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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Arakelov Theory and Applications (I): heights

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These are preliminary lecture notes, intended only for distribution to participants

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1. Introduction

These two talks are planned as a survey on Arakelov geometry and how it has been used so far. They are a sequel to U. Jannsen's talk "Arakelov Theory – the basic ideas".

To describe in general terms Arakelov geometry consider a system of diophantine equations:

$$\left.\begin{aligned} f_1(x_0,\dots,x_N) &= 0 \\ f_2(x_0,\dots,x_N) &= 0 \\ &\cdots \\ f_k(x_0,\dots,x_N) &= 0, \end{aligned}\right\}(*)$$

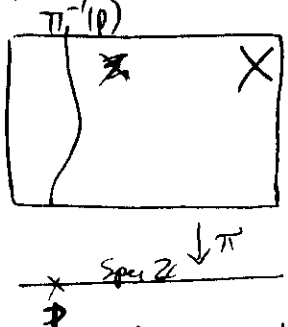
where $f_1,\dots,f_k \in \mathbb{Z}[X_0,\dots,X_N]$ are homogeneous polynomials with integral coefficients and where the unknowns $x_0,\dots,x_N$ are integers. The Grothendieck theory of schemes allows us to think of $(*)$ geometrically as follows. If I is the ideal in $\mathbb{Z}[X_0,\dots,X_N]$ generated by $f_1,\dots,f_k$, we may consider the graded ring $S = \mathbb{Z}[X_0,\dots,X_N]/I$ and the projective scheme

$$X = \mathrm{Proj}(S),$$

whose points are those homogeneous prime ideals in S which do not contain the augmentation ideal. There is a morphism

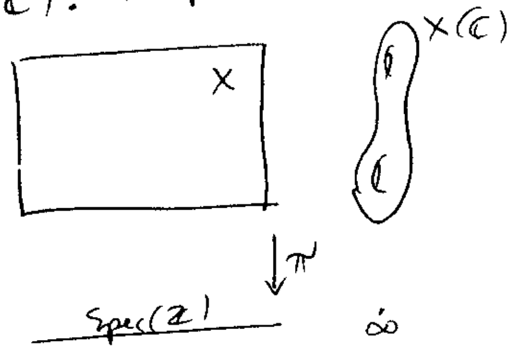
$$\pi: X \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

mapping the ideal $\mathfrak{p}$ to its intersection with $\mathbb{Z}$, and we get the following picture:



where the fiber $\pi^{-1}(p)$ over any prime p in $\mathbb{Z}$ is $\mathrm{Proj}(S/pS)$, i.e. the reduction modulo p of X, which would naturally come in when studying (*) modulo p.

We conclude from this that schemes is a geometric way to consider congruences imposed on solutions of diophantine equations. The basic remark leading to Arakelov geometric point of view is that looking at congruences between solutions of (*) might not be enough. In addition, one would like to have estimates on the size of the complex solutions of this system. And for this one plans to combine the algebraic theory of schemes with the complex analytic hermitian geometry of the complex variety $X(\mathbb{C})$. The picture then becomes the following:



where ∞ is the place at infinity on Spec($\mathbb{Z}$), corresponding to the archimedean absolute value. In other words the noncomplete scheme X is "compactified" at infinity by $X(\mathbb{C})$.

In this new framework many (all?) notions from algebraic geometry are supposed to find analogs. The basic ones are given in the following table:

Algebraic geometry	Arakelov geometry
X = projective variety over an algebraically closed field k (of char. 0).	X = (regular) projective flat scheme over $\mathbb{Z}$
E = algebraic vector bundle on X, i.e. a coherent locally free $\mathcal{O}_X$-module	(E, h) = hermitian vector bundle, i.e. E is an algebraic vector bundle on X, h is a smooth hermitian metric on the holomorphic vector bundle $E_{\mathbb{C}}$ induced by E on $X(\mathbb{C})$.
Z = algebraic cycle of codimension p on X, i.e. Z is a formal sum $\sum_\alpha m_\alpha Y_\alpha$, where $m_\alpha \in \mathbb{Z}$ and $Y_\alpha \subset X$ is a closed irreducible subvariety of codimension p.	(Z, g) = arithmetic cycle on X, i.e. $Z = \sum_\alpha m_\alpha Y_\alpha$ is an algebraic cycle on X and g is a Green current for Z on $X(\mathbb{C})$

To explain the notion of Green current, assume that all components of $X(\mathbb{C})$ have the same dimension d, and let $D^{pq}(X(\mathbb{C}))$ be the space of currents of type (p,q) on $X(\mathbb{C})$, i.e. differential forms of type (p,q) with distribution coefficients. If $(z_1, \dots, z_d)$ are local holomorphic coordinates for $X(\mathbb{C})$, any $S \in D^{pq}(X(\mathbb{C}))$ will be written locally as a sum of expression like $f\, dz_1 \dots dz_p\, d\bar{z}_{p+1} \dots d\bar{z}_{p+q}$, where f is a distribution on $\mathbb{C}^d$. In particular, when f is smooth, we recover ~~[illegible]~~ usual differential forms of type (p,q), which are ~~also~~ "smooth currents", a subspace denoted $A^{pq}(X(\mathbb{C})) \subset D^{pq}(X(\mathbb{C}))$.

By Poincaré duality, $D^{pq}(X(\mathbb{C}))$ can also be viewed as the topological dual of $A^{d-p,d-q}(X(\mathbb{C}))$.

Now, given a cycle Z on X, we may attach to it a current of integration $\delta_Z \in D^{pp}(X(\mathbb{C}))$ defined as follows: if η is a form of type $(d-p, d-p)$ and $Z = \sum_\alpha n_\alpha Y_\alpha$,

$$\delta_Z(\eta) = \sum_\alpha n_\alpha \int_{\widetilde{Y_\alpha(\mathbb{C})}} \eta \,,$$

where $\widetilde{Y_\alpha(\mathbb{C})}$ is a desingularization of the set of complex points of Y_α.

Since distributions admit (by duality) derivatives, we have an operator

$$dd^c = \frac{1}{2\pi i}\bar\partial\partial : D^{p-1,p-1}(X(\mathbb{C})) \to D^{p,p}(X(\mathbb{C})).$$

The definition of a Green current is then the following:

<u>Definition 1</u>: Given an algebraic cycle Z of codimension p on X, a Green current for Z is an element $g \in D^{p-1,p-1}(X(\mathbb{C}))$ such that g is real, $F_\infty^*(g) = (-1)^{p-1} g$ and

$$\boxed{dd^c g + \delta_Z = \omega}$$

is a <u>smooth</u> form of type (p,p).

Here $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the involution obtained by conjugation of the coordinates of complex points in X.

2. The height of a point:

2.1. Rather than developping the properties of the notions introduced so far, our goal will now be to explain how they are related to a basic classical notions in diophantine equations, the notion of height.

Given a point $P \in \mathbb{P}^N(\mathbb{Q})$ we define as follows a real number $h_n(P)$ called its "naive height" (or "Weil height"). Choose homogeneous coordinates $(x_0, \dots, x_N) \in \mathbb{Z}^{N+1}$ for P in such a way that $g.c.d.(x_0, \dots, x_N) = 1$. Then

$$h_n(P) = \log \max_{0 \leq i \leq N} (|x_i|) .$$

Since $(x_0, \dots, x_N)$ is unique up to sign, this number is well-defined. Its main property is that for ~~all~~ any real number $T > 0$ the set of points $P \in \mathbb{P}^N(\mathbb{Q})$ such that $h(P) \leq T$ is finite.

The idea is now to give a "geometric interpretation" of the height with the hope of proving finiteness results about diophantine equations. Examples of such statements are the following:

Mordell's conjecture: Any curve X of genus ≥ 2 over a number field K has only finitely many points (in K).

Shafarevich conjecture: The set of abelian varieties of fixed dimension over K, having good reduction outside a fixed set of places of K and equipped with a polarization of fixed degree, is finite.

Asymptotic Fermat. The set of integers $(x, y, z, n) \in \mathbb{N}^4$ such that $g.c.d.(x,y,z) = 1$ and $x^n + y^n = \pm z^n$ is finite.

In 1983, Faltings proved Shafarevich conjecture, from which Mordell follows. Vojta gave another proof of Mordell in 1989. In both works, Arakelov geometric notions have been used as one of the tools (among many). Asymptotic Fermat is weaker than Fermat's last theorem, but still unproved; it is known that a Miyaoka-Yau inequality for arithmetic surfaces would imply it. Notice that other approaches to Fermat last theorem have been proposed in recent years, one of them being via the Taniyama-Weil conjecture, which would imply it completely, not just asymptotically (Ribet's theorem). Also Bombieri gave a proof of Mordell's conjecture which does not use Arakelov theory.

2.2. Let's recall the notion of <u>degree</u> of a line bundle on a curve. Let X be a smooth projective curve over $k = \bar{k}$ and L a line bundle on X. One defines

$$\deg(L) = \sum_{x \in X(k)} v_x(s) \quad \in \mathbb{Z}$$

where s is any section of L on a nonempty open subset of X and

$$v_x(s) = \begin{pmatrix} \text{order of zero} \\ \text{or} \\ -\text{order of pole} \end{pmatrix} \text{ of } s \text{ at the point } x.$$

This integer does not depend on s. When $k = \mathbb{C}$ one has also

$$\deg(L) = \int_{X(\mathbb{C})} c_1(L),$$

where $c_1(L)$ is the first Chern form of L.

2.3. Let $X = \mathrm{Spec}(\mathbb{Z})$ and let (L, h) be an hermitian line bundle on X. Then

L = rank one vector bundle on X = free rank one $\mathbb{Z}$-module,
= projective rank one $\mathbb{Z}$-module

and h = hermitian metric on $L_{\mathbb{C}}$ = hermitian scalar product of $L \otimes_{\mathbb{Z}} \mathbb{C}$
$= \|\cdot\|$, a norm on $L \otimes_{\mathbb{Z}} \mathbb{C}$.

Let's define the <u>arithmetic degree</u> of (L, h) to be the real number

$$\widehat{\deg}(L, h) = -\log \|s_0\|,$$

where s_0 is a basis of L on $\mathbb{Z}$ $(L \simeq \mathbb{Z}s_0)$.

It is well defined (the other basis is $-s_0$) and can also be written

$$\widehat{\deg}(L, h) = \log \# \left(\frac{L}{\mathbb{Z}s}\right) - \log\|s\|$$

$$= \sum_{\substack{\text{prime } p \\ \text{integer}}} v_p(s) \log(p) - \log\|s\|,$$

where s is any non zero element in L, $\#(L/\mathbb{Z}s)$ is the cardinality of the finite group $L/\mathbb{Z}s$, and $v_p(s)$ is the exact power of the prime p dividing this number. If we define $v_\infty(s)$ to be $-\log\|s\|$, we see that the last formula above gets similar to the definition given in 2.2, with the complete curve being replaced by $\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$.

2.4. Let us come back to the situation of 2.1. Given $P \in \mathbb{P}^N(\mathbb{Q})$, we define an hermitian line bundle (L, h) as follows.

L consists of all choices of integral homogeneous coordinates of P and 0:

$$L = \{(x_0, \dots, x_N) \in \mathbb{Z}^{N+1},\ \text{hom. coord. of } P\} \cup \{(0, \dots, 0)\}.$$

On $L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \subset \mathbb{C}^{N+1}$ choose h to be the restriction of the standard hermitian scalar product of $\mathbb{C}^{N+1}$. Then define

$$h(P) = -\widehat{\deg}(L, h).$$

We obtain from definitions

$$h(P) = \log \sqrt{x_0^2 + \dots + x_N^2}, \quad (x_0, \dots, x_N) \in \mathbb{Z}^{N+1} \text{ coordinates of } P,\ \mathrm{g.c.d.}(x_0, \dots, x_N) = 1.$$

Thus $h(P)$ is (essentially) the naive height of P:

$$h_n(P) \le h(P) \le h_n(P) + \frac{1}{2}\log(N+1).$$

In other words, the height can be viewed as an arithmetic analogy of the degree:

$$\boxed{\text{height} = \text{arithmetic degree.}}$$

2.5. Let K be a number field, R its ring of integers, $X = \mathrm{Spec}(R)$, and (L, h) an hermitian line bundle on X. Then L is a projective R-module of rank one and, for any complex embedding $\sigma : K \to \mathbb{C}$, h defines a norm $\|\cdot\|_\sigma$ on the complex line $L \otimes_R \mathbb{C}$. The definition of 2.4. extends to that case:

$$\widehat{\deg}(L, h) = \log \# (L/Rs) - \sum_{\sigma : K \to \mathbb{C}} \log \|s\|_\sigma ,$$

for any $s \in L - \{0\}$.

To prove Shafarevich conjecture, Faltings introduced a new notion of heights for abelian varieties. Given a (semi) abelian scheme A over $\mathrm{Spec}(R)$, with generic fiber proper over K, let ω_A be the restriction to the zero section of (algebraic) differential forms on A of degree the dimension of A over R, equipped with the L^2-scalar product h_{L^2}. Faltings height is

$$h_{\text{Faltings}}(A) = \widehat{\deg}(\omega_A, h_{L^2}).$$

3. The height of projective varieties:

3.1. Degree of projective varieties:

Let $X \subset \mathbb{P}^N_k$, $k = \bar{k}$, char $k = 0$, be a projective variety. Its degree is the integer

$$\deg(X) = \#(\Lambda \cap X),$$

where $\Lambda \subset \mathbb{P}^N_k$ is a general ~~linear~~ projective subspace of codimension $d = \dim(X)$ (points in the finite set $\Lambda \cap X$ are counted with multiplicities).

For example, if X is an hypersurface of equation $f(x_0, \dots, x_N) = 0$, $\deg(X)$ is just the degree of f. When $k = \mathbb{C}$ we also have

$$\deg(X) = \int_{X(\mathbb{C})} c_1(\mathcal{O}(1)_X)^d$$

where $c_1(\mathcal{O}(1)_X)$ is ~~the restriction to X~~ the first Chern form of the restriction to X of the canonical line bundle on $\mathbb{P}^N_k$.

The Bézout theorem states that if $X \subset \mathbb{P}^N_k$ and $Y \subset \mathbb{P}^N_k$ meet properly (i.e. all the components of $X \cap Y$ have the ~~expected~~ smallest dimension ~~possible~~),

$$\deg(X \cap Y) = \deg(X) \deg(Y)$$

(where deg has been extended by linearity to cycles, in order to make sense on $X \cap Y$).

For example, in $\mathbb{P}^2$, two conics meet at four points.

3.2. The intersection theory of arithmetic cycles can be used to define the height of projective varieties over $\mathbb{Z}$ in a way which is very similar to the definition of degree in 3.1..

Let X be a regular flat projective scheme over $\mathbb{Z}$. Two arithmetic cycles (Z,g) and (Z',g') are said to be linear equivalent, when their difference is a sum of cycles of the form $(\operatorname{div}(f), -\log|f|^2 + \partial u + \bar\partial v)$, where u and v are currents of type $(p-1,p)$ and $(p,p-1)$ respectively, f is a nonzero rational function on an irreducible subvariety $Y \subset X$ of codimension $p-1$, and $\operatorname{div}(f)$ its divisor (a cycle of codimension p on X) and $-\log|f|^2$ is the current obtained by mapping any form η to the integral

$$(-\log|f|^2)(\eta) = -\int_{\widetilde{Y(\mathbb{C})}} (\log|f|^2)\,\eta$$

where $\widetilde{Y(\mathbb{C})}$ is a desingularization of $Y(\mathbb{C})$.

The arithmetic Chow group $\widehat{CH}^p(X)$ is the quotient of the group of arithmetic cycles by linear equivalence.

One can define an intersection product

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \longrightarrow \widehat{CH}^{p+q}(X) \otimes \mathbb{Z}^{\mathbb{Q}}.$$

It maps the classes $x \in \widehat{CH}^p(X)$ and $y \in \widehat{CH}^q(X)$

to the element $x \cdot y$ defined as follows. Represent x by (Z, g) and y by (Z', g') in such a way that Z and Z' meet properly on the generic fiber $X_{\mathbb{Q}} = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q})$, and the green current g is smooth outside the support of $Z(\mathbb{C})$, with logarithmic growth along $Z(\mathbb{C})$. Then Fulton's technic on K-theory can be used to define a cycle $Z \cap Z'$ of codimension $p+q$. This cycle is supported on the intersection $|Z| \cap |Z'|$ of the supports of Z and Z'. It is well defined up to linear equivalence on $|Z| \cap |Z'|$ and up to torsion. A green current for $Z \cap Z'$ is

$$g * g' = g\, \delta_{Z'} + \omega\, g',$$

where ω is the form $dd^c g + \delta_Z$, and the product of currents $g \delta_{Z'}$ is defined as the current mapping a form η to the integral of $g\eta$ on $Z'(\mathbb{C}) - Z(\mathbb{C})$ (one has to show that this integral makes sense because of the growth condition on g). The product xy is the class of $(Z \cap Z', g * g')$.

Let d be the (Krull) dimension of the scheme X. If $\pi : X \to \mathrm{Spec}(\mathbb{Z})$ is the projection and (Z, g) is a cycle of codimension d on X, we can define as follows a real number

$$\pi_*(Z, g) \in \mathbb{R}.$$

Since $Z = \sum_\alpha m_\alpha Y_\alpha$ has dimension 0 the rings of functions $\Gamma(Y_\alpha, \mathcal{O}_{Y_\alpha})$ are finite and g is of type $(d-1, d-1)$, where $d-1 = \dim X(\mathbb{C})$ (since $\mathrm{Spec}(\mathbb{Z})$ has dimension one). So we may consider

$$\pi_*(Z, g) = \sum_\alpha m_\alpha \log \# \Gamma(Y_\alpha, \mathcal{O}_{Y_\alpha}) + \frac{1}{2}\int_{X(\mathbb{C})} g .$$

This real number is invariant under linear equivalence. The Arithmetic intersection numbers are obtained from the pairing

$$\begin{aligned} \widehat{CH}^p(X) \otimes \widehat{CH}^{d-p}(X) &\longrightarrow \mathbb{R} \\ x \otimes y &\longmapsto \pi_*(xy) . \end{aligned}$$

3.3. Let $X \subset \mathbb{P}^N_{\mathbb{Z}}$ be a projective variety of Krull dimension d, and $\Lambda \subset \mathbb{P}^N_{\mathbb{Z}}$ a projective subspace of codimension d such that X and Λ meet properly on $\mathbb{P}^N_{\mathbb{Q}}$. A green current g_X (resp. g_Λ) for X (resp. Λ) can be chosen as follows. If $\underset{c_1(\mathcal{O}(1))}{\mu}$ is the standard Fubini-Study (1,1) form on $\mathbb{P}^N(\mathbb{C})$, g_X is chosen so that $dd^c g_X + \delta_X$ is a multiple of μ^d, and

$$\int_{\mathbb{P}^N(\mathbb{C})} g_X \, \mu^{N+1-d} = 0 .$$

These two conditions fix g_X up to the addition of a term of the form $\partial u + \bar\partial v$.

On the other hand, we may choose coordinates $x_0, \dots, x_N$ on $\mathbb{P}^N(\mathbb{C})$ in such a way that $\Lambda(\mathbb{C})$ has equation $x_0 = \dots = x_{d-1} = 0$. We then consider the functions

$\tau = \log(|x_0|^2 + \dots + |x_N|^2)$,

$\sigma = \log(|x_0|^2 + \dots + |x_{d-1}|^2)$,

and the 1-1 forms $\alpha = dd^c\tau$, $\beta = dd^c\sigma$ on $\mathbb{P}^N(\mathbb{C}) - \Lambda(\mathbb{C})$. H. Levine showed (in 1960) that the form

$$(\tau - \sigma)\left(\sum_{i=0}^{d-1} \alpha^i \wedge \beta^{d-1-i}\right)$$

on this open set is integrable on $\mathbb{P}^N(\mathbb{C})$, and defines a Green current g_Λ for Λ.

We may now define

$$h(X) = \pi_*\left((X, g_X) \cdot (\Lambda, g_\Lambda)\right) \in \mathbb{R}$$

to be the height of X.

This definition is a variant of a definition introduced by Faltings in his work on the Lang's conjecture for abelian varieties, which extended Vojta's work on curves. Bost, Gillet and myself proved that $h(X) \geqslant 0$ and we obtained an arithmetic Bézout theorem of the form

$$h(X \cap Y) \leq h(X)\deg(Y_{\mathbb{Q}}) + \deg(X_{\mathbb{Q}})h(Y) + c\,\deg(X_{\mathbb{Q}})\deg(Y_{\mathbb{Q}}),$$

where $\deg(X_{\mathbb{Q}})$ is the degree on $\mathbb{P}^N_{\mathbb{Q}}$ of $X_{\mathbb{Q}}$ (in the sense of 3.1), and c is a simple constant (which we would like to replace by zero). Notice that statements of this kind for hypersurfaces were already known to Nesterenko and Philippon, who defined $h(X)$ to be the naive height of the Chow point attached to X.

When X has codimension one, we choose its equation to be $f(x_0, .., x_N) = 0$, where f has integral coefficients with g.c.d. equal to one. Then

$$h(X) = \int_{S^{2N-1}} \log|f| \, d\sigma,$$

where S^{2N-1} is the sphere of equation $|x_0|^2 + .. + |x_N|^2 = 1$, and $d\sigma$ is the invariant measure of volume one on S^{2N-1}.

It would be interesting to have more cases where this kind of integrals is computed. For instance the integral

$$\int_{|x|^2+|y|^2+\dots+|v|^2=1} \log|xy + zt + uv| \, d\sigma$$

is the height of a quadric in $\mathbb{P}^5$, hence knowing its value would be progress in the "arithmetic Schubert calculus" on the grassmannian of 2-planes in dimension 4. To it a rational number? Are all the Chern numbers on Grassmannians rational numbers (this is the case for projective spaces)?

Reference:

→ Intersection theory as in 3.2. in joint work with H. Gillet.
A description of it can also be found in
"Lectures on Arakelov geometry", Soulé, Abramovich, Burnol, Kramer, Cambridge studies in advanced mathematics 33.

→ Bézout theorem in 3.3. can be found in
"Un analogue arithmétique du théorème de Bézout", Note CRAS Paris 312 (1991) 845-848.
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→ Grassmannians are studied in
"Eigenwerte des Laplace-Operators für Graßmann-Varietäten", C. Wirsching, (1992), preprint, München Universität.

→ Arakelov papers should probably be read first:
"An intersection theory for divisors on an arithmetic surface", Math. USSR Izv. 8 (1974), 1167-1180,
and
"Theory of intersections on the arithmetic surface" Proceedings International Congress of Math., Vancouver (1974), 405-408.