INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.637/24

ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

(31 August - 11 September 1992)

Fibred Surfaces

F. Catanese
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2
56127 Pisa
Italy

These are preliminary lecture notes, intended only for distribution to participants

LECTURES BY FABRIZIO CATANESE

7.1. The Albanese variety and the Albanese map.

In this lecture we will introduce an important tool for the classification of surfaces or, more generally, of complex manifolds, namely the study of a variety with the help of its Albanese map. Even if we only apply these methods to the case of complex projective surfaces we prefer to develop the results in more generality (cf. [Ue],...).

In the following, for the sake of brevity, assume X to be a compact Kähler manifold.
We recall that a complex manifold is called a Kähler manifold if and only if there exists a hermitian metric on X whose associated (1,1)-form is closed, i.e., if in local coordinates the metric h is given by $\Sigma_{i,j=1,\dots,\dim X} h_{i\bar{j}}(z)dz_i \otimes d\bar{z}_j$, then the associated (1,1)-form ξ is given by $\Sigma\, h_{i\bar{j}}(z)dz_i \wedge d\bar{z}_j$, and ξ is called closed iff $d\xi=0$. Such an hermitian metric on X is called a Kähler metric on X.

(7.1) Remark. A submanifold Y of a Kähler manifold X is again a Kähler manifold.

This is clear, because the restriction of a Kähler metric of X to Y gives a hermitian metric on Y, whose associated (1,1)-form is closed.

(7.2) Example. 1) A smooth projective variety X is a Kähler manifold.

2) A complex torus $T=\mathbb{C}^n/\Gamma$, where Γ is a lattice (i.e. a discrete subgroup of maximal rank) in $\mathbb{R}^{2n}$, is a Kähler manifold.

Proof. 1) We set $\xi := (\partial\bar{\partial}\log\|Z\|^2)/2\pi i$ on $\mathbb{P}^r$, where Z is a homogeneous coordinate vector. ξ is well defined (since if one replaces Z by fZ with f holomorphic $\neq 0$, then $\partial\bar{\partial}\log|f|^2 = 0$) and is the (1,1)-form associated to the Fubini-Study metric on $\mathbb{P}^r$ (cf. []). Moreover, remembering the relations $d = \partial + \bar{\partial}$, $\partial^2 = \bar{\partial}^2$, $\partial\bar{\partial} = -\bar{\partial}\partial$, we see that ξ is closed. Therefore $\mathbb{P}^r$ is Kähler and by (7.1) also a projective manifold is Kähler.

2) The (1,1)-form $\xi = (\Sigma\, dz_i \wedge d\bar{z}_j)/2\pi i$, coming from the standard Euclidean metric on $\mathbb{C}^n$, is obviously closed on T. Therefore T is Kähler. Q.E.D.

In general a complex torus of dimension ≥ 2 is not a projective variety. But as the following result shows there are several equivalent conditions which describe when a torus is algebraic.

(7.3) Theorem. Let T be a complex torus of dimension n. Then the following statements are equivalent:

1) The transcendence degree of $\mathbb{C}(T)$ over the complex numbers is n.

2) T is projective.

3) There exists a meromorphic function $f \in \mathbb{C}(T)$ without periods , i.e. $\Gamma_f := \{t \in T : f(x+t) = f(x) \text{ for all } x\} = \{0\}$. (Notice that Γ_f is always a closed subgroup of T).

4) There exists a positive definite hermitian form H on $\mathbb{C}^n$ such that $\mathrm{im}H|_{\Gamma\times\Gamma}$ takes integral values (Riemann conditions).

For a proof of this result we refer for example to [Mumford].

(7.4) Remark. 1) By a result of L. Siegel it holds for any compact complex manifold X:

$\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}(X) \leq \dim X$.

2) We recall that the imaginary part of a hermitian form is alternating and the real part is symmetric.

3) We want to point out that the equivalence of the conditions 1) and 2) in the theorem holds more generally for Kähler manifolds (cf. [Moisezon])

(7.5) Definition. A complex torus with one (or all) of the properties 1)-4) of (7.4) is called an abelian variety.

Therefore by definition an abelian variety admits an embedding into projective space.

If X is a Kähler manifold, then by Hodge theory (cf. [Griffiths-Harris]), holds a fact we already proved for projective surfaces (cf. (6.20), (6.24)):

(7.6) Remark. 1) $H^1_{DR}(X,\mathbb{C}) = H^0(X,\Omega^1_X) \oplus H^0(X,\Omega^1_X)^-$.

2) In particular, the complex vector space $H^0(X,\Omega^1_X)$ is isomorphic as a real vector space to $H^1_{DR}(X,\mathbb{R})$ by the map $\eta \to (\eta+\bar{\eta})/2$.

3) We recall that $H_1(X,\mathbb{R})$ is isomorphic to $H_1(X,\mathbb{Z})\otimes\mathbb{R}$. So, if j: $H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{R})$ is given by $c \to c\otimes 1$, then $H_1(X,\mathbb{R})/j(H_1(X,\mathbb{Z}))$ is isomorphic to $(\mathbb{R}/\mathbb{Z})^{2q}$ as a differentiable manifold.

$H_1(X,\mathbb{Z})$ is a finitely generated abelian group, hence consists of a free subgroup and a torsion subgroup; the last one is killed by the map j (j is in general not a monomorphism !).

4) The real vector spaces $H^1_{DR}(X,\mathbb{R})$ and $H_1(X,\mathbb{R})$ are naturally dual and the duality is given by integration, i.e. if $c\in H_1(X,\mathbb{R})$ and $\varphi\in H^1_{DR}(X,\mathbb{R})$ then $\langle\varphi,c\rangle = \int_c \varphi$.

In the sequel we denote the $\mathbb{R}$-dual of a (real) vector space W by W^*, the $\mathbb{C}$-dual of a complex vector space V by $V^\vee$.

(7.7) Remark. Let V be a complex vector space. Then $V^\vee := \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ is naturally $\mathbb{R}$-isomorphic to V^* $(=\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}))$ by the ($\mathbb{R}$-linear) map $\psi \longmapsto \mathrm{Re}(\psi)$.

As a consequence of (7.7) we get (using (7.6)):

$$H_1(X,\mathbb{C}) \cong H^0(X,\Omega^1{}_X)^\vee \oplus (H^0(X,\Omega^1{}_X)^-)^\vee.$$

In this way we obtain:

$$j(H_1(X,\mathbb{Z})) \subset H_1(X,\mathbb{R}) \cong H^0(X,\Omega^1{}_X)^\vee,$$

where the last equality follows from (7.6), 2) by (7.7), and we get a q-dimensional complex torus

$$A := \mathrm{Alb}(X) \cong H^0(X,\Omega^1{}_X)^\vee / j(H_1(X,\mathbb{Z})),$$

the <u>Albanese variety of X</u>.

If we fix a base point $x_0 \in X$, then $\alpha(x) := \int_{x_0}^{x}$ defines a map

$$\alpha : X \to H^0(X,\Omega^1{}_X)^\vee / j(H_1(X,\mathbb{Z})) = \mathrm{Alb}(X),$$

since $\int_{x_0}^{x}$ is only well defined modulo $\int_c$, $c \in H_1(X,\mathbb{Z})$.

α is called the <u>Albanese map of X</u>.

<u>(7.8) Remark.</u> The Albanese map is defined up to translations, i.e. changing the base point $x_0 \in X$, α changes by a translation on A.

<u>(7.9) Proposition</u> (universal property of the Albanese variety). Let $f: X \to T$ be a holomorphic map from a compact Kähler manifold X to a complex torus T. Then there exists a unique affine homomorphism $\beta: \mathrm{Alb}(X) \to T$ (i.e., β is a homomorphism for a suitable choice of the origin in T) and a unique factorization of f through Alb(X), i.e., the following diagram commutes

f

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\alpha"'] & & T \\ & \mathrm{Alb}(X) \arrow[ur, "\beta"'] & \end{tikzcd} .$$

<u>Proof.</u> Let $T=\mathbb{C}^n/\Gamma$ be a complex torus and $f:X \to T$ be a holomorphic map. Since $\Omega^1{}_T \cong (\mathcal{O}_T)^{\dim T}$, we have:

$$T \cong H^0(T,\Omega^1{}_T)^\vee / H_1(T,\mathbb{Z}).$$

Let β be the affine homomorphism provided by the linear map $(f^*)^\vee$ (where $f^*:H^0(T,\Omega^1{}_T) \to H^0(X,\Omega^1{}_X)$), (noting that $(f^*)^\vee = f_*$ on $H_1(X,\mathbb{Z})$ and has image in $H_1(T,\mathbb{Z})$).
We show that f factors through α.
Given a base point x_0 we can assume to have chosen the origin in T so that $f(x_0)=0\in T$.

Then f and $\beta\circ\alpha$ coincide at x_0. By definition $\beta^*=f^*$, hence $(f-\beta\circ\alpha)^* = f^*-\alpha^*\circ f^*$. Moreover, $\int_{x_0}^{x} \eta$, with $\eta\in H^0(\Omega^1{}_X)$, is a multivalued function such that $d(\int_{x_0}^{x}\eta)=\eta(x)$, because η is closed, i.e., $d\eta=0$, and therefore $\alpha^*(\eta)=\eta$ for all $\eta\in H^0(\Omega^1{}_X)$. Hence $(f-\beta\circ\alpha)^*(\omega)=0$ for all $\omega\in H^0(\Omega^1{}_T)$ and $f-\beta\circ\alpha$ is constant. This implies that f and $\beta\circ\alpha$ are equal.

Conversely, if β exists, then necessarily $\beta^*=f^*$ by the above argument, hence β is unique. Q.E.D.

<u>(7.10) Proposition.</u> Let X be a compact (connected) Kähler manifold and $\alpha:X \to A$ the Albanese map. Then $\alpha(X)$ generates $A=\mathrm{Alb}(X)$, i.e. there exists a natural number m such that the map

$$u_m: X\times\ldots\times X = X^m \longrightarrow A,$$

given by $(x_1,\ldots,x_m) \longmapsto \alpha(x_1)+\ldots+\alpha(x_m)$ is surjective.

Proof. Since $\alpha(x_0)=0$, we have $\alpha(X) \subset \mathrm{im}(u_2) \subset \mathrm{im}(u_3) \subset \ldots$.

Since X is compact, also X^m is compact and therefore $Y_m :=$ $\mathrm{im}(u_m)$ is a closed subvariety of A by Remmert's proper mapping theorem (cf. [Re]). Since Y_m is irreducible, the above sequence of inclusions stabilizes, i.e. there exists a m such that $Y_m = Y_{m+1} = \ldots =: Y \subset \mathrm{Alb}(X)$.

CLAIM: Y is a subtorus of A.
We assume for the time being the validity of the above claim.

Then, by definition $\alpha : X \to Y \subset A$ and therefore $H^0(\Omega^1{}_X) =$ $\alpha^*(H^0(\Omega^1{}_A)) \subset \alpha^*(H^0(\Omega^1{}_Y))$. But $\dim H^0(\Omega^1{}_A) = q = \dim H^0(\Omega^1{}_X)$ $\leq \dim H^0(\Omega^1{}_Y) = \dim Y$, hence $Y=A$. Q.E.D.

It remains to prove that Y is a subtorus of Alb(X).

Proof of the claim. Let $\pi : \mathbb{C}^q \to A$ be the universal cover of A. Since Y is obviously a semigroup (note that $0 \in Y$), also $\tilde{Y} :=$ $\pi^{-1}(Y)$ is so. We take the irreducible component of $\tilde{Y}$ containing the origin of $\mathbb{C}^q$ (we want to point out that $\tilde{Y}$ is a priori not necessarily irreducible), which for brevity we will again call $\tilde{Y}$. For all m it holds $mY \subset Y$ and since they are both irreducible of the same dimension, we have in fact $mY=Y$, and therefore also $m\tilde{Y}=\tilde{Y}$. Therefore if $\xi \in \tilde{Y}$, also $\xi/m \in \tilde{Y}$ and we have shown that $(\mathbb{Q}^+)\cdot\tilde{Y} \subset \tilde{Y}$. So every holomorphic function vanishing on $\tilde{Y}$, vanishes on $\mathbb{Q}^+\xi$ and therefore vanishes on $\mathbb{C}\xi$. This shows that $\tilde{Y}$ is a complex vector subspace of $\mathbb{C}^q$ and therefore $\pi(\tilde{Y})=Y$ is a subtorus of A. Q.E.D.

From the construction we gave for the Albanese variety it is a priori not clear that Alb(X) for a projective manifold X is an abelian variety. But from (7.10) we obtain immediately this property.

<u>(7.11) Corollary.</u> Let X be a projective manifold. Then the Albanese variety of X, Alb(X), is an abelian variety.

<u>Proof.</u> By (7.10) There exists a natural number such that $u_m:X^m \to A$ is surjective, hence there exists a closed subvariety $Z\subset X^m$ of dimension q(X) (=dimA) such that $u_m:Z \to A$ is surjective and generically finite; (this can be seen easily by inductively taking generic hyperplane sections of X^m, $X^m\cap H,\ldots$). Therefore $\mathbb{C}(Z)$ is a finite extension of $\mathbb{C}(A)$, hence $\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}(A)=\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}(Z)=q$ (since Z is projective). This implies by (7.3) that A is projective. Q.E.D.

In the following lectures we will see how important for the classification of projective surfaces it is to study the Albanese variety and Albanese map. In this lecture we will only give one application of the methods introduced so far.

<u>(7.12) Theorem.</u> Let X be a smooth projective variety and assume that the image of the Albanese map $\alpha(X)=:Y\subset A$ is a curve. Then Y is smooth and α has connected fibres.

Before proving this result we want to recall a classical result, which says essentially that we can factor each morphism between projective varieties as the composition of a morphism with connected fibres followed by a finite morphism.

<u>(7.13) Theorem</u> (Stein factorization). Let $f:X \to Y$ be a morphism of projective varieties (or more generally of reduced compact complex spaces). Then there exists a complete algebraic variety Z (resp. a reduced compact complex space), a finite morphism $h:Z \to Y$ and a morphism $g:X \to Z$ with connected fibres such that $f=h\circ g$.

<u>Proof.</u> We want only to give an idea of the proof, for more details we refer to [Hartshorne].

The main point is that the sheaf $f_*\mathcal{O}_X$ (given by $(f_*\mathcal{O}_X)_y = \lim_{y\in V}\mathcal{O}_X(\varphi^{-1}(V)))$ is a coherent sheaf of $\mathcal{O}_Y$-algebras on Y (cf []). Then we define $Z := \mathrm{Spec}(f_*\mathcal{O}_X)$, which is obtained by glueing affine schemes (over Y) given as follows: for each open affine subset $U\subset Y$ $f_*\mathcal{O}_X(U)$ is a finite integral extension of $\mathcal{O}_Y(U)$, therefore $f_*\mathcal{O}_X(U) \cong \mathcal{O}_Y(U)[z_1,\dots,z_r]/I$. Then $Z_U \subset U\times\mathbb{C}^r$ is defined by the ideal I. The maps g and h are then naturally given and fulfill the desired properties. Q.E.D.

With the help of this reult we are now able to prove (7.12).

<u>Proof of (7.12).</u> Let $\pi: C \to Y$ be the normalization of Y.

CLAIM: There exists a map $f: X \to C$ such that the following diagram is commutative:

$$\begin{tikzcd} \alpha: X \arrow[r] \arrow[dr, "f"'] & \alpha(X)=Y \subset A \\ & C . \arrow[u, "\pi"'] \end{tikzcd}$$

<u>Proof of the claim.</u> The surjective map $\alpha: C \to Y$ induces on each affine open set $U\subset Y$ an inclusion

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(\alpha^{-1}(U)).$$

Since X is smooth, $\mathcal{O}_X(\alpha^{-1}(U))$ is integrally closed in its field of fractions. Therefore we obtain an inclusion

$$\mathcal{O}_C(\pi^{-1}(U)) \to \mathcal{O}_X(\alpha^{-1}(U)),$$

(because $\mathcal{O}_C(\pi^{-1}(U))$ is the integral closure of $\mathcal{O}_Y(U)$ in its field of fractions) which gives rise to a map $f: X \to C$. Q.E.D.

If f has not yet connected fibres we consider the Stein factorization

$$\begin{tikzcd} f: X \arrow[rr] \arrow[dr, "g"'] & & C \\ & C', \arrow[ur, "h"'] & \end{tikzcd}$$

$f = h \circ g$. Since C' is a smooth curve, we can consider the the Albanese map (also called the Jacobi map) $C' \rightarrow J(C') =: A'$ of C', (we recall that dim $A' = \text{genus}(C)$).
Now we have the following commutative diagram:

$$\begin{tikzcd} \alpha: X \arrow[r] \arrow[dr, "f"'] \arrow[dd] & Y \subset A \\ & C \arrow[u, "\pi"'] \\ C' \arrow[r, "u"'] & A'. \end{tikzcd}$$

Applying the universal property of the Albanese map to $\pi \circ h$ and $\alpha' := u \circ g$ we obtain maps $\varphi: A \rightarrow A'$ and $\psi: A' \rightarrow A$, such that $\alpha = \psi \circ \alpha'$ and $\alpha' = \varphi \circ \alpha$. Then $\varphi \circ \psi$ resp. $\psi \circ \varphi$ is the identity on $\text{im}(\alpha')$ respectively on $\text{im}(\alpha)$, hence everywhere (cf. (7.10)).
If C' has genus zero, then $A' = 0$, hence $\alpha(X)$ is a point which is a contradiction. Therefore the genus of C' has to be bigger or equal to one, which implies that C' is embedded in its Jacobian and so by the above we obtain that $C' \cong \alpha(X) = Y$. This proves the assertion (note that, since we have proved now that the degree of $C' \rightarrow C$ is one, C' is equal to C, hence the fibres of α are connected). Q.E.D.

We will show now that sometimes we can choose two one-forms in such a way that their wedge product is zero in $H^0(S', \Omega^2_{S'})$.

<u>(8.13) Lemma.</u> Let S be an algebraic surface with $p_g \le 2q-4$. Then there exist two $\mathbb{C}$-linearly independent one-forms ω_1, $\omega_2 \in H^0(\Omega^1_S)$ such that $\omega_1 \wedge \omega_2 = 0$ in $H^0(\Omega^2_S)$.

<u>Proof.</u> We consider the linear map

$$\Lambda^2(H^0(\Omega^1_S)) \to H^0(\Omega^2_S),$$

obtained from the bilinear, alternating map $(\omega_1, \omega_2) \to \omega_1 \wedge \omega_2$. In the following we shall denote by $(\omega_1) \wedge (\omega_2)$ the element of $\Lambda^2(H^0(\Omega^1_S))$ corresponding to the pair $(\omega_1, \omega_2) \in H^0(\Omega^1_S) \times H^0(\Omega^1_S)$, which should be distinguished from the section $\omega_1 \wedge \omega_2$ in $H^0(\Omega^2_S)$. Moreover let

$$\beta: \mathbb{P}(\Lambda^2(H^0(\Omega^1_S))) \dashrightarrow \mathbb{P}(H^0(\Omega^2_S)) \cong \mathbb{P}^{p_g-1}$$

be the corresponding rational map of projective spaces and $\mathbb{G}(2,q) \subset \mathbb{P}(H^0(\Omega^1_S))$ the Grassmann manifold of 2-dimensional subspaces of $H^0(\Omega^1_S)$. We have the Plücker embedding

$$\mathbb{G}(2,q) \to \mathbb{P}(\Lambda^2(H^0(\Omega^1_S))),$$

which sends a 2-plane in $H^0(\Omega^1_S)$, given by $\mathbb{C}\omega_1 \oplus \mathbb{C}\omega_2$, to $(\omega_1) \wedge (\omega_2)$, i.e. in particular we have

$$\mathbb{C}(\omega_1 \wedge \omega_2) = \mathbb{C}(\eta_1 \wedge \eta_2) \Leftrightarrow \mathbb{C}\omega_1 + \mathbb{C}\omega_2 = \mathbb{C}\eta_1 + \mathbb{C}\eta_2.$$

Now we want to find a plane $\pi := \mathbb{C}\omega_1 \oplus \mathbb{C}\omega_2$ in $H^0(\Omega^1{}_S)$ for which $\omega_1 \wedge \omega_2 = 0$ in $H^0(\Omega^2{}_S)$, i.e. a point which lies in the base locus of β restricted to $\mathbb{G}(2,q)$. But this is given by the intersection of $\mathbb{G}(2,q)$ with p_g hyperplane sections, which has dimension bigger or equal to $\dim \mathbb{G}(2,q) - p_g = 2(q-2) - p_g$ and this is bigger or equal to zero by our assumption. Hence such a plane π exists and the lemma is proven. Q.E.D.

<u>(8.14) Theorem</u> (Castelnuovo, de Franchis). Let X be a compact Kähler manifold. We assume that there exist linearly independent holomorphic one-forms $\omega_1, \dots, \omega_r \in H^0(\Omega^1{}_X)$ $(r \geq 2)$, such that $\omega_i \wedge \omega_j = 0$ in $H^0(\Omega^2{}_X)$ for all $i,j \in \{1,\dots,r\}$. Then there exists a holomorphic map $f: X \to C$ from X to a curve C, such that f has connected fibres. Furthermore there exist holomorphic one-forms $\eta_1, \dots, \eta_r \in H^0(\Omega^1{}_C)$ such that $\omega_i = f^*(\eta_i)$ for all $i \in \{1,\dots,r\}$.

<u>(8.15) Remark.</u> The genus g(C) of the above curve is then at least r.

<u>Proof of (8.14).</u> Let $\omega_1, \dots, \omega_r \in H^0(\Omega^1{}_X)$ be linearly independent holomorphic one-forms, such that $\omega_i \wedge \omega_j = 0$ in $H^0(\Omega^2{}_X)$ for all i,j. In particular $\omega_1 \wedge \omega_2 = 0$, which is equivalent to $\omega_1 = g\omega_2$ for some $g \in \mathbb{C}(X)$. Since X is a Kähler manifold, we have $d\omega = 0$ for all $\omega \in H^0(\Omega^1{}_X)$ and therefore we get

$$0 = d\omega_1 = dg \wedge \omega_2 + g\,d\omega_2 = dg \wedge \omega_2.$$

We consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}^1 \\ \tilde{X} \arrow[u, "\pi"] \arrow[r, "f"'] \arrow[ur, dashed] & C \arrow[u] \end{tikzcd},$$

where π is a blow-up of X, such that $\tilde{g}$ is a morphism, and $\tilde{g}$ = $h \circ f$ is the Stein factorization. Since h is finite, C has dimension one and therefore it is smooth (because it is normal).

We denote $\pi^*(\omega_i)$ by $\tilde{\omega}_i$ for $i \in \{1,...,r\}$ and we remark that $\tilde{\omega}_i \wedge d\tilde{g} = 0$ (note that $\omega_i = \lambda'_i \omega_2$). Therefore $\tilde{\omega}_i = \lambda_i d\tilde{g}$.

CLAIM: $\lambda_i \in f^*(\mathbb{C}(C))$.

PROOF (of the claim). Since $d\tilde{\omega}_i = 0$, it follows that $d\lambda_i \wedge d\tilde{g} = 0$ and therefore the differential of (f,λ_i): $\tilde{X} \rightarrow C \times \mathbb{P}^1$ has rank one. This implies that the image of (f,λ_i) is a curve $C'_i \subset C \times \mathbb{P}^1$. Because $f = pr_1 \circ (f,\lambda_i)$ has connected fibres, $pr_1|C'_i$: $C'_i \rightarrow C$ has degree one and is therefore an isomorphim (since C is smooth). So we have proven that C'_i is the graph of a map $\hat{\lambda}_i$: $C \rightarrow \mathbb{P}^1$ with $\lambda_i = \hat{\lambda}_i \circ f$, which shows the CLAIM.

With this we get

$$\tilde{\omega}_i = \lambda_i d\tilde{g} = f^*(\hat{\lambda}_i) f^*(dh) = f^*(\hat{\lambda}_i dh)$$

for all $i \in \{1,...,r\}$.

Setting $\eta_i := \hat{\lambda}_i dh$ we have found <u>rational</u> one-forms η_i on C with $\omega_i = f^*\eta_i$. We are now going to prove that $\eta_1,...,\eta_r$ are in fact <u>holomorphic</u> one-forms on C.

CLAIM: η_i has no poles.

PROOF (of the claim): Let $p \in C$ and let t be a local coordinate at p. Furthermore we choose a smooth point $x \in (f^{-1}(p))_{red}$. Then there are local coordinates $(z_1,\dots,z_n)$ at x such that $t \circ f = (z_1)^m$. If $\eta_i = \varphi_i(t)dt$ with $\varphi_i(t) \in \mathbb{C}(C)$, then $\omega_i = g^*(\eta_i) = \varphi_i(z_1{}^m) m z_1{}^{m-1} dz_1$. This implies that φ_i cannot have a pole, because then also ω_i would have a pole. Hence the CLAIM follows.

Since $\eta_1,\dots,\eta_r \in H^0(\Omega^1{}_C)$ are linearly independent, the genus of C is bigger or equal to r (≥ 2). This implies that every map from $\mathbb{P}^1$ to C is constant and therefore f maps the fibres of π to points. Hence f factors through X, which proves the theorem. Q.E.D.

In order to conclude the proof of (8.7) we need another result, which we will state in the sequel, whereas the proof will be postponed to the next lectures.

(8.16) Theorem (Zeuthen-Segre formula). Let $f: S \to C$ be a fibration (i.e., surjective with connected fibres) of a smooth projective surface S over a curve C and F a smooth fibre of f. Then we have the following identity for the topological Euler characteristics:

$$e(S) = e(C)e(F) + \mu,$$

where $\mu \geq 0$, $\mu = \Sigma_{y \in C}\, \delta(y)$ with $\delta(y) \geq 0$ and $\delta(y) > 0$ iff $F_y := f^{-1}(y)$ is singular and not equal to the multiple of a smooth elliptic curve.

Proof. Cf. lecture 9.

(8.17) Remark. If f is a Lefschetz pencil (i.e. F_y is either smooth or has at most one node as a singularity), then μ is the number of singular fibres of f, and this enumerative formula was ssically taken as the definition of $e(S) = I+4$ (I:=Zeuthen-Segre invariant).

(8.18) Corollary. Let $f: S \to C$ be as above, $e(S)<0$ and $g(C)\geq 2$. Then it follows that $F\cong \mathbf{P}^1$.

Proof. We recall that $g(C)\geq 2$ if and only if $e(C)<0$. Hence (by (8.17)) $e(F)>0$, but $e(F) = 2-2g(F)$. Therefore $g(F)=0$ and $F\cong \mathbf{P}^1$. Q.E.D.

We are now finally able to conclude the proof of the theorem of Castelnuovo.

Proof of (8.7). Let S be a smooth projective surface with $e(S)<0$. Then by (8.12) there exists a connected unramified covering $f: S' \to S$, such that $e(S')\leq -5$, which implies that $p_g(S')\leq 2q(S')-4$. Therefore by lemma (8.13) there exist two $\mathbf{C}$-linearly independent holomorphic 1-forms $\omega_1, \omega_2 \in H^0(\Omega^1{}_{S'})$ such that $\omega_1 \wedge \omega_2 = 0$ in $H^0(\Omega^2{}_{S'})$, which implies (by the theorem of Castelnuovo-de Franchis, cf. (8.14)) that there exists a fibration $f: S' \to C$ with $g(C)\geq 2$. Using (8.18) we get that the general fibre of f is isomorphic to $\mathbf{P}^1$, which implies by (8.11) that S' is ruled. Finally by (8.10) (note that $e(S)<0$ implies $q(S)\geq 1$) we conclude that also S is ruled and hence the theorem is proven. Q.E.D.

Surfaces fibred over a curve.

In this lecture we will study surfaces S, fibred over a (smooth) algebraic curve B. This means we consider fibrations $f: S \to B$ (i.e. f is surjective and has connected fibres), where S is a smooth projective surface and B a smooth curve. By Bertini's theorem (cf. (2.32)) we know that for generic $y \in B$ the fibre $F_y := f^{-1}(y)$ is a smooth curve of genus $g(F)$.
By the adjunction formula we have:

$$\omega_F = \mathcal{O}_F(K_S),$$

$$2g(F)-2 = -e(F) = K_S.F.$$

An important and not at all trivial problem is now to determine how special fibres can degenerate, i.e. which singularities will occur on the singular fibres of f.

(9.1) Zariski's lemma. Let $f: S \to B$ be a fibration of the surface S over the curve B and let $F=\Sigma_{i=1,\ldots,k}\, n_iC_i$ be a fibre of f (C_i are irreducible and all $n_i>0$). Furthermore let $D=\Sigma_{i=1,\ldots,k}\, m_iC_i$, $m_i \in \mathbb{Z}$, be a divisor on S with support on F. Then

1) $D^2 \le 0$,

2) $D^2=0$ if and only if $D \in \mathbb{Q}\cdot F$.

Proof. We keep in mind the following facts:

a) we have $C_i.C_j \ge 0$ for $i \ne j$ (since the C_i's are irreducible);

b) $n_i>0$ for all $i \in \{1,\ldots,k\}$;

c) we have $F.C_i=0$ for all $i\in\{1,\ldots,k\}$, since $\mathcal{O}_{C_i}(F) = \mathcal{O}_{C_i}$;

d) for each pair $i,j\in\{1,\ldots,k\}, i\neq j$, there exist $i_0=i,\ i_1,\ldots,i_r=j \in \{1,\ldots,k\}$, such that $C_{i_l}.C_{i_{l+1}}>0$, i.e. there is a chain of curves sucessively intersecting each other, connecting C_i and C_j (since F is connected).

We calculate the selfintersection of D:

$$D^2 = (\Sigma_{i=1,\ldots,k}\, m_iC_i)^2 = \Sigma_{i,j}\, \frac{m_i}{n_i}\,\frac{m_j}{n_j}\, n_in_jC_i.C_j =$$

$$= 2\Sigma_{i<j}\, \frac{m_i}{n_i}\,\frac{m_j}{n_j}\, n_in_jC_i.C_j + \Sigma_i\, (\frac{m_i}{n_i})^2\, n_i^2C_i^2 \leq$$

$$\leq \Sigma_{i<j}(\frac{m_i^2}{n_i^2}) + (\frac{m_j^2}{n_j^2})n_in_jC_i.C_j + \Sigma_i(\frac{m_i}{n_i})^2\, n_i^2C_i^2 =$$

$$= \Sigma_{i\neq j}(\frac{m_i^2}{n_i^2})n_in_jC_i.C_j + \Sigma_{i=j}(\frac{m_i^2}{n_i^2})n_in_jC_i.C_j =$$

$$= \Sigma_{i,j}\, (\frac{m_i^2}{n_i^2})n_in_jC_i.C_j = \Sigma_i(\frac{m_i^2}{n_i^2})n_iC_i(\Sigma_jn_jC_j) =$$

$$= \Sigma_i(\frac{m_i^2}{n_i^2})n_iC_iF = 0,$$

and we have proved 1).

Equality above holds iff for each pair (i,j) with $i\neq j$ holds:

either $C_i.C_j = 0$ or $\dfrac{m_i}{n_i} = \dfrac{m_j}{n_j}$.

By property 4) we see:

$$\frac{m_{i_n}}{n_{i_n}} = \frac{m_{i_{n+1}}}{n_{i_{n+1}}}$$

and therefore

$$\frac{m_i}{n_i} = \frac{m_j}{n_j} = \frac{p}{q} \in \mathbb{Q}, \text{ for all } i,j. \text{ Q.E.D.}$$

Analogously as for surfaces we have in the relative situation of fibrations of surfaces the notion of minimality.

<u>(9.2) Definition.</u> A fibration $f: S \to B$ of the surface S over the curve B is called <u>relatively minimal</u> if and only if no fibre of f contains an exceptional curve of the first kind.

<u>(9.3) Remark.</u> If $f: S \to B$ is an arbitrary fibration, then by the theorem of Castelnuovo-Enriques (cf. (3.18)) there exists a relatively minimal fibration $f': S' \to B$ and a sequence of blow-downs $\pi: S \to S'$ such that $f = f' \circ \pi$.

As a first example of how convenient it is to assume a fibration to be relatively minimal, we give the following result.

<u>(9.4) Proposition.</u> Let $f: S \to B$ be a relatively minimal fibration.

If the generic fibre F of f has genus zero, then $F_y \cong \mathbb{P}^1$ for all $y \in B$.

For the proof of the above proposition we need some auxiliary results.

<u>(9.5) Lemma.</u> Let $f: S \to B$ be a relatively minimal fibration and $F = \Sigma_{i=1,\dots,k}\, n_i C_i$ a reducible fibre of f (i.e. $\Sigma_{i=1,\dots,k}\, n_i > 1$). Then $K_S.C_i \geq 0$ for all $i \in \{1,\dots,k\}$.

<u>(9.6) Remark.</u> The above statement is obviously wrong if f is not relatively minimal (since for an exceptional curve of the first kind E we have K.E=-1).

<u>Proof of (9.5).</u> Let $F=\Sigma_{i=1,\dots,k}\, n_iC_i$ ($n_i>0$ for all i) be a reducible fibre of f.
1.case: $k\geq 2$.

Since $p(C_i)\geq 0$, we have $C_i^2 + K.C_i = 2p(C_i)-2 \geq -2$. Moreover $C_i^2<0$ by (9.1), therefore $K.C_i<0$ implies $C_i^2 + K.C_i = -2$. So we obtain $C_i^2 = K.C_i = -1$ and $p(C_i)=0$, which implies that C_i is an exceptional curve of the first kind (contained in F) contradicting the relative minimality of f.

2.case: k=1.
Then F=nC, n>1 and C irreducible. If K.C<0, then $2p(C)-2 = C^2+K.C = K.C < 0$ and therefore $C\cong\mathbb{P}^1$, K.C=-2. This implies $2p(F)-2 = K.F+F^2 = K.F = -2n$ and so n=1, which is a contradiction. Q.E.D.

<u>(9.7) Corollary.</u> Let f: S $\rightarrow$ B be a relatively minimal fibration and $F=\Sigma_{i=1,\dots,k}\, n_iC_i$ a reducible fibre of f.

X 1) If $k\geq 2$ and $K.C_i = 0$ ~~for all~~ i, then $C_i^2=-2$ and $C_i\cong\mathbb{P}^1$ ~~for all~~

2) If k=1 (i.e.F=nC), then K.C = 0 if and only if p(C)=1.

<u>Proof.</u> 1) In this case $C_i^2 = C_i^2 + K.C_i = 2p(C_i)-2 = -2$ and therefore $p(C_i)=0$, which implies that $C_i\cong\mathbb{P}^1$.

since $F^2=n^2C^2=0$
2) $C^2 = 0$ ~~[illegible]~~ and therefore $C^2 + K.C = 0$, which implies p(C)=0 by the adjunction formula. Q.E.D.

(9.8) Remark. In (9.7), 2) C is either a smooth elliptic curve or has an ordinary double point or a cusp as a singularity.

Proof of (9.4). Let F_y be an arbitrary fibre of f. By (6.5) F_y is irreducible, because otherwise $K.F_y = \Sigma n_i K.C_i \geq 0$, which contradicts $-2 = K.F = K.F_y$. Furthermore $p(F_y)=0$ and therefore $F_y \cong \mathbb{P}^1$. Q.E.D.

In the remaining part of the lecture we will essentially proof the (already stated) theorem of Noether and Enriques (cf. (8.11)) as well as the Zeuthen-Segre formula (cf. (8.16)).

(9.9) Theorem (Noether, Enriques). Let $f: S \to C$ be a fibration of a projective surface S over a curve C such that the generic fibre has genus zero. Then there exists a rational map

$$\psi: S \dashrightarrow \mathbb{P}^1,$$

such that

$$\psi \times f: S \dashrightarrow \mathbb{P}^1 \times B$$

is a birational map.

Proof. Since $g(F)=0$, $K.F+F^2 = K.F = -2$. Therefore K cannot be effective (since $F^2=0$, F irreducible implies that D.F≥0 for each effective divisor). This implies that $p_g(S)=\dim H^0(S,\Omega^2{}_S)=0$, which is by Serre duality (cf. (2.38)) equivalent to $H^2(S,\mathcal{O}_S)=0$. Therefore we obtain by the exponential sequence (cf. proof of (4.1)) that the morphism $c_1: H^1(S,\mathcal{O}_S{}^*)\to H^2(S,\mathbb{Z})$ is surjective, which means that every cohomology class in $H^2(S,\mathbb{Z})$ is the class of a divisor on S. We observe that the intersection form on $H^2(S,\mathbb{Z})$ is unimodular (i.e. the determinant of the associated matrix is 1 or -1) by Poincare' duality. F is indivisible (because

if $F=nF'$, $K.F'=-2$ and therefore $n=1$) and so there exists a divisor D such that $D.F=1$.
We consider the long exact cohomology sequence (noting that $H^0(\mathcal{O}_F(D+nF)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, since $F \cong \mathbb{P}^1$ and $F.(D+nF)=1$)

$$0 \to H^0(D+(n-1)F) \to H^0(D+nF) \to H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2 \to$$

$$H^1(D+(n-1)F) \to H^1(D+nF) \to H^1(\mathcal{O}_{\mathbb{P}^1}(1)) = 0.$$

Therefore the map $H^1(\rho_n)$: $H^1(D+(n-1)F) \to H^1(D+nF)$ is surjective for all n and so the dimension of $H^1(D+nF)$ decreases with increasing n. This implies that there exists a $n_0>0$ such that $H^1(\rho_n)$ is an isomorphism for all $n \geq n_0$ and so for $n \geq n_0$ there exist sections σ_0, $\sigma_1 \in H^0(D+nF)$ which induce a basis of $H^0(\mathcal{O}_F(D+nF)) \cong \mathbb{C}^2$.

We consider the rational map $\psi=(\sigma_0,\sigma_1)$: $S \dashrightarrow \mathbb{P}^1$; then $\psi|F$: $F \to \mathbb{P}^1$ is an isomorphism.

CLAIM: $\psi \times f$: $S \dashrightarrow \mathbb{P}^1 \times B$ is birational.

Proof of the claim: It is enough to show that the degree of $\psi \times f$ is one. We fix a general point $(t,y) \in \mathbb{P}^1 \times B$, then

$$(\{t\} \times B).(\mathbb{P}^1 \times \{y\}) = 1.$$

We recall that for any two divisors C,D on $\mathbb{P}^1 \times B$ we have $f^*(D).f^*(C) = \deg(f)(C.D)$ (cf. remarks after (3.12)) and therefore

$$\deg(f) = \psi^{-1}(t).F_y = (D+nF).F_y = 1.$$

So we have proven the claim and the theorem. Q.E.D.

(9.10) Definition. Let $f: S \to B$ be a fibration of the surface S over the curve B. A fibre $F=\Sigma_{i=1,\dots,k}\, n_iC_i$ of f is called multiple fibre iff $m:=GCD(n_i) > 1$.

Then we can write $F = mF'$, and $F'^2=0$.

(9.11) Example. Let $f: S \to B$ be a fibration of the surface S over the curve B and $F=\Sigma_{i=1,\dots,k}\, n_iC_i$ a fibre. Furthermore let

$D=\Sigma_{i=1,\dots,k}\, m_iC_i$ with $m_i \in \mathbb{Z}$ and $D^2=0$. Then by Zariski's lemma we obtain that there exists a $r \in \mathbb{Z}$ such that $D=rF'$.

(9.12) Remark. Let $f: S \to B$ be a fibration of the surface S over the curve B and $g=g(F)$ the genus of the general fibre F of f. Let $F_y = mF'$ be a multiple fibre. Then $2g-2 = K.F = K.F_y = mK.F' = m(2p(F')-2)$. From this we see that $g=0$ implies $m=1$, i.e. there don't exist multiple fibres.
If $g \geq 2$, then $2p(F')-2 \geq 2$, hence $m \leq g-1$.
If $g=1$, then m can be arbitrarily large and in fact an essential tool of surface classification is the study of elliptic fibrations (i.e. fibrations whose general fibre has genus one).

We will now prove the formula of Zeuthen-Segre which we already formulated (cf. (8.16)) in order to use the result for the proof of the theorem of Castelnuovo.

(9.13) Theorem (Zeuthen-Segre formula). **Let $f: S \to C$ be a** fibration of a smooth projective surface S over a curve C and F a smooth fibre of f. Then we have the following identity for the topological Euler characteristics:

$$e(S) = e(C)e(F) + \mu,$$

where $\mu \geq 0$, actually $\mu = \Sigma_{y \in C}\, \mu(y)$ with $\mu(y) \geq 0$ and $\mu(y)>0$ iff $F_y := f^{-1}(y)$ is singular and not equal to the multiple of a smooth elliptic curve.

For the proof we need to recall some facts about Chern classes of vector bundles. A general reference for this topic is for example [ ]. Hartshorn Grothendie

(9.14) Remark. 1) To any locally free sheaf $\mathcal{F}$ of rank r on a nonsingular projective variety X we can associate integral cohomology classes $c_i(F) \in H^{2i}(X,\mathbb{Z})$, $i=0,\dots,r$, the Chern classes of $\mathcal{F}$. For convenience we put the Chern classes together to form the total Chern class $c(F) := 1+c_1(\mathcal{F})+\dots+c_r(\mathcal{F}) \in \oplus H^{2i}(X,\mathbb{Z})$.
This cohomology classes exist and are uniquely determined by the following properties:

a) $c_0(\mathcal{F})=1$.

b) If $\mathcal{O}(D)$ is the line bundle of a divisor D, then $c(D) = 1+c_1(D) = 1+c_1(\mathcal{O}(D))$ is given by the image of $\mathcal{O}(D)$ under the map $H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$ arising from the exponential sequence on X (cf. proof of (4.1)).

c) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of locally free sheaves on X, then $c(\mathcal{G})=c(\mathcal{F}).c(\mathcal{H})$.

2) Let $\mathcal{E}$ be a locally free sheaf on X, then $c_1(\mathcal{E})=c_1(\det\mathcal{E})$.

3) Let S be a projective surface, then $c_2(S):=c_2(\Omega_S^1)=e(S)$, where e(S) is the topological Euler characteristic of S.

(9.15) Example. Let S be a projective surface and $\mathcal{E}$ a locally free sheaf of rank 2 on S. Furthermore let σ be a section of $\mathcal{E}$ vanishing on a finite set. We get an exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\sigma} \mathcal{E} \xrightarrow{{}^t\sigma} \mathcal{J}_Z\mathcal{L} \to 0,$$

where $\mathcal{L}$ is the determinant of $\mathcal{E}$, ${}^t\sigma$ is the transposed of σ (i.e. if σ is locally given by $\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$ then ${}^t\sigma=(-\sigma_2,\sigma_1)$) and $\mathcal{J}_Z$ is the ideal of the 0-dimensional subscheme Z of S locally given by $\{\sigma_1=\sigma_2=0\}$. From this we obtain the following:

$$c_1(\mathcal{L}) = c_1(\mathcal{E}),$$

$$c_2(\mathcal{E}) = \deg Z = h^0(\mathcal{O}_Z).$$

Furthermore the total Chern class of $\mathcal{O}_Z$ is given by

$$c(\mathcal{O}_Z) = c(\mathcal{L}/\mathcal{J}_Z\mathcal{L}) = 1-\deg Z.$$

Obviously the degree of Z is always bigger or equal to zero and it is equal to zero if and only if $Z=\emptyset$. This is a completely elementary fact, but it will be very useful for the proof of (9.13).

Proof of (9.13). We can assume f to be relatively minimal, since otherwise by (9.3) there exists a relatively minimal fibration f': $S' \rightarrow B$ and a sequence of blow-downs π: $S \rightarrow S'$ such that $f=f'\circ\pi$ and then $e(S) = e(S')$ + number of blow-ups.
We have, by definition of the sheaf of relative Kaehler differentials $\Omega^1{}_{S/B}$, we have the following exact sequence:

$$0 \rightarrow f^*(\Omega^1{}_B) \rightarrow \Omega^1{}_S \rightarrow \Omega^1{}_{S/B} \rightarrow 0.$$

We observe that we have the following relation between the sheaf of Kaehler differentials of a fibre F of f and the relative Kaehler differentials $\Omega^1{}_{S/B}$:

$$\Omega^1{}_F = \Omega^1{}_{S/B} \otimes \mathcal{O}_F.$$

By the above exact sequence we see, that $\omega_{S/B}$ $(:=\det(\Omega^1{}_{S/B}))$ $= \Omega^2{}_S \otimes (f^*(\Omega^1{}_B))^{-1} \cong \mathcal{O}_S(K_S - f^*K_B)$.

We define a map $\xi': \Omega^1{}_S \to \omega_{S/B}$, locally given by $\xi'(\eta) = (\eta \wedge dt) \otimes (dt)^{-1}$, and it is easy to verify that ξ' is welldefined.
Let (x,y) be local coordinates around $p \in S$ and t a local parameter of B at $f(p)$. Then locally we have $\Omega^1{}_S = \mathcal{O}_S dx + \mathcal{O}_S dy$ and since $dt = \frac{\partial t}{\partial x} dx + \frac{\partial t}{\partial y} dy$, $\xi'(dx) = \frac{\partial t}{\partial y}(dx \wedge dy) \otimes (dt)^{-1}$ and $\xi'(dy) = -\frac{\partial t}{\partial x}(dx \wedge dy) \otimes (dt)^{-1}$ (note that $(dx \wedge dy) \otimes (dt)^{-1}$ is a local generator of $\omega_{S/B}$). Therefore we see that $\mathrm{im}(\xi') = \mathcal{J}_{\mathfrak{C}} \omega_{S/B}$, where $\mathcal{J}_{\mathfrak{C}}$ is the ideal sheaf of the critical set of f (i.e. $\mathcal{J}_{\mathfrak{C}}$ is locally given by $(\frac{\partial t}{\partial x}, \frac{\partial t}{\partial y})$.
The critical set $\mathfrak{C}$ of f is in general not a divisor, it can have zero-dimensional components. We consider the <u>divisorial part $\mathcal{S}$ of $\mathfrak{C}$</u>, locally defined by $\sigma = \mathrm{G.C.D}(\frac{\partial t}{\partial x}, \frac{\partial t}{\partial y})$. Then $\mathcal{S} = \Sigma_{y \in B}\, \mathcal{S}_y$, where $\mathcal{S}_y = \Sigma(n_i - 1)C_i$ if $F_y = \Sigma n_i C_i$ (note that this fact is not true in positive characteristics).
Therefore we get: $\frac{\partial t}{\partial x} = \gamma_x \sigma$, $\frac{\partial t}{\partial y} = \gamma_y \sigma$ with γ_x, γ_y relatively prime regular functions.

CLAIM: $\ker(\xi') \cong f^*(\Omega^1{}_B)(\mathcal{S}) = \mathcal{O}_S(f^*(K_B) + \mathcal{S})$.

Proof (of the claim): We calculate in local coordinates (x,y) around a point p of S, then an element of $\Omega^1{}_S$ is given by $adx+bdy$, where a,b are regular functions around p. Obviously $\xi'(adx+bdy) = 0$ if and only if $a\frac{\partial t}{\partial y} - b\frac{\partial t}{\partial x} = 0$, which is again equivalent that $a\gamma_y = b\gamma_x$. This means that $a=u\gamma_x$ and $b=u\gamma_y$ and so $adx+bdy = u(\frac{dt}{\sigma})$, which proves the claim. Q.E.D.

Putting the knowledge about ξ' together we obtain an exact sequence of sheaves on S:

$$(*) \qquad 0 \to \mathcal{O}_S(f^*K_B+\mathcal{S}) \to \Omega^1{}_S \to \omega_{S/B} \to \mathcal{O}_{\mathcal{C}}(\omega_{S/B}) \to 0.$$

Since the ideal of $\mathcal{C}$ is contained in the ideal of $\mathcal{S}$, we also have the exact sequence

$$(**) \qquad 0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{S}} \to 0,$$

where the support of $\mathcal{F}$ has dimension zero (i.e. $\mathcal{F}$ is concentrated in finitely many points). In fact locally we have: $\mathcal{O}_{\mathcal{S}}=\mathcal{O}_S/(\sigma)$, $\mathcal{O}_{\mathcal{C}}=\mathcal{O}_S/(\sigma\gamma_x,\sigma\gamma_y)$ and the kernel of the natural quotient map $\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{S}}$ is given by $(\sigma)\mathcal{O}_S/(\sigma\gamma_x,\sigma\gamma_y) \cong \mathcal{O}_S/(\gamma_x,\gamma_y)$ $=:\mathcal{F}$. Moreover the stalk $\mathcal{F}_p \neq 0$ if and only if p is a singular point of the reduction F_{red} of a fibre F of f. Therefore $\mathcal{F}$ is concentrated in finitely many points.

Tensoring $(**)$ by $\omega_{S/B}$ we obtain the exact sequence:

$$(***) \qquad 0 \to \mathcal{F} \to \mathcal{O}_{\mathcal{C}}(\omega_{S/B}) \to \mathcal{O}_{\mathcal{S}}(\omega_{S/B}) \to 0.$$

With the help of the above exact sequences and continuously using (9.15) we will now calculate $e(S)=c_2(S)$ and $e(F)$.

By (9.15), 1c) we obtain from $(*)$:

$$c(\Omega^1{}_S) = c(\mathcal{O}_S(f^*K_B+\mathcal{S}))c(\omega_{S/B})c(\mathcal{O}_{\mathcal{C}}(\omega_{S/B}))^{-1}.$$

By the exact sequence $(***)$ we know on the other hand:

$$c(\mathcal{O}_{\mathcal{C}}(\omega_{S/B})) = c(\mathcal{F})c(\mathcal{O}_{\mathcal{S}}(\omega_{S/B})),$$

and therefore we get:

$$c(\Omega^1{}_S) = c(\mathcal{O}_S(f^*K_B+\mathcal{S}))c(\omega_{S/B})c(\mathcal{F})^{-1}c(\mathcal{O}_{\mathcal{S}}(\omega_{S/B}))^{-1} =$$

$$= c(\mathcal{O}_S(f^*K_B+\mathcal{D}))c(\mathcal{F})^{-1}c(\omega_{S/B}(-\mathcal{D})) =$$

$$= (1+f^*K_B+\mathcal{D})(1+\deg\mathcal{F})(1+K_S-f^*K_B-\mathcal{D}).$$

By definition $c(\Omega^1{}_S) = 1+c_1(\Omega^1{}_S)+c_2(\Omega^1{}_S) = 1+K_S+c_2(S)$ and therefore we see from the above equality:

$$c_2(S) = \deg\mathcal{F} + (f^*K_B+\mathcal{D})(K_S-f^*K_B-\mathcal{D}) = \deg\mathcal{F} + f^*K_B.K_S + \mathcal{D}.K_S,$$

where the last equality holds by Zariskis lemma, since f^*K_B is a sum of fibres and $\mathcal{D}$ is contained in a sum of fibres.
The canonical divisor K_B of the curve B is linearly equivalent to $2g(B)-2$ points, therefore f^*K_B is linearly equivalent to $2g(B)-2$ fibres. Furthermore $\mathcal{O}_F(K_S) = \omega_F$, hence $K_S.F = 2g(F)-2$.
Putting these observations together we obtain:

$$c_2(S) = (2g(F)-2)(2g(B)-2) + \deg\mathcal{F} + \mathcal{D}.K_S =$$

$$= (-e(F))(-e(B)) + \deg\mathcal{F} + \mathcal{D}.K_S =$$

$$= e(F)e(B) + \mu,$$

where $\mu := \deg\mathcal{F} + \mathcal{D}.K_S$.
Furthermore $\mu = \Sigma_y\mu_y$, where $\mu_y = \mu(F_y) = \deg(\mathcal{F}\cap F_y) + \mathcal{D}_y.K_S$.
Let $F_y = \Sigma_{i=1,\dots,k}\, n_iC_i$ be a fibre of f. If $\mathcal{D}_y\ (=\Sigma_{i=1,\dots,k}\,(n_i-1)C_i) \neq 0$, then F_y is not irreducible and (9.5) implies that $K_S.C_i \geq 0$ and so also $\mathcal{D}_y.K_S \geq 0$.
On the other hand $\deg(\mathcal{F}\cap F_y) > 0$, unless $(F_y)_{red}$ is smooth or equivalently $F_y = mC$, where C is a smooth curve. If $m=1$, then F_y is smooth and if $m \geq 2$, if moreover $\mathcal{D}_y.K_S = 0$, then $C.K_S = C^2 = 0$ and therefore by the adjunction formula C is a smooth elliptic curve. This proves the theorem. Q.E.D.

Elliptic fibrations (and their role in the classification theory of surfaces).

In this lecture we will study elliptic fibrations, i.e.,fibrations f: S → B of a smooth projective surface S over a smooth curve B such that the general fibre is a (smooth) elliptic curve (i.e., has genus one). Furthermore we will always assume f to be relatively minimal (cf. (9.2)).
If F is any fibre of f, then it follows from the adjunction formula together with the fact that F has selfintersection zero, that

$$K_S.F = 0.$$

The first aim of this lecture will be to give a complete classification of all possible singular fibres of f.

(10.1) Remark. If the fibre $F=\Sigma_{i=1,\dots,k}\, n_iC_i$ (note that we have adopted the convention $n_i \geq 1$ for all i) is not irreducible (i.e., there exists an i such that $n_i \geq 2$, or $k \geq 2$), then by (9.5) we have $K_S.C_i \geq 0$ for all $i \in \{1,\dots,k\}$. Since

$$0 = K_S.F = \Sigma_{i=1,\dots,k}\, n_i(K_S.C_i) \geq 0,$$

it follows that

$$K_S.C_i = 0 \text{ for all } i \in \{1,\dots,k\}.$$

(10.2) Definition. Let S be a smooth projective surface and let $D=\Sigma_{i=1,\dots,k}\, n_iC_i$ an effective divisor on S. D is called of elliptic type if and only if the following is fulfilled:

1) $D.C_i = 0$ for all i,

2) $K_S.F = 0$.

D is called an <u>indecomposable divisor of elliptic type</u> if D is of elliptic type and cannot be decomposed in a sum of two divisors of elliptic type.

<u>(10.3) Remark.</u> 1) Obviously an indecomposable divisor of elliptic type is connected.
2) If we look at the proof of Zariski's lemma (cf. lecture 9) we see that the statement remains true if we replace the fibre F of the elliptic fibration by an indecomposable divisor of elliptic type. This means that in the following we will not only give a classification of all singular fibres of an elliptic fibration, but we give a classification of the indecomposable divisors of elliptic types(i.e., ~~we [illegible]~~ as in 9.12, we replace a multiple fibre $F = mF'$ by F').

<u>I. CASE: $k \geq 2$.</u>

Let $F = \Sigma_{i=1,\dots,k}\, n_i C_i$ be a fibre of f with $k \geq 2$. Then by Zariski's lemma we know:

$$C_i^2 < 0 \text{ for all } i \in \{1,\dots,k\}.$$

So we obtain ~~with~~ by the adjunction formula:

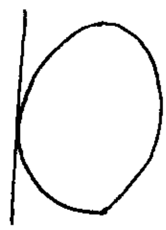
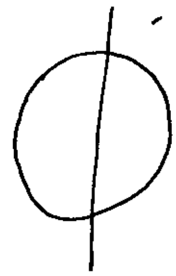
$$-2 \leq 2p(C_i)-2 = K.C_i + C_i^2 = C_i^2 < 0,$$

hence $C_i^2 = -2$ and $C_i \cong \mathbb{P}^1$ for all i.

If there exist $i \neq j \in \{1,\dots,k\}$ such that $C_i.C_j \geq 2$, then

$$(C_i+C_j)^2 = -4+2C_i.C_j \geq 0.$$

Hence by Zariski's lemma $C_i.C_j = 2$ and there exists a natural number m such that $F = m(C_i+C_j) = mF'$ (F' as in (9.10)). In this case we have the following two types of intersection (of C_i and C_j):

$\tilde{A}_{1a}$:  and $\tilde{A}_1$:  ,

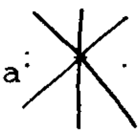
i.e., F' is of type $\tilde{A}_{1a}$ or $\tilde{A}_1$.

Therefore we can assume in the following:

$C_i.C_j \leq 1$ for all $i \neq j \in \{1,\dots,k\}$.

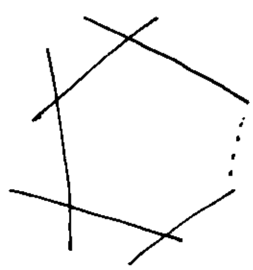
If there exist three different elements i,j,l of $\{1,\dots,k\}$ such that $C_i \cap C_j \cap C_l \neq \emptyset$ (in particular $C_i.C_j = C_i.C_l = C_j.C_l = 1$), then

$$(C_i+C_j+C_l)^2 = -6 + 2(C_i.C_j+C_i.C_l+C_j.C_l) = 0,$$

and again by Zariski's lemma we see that $F' = C_i+C_j+C_l$, i.e. F' is of type $\tilde{A}_{2a}$:  .

So we can also assume in the following that for all pairwise different $i,j,k \in \{1,\dots,k\}$ we have $C_i \cap C_j \cap C_l = \emptyset$.

Finally if we assume that there exists a cycle of b (≥ 3) irreducible curves $C_1,\dots,C_b$ contained in F, i.e. $C_1.C_2 \geq 1,\dots,C_{b-1}.C_b \geq 1$, $C_b.C_1 \geq 1$ (note that by the above argument we know that $C_i.C_{i+1} = 1$), we can conclude analogously that $F' = C_1+\dots+C_b$, i.e. F' is of type $\tilde{A}_{b-1}$:

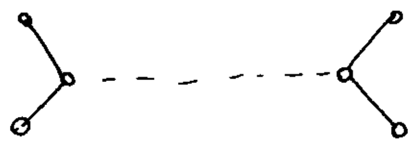
 .

(Polygon with n sides)

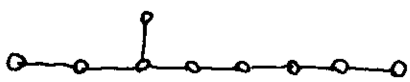
If $F' = \Sigma_{i=1,\dots,k}\, n_i C_i$ (G.C.D$(n_i)=1$) is different from the above configurations (as we already saw F' then also does'nt contain one of the preceding configurations $\tilde{A}_{1a}$, $\tilde{A}_1$, $\tilde{A}_{2a}$, $\tilde{A}_{b-1}$), then we associate to F' a graph, called the Dynkin-graph, of F'.
The vertices of the graph correspond to the irreducible components C_i of F' and two vertices (corresponding to C_i and C_j) are connected by an edge if and only if $C_i.C_j=1$.
Moreover the vertices are labelled by the multiplicities n_i of C_i.
Since F' does not contain cycles of curves, the corresponding Dynkin-graph is simply connected, hence is a vertex labelled tree.

(10.4) Theorem. Let $f: S \to B$ be a relatively minimal elliptic fibration and let $F = mF' = m(\Sigma_{i=1,\dots,k}\, n_i C_i)$, (G.C.D$(n_i)=1$), be a fibre of f.
We assume furthermore that F' is not of type $\tilde{A}_{1a}$, $\tilde{A}_1$, $\tilde{A}_{2a}$, $\tilde{A}_{b-1}$. Then the Dynkin-graph of F' is one of the following trees:

$\tilde{D}_{4+b}$:

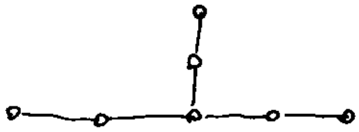


$\tilde{E}_8$:



$\tilde{E}_7$:

$\tilde{E}_6$:



<u>(10.5) Remark.</u> Vice versa all the above graphs ($\tilde{A}_{1a}$, $\tilde{A}_1$, $\tilde{A}_{2a}$, $\tilde{A}_{b-1}$, $\tilde{D}_{4+b}$, $\tilde{E}_8$, $\tilde{E}_7$, $\tilde{E}_6$) occur as Dynkin-graphs of elliptic fibrations (even with S being an appropriate blow-up of the plane) (cf. [Miranda]).

<u>Proof</u> (of (10.4)). It is easy to verify that for each of the above graphs the associated divisor $F'=\Sigma_{i=1,\dots,k}\, n_iC_i$ is indecomposable of elliptic type. Therefore (by Zariski's lemma) a Dynkin graph which is a subgraph of one of the above trees or contains one of the above trees must coincide with it.
Let D be a Dynkin-graph arising from an elliptic fibration. Then any vertex touches at most four edges and if there exists a vertex touching four edges then $D=\tilde{D}_4$.
Furthermore D has at most two <u>nodes</u> (i.e. vertices through which pass three edges) and if D has two nodes, then $D=\tilde{D}_{4+b}$ with $b\geq 1$.
In the case that D has exactly one node, we get by removing this node three connected components with a_1, a_2, a_3 vertices ($a_1\leq a_2\leq a_3$).

If $a_1\geq 2$, then D contains $\tilde{E}_6$ and therefore $D=\tilde{E}_6$.

If $a_1=1$ and $a_2\geq 3$, then D contains $\tilde{E}_7$, hence $D=\tilde{E}_7$.
On the other hand if $a_1=1$, $a_2=2$, then we distinguish two cases:

α) $a_3\geq 5$.

In this case D contains $\tilde{E}_8$, hence $D=\tilde{E}_8$.

β) $a_3\leq 4$.

Here D is a proper subtree of $\tilde{E}_8$, which is a contradiction, so this case cannot occur.

The remaining case $a_1 = a_2 = 1$ is not possible, since then D would be a proper subtree of $\tilde{D}_{4+b}$ for an appropriate b.

If D had no nodes, D would be a proper subtree of $\tilde{D}_{4+b}$ for an appropriate b, hence also this case cannot occur and we have proven the theorem. Q.E.D.

II. CASE: $k=1$.

In this case $F = mC$, $m \geq 1$, where C is an irreducible curve. Moreover $p_a(C) = 1$ and (using Kodaira's notation (cf. [])) for C only the following cases can occur:

I_0: smooth elliptic curve,

I_1: nodal cubic,

II: cuspidal cubic.

We have now completely classified the non multiple fibres of a relatively minimal elliptic fibration and the following theorem will conclude the classification of all possible singular fibres of an elliptic fibration.

(10.6) Theorem. Let $F = mF'$ be a multiple fibre (i.e. $m \geq 2$) of a relatively minimal elliptic fibration. Then F is of the form mI_0, mI_1 or $m\,\tilde{A}_{b-1}$ $(b \geq 2)$.

This result is an immediate consequence of the following proposition together with the classification above.

(10.7) Proposition. Let $F = mF'$ be a multiple fibre of a relatively minimal elliptic fibration $f\colon S \to B$. Then the following assertions hold:

1) F' is not simply connected.

TOPOLOGY AND THE EXISTENCE OF IRRATIONAL PENCILS.

Let X be a compact Kähler manifold, and let W be a $\mathbb{C}$-vector subspace of $H^1(X,\mathbb{C})$.
There is a natural bilinear, alternating map:

$$H^1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \longrightarrow H^2(X,\mathbb{C}),$$
$$(\eta_1, \eta_2) \longmapsto \eta_1 \wedge \eta_2$$

(here we interpret cohomology as de Rham cohomology, so, since X is Kähler,

(11.1) $$H^1(X,\mathbb{C}) = H^0(\Omega^1_X) \oplus \overline{H^0(\Omega^1_X)},$$

and these cohomology classes are uniquely represented by differential forms).

(11.2) DEFINITION. $W \subset H^1(X,\mathbb{C})$ is said to be isotropic iff $W \wedge W = 0$ in $H^2(X,\mathbb{C})$.

(11.3) REMARK. 1) If $f: X \longrightarrow B$ is a genus b pencil (i.e., a holomorphic map to a smooth curve of genus b with connected fibres), then $U_f := f^*(H^0(\Omega^1_B))$ is a b-dimensional isotropic subspace of $H^0(\Omega^1_X)$.
2) Conversely, the theorem of Castelnuovo-de Franchis shows that the correspondence which associates to f $U_f = f^*(H^0(\Omega^1_B))$ yields a bijective correspondence between:

$$\{\text{genus } b \text{ pencils}\} \longleftrightarrow \{\text{maximal isotropic subspaces } U \subset H^0(\Omega^1_X), \text{ of dim } b\}$$

PROOF of 2). There is only to prove that $U_f = f^*(H^0(\Omega^1_B))$ is maximal isotropic. But if $U \supsetneq U_f$, then there exists $g : X \longrightarrow C$ such that $U \subset g^*(H^0(\Omega^1_C))$, and the fibres of g would be contained in the fibres of f, whence $g = f$ (f has connected fibres). Contradiction. ■

On the other hand, $H^1(B, \mathbb{C})$ contains an infinite family of maximal isotropic subspaces, and one can pull them back to obtain isotropic subspaces of $H^1(X, \mathbb{C})$.

(11.4.) THEOREM. There exist a genus b pencil f with $b \geq 2$ $\Longleftrightarrow$ there exist a maximal isotropic subspace W of $H^1(X, \mathbb{C})$, having dimension b.
Moreover, any such subspace W determines a unique genus b pencil f, and, being contained in $f^* H^1(B, \mathbb{C})$, is the pull-back of a maximal isotropic subspace of $H^1(X, B)$.

PROOF. By (11.3) it suffices to show the implication "$\Longleftarrow$".
Let $W = \langle \varphi_1, \dots, \varphi_r \rangle$ with $r = \dim W$.
Write $\varphi_i = \omega_i + \bar{\eta}_i$, where $\omega_i, \eta_i \in H^0(\Omega^1_X)$.

Decomposing $0 = \varphi_i \wedge \varphi_j$ into types, we obtain:

$$(11.5.) \quad \begin{cases} \omega_i \wedge \omega_j = 0 & \text{(part of type (2,0))}, \\ \eta_i \wedge \eta_j = 0 & \text{(conjugate of part of type (2,0))}, \\ \omega_i \wedge \bar{\eta}_j + \bar{\eta}_i + \omega_j = 0 & \text{(part of type (1,1))} \end{cases}$$

Set $U := \langle \omega_1, \dots, \omega_r \rangle$ and $V := \langle \eta_1, \dots, \eta_r \rangle$.

CLAIM $U+V$ is an isotropic subspace of $H^0(\Omega^1_X)$.

PROOF of the claim.

Case I. $\dim U = 1$.

Then, after a change of basis we can assume

$$\omega_2 = \dots = \omega_r = 0.$$

Then by (11.5.) $\omega_1 \wedge \bar{\eta}_j = 0$, whence $\omega_1 \wedge \eta_j = 0$ (else, if ξ is the Kähler (1,1)-form

$$\int_X \underbrace{(\omega_1 \wedge \eta_j) \wedge (\bar{\omega}_1 \wedge \bar{\eta}_j) \wedge \xi^{n-2}} \neq 0$$

$$= (\omega_1 \wedge \bar{\eta}_j) \wedge (\omega_1 \wedge \bar{\eta}_j) \wedge \xi^{n-2},$$

contradicting $\omega_1 \wedge \bar{\eta}_j = 0$).

Thus U and V generate an isotropic subspace.

Case II. If $\dim V = 1$, same conclusion (replace W by $\bar{W}$!).

Case III. $\dim U \geq 2$, $\dim V \geq 2$.

Same conclusion, otherwise by Castelnuovo-de Franchis exist pencils $f_1 : X \longrightarrow C_1$, $f_2 : X \longrightarrow C_2$ with $\begin{cases} U \subset f_1^*(H^0(\Omega^1_{C_1})), \\ V \subset f_2^*(H^0(\Omega^1_{C_2})). \end{cases}$

Let $F := (f_1 \times f_2) : X \longrightarrow C_1 \times C_2$.

Since $U+V$ is not isotropic, the generic rank of F is 2, thus F is onto.

Therefore F^* is injective on de Rham cohomology (this follows from the projection formula).

But $H^*(C_1 \times C_2, \mathbb{C}) \cong H^*(C_1, \mathbb{C}) \otimes H^*(C_2, \mathbb{C})$ (Künneth formula) therefore from $\bar{\eta}_i \wedge \omega_j + \omega_i + \bar{\eta}_j = 0$ we derive that F^*

not injective, whence a contradiction.
So we have proven the claim.

CONCLUSION. $U+V$ is isotropic, whence there exists a pencil $f: X \longrightarrow B$, such that $U+V \subset f^*(H^0(\Omega^1_B))$.
Thus $W \subset f^*(H^1(B, \mathbb{C}))$.
Being maximal isotropic, W is the pull-back of a maximal isotropic subspace (f^* is injective), whence $r = \dim W = b = \text{genus}(B)$.
Unicity follows, since then $U+V = f^*(H^0(\Omega^1_B))$ and we use (11.3.). ∎

§ 12 : Inequalities for fibrations $f: S \longrightarrow B$

Let $f: S \longrightarrow B$ be a fibration with fibres F_y $(y \in B)$ of genus g.
Except for a finite set $\Sigma \subset B$, all the fibres $F_y = f^{-1}(y)$ are smooth and connected for $y \notin \Sigma$

<u>Definition 12.1</u> f is said to have <u>constant moduli</u> if all the smooth fibres are isomorphic.

<u>Def. 12.2</u> f is said to be <u>semi-stable</u> if (it is relatively minimal and) every singular fibre $F_y = \sum n_i C_i$ is <u>reduced</u> (i.e., $n_i = 1 \ \forall i$), and the only singularities of F are ordinary double points (i.e., there are local coordinates (x, y) on S at a singular point P of F and t on B at $f(P)$ such that $t(x,y) = xy$).

<u>12.3</u> (Base - change) Let $g : B' \rightarrow B$ be a finite ramified covering and let S' be a minimal resolution of $B' \times_B S =$
$= \{ (b', s) \in B' \times S$ such that $g(b') = f(s) \}$.

Then we have a diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[dr, "f'"'] & B' \times_B S \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ & B' \arrow[r, "g"] & B \end{tikzcd}$$

and f' is again a fibration with genus g fibres.

<u>Definition 12.4.</u> We shall say that $f' : S' \longrightarrow B'$ is obtained from f by <u>base change</u> through $g : B' \rightarrow B$.

Def. 12.5. f is said to be isotrivial if there is a base change yielding a product fibration birational fibration $f' : B' \times F \longrightarrow B'$ (in particular isotrivial $\Rightarrow$ constant moduli).

Def. 12.6. f is said to be a holomorphic bundle if $\forall y \in B$, there exists an open set U in the Hausdorff topology and a biholomorphism $f^{-1}(U) \cong U \times F_y$ compatible with the projections, f, p_U.

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "f"'] & & U \times F_y \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

Remark 12.7. Kodaira (cf Collected Works) has given nice examples of fibrations ~~with~~ where all the fibres are smooth, but which are not holomorphic bundles (there f does not have constant moduli).

The importance of semistable fibrations lies in the following (non trivial) theorem, for the proof of which the reader can consult [B-P-V] (Barth-Peters-Van de Ven)

Theorem 12.8. If f is semistable and has constant moduli, then f is a holomorphic bundle.

This is one reason why it is important to reduce the study of a fibration to the study of a semi stable fibration.

Theorem 12.9. For each fibration $f : S \to B$, there exists a covering $g : B' \to B$ yielding a semistable fibration $f' : S' \to B'$.

Proof. ~~...~~ Write every singular fibre F_y as $\sum n_i(y) C_i$, and let m' be ~~...~~ a common multiple of all the $n_i(y)$.
Let $y_1, \dots, y_r$ be the points of B for which F_y is singular, and let $z_1, \dots, z_p$ be other points of B such that $r+p \equiv 0 \pmod{m'}$.
Since $\mathrm{Pic}^0(B)$ is a divisible group, ~~if~~ if $r+p = m'k$, there exists a divisor D of degree k such that $\mathcal{O}_B(m'D) \cong \mathcal{O}_B(\sum y_i + \sum z_j)$.

To D we associate a line bundle L such that $\mathcal{O}_B(D)$ is the sheaf of sections of L.
(I.e., if $g_{\alpha\beta} \in H^1(B, \mathcal{O}_B^*)$ is a cocycle associated to $\mathcal{O}_B(D)$, then L is obtained by glueing $\coprod (U_\alpha \times \mathbb{C})$, ~~where U_α~~ identifying $((U_\alpha \cap U_\beta) \times \mathbb{C}) \subset U_\alpha \times \mathbb{C}$, to

$$((U_\alpha \cap U_\beta) \times \mathbb{C}) \subset U_\beta \times \mathbb{C} \quad \text{via the formula}$$

$$(x, z_\alpha) \sim (x, z_\beta) \iff z_\alpha = g_{\alpha\beta}(x)\, z_\beta \text{).}$$

Inside L we can take the $\sqrt[m']{}$ of σ, where σ is the section defining the divisor $\sum y_i + \sum z_j$ (in fact, $\sigma_\alpha = g_{\alpha\beta}^{m'}\, \sigma_\beta$).

$B' \subset L$ is thus obtained as a cyclic covering of B, and is therefore locally defined by the equations

$$z_\alpha^{m'} = \sigma_\alpha(x).$$

There remains to prove that, for a suitable m' the minimal resolution S' of $B' \times_B S$ yields a semistable f'.

To obtain S', first of all we blow-up points in S in order to make the reduced fibres have only double points.
~~That is~~, This is possible by the theorem on resolution of curves inside surfaces, and we obtain $\Pi : \hat{S} \to S$ such that, setting $\hat{f} = f \circ \Pi$,
$\forall$ point P of $\tilde{S}$ there exist local coordinates (x, y) such that $\hat{f}$ is expressed either by

1) $t = x^{\tilde{n}_i}$ $\quad (P \in \tilde{C}_i)$
2) $t = x^{\tilde{n}_i} y^{\tilde{n}_j}$

<u>Choice</u> : Let m' be a common multiple of all the $\hat{n}_i$'s.

What are the singularities of $B' \times_B S$?
They are exactly of the form

3) $$t'^{m'} = x^{a'} y^{b'}$$, where $a' + b' \geq 1$, and m' is a common multiple of a', b'.

Thus, if $d = G.C.D.\ (a', b')$, $\begin{cases} a' = a d \\ b' = b d \\ m' = m d \end{cases}$

The equation $(t'^m)^d = (x^a y^b)^d$ yields a surface singularity which has d components having equation

4) $$t'^m = \varepsilon\, x^a y^b$$, where $\varepsilon^d = 1$.

We try to normalize these singularities, which are isomorphic varying ε.

Write $m = a\,b\,n$, and assume the singularity is $t^m = x^a y^b$.

Then if $(t'^n) = \tau$, we have $\tau^{ab} = x^a y^b$.

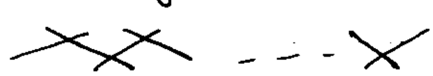
Set $z = \tau^b / x$: then $z^a = y^b$, so the function z satisfies an integral equation and is holomorphic on the normalization.
Normalizing the equation $z^a = y^b$, we obtain a function u ~~with~~ on the normalization (notice that a, b, are relatively prime) such that $z = u^b$, $y = u^a$.
We finally get a partial normalization with functions u, x, t such that
$u^b x = \tau^b$, $\tau = t'^n$.
Again $\xi = \tau / u$ satisfies $\xi^b = x$, whence on the normalization we have functions ξ, u, t' such that

5) $t'^n = u\,\xi$.

5) Defines a hypersurface in $\mathbb{C}^3$ with isolated singularities, whence a normal singularity (if $n = 1$ we do not have a singularity).

Blowing up $\left[\frac{n-1}{2}\right]$ the singularity in 5

(called of type A_{n-1}) one obtains a resolution with a string of $(n-1)$ $\mathbb{P}^1$'s with self-intersection
(-2)



We leave this assertion as an exercise with hint:
if $n \geq 3$, the blow-up of
$t'^n = u\,\xi$ yields ~~a~~ a singularity of type A_{n-3}.

Let us change the name of the coordinates on $\mathbb{C}^3$ and let A_{n-1} be defined by

6) $y_2^n = y_0 y_1$.

Notice that $\mathbb{P}^2 = \mathbb{C}^3 - \{0\}/\mathbb{C}^*$ with $\lambda \in \mathbb{C}^*$ acting by $(x_0, x_1, x_2) \longmapsto (\lambda x_0, \lambda x_1, \lambda x_2)$.

The blow up $\tilde{\mathbb{C}}^3$ of $\mathbb{C}^3$ at the origin, traditionally contained in $\mathbb{C}^3 \times \mathbb{P}^2$, can be conveniently described as

$\tilde{\mathbb{C}}^3 = (\mathbb{C}^3 - \{0\}) \times \mathbb{C}/\mathbb{C}^*$, where λ acts on (x_0, x_1, x_2, z) by sending it to $(\lambda x_0, \lambda x_1, \lambda x_2, \lambda^{-1} z)$.

The map to $\mathbb{P}^2$ is given by $\tilde{\mathbb{C}}^3 \longrightarrow \mathbb{C}^3 - \{0\}/\mathbb{C}^*$, i.e., $(x_0, x_1, x_2, z) \longmapsto (x_0, x_1, x_2)$, whereas the map to $\mathbb{C}^3$ is defined by

$y_j = x_i z$. It is now obvious that $z = 0$ is the equation of the exceptional divisor E.

Taking the proper transform of equation 6) yields

7) $\Gamma = \{ x_2^{n} \cdot z^{n-2} = x_0 x_1 \}$, and for $x_0 = x_1 = 0$ we get an A_{n-3} singularity.

Whereas intersecting with $E \cong \mathbb{P}^2$ (set $z = 0$), we obtain two transversal lines in $\mathbb{P}^2$, given by $x_0 x_1 = 0$.

8) There remains to prove that f' is semistable.

To this purpose, we pull back the function t' $(= y_2)$ to S'.

Each time, since $y_2 = x_2 z$, we obtain the exceptional divisor with multiplicity 1 plus the curve $x_2 = 0$.

Therefore all the multiplicities are 1, and since the lines $x_0 = 0$, $x_1 = 0$, $x_2 = 0$ are transversal we obtain at worst a double point. Finally, if E_{new} is of the I kind, $E \cdot (\text{fibre} - E) = 1$ $\cancel{\ }$, whence if you contract it the new fibre does not get singularities QED

Theorems 12.8 and 12.9 imply that if f has constant moduli, then there is a base change yielding a holomorphic bundle.

<u>Theorem 12.10.</u> A holomorphic bundle with $g \geq 2$ is isotrivial. More precisely, there exists an unramified covering $B' \to B$ such that $S' = B' \times_B S$ is a ~~trivial~~ product fibration $S' \cong B' \times F \to F$.

<u>Corollary 12.11</u> If $g \geq 2$ every fibration with constant moduli is isotrivial.

<u>Proof of 12.10.</u> The main point is the theorem of Schwartz-Klein-Hurwitz, by which, if a smooth curve F has genus $g \geq 2$, then the group $\mathrm{Aut}(F)$ of ~~ho~~holomorphic automorphisms of F is finite, and has indeed cardinality $\leq 84(g-1)$.
Now, the fundamental group $\pi_1(B)$ ~~acts~~ has a homomorphism $\mu : \pi_1(B) \to \mathrm{Aut}(F)$ given by the monodromy transformations (μ is trivial $\iff$ the bundle is trivial, i.e., $\cong B \times F$).
The covering $B' \to B$ is the covering associated to the normal subgroup $\ker(\mu)$, which has finite index.
Then $S' \to B'$ has trivial monodromy and is trivial.
■

We now come to a sequel of results, which give some inequalities for a fibration $f : S \to B$.
In the following, we shall restrict ourselves (also in view of 12.11) to the case where the genus g of the fibres is ≥ 2. One of these results was already encountered, namely the Zeuthen-Segre formula.
Recall that $q = \dim H^1(\mathcal{O}_S) = \dim H^0(\Omega^1_S)$

Topics on surfaces of general type

Let $f : S \longrightarrow B$ be a fibration, with $g = g(F_t) \geq 2$ (arithmetic genus $g(F_t) = g(F_{t'})$).
Let $b =$ genus (B).

Assume $f : S \longrightarrow B$ is relatively minimal (no I kind curve in F_t

<u>Thm.</u> $b \leq q(S) \leq b + g$, and we have equality
$\iff S = B \times F$.

<u>Proof.</u>

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathrm{Alb}(S) = A \arrow[d, "\pi"] \\ B \arrow[r, "g"] & \mathrm{Alb}(B) = A/K \\ & \overset{\shortparallel}{J(B)} \end{tikzcd}$$

To S we attach $\begin{matrix} J \\ \downarrow \\ B \end{matrix}$ s.t. $J_t = J(F_t)$. Then

$F_t \longrightarrow A$ induces $\begin{matrix} J \to A \\ \downarrow \\ B \end{matrix}$ and by prop. 1, the image is indep. of t

Set $K' =$ image.

Clearly $K' \subset K$ ($f(F_t) = t$, a point).

But

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d] & A \arrow[d, "\pi'"] \\ B \arrow[r] \arrow[dr, "g"'] & A/K' \arrow[d, shift left] \\ & A/K \arrow[u, shift left] \end{tikzcd}$$

factors through f $\quad K = K'$

Whence ~~$\dim K = \dim K'$, and $K = K'$~~.

$q = \dim A \overset{\dim A/K +}{=\!\sqrt{\dim}} K = b + \dim K \leq b + g$.

Assume $=$: then $J_t \longrightarrow K$ is a finite covering.

$\Rightarrow J_t \cong \tilde{K}$ covering of K, and the family $J \cong B \times \tilde{K}$.

$J \cong B \times \underset{\mathrm{Pic}^0(F)}{\overset{\shortparallel}{\mathrm{Alb}(F)}}$, by the way.

By the theorem of Torelli $\Longrightarrow$ all F_t are isomorphic $\Longrightarrow$
$\begin{matrix} S \\ \downarrow \\ B \end{matrix}$ is an F-bundle (after taking the pull-back by a covering of B)

Since $g(F) \geq 2$, by Hurwitz, $\#(\mathrm{Aut}(F)) < +\infty$

and since the bundle is repr. by $\pi_1(B) \to Aut(F)$, $\exists$ an etale G-cover $B' \to B$ s.t. the pull back is trivial

$$\begin{array}{ccc} B' \times F & \longrightarrow & S \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

Now, $H^0(\Omega^1_S) = H^0(\Omega^1_{B' \times F})^G =$

$$= \underbrace{H^0(\Omega^1_{B'})^G}_{H^0(\Omega^1_B)} + H^0(\Omega^1_F)^G .$$

Since $q = b + g$, G acts trivially on $H^0(\Omega^1_F)$, hence ~~also~~ on Alb(F).

But $\gamma \in G$ acts trivially on cohomology, hence its Lefschetz # is < 0 $\Rightarrow \gamma = id$.

Conclusion : $G = id.$, and $S \cong B \times F$.

Q.E.D.

Another way of looking at the previous theorem

Consider $\omega_{S/B} = K_S - f^* K_B$.

Then, by Fujita, $V = f_* \omega_{S/B}$ is semipositive,

write $V = \oplus W_i$, W_i indecomposables of rank r_i, $\sum r_i = g$.

(Rel. duality) $V^\vee = R^1 f_* \mathcal{O}_S$ is semi negative. (any inv. subsheaf has degree ≤ 0).

Cor. 1 Any section of $V^\vee$ ~~with a zero~~ vanishing somewhere is identically zero. $\Rightarrow$

$$0 \to H^0(W_i^\vee) \otimes \mathcal{O}_B \to W_i^\vee \to U_i \to 0$$

Thus Cor. 2 $h^0(V^\vee) \leq g$, and equality $\iff V^\vee \cong \mathcal{O}^g$.

Spectral seq. $\cdot H^0(R^1 f_* \mathcal{O}_S)$ $\cdot H^1(R^1 f_* \mathcal{O}_S)$

$$\Rightarrow \quad h^1(V^\vee) = p_g$$

$H^0(\mathcal{O}_B)$ $\quad H^1(\mathcal{O}_B)$ $\qquad b + h^0(V^\vee) = q$

$$\chi(\mathcal{O}_S) = 1 - q + p_g = \chi(\mathcal{O}_B) - \chi(V^\vee) = (1-g)(1-b) + \deg V$$

Then we get C) $\chi(\mathcal{O}_S) \geq (1-g)(1-b)$.

Let us introduce e ~~=~~ $=$ topological Euler Poincaré :
then $e(B) = 2-2b$, and e, χ, K^2 of S are related by

$$12\ \chi(\mathcal{O}_S) = (K^2_S + e(S)).$$

We have two basic inequalities, to be obtained in several ways :

A) $K^2_S \geq 8(b-1)(g-1)$

B) $e(S) \geq 4(b-1)(g-1) = e(B)\cdot e(F)$

[$\chi(\mathcal{O}_S) \geq (b-1)(g-1)$ follows from the previous, and equality $\Rightarrow$
$\Rightarrow$ equality then in both the previous]

A) $\Longleftrightarrow$ Thm. (ARAKELOV) S minimal, $g \geq 2$.

1) $(\omega_{S/B})^2 \geq 0$, and $\omega_{S/B}$ is nef $(\omega_{S/B}\cdot C \geq 0)$.

2) If the smooth fibres are ~~in~~ not isomorphic ,
$(\omega_{S/B})^2 > 0$ and $\omega_{S/B}\cdot C > 0$ except if $C \cong \mathbb{P}^1$, $C \subset F_t$,
$C^2 = -2$.

(A) cf. Szpiro, Beauville, [B-P-V]

(B) in the classical

Thm. (ZEUTHEN-SEGRE)
$f: S \to B$ as above, then

$$e(S) = e(B)\cdot e(F) + \mu \quad , \text{ where } \mu \geq 0 \text{ and}$$

$\mu = \sum_{t\in B} \mu_t$, $\mu_t \geq 0$, ~~...~~ $\mu_t = 0 \Longleftrightarrow F_t$ smooth

$F_t =$ ~~in~~ smooth elliptic curve.

(if $g \geq 2$ as always)

Corollary. $\forall \chi \geq (b-1)(g-1)$ except if
$f: S \to B$ is an F bundle.

With these, we proceed to the proof of another prototype theorem
($\geq$, and char. of equality)

(Classical + Beauville)

Thm. $\forall S$ minimal of gen. type . Then $p_g \geq 2q-4$, equality
$\Leftrightarrow S = C_1 \times C_2$, $g_1 = 2$, $g = g_2 \geq 2$.

Proof. $\Leftarrow$ $q = 2+g$, $p_g = 2g$.