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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Abelian Varieties

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These are preliminary lecture notes, intended only for distribution to participants

ABELIAN VARIETIES

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k a field, $\bar{k}$ its algebraic closure.
A variety V over k is a separated scheme of finite type over $\mathrm{Spec}(k)$ which is geometrically integral.

§1 Basic definitions

A <u>group variety over k</u> is a variety G over k together with k-morphisms

$$\mu : G \times G \longrightarrow G \qquad \text{(multiplication)}$$
$$\iota : G \longrightarrow G \qquad \text{(inverse)}$$
$$\varepsilon : \mathrm{Spec}(k) \longrightarrow G \qquad \text{(unit section)}$$

such that the following diagrams commute:

(associativity)
$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times id"] \arrow[d, "id \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd},$$

(unit element)
$$\begin{tikzcd} & G \times G \arrow[dr, "\mu"] & \\ G \arrow[ur, "(id, \varepsilon)"] \arrow[rr, equal] & & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & G \times G \arrow[dr, "\mu"] & \\ G \arrow[ur, "(\varepsilon, id)"] \arrow[rr, equal] & & G \end{tikzcd},$$

(inverse)
$$\begin{tikzcd} & G \times G \arrow[dr, "\mu"] & \\ G \arrow[ur, "(id, \iota)"] \arrow[r] & \mathrm{Spec}(k) \arrow[r, "\varepsilon"'] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & G \times G \arrow[dr, "\mu"] & \\ G \arrow[ur, "(\iota, id)"] \arrow[r] & \mathrm{Spec}(k) \arrow[r, "\varepsilon"'] & G \end{tikzcd}.$$

A <u>homomorphism</u> $f : G \longrightarrow H$ of group varieties G and H over k is a k-morphism such that the diagram

$$\begin{tikzcd} G \times G \arrow[r, "f \times f"] \arrow[d, "\mu"'] & H \times H \arrow[d, "\mu"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

commutes.

<u>Note</u>: For any k-scheme S we have a group structure on the set $G(S)$ of S-valued points of G: Given $a, b \in G(S)$, i.e., $a, b : S \longrightarrow G$ define

$$a \cdot b : S \xrightarrow[(a,b)]{} G \times G \xrightarrow[\mu]{} G$$

$$a^{-1}: S \xrightarrow[a]{} G \xrightarrow[i]{} G$$

and $$1.: S \longrightarrow \mathrm{Spec}(k) \xrightarrow[\varepsilon]{} G \quad .$$

Homomorphisms induce group homomorphisms on $G(S)$.

Translation maps: For any $a \in G(k)$ there is the isomorphism

$$t_a : G \xrightarrow[(a,id)]{} G \times G \xrightarrow[\mu]{} G$$

$$\left(t_a : \begin{array}{ccc} G(S) & \longrightarrow & G(S) \\ b & \longmapsto & a \cdot b \end{array} \right) .$$

Fact: Any group variety G is smooth.
Proof: G is smooth if $G_{/\bar{k}}$ is smooth,
$G_{/\bar{k}}$ integral $\Longrightarrow$ $G_{/\bar{k}}$ has nonempty open subvariety U which is smooth
$\longrightarrow t_a(U)$ is smooth $\longrightarrow G_{/\bar{k}} = \bigcup_{a \in G(\bar{k})} t_a(U)$ is smooth.

Definition: An abelian variety A over k is a group variety over k which is complete, i.e., proper over $\mathrm{Spec}(k)$.
(We will write μ always as $+$ $\longrightarrow$ see later for reason.)

§2 Rigidity

Proposition: Let $f: V \times W \longrightarrow U$ be a morphism of k-varieties where V is complete; if $f(V \times \{w_0\}) = \{u_0\} = f(\{v_0\} \times W)$ holds for some $v_0 \in V(k)$, $w_0 \in W(k)$ and $u_0 \in U(k)$ then f is constant.
Proof: We may assume $k = \bar{k}$.
The reason is: There are no nonconstant maps from a complete to an affine variety.
How do we use that?
Let $p: V \times W \longrightarrow W$ be the projection map: is proper and therefore closed;
let $U_0 \subseteq U$ be an open affine neighbourhood of u_0.
For any $w \in W(k)$ we have

$$f(V \times \{w\}) \subseteq U_0 \iff V \times \{w\} \cap f^{-1}(U \setminus U_0) = \emptyset \iff w \notin \underbrace{p(f^{-1}(U \setminus U_0))}_{\substack{\| \\ Z \text{ closed in } W!}} .$$

"reason" implies now: $f \mid V \times \{w\} =$ constant for any closed point in $W \setminus Z$,
Hilbert's Nullstellensatz $\Longrightarrow$ closed points are dense in $W \setminus Z$,

moreover $f(\{v_0\}\times\{w\}) = \{u_0\}$ by assumption,
hence $f|V\times(W\setminus Z)$ = constant with value u_0.

$$\left.\begin{array}{l} \text{But } W\setminus Z \text{ open in } W \\ v_0 \notin Z \Rightarrow W\setminus Z \neq \emptyset \end{array}\right\} \Rightarrow W\setminus Z \text{ dense in } W \text{ (since } W \text{ irreducible)}$$

$$\Rightarrow V\times(W\setminus Z) \text{ dense in } V\times W$$

$$\Rightarrow f = \text{constant}.$$

Corollary: Every morphism $f: A \to B$ of abelian varieties is the composite of a homomorphism $g: A \to B$ and a translation t_b where $b := +f(0) \in B(k)$

$$f = t_b \circ g .$$

Proof: Replace f by $t_{-f(0)} \circ f$; it remains to show:

$f(0) = 0 \Rightarrow f$ homomorphism.

Consider the morphism

$$\varphi := f\circ\mu_A - \mu_B\circ(f\times f) : A\times A \longrightarrow B$$

$$(a,a') \longmapsto f(a+a') - f(a) - f(a'),$$

we have to show that $\varphi = 0$.

But $\varphi(A\times\{0\}) = \{0\} = \varphi(\{0\}\times A$ so that rigidity applies.

Corollary: Any abelian variety is commutative.

Proof: Previous Corollary $\Rightarrow$ the inverse $i: A\to A$ is a homomorphism, but this can only be true for a commutative group law.

§3 Abelian varieties over the complex numbers

$k = \mathbb{C}$, A an abelian variety over $\mathbb{C}$ of dimension g,

$A(\mathbb{C})$ is a complex manifold of dimension g

(write small open subsets by equations for which the Jacobi matrix has maximal rank – possible by smoothness, use implicit function theorem to get a chart),

A irreducible $\Rightarrow A(\mathbb{C})$ connected,

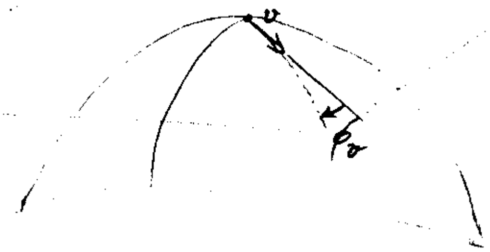
A complete $\Rightarrow A(\mathbb{C})$ compact,

A group variety $\Rightarrow A(\mathbb{C})$ is a complex Lie group.

How do compact connected complex Lie groups look like?

Let $V_0 := T_0 A \cong \mathbb{C}^g$ be the tangent space of A in O.

For any $v \in T_0 A$ we have a unique holomorphic homomorphism
$$\phi_v : \mathbb{C} \longrightarrow A(\mathbb{C})$$
such that
$$d\phi_v(1) = v \, .$$



The exponential map is defined by
$$\begin{aligned} \exp : T_0 A &\longrightarrow A(\mathbb{C}) \\ v &\longmapsto \phi_v(1) \end{aligned} \, .$$

$$\text{Unicity} \Longrightarrow \begin{cases} \phi_{sv}(t) = \phi_v(st) & \Longrightarrow \exp(tv) = \phi_v(t), \\ \phi_v + \phi_{v'} = \phi_{v+v'} & \Longrightarrow \exp \text{ is a homomorphism} \end{cases} .$$

It follows that exp is a holomorphic homomorphism with $d\exp = \mathrm{id}$

$\Longrightarrow$ exp is a local homeomorphism (by implicit function theorem!)

$\Longrightarrow$ $\exp : V_0 \twoheadrightarrow A(\mathbb{C})$ is the universal covering of $A(\mathbb{C})$

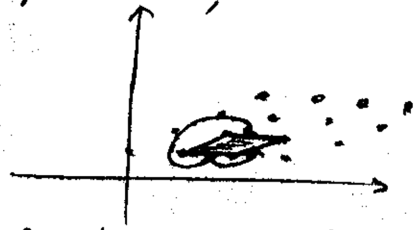
$\Longrightarrow$ $\ker(\exp)$ is a discrete subgroup in $V_0 \cong \mathbb{C}^g$ with compact quotient

$\Longrightarrow$ $\Lambda := \ker(\exp)$ is a lattice of rank $2g$.

<u>Result</u>: $A(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$ is a complex torus of dimension g .

<u>Example</u>: $g = 1$, A an elliptic curve,

$\mathbb{C}/\Lambda$



<u>Warning</u>: Not every complex torus is an abelian variety ($\to$ see later).

<u>Consequence about n-torsion</u>:

$k = \bar{k}$ of char. 0, A an abelian variety over k of dimension g, $n \in \mathbb{N}$, then
$$A(k)_n := \{ a \in A(k) : na = 0 \} \cong (\mathbb{Z}/n\mathbb{Z})^{2g} \, .$$

(This is not true if char $k \mid n$!)

§4 Some fundamental technical theorems

V a complete variety over k.
What is a "family" of line bundles on V?

T an integral scheme of finite type over k,
$\mathcal{L}$ a line bundle on $V \times T$;
for any point $t \in T$ we then have the inverse image line bundle $\mathcal{L}_t$ on $V_t := V \times \{t\}$ (this is a complete variety over $k(t)$) – the "fiber" in t –

Seesaw principle: Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on $V \times T$ such that:

a. $\mathcal{L}_t \cong \mathcal{M}_t$ for any t in some dense subset $\mathcal{U} \subseteq T$,

b. $\mathcal{L}_a \cong \mathcal{M}_a$ for some $a \in V(k)$;

then $\mathcal{L} \cong \mathcal{M}$.

Proof: Replacing $\mathcal{L}$ by $\mathcal{L} \otimes \mathcal{M}^{-1}$ it suffices to prove: If

a'. $\mathcal{L}_t$ is trivial for $t \in \mathcal{U}$, and

b'. $\mathcal{L}_a$ is trivial for some $a \in V(k)$

holds then $\mathcal{L}$ is trivial.

Let $q : V \times T \longrightarrow T$ be the projection map: it is proper hence closed since V is complete.

We need (part of) the cohomological base change theorem:

i. The function $t \longmapsto \dim_{k(t)} H^0(V_t, \mathcal{L}_t)$ is upper semicontinuous, i.e., for any $c \in \mathbb{R}_{\geq 0}$ the set $\{t \in T : \dim H^0(V_t, \mathcal{L}_t) > c\}$ is closed in T,

ii. if the above function is constant of value r then $q_*\mathcal{L}$ is a locally free $\mathcal{O}_T$-module of rank r and $q_*\mathcal{L} \otimes_{\mathcal{O}_T} k(t) = H^0(V_t, \mathcal{L}_t)$.

Step 1: a' $\Rightarrow$ $\mathcal{U} \subseteq \{t : H^0(V_t, \mathcal{L}_t)$ and $H^0(V_t, \mathcal{L}_t^{-1}) \neq 0\}$ closed in T,

$\mathcal{U}$ dense $\Rightarrow$ — " — $= T$,

for arbitrary t we then have nonzero sections $s_1 \in \mathcal{L}_t(V_t)$ and $s_2 \in \mathcal{L}_t^{-1}(V_t)$, i.e., nonzero homomorphisms $s_1 : \mathcal{O}_{V_t} \to \mathcal{L}_t$ and $s_2 : \mathcal{O}_{V_t} \to \mathcal{L}_t^{-1}$,

then
$$\mathcal{O}_{V_t} \xrightarrow{s_1} \mathcal{L}_t \xrightarrow{s_2^{dual}} \mathcal{O}_{V_t}$$

being nonzero is an isomorphism ($\mathrm{Hom}(\mathcal{O}_{V_t},\mathcal{O}_{V_t}) = \mathcal{O}_{V_t}(V_t) = k(t)$ by the completeness of V_t)

$\Longrightarrow s_2^{dual}$ epimorphism $\Longrightarrow s_2^{dual}$ isomorphism;

we therefore have:

a''. $\mathcal{L}_t$ is trivial for any $t \in T$.

Step 2: a'' $\Longrightarrow \dim H^0(V_t,\mathcal{L}_t) = \dim H^0(V_t,\mathcal{O}_{V_t}) = \dim k(t) = 1$ is constant in t,

ii. $\Longrightarrow q_*\mathcal{L}$ is a line bundle with $q_*\mathcal{L} \underset{\mathcal{O}_T}{\otimes} k(t) = H^0(V_t,\mathcal{L}_t)$;

consider the natural adjunction homomorphism

$$\alpha : q^*q_*\mathcal{L} \longrightarrow \mathcal{L} ;$$

on fibers we have

$$\begin{tikzcd}
\alpha_t : (q^*q_*\mathcal{L})_t \arrow[r] \arrow[d, phantom, "\Vert"] & \mathcal{L}_t \arrow[dd, equal] \\
\mathcal{O}_{V_t} \underset{k(t)}{\otimes} (q_*\mathcal{L})_t \arrow[d, phantom, "\Vert"] & \\
\mathcal{O}_{V_t} \underset{k(t)}{\otimes} \mathcal{L}_t(V_t) \arrow[d, phantom, "\wr\Vert"] \arrow[r, phantom, "\leftarrow \text{trivialization} \rightarrow"] & \mathcal{L}_t \arrow[d, phantom, "\wr\Vert"] \\
\mathcal{O}_{V_t} \underset{k(t)}{\otimes} k(t) \arrow[r, equal] & \mathcal{O}_{V_t}
\end{tikzcd}$$

bijectivity,
Nakayama lemma $\Longrightarrow \alpha$ is surjective,
both sides are line bundles $\longrightarrow \alpha$ is an isomorphism.

Step 3: consider

$$\begin{tikzcd}
T = \{a\} \times T \arrow[r, hook] \arrow[dr, equal] & V \times T \arrow[d, "q"] \\
& T
\end{tikzcd}$$

therefore $\mathcal{L}_a \overset{\alpha}{\cong} (q^*q_*\mathcal{L})_a = q_*\mathcal{L}$ which is trivial by b'.,

$\longrightarrow q^*q_*\mathcal{L} \cong \mathcal{L}$ is trivial.

Theorem of the cube: Let U, V, W be complete varieties over k and $u_0 \in U(k)$, $v_0 \in V(k)$, $w_0 \in W(k)$; a line bundle $\mathcal{L}$ on $U \times V \times W$ is trivial if its restrictions to $\{u_0\} \times V \times W$, $U \times \{v_0\} \times W$, and $U \times V \times \{w_0\}$ are trivial.

Remarks on the proof: Step 1: Reduce to $k = \bar{k}$.

Step 2: $\mathcal{L} \,/\, \underbrace{U \times V}_{Z} \times \{w_0\}$ is trivial,

therefore it suffices by the seesaw principle to show that $\mathcal{L} / \{z\} \times W$ is trivial for z in a dense subset of $Z = U \times V$.

Step 3: Reduce to case $\dim U = 1$ by showing that the union of all irreducible curves in U which contain u_0 is dense in U.

Step 4: (technical part) Use the Riemann-Roch theorem for the curve U.

Corollary: Let f, g, l be k-morphisms from a variety V into an abelian variety A over k; for any line bundle $\mathcal{L}$ on A the line bundle

$$\mathcal{L}_{f,g,l} := (f+g+l)^*\mathcal{L} \otimes (f+g)^*\mathcal{L}^{-1} \otimes (g+l)^*\mathcal{L}^{-1} \otimes (f+l)^*\mathcal{L}^{-1} \otimes f^*\mathcal{L} \otimes g^*\mathcal{L} \otimes l^*\mathcal{L}$$

on V is trivial.

Proof: Let $\varphi : V \xrightarrow{(f,g,l)} A \times A \times A$

and $p_i : A \times A \times A \longrightarrow A$ be the projection maps,

then $\mathcal{L}_{f,g,l} = \varphi^* \mathcal{L}_{p_1,p_2,p_3}$,

but the theorem of the cube applies to $\mathcal{L}_{p_1,p_2,p_3}$.

Theorem of the Square: For any line bundle $\mathcal{L}$ on an abelian variety A over k and any two points $a, b \in A(k)$ we have

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L} \cong t_a^* \mathcal{L} \otimes t_b^* \mathcal{L}.$$

Proof: Corollary $\Longrightarrow$

(left side) $\otimes$ (right side)$^{-1}$ $\cong \mathcal{L}_{\mathrm{id},\, t_a \circ (\text{zero map}),\, t_b \circ (\text{zero map})}$ is trivial.

Important application: Tensoring the above isomorphism with $\mathcal{L}^{\otimes -2}$ shows that

$$\begin{aligned} \varphi_{\mathcal{L}} : A(k) &\longrightarrow \mathrm{Pic}(A) \\ a &\longmapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1} \end{aligned}$$

is a group homomorphism. Put $K(\mathcal{L})(k) := \ker \varphi_{\mathcal{L}}$.

Moreover: $\varphi_{\mathcal{L}\otimes\mathcal{L}'} = \varphi_{\mathcal{L}} + \varphi_{\mathcal{L}'}$, $\varphi_{T_b^*\mathcal{L}} = \varphi_{\mathcal{L}}$ for any $b \in A(k)$.

Now consider the line bundle $\mu^*\mathcal{L} \otimes p_2\mathcal{L}^{-1}$ on $A \times A$; from an argument in the proof of the seesaw principle we know that

$$K(\mathcal{L}) := \{ t \in A : (\mu^*\mathcal{L} \otimes p_2\mathcal{L}^{-1})_t \text{ is trivial} \}$$

is closed in A; but for $a \in A(k)$ we have

$$(\mu^*\mathcal{L} \otimes p_2\mathcal{L}^{-1})_a = T_a^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Therefore we see (at least if char $k = 0$) that $K(\mathcal{L})(k)$ indeed is the set of k-rational points of the closed subgroup scheme $K(\mathcal{L})$ in A.

(in general not integral, nor even reduced in char > 0 !!)

§5 Projectivity of abelian varieties

Let D be an effective divisor on the abelian variety A,
let $\mathcal{L} := \mathcal{L}(D)$ be the associated line bundle,
$|D| := \{ (f) + D \geq 0 : f \in k(A)^* \}$ be the associated linear system.

Try to define a morphism $A \to \mathbb{P}^N$ in the following way:
Choose a k-basis $s_1, \dots, s_N$ of $H^0(V, \mathcal{L}(D))$ and put

$$\begin{aligned} A &\longrightarrow \mathbb{P}^N \\ a &\longmapsto [s_1(a) : \dots : s_N(a)] \end{aligned};$$

this only makes sense if it cannot happen that $s_1(a) = \dots = s_N(a) = 0$,
but $\mathrm{div}(s_i) \in |D|$;
therefore the above morphism is defined if $|D|$ has no base points, i.e., no points a such that $a \in \operatorname{supp} E$ for any $E \in |D|$.

Proposition: The following assertions are equivalent,
i. $H := \{ b \in A(\bar{k}) : T_b^* D = D \}$ is finite;
ii. $K(\mathcal{L})$ is a finite group scheme;
iii. $|2D|$ has no base points and defines a finite morphism $A \to \mathbb{P}^N$;
iv. $\mathcal{L}$ is an ample line bundle.

Proof of i. $\Rightarrow$ first part of iii.: We may assume that $k = \bar{k}$;

theorem of the square $\Longrightarrow T_b^*\mathcal{L} \otimes T_{-b}^*\mathcal{L} \cong \mathcal{L}\otimes\mathcal{L} = \mathcal{L}(2D)$

$\Longrightarrow T_b^*D + T_{-b}^*D$ is linearly equivalent to $2D$

$\Longrightarrow T_b^*D + T_{-b}^*D \in |2D|$ for any $b \in A(k)$,

now assume that $a \in A(k)$ is a base point of $|2D|$,

$D = \sum n_i C_i \Longrightarrow T_b^*D + T_{-b}^*D = \sum n_i\big((C_i - b) + (C_i + b)\big)$

$\Longrightarrow a \in \operatorname{supp}(T_b^*D + T_{-b}^*D) \subseteq \{(C_i \pm b)\}$

$\Longrightarrow a+b$ or $a-b \in \operatorname{supp} D$ for any $b \in A(k)$

$\Longrightarrow A = (\operatorname{supp} D - a) \cup (a - \operatorname{supp} D)$ $\lightning$

<u>Theorem:</u> Any abelian variety A is projective.

Proof: By the general theory one knows:

If $\mathcal{L}$ is an ample line bundle on A then $\mathcal{L}^{\otimes m}$, for $m \gg 0$, is very ample and defines an embedding $A \hookrightarrow \mathbb{P}^N$.

By the Proposition it therefore suffices to find an effective divisor D on A such that i. holds:

Let U be an affine open neighbourhood of $0 \in A$,

let $D_1, \dots, D_r$ be the irreducible components of $A \setminus U$; $D := \sum D_i$ is an effective divisor (since U is affine!);

for $b \in H$ we have $T_b(U) = U$, in particular $T_b(0) = b \in U \Longrightarrow H \subseteq U$,

we have $H = \mathcal{H}(\bar{k})$ for the closed subscheme

$$\mathcal{H} := \bigcap_{a \in (A\setminus U)(\bar{k})} \text{preimage of } A\setminus U \text{ under } A \xrightarrow{a+} A$$

$\longrightarrow \mathcal{H}$ complete and affine $\longrightarrow \mathcal{H}$ finite $\Longrightarrow H$ finite.

<u>Corollary:</u> There are line bundles $\mathcal{L}$ on A (the ample ones) such that $K(\mathcal{L})$ is finite.

<u>Application to complex tori:</u>

Since any holomorphic line bundle on $\mathbb{C}^N$ is trivial we have, for a line bundle L (here as a scheme, not as invertible sheaf) on A a diagram

$$\begin{array}{ccccc} \mathbb{C}\times V_0 & \longrightarrow & (\mathbb{C}\times V_0)/\Lambda & = & L(\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ V_0 & \xrightarrow{\pi} & V_0/\Lambda & = & A(\mathbb{C}) \end{array}$$

where the action of $\lambda \in \Lambda$ on $\mathbb{C}\times V_0$ is of the form

$$\lambda : (c, v_0) \longmapsto (e_\lambda(v_0)\cdot c,\ v_0 + \lambda) \quad \text{for some "1-cocycle" } e_\lambda \in \mathcal{O}^*_{hol}(V_0)$$

Theorem of Appell-Humbert:

Up to a "1-coboundary" e_λ is of the form

$$e_\lambda(v_0) = \alpha(\lambda)\cdot e^{\pi H(v_0,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)}$$

where H is a positive Hermitian form on V_0 such that $(\mathrm{im}\, H)/\Lambda\times\Lambda \subseteq \mathbb{Z}$ and $\alpha : \Lambda \longrightarrow \{|c| = 1\}$ is an appropriate homomorphism.

By explicit computation one checks that for $a = \pi(v_0) \in A(\mathbb{C})$ the line bundle

$$T_a^* L \text{ is given by } (H,\ \alpha\cdot e^{2\pi i (\mathrm{im}\, H)(v_0,\,.)}) .$$

Consequence: $K(L) = \{\underbrace{v \in V_0 : (\mathrm{im}\, H)(v,\Lambda) \subseteq \mathbb{Z}}_{\overset{\shortparallel}{\Lambda^\perp}}\}/\Lambda$;

in particular: $K(L)$ is finite $\iff \Lambda^\perp/\Lambda$ is finite

$\iff \mathrm{im}\, H$ nondegenerate

$\iff H$ nondegenerate

$\iff H$ positive definite .

Theorem: A complex torus V_0/Λ is an abelian variety if and only if there is a positive definite Hermitian form H on V_0 such that $(\mathrm{im}\, H)/\Lambda\times\Lambda \subseteq \mathbb{Z}$.

Proof: "$\Longrightarrow$"

V_0/Λ abelian variety $\Longrightarrow$ exists L such that $K(L)$ is finite

$\Longrightarrow$ the associated H has the required properties ;

"$\Longleftarrow$"

If $K(L)$ is finite one shows that the holomorphic sections of L (theta functions) define ~~a~~ projective embedding .

Corollary: Any 1-dim. complex torus $\mathbb{C}/\Lambda$ is an abelian variety (elliptic curve).

Proof: We may assume $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ with $\mathrm{im}(\lambda_1/\lambda_2) > 0$, define a $\mathbb{R}$-linear alternating form $E : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{R}$ by

$$v \wedge v' = E(v,v')\cdot \lambda_1\wedge\lambda_2 ,$$

then $H(v,v') := E(iv,v') + i\,E(v,v')$ has the required properties .

§6 The dual abelian variety

Recall the homomorphism

$$\varphi_{\mathcal{L}} : A(\bar{k}) \longrightarrow \operatorname{Pic}(A_{/\bar{k}})$$
$$a \longmapsto T_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

and: i. $\varphi_{\mathcal{L}\otimes\mathcal{L}'} = \varphi_{\mathcal{L}} + \varphi_{\mathcal{L}'}$,

ii. $\varphi_{T_b^*\mathcal{L}} = \varphi_{\mathcal{L}}$ for any $b \in A(\bar{k})$.

<u>Definition</u>: $\operatorname{Pic}^\circ(A) := \{\mathcal{M} \in \operatorname{Pic}(A) : \varphi_{\mathcal{M}} \equiv 0\}$.

i. $\Rightarrow$ $\operatorname{Pic}^\circ(A)$ is a subgroup;

i, ii. $\Rightarrow$ $\operatorname{im} \varphi_{\mathcal{L}} \subseteq \operatorname{Pic}^\circ(A_{/\bar{k}})$ for any $\mathcal{L}$;

we therefore have an exact sequence

$$0 \to \operatorname{Pic}^\circ(A_{/\bar{k}}) \xrightarrow{\subseteq} \operatorname{Pic}(A_{/\bar{k}}) \longrightarrow \operatorname{Hom}(A(\bar{k}), \operatorname{Pic}^\circ(A_{/\bar{k}})) .$$
$$\mathcal{L} \longmapsto \varphi_{\mathcal{L}}$$

Now assume $\mathcal{L}$ to be <u>ample</u>, then $K(\mathcal{L})$ is a finite subgroup scheme in A. For general reasons the quotient

$$\hat{A} := A/K(\mathcal{L})$$

in the category of schemes exists and is again an abelian variety (which for the moment being depends on $\mathcal{L}$).

<u>Theorem</u>: If $\mathcal{L}$ is ample then

$$\varphi_{\mathcal{L}} : A(\bar{k}) \longrightarrow \operatorname{Pic}^\circ(A_{/\bar{k}}) \quad \text{is surjective}$$

and therefore induces an isomorphism $\hat{A}(\bar{k}) \xrightarrow{\cong} \operatorname{Pic}^\circ(A_{/\bar{k}})$.

More generally one has that $\hat{A}$ represents the functor

$$k\text{-schemes} \longrightarrow \text{abelian groups}$$
$$S \longmapsto \operatorname{Pic}^\circ(A \times S_{/S}) .$$

This shows that $\hat{A}$ is up to unique isomorphism independent of the choice of the ample line bundle $\mathcal{L}$. We also see that

$$\varphi_{\mathcal{L}} : A \longrightarrow \hat{A}$$

actually is a homomorphism of abelian varieties over k (corresponding to the quotient homomorphism $A \longrightarrow A/K(\mathcal{L})$); it is called a <u>polarization</u> of A and, of course, depends on $\mathcal{L}$.

<u>Definition</u>: $\hat{A}$ is called the <u>dual abelian variety</u> of A.

Why 'dual'?
Any homomorphism $f: A \longrightarrow B$ of abelian varieties over k induces (by pulling back line bundles) a natural transformation of functors

$$\mathrm{Pic}^\circ(B\times ./.) \longrightarrow \mathrm{Pic}^\circ(A\times ./.)$$

and hence a homomorphism of abelian varieties

$$\hat{f}: \hat{B} \longrightarrow \hat{A} \,.$$

So we see that $A \longmapsto \hat{A}$ actually is a contravariant functor.
What about the double dual $\hat{\hat{A}}$?
One has an additional structure, namely a line bundle – the Poincaré bundle – $\mathcal{P}$ on $A\times\hat{A}$ such that

a. for any $\alpha \in \hat{A}(\bar{k})$ the isomorphism class of the line bundle $\mathcal{P}_\alpha$ on $A_{/\bar{k}} \times \{\alpha\} = A_{/\bar{k}}$ is the image of α under $\hat{A}(\bar{k}) \xrightarrow{\sim} \mathrm{Pic}^\circ(A_{/\bar{k}})$

b. $\mathcal{P}/\{0\}\times\hat{A}$ is trivial.

Note that by the seesaw principle the properties a. and b. determine $\mathcal{P}$ up to isomorphism.
But symmetrically to a. we may use $\mathcal{P}$ to define a homomorphism

$$i: A(\bar{k}) \longrightarrow \mathrm{Pic}^\circ(\hat{A}_{/\bar{k}}) = \hat{\hat{A}}(\bar{k})$$
$$a \longmapsto \text{isomorphism class of } \mathcal{P}_a .$$

<u>Proposition</u>: $i: A \xrightarrow{\sim} \hat{\hat{A}}$ is an isomorphism of abelian varieties.

LITERATURE:

D. Mumford "Abelian Varieties", Oxford Univ. Press 1974
J.S. Milne "Abelian Varieties", Chap. V of Arithmetic Geometry (Eds. Cornell, Silverman), Springer 1986