apply these results to manifolds. Let M be a compact Hermitian manifold. Then by doing the constructions of the last section fibrewise starting from $E = T^c_\xi M$, $\xi \in M$ the complex tangent space of M in ξ with its hermitian metric, we get bundle maps $L : \Lambda^p T^*(M)_{\mathbb{C}} \to \Lambda^{p+2} T^*(M)_{\mathbb{C}}$, L^*, B, $*$, J, $\#$ etc. Note that the pointwise fundamental forms Ω_ξ glue to a section $\Omega \in \Gamma(M, \Lambda^2 T^*(M)_{\mathbb{C}})$ the fundamental form of M. Also note that the above bundle maps induce maps of sections (differential forms)

$$L : \Gamma(M, \Lambda^p T^*(M)_{\mathbb{C}}) =: A^p(M) \to A^{p+2}(M) \quad \text{etc.}$$

It is easy to check that in this interpretation $L^* = w * L *$ becomes the adjoint of L with respect to the Hodge inner product.

On forms we also have exterior differentiation $d, \bar\partial$ together with their adjoints d^* with respect to the Hodge inner product. We now assume in addition that M is Kähler i.e. that $d\Omega = 0$ or in other words that $[L, d] = 0$. Then we have:

<u>(3.1) Theorem:</u> $\Delta = 2\bar\Box$ where $\Delta = d d^* + d^* d$ and $\bar\Box = \bar\partial \bar\partial^* + \bar\partial^* \bar\partial$.

The proof will be based on the relations between d, d^*, L, L^* given by the next lemma.

Set
$$d_c = J^{-1} d J = \omega J d J$$
$$d_c^* = J^{-1} d^* J = \omega J d^* J$$
then we have

<u>(3.2) Crucial Lemma:</u> a) $[L, d] = 0$, $[L^*, d^*] = 0$

b) $[L, d^*] = d_c$, $[L^*, d] = - d_c^*$.

<u>Proof</u>: a) is clear since M is Kähler. (Note that $[L^*, d^*] = 0$ is the adjoint assertion to $[L, d] = 0$.)

b) Since $[L, d^*]^* = [d, L]$ it is sufficient to prove $[L^*, d] = -d_c^*$ i.e. that
$$L^* d - d L^* = - J^{-1} d^* J .$$

We know that

(3.3) $$d^* = - * d * = (-1)^{p+1} * d *^{-1} \quad \text{on } A^p(M) .$$

For $\varphi \in A^p(M)$ we have by (2.8)
$$\# d \#^{-1} \varphi = i J \left((-1)^{p+1} * d *^{-1} \right) J^{-1} \varphi$$
$$\overset{(3.3)}{=} i J d^* J^{-1} \varphi = - i J^{-1} d^* J \varphi$$

Now let
$$d_t = \exp(it(L^* + L)) \circ d \circ \exp(-it(L^* + L)) .$$

Thus $\quad d_t = \mathrm{Ad}(\exp it(L^*+L))\circ d$

$$= \exp(it\ \mathrm{ad}(L^*+L))\circ d .$$

Note that:

$$d_0 = d$$

(3.4) $$d_{\frac{\pi}{2}} = \# d \#^{-1} = -i J^{-1} d^* J$$

$$\frac{d}{dt} d_t = i(\mathrm{ad}\, L^* + \mathrm{ad}\, L)\, d_t .$$

The relations

$(\mathrm{ad}\, L^*)^2 = 0$, $\mathrm{ad}\, L\circ d = [L, d] = 0$, $\mathrm{ad}(-B)\circ d = d$

and $[L, L^*] = -B$ show that

$$\tilde{d}_t := \cos t\cdot d + i\sin t\ \mathrm{ad}(L^*)\, d$$

is a solution of

$$\frac{d}{dt}\tilde{d}_t = i(\mathrm{ad}(L^*) + \mathrm{ad}(L))\,\tilde{d}_t .$$

Since $\tilde{d}_0 = d$ we must have $d_t = \tilde{d}_t$ i.e.

$$d_t = \cos t\cdot d + i\sin t\ \mathrm{ad}(L^*)\, d .$$

In particular

$$d_{\frac{\pi}{2}} = i\ \mathrm{ad}(L^*)\, d = i[L^*, d]$$

which implies the assertion in view of (3.4).

(3.5) Corollary: 1) $[L, d_c] = 0$, $[L^*, d_c^*] = 0$, $[L, d_c^*] = -d$, $[L^*, d_c] = d^*$.

2) $$d_c d^* + d^* d_c = 0 \ , \quad d d_c^* + d_c^* d = 0$$

Proof: 1) follows from (3.2) by noting that J commutes with the real operators L, L^*.

2) Since $d^* = [L^*, d_c]$ by 1) we have:

$$d^* d_c = L^* d_c d_c - d_c L^* d_c = - d_c L^* d_c$$
$$d_c d^* = d_c L^* d_c - d_c d_c L^* = d_c L^* d_c$$

so $d_c d^* + d^* d_c = 0$. The other equation follows by taking adjoints.

We can now prove Th. (3.1) : $\Delta = 2\overline{\Box}$ as follows:

$$\Delta = d d^* + d^* d = d [L^*, d_c] + [L^*, d_c] d$$
$$= d L^* d_c - d d_c L^* + L^* d_c d - d_c L^* d .$$

On the other hand

$$J^{-1} \Delta J = - d_c L^* d + d_c d L^* - L^* d d_c + d L^* d_c$$

so $\Delta = J^{-1} \Delta J$ since $d_c d = - d d_c$. On the other hand

$$J^{-1} \Delta J = J^{-1} d d^* J + J^{-1} d^* d J = d_c d_c^* + d_c^* d_c$$

and hence

$$2\Delta = \Delta + J^{-1} \Delta J = d d^* + d^* d + d_c d_c^* + d_c^* d_c .$$

On the other side the relations

$$d_c = i(\bar\partial - \partial) \quad , \quad d_c^* = -i(\bar\partial^* - \partial^*) \quad \text{imply}$$

$$\bar\partial = \frac{1}{2}(d - i d_c) \quad , \quad \bar\partial^* = \frac{1}{2}(d^* + i d_c^*)$$

and hence

$$4\bar\Box = 4(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = d d^* + i d d_c^* - i d_c d^* + d_c d_c^* + d^* d - i d^* d_c + i d_c^* d + d_c^* d_c$$

$$\overset{(3.5)\,2)}{=} d d^* + d^* d + d_c d_c^* + d_c^* d_c$$

$$= 2\Delta \qquad \text{q.e.d.}$$

References

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