INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.637/21

ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

(31 August - 11 September 1992)

Derived Categories and Functors

T. Fimmel
Mathematisches Institut
Universität zu Köln
Weyertal 86-90
5000 Köln 41
Germany

These are preliminary lecture notes, intended only for distribution to participants

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 224559 TELEX 460392

Derived categories and functors

This theory is contained in two fields of mathematics

1. Homological algebra
   Some ideas and questions have been considered in the lectures of A. Dold about cohomology theories.
2. Theory of categories = "abstract nonsense"
   This is essentially a language of mathematics which has been used implicitly in many lectures

Let us give some general definitions of 2.:

Structure	"Maps" between these structures

<u>category</u>: $\mathcal{C}$ is a class of objects $Ob(\mathcal{C})$ and a family of sets $Hom_{\mathcal{C}}(X,Y)$ for all $X,Y \in Ob(\mathcal{C})$ (the elements are called morphisms) and composition maps:

$$Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \to Hom_{\mathcal{C}}(X,Z)$$
$$\varphi \quad , \quad \psi \quad \mapsto \psi \circ \varphi$$

for all $X,Y,Z \in Ob(\mathcal{C})$ and furthermore we have identities

$$1_X \in Hom_{\mathcal{C}}(X,X)$$

satisfying the conditions.

- $(\varphi \circ \psi) \circ \eta = \varphi \circ (\psi \circ \eta)$
- $1 \circ \varphi = \varphi \circ 1 = \varphi$

<u>functor</u>: $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}, \mathcal{D}$ is a map on objects:

$$\mathcal{F}: Ob(\mathcal{C}) \to Ob(\mathcal{D})$$

and a map of sets:

$$\mathcal{F}: Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(\mathcal{F}X, \mathcal{F}Y)$$

for all $X,Y \in Ob(\mathcal{C})$ satisfying the following conditions:

$\rightsquigarrow$ compatibility with composition

$\rightsquigarrow$ compatibility with identity ex.: $\mathcal{C}$ = Sets the category of sets, this means

$Ob\,\mathcal{C}$ = the class of all sets

$Hom_{\mathcal{C}}(X,Y)$ = all maps from X to Y

In the same way we define:

$\mathcal{T}op$ .. the category of topological spaces

$\mathcal{S}ch$ .. the category of schemes

$\mathcal{A}b$ .. the category of abelian groups

$\mathcal{A}b(X)$ .. the category of sheafs of abelian groups over some topological space X.

ex.: - $\mathcal{F}: \mathcal{T}op \longrightarrow$ Sets

$$X \longmapsto X$$

we forget that X has a topological structure

- $\mathcal{F} = \Gamma: \mathcal{A}b(X) \longrightarrow \mathcal{A}b$

$$\underbrace{\mathcal{J}}_{\text{a sheaf}} \longmapsto \underbrace{\Gamma(X,\mathcal{J})}_{\text{the global sections}}$$

- If $f: X \to Y$ is a map of topological spaces, there has been defined a direct image functor:

$\mathcal{F} = f_*: \mathcal{A}b(X) \longrightarrow \mathcal{A}b(Y)$

$$\underbrace{\mathcal{J}}_{\text{a sheaf on } X} \longmapsto \underbrace{f_*\mathcal{J}}_{\text{a sheaf on } Y \text{ given by:}}$$

$$(f_*\mathcal{J})(\underbrace{V}_{\text{open in } Y}) := \mathcal{J}(f^{-1}(V)).$$

<u>Abelian category</u>: $\mathcal{A}$ is a category with additional structure:

- $\mathcal{O} \in Ob(\mathcal{A})$

with $Hom_{\mathcal{A}}(\mathcal{O}, A) = \{0\}$

$Hom_{\mathcal{A}}(A, \mathcal{O}) = \{0\}$

consist of one element (the morphism 0) for all $A \in Ob(\mathcal{A})$

- a direct sum $\oplus$:

$Hom(A \oplus B, C) = Hom(A,C) \times Hom(B,C)$

- a kernel:

for all $f: A \to B$ we have

$ker f \xrightarrow{\lambda} A \xrightarrow{f} B$ such that

$\lambda \circ f = 0$ and for all

$C \xrightarrow{\gamma} A \xrightarrow{f} B$ with $f \circ \gamma = 0$

<u>Exact functor</u>: $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ a functor between abelian categories, satisfying:

1. $\mathcal{F}(\mathcal{O}) = \mathcal{O}$
2. $\mathcal{F}(A \oplus B) \simeq \mathcal{F}(A) \oplus \mathcal{F}(B)$ in a natural way
3. $\mathcal{F}(ker(f)) \simeq ker\, \mathcal{F}(f)$

we get uniquely

$$\begin{tikzcd} \ker(f) \arrow[r, "\lambda"] & A \\ C \arrow[u, dashed] \arrow[ur, "\xi"'] & \end{tikzcd}$$

- a cokernel

for all $f: A \to B$ we have

$$A \xrightarrow{f} B \xrightarrow{\lambda} \operatorname{Coker}(f)$$

with the dual properties satisfying:

The natural morphism

$$\operatorname{Coker}(\ker(f)) \to \ker(\operatorname{Coker}(f))$$

is an isomorphism.

4. $\mathcal{F}(\operatorname{Coker}(f)) \simeq \operatorname{Coker} \mathcal{F}(f)$

A functor satisfying only 1. and 2. is called <u>additive</u>, if he also satisfied 3 (resp. 4) he is called <u>left exact</u> (resp <u>right exact</u>)

---

ex.: $\mathcal{A} = \mathcal{A}b$, $\mathcal{A}b(X)$

$$\ker(f: A \to B) = \{a \in A \mid f(a) = 0\}$$

$$\operatorname{Coker}(f) = B / \operatorname{Im}(f)$$

ex.: $\mathcal{A}b \longrightarrow \mathcal{A}b(X)$

$A \longmapsto \underline{A}$ the constant sheaf

$$\underline{A}(\mathcal{U}) := \operatorname{Hom}_{cont.}(\mathcal{U}, A) = A^{\pi_0(\mathcal{U})}$$

Most other functors are only left or right exact, for instance:

$\mathcal{F} = \Gamma : \mathcal{A}b(X) \to \mathcal{A}b$

$f_* : \mathcal{A}b(X) \longrightarrow \mathcal{A}b(Y)$

are left exact but in general not exact.

May the functor $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is not right (resp. left) exact but the condition 4. (resp. 3) can be satisfied for some morphisms f.

ex.: Suppose $\mathcal{F}$ is additive and $A = S$ is an injective object (this means that for every monomorphism $C \xrightarrow{\varphi} D$ ($\Leftrightarrow \ker(\varphi) = 0$) and every morphism $C \xrightarrow{\lambda} S$ we have a continuation to a commutative diagram:

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\lambda"'] & D \arrow[dl, dashed] \\ S & \end{tikzcd}$$

).

Let $f: A \to B$ be a monomorphism. Then
$\mathcal{F}(\mathrm{Coker}(f)) = \mathrm{Coker}(\mathcal{F}(f))$

This can easily be seen by:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"] & B \arrow[dl, dashed, "\lambda"] \\ & A & \end{tikzcd}$$

hence $B = A \oplus \ker(\lambda)$, $\ker(\lambda) = \mathrm{Coker}(f)$

$$\Rightarrow \mathcal{F}B = \mathcal{F}A \oplus \mathcal{F}\,\mathrm{Coker}(f)$$
$$\Rightarrow \mathrm{Coker}(\mathcal{F}(f)) = \mathcal{F}\,\mathrm{Coker}(f)$$

If we have sufficiently many "good" objects we can try to change an arbitrary object to a good object.

Let us realize this idea in an abstract way:

Suppose there is given some additive functor $\mathcal{F}: \mathcal{A} \to \mathcal{B}$. A class of objects $\mathcal{S} \subseteq Ob(\mathcal{A})$ is called a (right) <u>rich class</u> for $\mathcal{F}$ if the following conditions are satisfied:

1. $0 \in \mathcal{S}$
2. If $S, S' \in \mathcal{S}$ then $S \oplus S' \in \mathcal{S}$.
3. If $0 \to S^0 \to S^1 \to \dots$ is an exact sequence of objects of $\mathcal{S}$ then the sequence
   $0 \to \mathcal{F}S^0 \to \mathcal{F}S^1 \to \dots$
   is also exact (in $\mathcal{B}$)
4. Every object $A \in Ob(\mathcal{A})$ can be embedded into some $S \in \mathcal{S}$.

ex.: Suppose $\mathcal{F}$ is left exact (f.i. $\mathcal{F} = \Gamma, f_*, \dots$) and let $\mathcal{S}$ be the class of all injective objects of $\mathcal{A}$. Then properties 1., 2. and 3. can easily be verified. Property 4. is not true in general, but if this holds we say that $\mathcal{A}$ has <u>enough injectives</u>. This is true for $\mathcal{A} = \mathcal{Ab}, \mathcal{Ab}(X)$ and hence we see that the class of injective objects is a rich class for every left exact functor $\mathcal{F}$ from a category with enough injectives

Suppose now that $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a rich class $\mathcal{S}$. We want to give a new definition of $\mathcal{F}$, we call it $\mathbb{R}\mathcal{F}$ the (right) derived functor of $\mathcal{F}$ which is the old functor $\mathcal{F}$ on good objects and satisfies 1., 2., 3. and 4.

The idea is the following:

Change an object $A \in Ob(\mathcal{A})$ to a "resolution" in $\mathcal{S}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "f"] & S^0 \arrow[r, dashed, "d^0"] \arrow[dr] & S^1 \arrow[r, dashed] \arrow[dr] & S^2 \arrow[r, phantom, "\cdots"] & {} \\ & & & \mathrm{Coker}(f) \arrow[u, hook] & \mathrm{Coker}(d^0) \arrow[u, hook] & \end{tikzcd}$$

constructed by property 4.

This is an exact complex and we define:

$$(\mathbb{R}\mathcal{F})(A) := \mathcal{F}S^\bullet = 0 \to \mathcal{F}S^0 \to \mathcal{F}S^1 \to \cdots$$

But what does this mean?

1. $(\mathbb{R}\mathcal{F})(A)$ is not an element of $\mathcal{B}$ but an element of the category of (bounded to the left) complexes over $\mathcal{B}$, we will write $C(\mathcal{B})$ for this category
2. The resolution $S^\bullet$ is not uniquely defined, we get a completely different complex if we take an other resolution

The first problem can be solved in two ways:

First, we can return to $\mathcal{B}$ by taking cohomology:

$$C(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}$$

$$B^\bullet \longmapsto \ker(d^i) / \mathrm{Im}(d^{i-1})$$

This is the classical idea, we get a family of functors $\mathbb{R}^i\mathcal{F} := H^i\mathbb{R}\mathcal{F}$, which are called i-th derived functor of $\mathcal{F}$. We will return to this interpretation later

The second possibility is to pass to complexes:
If $A^\bullet$ is a (bounded below) complex we can also choose a resolution, this means we have some

$$A^\bullet \xrightarrow{f} S^\bullet$$

a map of complexes such that

$$H^i(A^\bullet) \xrightarrow{H^i(f)} H^i(S^\bullet)$$

is an isomorphism for all i. Such a map is called a <u>quasiisomorphism</u>. Note that if

$$A^\bullet = \ldots \to 0 \to 0 \to A \to 0 \to 0 \to \ldots$$

is a complex concentrated in one degree we get the old definition of resolution (supposing that $S^\bullet$ begins at the same degree as $A^\bullet$ which we can suppose).
Now we define

$$(\mathbb{R}\mathcal{F})(A^\bullet) = \mathcal{F}(S^\bullet)$$

So we get $\mathbb{R}\mathcal{F}$ which maps complexes over $\mathcal{A}$ to complexes over $\mathcal{B}$.

The second problem is more difficult.
Suppose we have two resolutions

$$\begin{array}{ccc} & & S'^\bullet \\ & \overset{f'}{\nearrow} & \\ A^\bullet & & \\ & \underset{f}{\searrow} & \\ & & S^\bullet \end{array}$$

then one shows using the properties of the rich class $\mathcal{S}$ that one can find a third resolution making the following diagram commutative:

$$\begin{tikzcd}[column sep=small]
 & S''^{\bullet} \arrow[dr, "\varphi"] & & & & \mathcal{F}S''^{\bullet} \arrow[dr, "\mathcal{F}\varphi"] & \\
A^{\bullet} \arrow[ur, "f'"] \arrow[rr, "f''"] \arrow[dr, "f"'] & & S'''^{\bullet} & \rightsquigarrow & & & \mathcal{F}S'''^{\bullet} \\
 & S'^{\bullet} \arrow[ur, "\psi"'] & & & & \mathcal{F}S'^{\bullet} \arrow[ur, "\mathcal{F}\psi"'] &
\end{tikzcd}$$

Which properties has $\mathcal{F}\varphi$ resp $\mathcal{F}\psi$. ?

First we have that φ (resp ψ) is a quasiisomorphism, because f' and f'' are quasiisomorphisms.
So may be $\mathcal{F}\varphi$ is a quasiisomorphism too?

Let us prove that this is really satisfied:

The main point is the following important construction from Homological algebra:

Let $f : A^{\bullet} \to B^{\bullet}$ be a map of complexes, then the following complex

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
\mathrm{Con}(f)^{n} \arrow[d, "d"] & := & B^{n} \oplus A^{n+1} \arrow[d, "d \;\; f \;\; -d"] \\
\mathrm{Con}(f)^{n+1} \arrow[d] & := & B^{n+1} \oplus A^{n+2} \\
\vdots & &
\end{tikzcd}$$

is called the <u>cone</u> of f. Its main properties are the following

1. We have an exact sequence of complexes (this means that they are exact in every degree):

$$\begin{array}{ccccccccc}
0 & \to & B^{\bullet} & \longrightarrow & \mathrm{Con}(f)^{\bullet} & \longrightarrow & A^{\bullet}[1]^{\bullet} & \longrightarrow & 0 \\
 & & b & \longmapsto & (b, 0) & & & & \\
 & & & & (b, a) & \longmapsto & a & &
\end{array}$$

where $A[1]^{\bullet}$ is the <u>shifted complex</u> of $A^{\bullet}$ given by

$$\begin{tikzcd}
A[1]^{n} \arrow[d, "d"] & := & A^{n+1} \arrow[d, "-d"] \\
A[1]^{n+1} & := & A^{n+2}
\end{tikzcd}$$

2. f is a quasiisomorphism $\iff$ $\mathrm{Con}(f)$ is exact.

3. If $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an additive functor, then
$$\mathcal{F}(\mathrm{Con}(f)) \simeq \mathrm{Con}\,\mathcal{F}(f)$$

These properties are an easy exercise.

Now let us return to our question:

φ is a quasiisomorphism $\overset{2.}{\Rightarrow}$ $\mathrm{Con}(\varphi)$ is exact. But $\mathrm{Con}(\varphi)$ is a complex with components in $\mathcal{S}$ so we can apply axiom 3. of rich classes to get that $\mathcal{F}\,\mathrm{Con}(\varphi)$ is exact. Hence $\mathrm{Con}\,\mathcal{F}(\varphi)$ is exact and this proves that $\mathcal{F}(\varphi)$ is a quasiisomorphism.

Okay, so we know that our definition of $(\mathbb{R}\mathcal{F})(A^\cdot)$ is unique up to quasiisomorphism.

If we have some quasiisomorphism $A^\cdot \xrightarrow{f} B^\cdot$ we can take a resolution of $B^\cdot$ $B^\cdot \xrightarrow{\lambda} S^\cdot$, then the composition $\lambda \circ f$ will be a resolution of $A^\cdot$, so their images under $\mathbb{R}\mathcal{F}$ should be the same.

Suppose now there is given some functor $Q: C(\mathcal{B}) \to \mathcal{C}$ with the property that every quasiisomorphism f maps to an isomorphism $Q(f)$. Then the composition
$$C(\mathcal{A}) \overset{``\mathbb{R}\mathcal{F}"}{\rightsquigarrow} C(\mathcal{B}) \xrightarrow{Q} \mathcal{C}$$
is a well defined functor.

If we take a "minimal" category $\mathcal{D}(\mathcal{B})$ with this property and the analoguous category $\mathcal{D}(\mathcal{A})$ we see that we have a continuation

$$\begin{tikzcd}
C(\mathcal{A}) \arrow[d] \arrow[dr] & & \\
\mathcal{D}(\mathcal{A}) \arrow[r, dashed, "\mathbb{R}\mathcal{F}"] & \mathcal{D}(\mathcal{B}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

This well defined functor $\mathbb{R}\mathcal{F}$ is called the derived functor of $\mathcal{F}$ and $\mathcal{D}(\mathcal{A})$ resp. $\mathcal{D}(\mathcal{B})$ is called the derived category of $\mathcal{A}$ resp. $\mathcal{B}$. One can show that the definition does not depend on the choice of the rich class $\mathcal{S}$.

The last idea of taking some "minimal" category in which some class of morphisms will be isomorphisms is a special case of some abstract categorial situation:

<u>Proposition</u>. Let $\mathcal{C}$ be some (small) category and $\mathcal{T}$ a class of morphisms (it means $\mathcal{T} \subseteq \bigcup_{X,Y} \mathrm{Hom}_{\mathcal{C}}(X,Y) =: \mathrm{Mor}(\mathcal{C})$)

a) Then there exists a category $\mathcal{C}[\mathcal{T}^{-1}]$ together with a functor
$$Q : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{T}^{-1}]$$
satisfying the following properties:
1. For all $t \in \mathcal{T}$ $Q(t)$ is an isomorphism.
2. If $\mathcal{D}$ and $\mathcal{C} \xrightarrow{Q'} \mathcal{D}$ are another category and functor satisfying 1. then we have a uniquely defined commutative diagram:
$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q'"] \arrow[dr, "Q"'] & \mathcal{D} \\ & \mathcal{C}[\mathcal{T}^{-1}] \arrow[u, dashed] \end{tikzcd}$$

The category $\mathcal{C}[\mathcal{T}^{-1}]$ is up to equivalence uniquely defined.

b) If the class $\mathcal{T}$ has the following properties:

1. $1_X \in \mathcal{T}$ for all $X \in \mathrm{Ob}(\mathcal{C})$

2. $X \xrightarrow{t} Y \xrightarrow{t'} Z$ $t, t' \in \mathcal{T}$, then $t' \cdot t \in \mathcal{T}$

3. $\begin{tikzcd} & Z \arrow[d, "t \in \mathcal{T}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$ can be continued to a commutative square:

$$\begin{tikzcd} T \arrow[r, "f'"] \arrow[d, "\mathcal{T} \ni t'"'] & Z \arrow[d, "t"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

analoguously if we change the direction of all arrows.

4. $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ Then

ex. $t \in \mathcal{T}$ with $t \circ f = t \circ g \iff$ ex. $t' \in \mathcal{T}$ with $f \circ t' = g \circ t'$.

(such a class is called a <u>localizing class</u>), then we can give the following explicit description of $\mathcal{C}[\mathcal{T}^{-1}]$:

$$\mathrm{Ob}\, \mathcal{C}[\mathcal{T}^{-1}] := \mathrm{Ob}\, \mathcal{C}$$

$$\mathrm{Hom}_{\mathcal{C}[\mathcal{T}^{-1}]}(X, Y) = \left\{ \begin{tikzcd} & Z \arrow[dl, "\mathcal{T} \ni t"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \right\} \Big/ \sim$$

where the equivalence relation $\sim$ is given by:

$$\begin{tikzcd} & Z \arrow[dl, "t"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \sim \begin{tikzcd} & Z' \arrow[dl, "t'"'] \arrow[dr, "f'"] & \\ X & & Y \end{tikzcd}$$

if it exists a commutative diagram:

$$\begin{tikzcd} & & Z'' \arrow[dl, "t''"'] \arrow[dr, "f''"] & & \\ & Z \arrow[dl, "t"'] \arrow[drrr, "f", near end] & & Z' \arrow[dlll, "t'"', near end] \arrow[dr, "f'"] & \\ X & & & & Y \end{tikzcd}$$

The composition is given by:

$$\left(\begin{tikzcd} & Z' \arrow[dl, "t'"'] \arrow[dr, "f'"] & \\ Y & & T \end{tikzcd} \right) \circ \left(\begin{tikzcd} & Z \arrow[dl, "t"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \right) := \begin{tikzcd} & Z'' \arrow[dl, "t \circ t''"'] \arrow[dr, "f' \circ f''"] & \\ X & & T \end{tikzcd}$$

with:

$$\begin{tikzcd} & & Z'' \arrow[dl, "t''"'] \arrow[dr, "f''"] & & \\ & Z \arrow[dl, "t"'] \arrow[dr, "f"] & & Z' \arrow[dl, "t'"'] \arrow[dr, "f'"] & \\ X & & Y & & T \end{tikzcd}$$

commutative.

constructed by property 3. of the localizing class

Let me give some ideas of proof:

1. If you add to $\mathcal{T}$ all 1_X and all compositions of elements of $\mathcal{T}$ you get some new $\mathcal{T}$ but the localized category will be the same.
2. Take $Ob\, \mathcal{C}[\mathcal{T}^{-1}] := Ob\, \mathcal{C}$, suppose $\mathcal{T}$ has properties 1. and 2. of localizing classes
3. Let us extend $Hom_{\mathcal{C}}(X,Y)$ by the following procedure:

   If $t: Y \to X \in \mathcal{T}$ we take symbol

   $$t^{-1} \in Hom_{\mathcal{C}[\mathcal{T}^{-1}]}(X,Y)$$

   The composition we define also formally, this means compositions are new morphisms. So we get something like:

   $$t_1^{-1} \circ f_1 \circ \ldots \circ f_i \circ t_2^{-1} \circ \ldots \circ t_k^{-1} f_{i+1} \circ \ldots \qquad (*)$$

   This defines a category (without 1).
4. Factorize all morphisms by the minimal equivalence relation generated by:

   $$t \circ t^{-1} = 1, \quad t^{-1} \circ t = 1, \quad t^{-1} \circ t'^{-1} = (t' \circ t)^{-1}$$
   $$f \circ f' = f \circ f'$$

   composition in $\mathcal{C}$.

   This defines $\mathcal{C}[\mathcal{T}^{-1}]$, the functor Q is obvious.
5. If $\mathcal{T}$ has also properties 3. and 4. of localizing class we can change with the help of 3. $f \circ t^{-1}$ to $t'^{-1} \circ f'$ and so we order $(*)$ and we get the general form of a morphism: $t^{-1} \circ f$. Using 4. we see that the relations are exactly those written in the proposition.

Let us return to our derived categories:

We have $\mathcal{C} = C(\mathcal{A})$ the category of complexes over $\mathcal{A}$ and $\mathcal{T} =$ Quasiiso. the class of quasiisomorphisms, we can give now an exact definition of the derived category $\mathcal{D}(\mathcal{A})$:

$$D(\mathcal{A}) := C(\mathcal{A})[\text{Quasiiso}^{-1}]$$

Unfornutely the class of quasiisomorphisms is not a localizing class, for instance we take:

$\mathcal{A} = \mathcal{A}b$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[d] \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[d] \arrow[r] & 0 \arrow[d] \\
0 \arrow[d] \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[d] \arrow[r] & 0 \arrow[d] \\
\mathbb{Z}/2\mathbb{Z} \arrow[d] \arrow[r, shift left, "0"] \arrow[r, shift right, "id"'] & \mathbb{Z}/2\mathbb{Z} \arrow[d] \arrow[r] & 0 \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

$$X \underset{g}{\overset{f}{\rightrightarrows}} \underbrace{Y}_{\text{exact}} \xrightarrow[\text{quasiisomorphism}]{t'} 0 \quad , \quad t' \circ f = t' \circ g$$

If t is a map with $f \circ t = g \circ t$ then one easily sees that $t = 0$ but this is not a quasiisomorphism.

This difficulty to give an explicite description can be solved using the following lemma:

<u>Lemma</u>: If $f, g : A^{\cdot} \to B^{\cdot}$ are homotopic maps (remember that this means there exist a family of maps $s^i : A^i \to B^{i-1}$ such that $f - g = d \circ s + s \circ d$) then

$$Q(f) = Q(g) \qquad Q : C(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$$

Proof: We need a second important construction of Homological algebra:

If $h : A^{\cdot} \to B^{\cdot}$ is a morphism of complexes we define the cylinder of h $Cyl(h)^{\cdot}$ as the following complex:

$$\begin{array}{ccccccc}
\downarrow & & & & & & \\
Cyl(h)^n & := & A^n & \oplus & A^{n+1} & \oplus & B^n \\
\downarrow & & \downarrow d & \swarrow id & \downarrow -d & \searrow f & \downarrow d \\
Cyl(h)^{n+1} & := & A^{n+1} & \oplus & A^{n+2} & \oplus & B^{n+1} \\
\downarrow & & & & & & \\
\vdots & & & & & &
\end{array}$$

(verify that this defines a complex)
This object has the following properties:

1. We have morphisms of complexes

$$B^\cdot \xrightarrow{\alpha_h} Cyl(h)^\cdot \xrightarrow{\beta_h} B^\cdot$$
$$b \longmapsto (0,0,b)$$
$$(a_n, a_{n+1}, b_n) \longmapsto b_n + h(a_n)$$

satisfying: $\beta_h \circ \alpha_h = 1_{B^\cdot}$

$\alpha_h \circ \beta_h \sim 1_{Cyl(h)}$

homotopic

Especially, because homotopic maps define the same maps on cohomology of the complexes we have that $H^i(\alpha_h)$ and $H^i(\beta_h)$ are invers to each other and hence are quasiisomorphisms.

2. We have an exact sequence of complexes

$$0 \to A^\cdot \xrightarrow{\bar{h}} Cyl(h) \longrightarrow Con(h) \longrightarrow 0$$
$$a \longmapsto (a,0,0)$$
$$(a_n, a_{n+1}, b_n) \longmapsto (b_n, a_{n+1})$$

Further we see that $h = \beta_h \circ \bar{h}$.

It is a good exercise to verify these properties.

Let us begin the proof of the lemma. We define a map of complexes:

$$\begin{array}{ccc} Cyl(f)^{\cdot} & \xrightarrow{\lambda} & Cyl(g)^{\cdot} \\ A^n \oplus A^{n+1} \oplus B^n & \longrightarrow & A^n \oplus A^{n+1} \oplus B^n \\ (a_n, a_{n+1}, b_n) & \longmapsto & (a_n, a_{n+1}, b_n + s a_{n+1}) \end{array}$$

It is easy to verify that this really a map of complexes We get the following diagram:

$$\begin{tikzcd} & B^{\cdot} \arrow[d, "\alpha_f"] \\ A^{\cdot} \arrow[ur, "f"] \arrow[r, "\bar{f}"] \arrow[d, equal] & Cyl(f)^{\cdot} \arrow[d, "\lambda"] \\ A^{\cdot} \arrow[r, "\bar{g}"] \arrow[dr, "g"'] & Cyl(g)^{\cdot} \arrow[d, "\beta_g"] \\ & B^{\cdot} \end{tikzcd}$$

The square and the lower triangle are commutative and $\beta_g \circ \lambda \circ \alpha_f = 1_B$

The upper triangle is not commutative but if we go the derived category it will be commutative because we have:

$$\beta_f \circ \alpha_f = 1 \quad \rightsquigarrow \quad \underbrace{Q(\beta_f)}_{\text{isomorphism}} \circ Q(\alpha_f) = 1$$

$$\hookrightarrow Q(\alpha_f) = Q(\beta_f)^{-1}$$

$$\hookrightarrow Q(\alpha_f) \circ Q(\beta_f) = 1$$

$$\beta_f \circ \bar{f} = f \quad \rightsquigarrow \quad Q(f) = Q(\beta_f) \circ Q(\bar{f})$$

$$\hookrightarrow Q(\alpha_f) \circ Q(f) = Q(\alpha_f) \circ Q(\beta_f) \circ Q(\bar{f}) = Q(\bar{f})$$

So we get the commutative diagram in the derived category:

$$\begin{tikzcd} & B^{\cdot} \arrow[d, "Q(\alpha_f)"] \\ A^{\cdot} \arrow[ur, "Q(f)"] \arrow[r, "Q(\bar{f})"] \arrow[d, equal] & Cyl(f)^{\cdot} \arrow[d, "Q(\lambda)"] \\ A^{\cdot} \arrow[r, "Q(\bar{g})"] \arrow[dr, "Q(g)"'] & Cyl(g)^{\cdot} \arrow[d, "Q(\beta_g)"] \\ & B^{\cdot} \end{tikzcd}$$

But we have

$$Q(\beta_g) \circ Q(\lambda) \circ Q(\alpha_f) = 1$$

$$\hookrightarrow \quad Q(f) = Q(g).$$

This lemma allows us to consider some category between $C(\mathcal{A})$ and $D(\mathcal{A})$ the <u>homotopy category</u> $K(\mathcal{A})$ which is defined as

$$\mathrm{Ob}\, K(\mathcal{A}) = \mathrm{Ob}\, C(\mathcal{A})$$

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\cdot, B^\cdot) = \mathrm{Hom}_{C(\mathcal{A})}(A^\cdot, B^\cdot)/\sim$$

and $\sim$ is the equivalence relation given by homotopy.

The composition is given by the composition in $C(\mathcal{A})$ (verify correctness).

We have the following commutative diagram of categories:

$$\begin{tikzcd} C(\mathcal{A}) \arrow[r] \arrow[d, "Q"'] & K(\mathcal{A}) \arrow[dl, dashed, "Q"] \\ D(\mathcal{A}) & \end{tikzcd}$$

If we take the class of quasiisomorphisms in $K(\mathcal{A})$ we have obviously:

$$D(\mathcal{A}) = K(\mathcal{A})[\mathrm{Quasiiso}^{-1}]$$

The advantage of this description is that the class of quasiisomorphisms form a localizing class in $K(\mathcal{A})$ (the proof is not so easy) and hence we obtain from our proposition the following explicit description of the derived category:

$$\mathrm{Ob}\, D(\mathcal{A}) = \mathrm{Ob}\, C(\mathcal{A})$$

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\cdot, B^\cdot) = \left\{ A \xleftarrow{t} Z^\cdot \xrightarrow{f} B \quad t \text{ quasiisomorphism} \right\} / \sim$$

where $A^\cdot \xleftarrow{t} Z^\cdot \xrightarrow{f} B^\cdot \;\sim\; A^\cdot \xleftarrow{t'} Z'^\cdot \xrightarrow{f'} B^\cdot$ if we have

a commutative up to homotopy commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & & Z''' \arrow[dl, "t''"'] \arrow[dr, "f''"] & & \\
 & Z' \arrow[dl, "t"'] \arrow[rrrd, "f"] & & Z'' \arrow[llld, "t'"] \arrow[dr, "f'"] & \\
A^{\cdot} & & & & B^{\cdot}
\end{tikzcd}$$

Last let us consider how to return to our abelian categories $\mathcal{A}$ and $\mathcal{B}$.

We have by universal property:

$$\begin{tikzcd}
C(\mathcal{B}) \arrow[r, "H^i"] \arrow[d, "Q"'] & \mathcal{B} \\
D(\mathcal{B}) \arrow[ur, dashed, "H^i"'] &
\end{tikzcd}$$

Now we can consider the composition:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, dashed] \arrow[d] & \mathcal{B} \\
D(\mathcal{A}) \arrow[r, "\mathbb{R}\mathcal{F}"] & D(\mathcal{B}) \arrow[u, "H^i"']
\end{tikzcd}$$

This is the so called i-th derived functor of $\mathcal{F}$.

ex.: $f_* : \mathcal{A}b(X) \longrightarrow \mathcal{A}b(Y)$, $\Gamma : \mathcal{A}b(X) \longrightarrow \mathcal{A}b$

(or an arbitrary left exact functor) as rich class we take the class of injective sheafs on X. We get:

$\mathbb{R}^i\Gamma$, $\mathbb{R}^i f_* = 0$ for $i<0$

$\mathbb{R}^0\Gamma = \Gamma$, $\mathbb{R}^0 f_* = f_*$

$i>0$: $\mathbb{R}^i f_*$ ... are the so called <u>higher direct images</u>.

$\mathbb{R}^i\Gamma$ .. is the <u>cohomology</u>, this means

$(\mathbb{R}^i\Gamma)(\underbrace{\mathcal{G}}_{\text{a sheaf}}) =: H^i(X, \mathcal{G})$

If we start not with a sheaf $\mathcal{G}$ but with a complex of sheafs we get the so called <u>hypercohomology</u>

$\mathbb{H}^i(X, \mathcal{G}^{\cdot}) = H^i(R\Gamma(\mathcal{G}^{\cdot}))$

Unfortunately there is no time to speak about such interesting things as:

What categorial properties has $\mathcal{D}(\mathcal{A})$?

It is in almost all situations non abelian. This leads to triangulated categories.

Further we have $\mathcal{A} \subset \mathcal{D}(\mathcal{A})$ as complexes in degree zero. Which special properties has this abelian subcategory?

This leads to <u>hearts</u> in triangulated categories. In the triangulated category $\mathcal{D}(\mathcal{A}b(X))$ we have for instance is some situations another abelian subcategory, the so called <u>perverse sheafs</u>, with very interesting properties.

Literature:

Iversen B., "Cohomology of sheaves", Berlin, New York, Heidelberg Springer, 1986

Verdie J.-L., "Categories derivées", etat 0, in SGA 4 1/2 Lecture Notes in Math. v.569 p.262-311

Borel A. (Ed.) "Seminar on Intersection Homology" Boston, Birkhäuser, 1984

Beilinson A.A., Bernstein J., Deligne P. "Faiseaux Pervers" Asterisque, 1982 v 100