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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Arakelov Theory and applications (II): ampleness

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In this talk we shall see how the algebraic geometric notion of ampleness has an arithmetic counterpart in Arakelov geometry.

1. Dimension one:

1.1. Let E be a rank $r>0$ vector bundle on a smooth projective curve X over $k (= \bar{k})$. Its degree is defined as

$$\deg(E) = \deg(\Lambda^r E) \in \mathbb{Z},$$

where $\Lambda^r E$ is the ~~other~~ top exterior power of E.

Proposition 1: If $\deg(E) \gg 0$, E has a non trivial section on X.

Proof: The Riemann-Roch theorem

$$h^0(X,E) - h^1(X,E) = \deg(E) + r(1-g)$$

implies that, if $\deg(E)$ is big enough, more precisely if $\deg(E) > r(g-1)$, then $h^0(X,E) \neq 0$. q.e.d.

1.2. Let now $X = \mathrm{Spec}(\mathbb{Z})$ and let (E,h) be an hermitian vector bundle on $\mathrm{Spec}(\mathbb{Z})$. In other words E is a rank r free $\mathbb{Z}$-module, and h is an hermitian metric on $E \otimes_{\mathbb{Z}} \mathbb{C}$, invariant under the complex conjugation F_∞.

We define

(1) $\widehat{\deg}(E,h) = \widehat{\deg}(\Lambda^r E, \Lambda^r h) \in \mathbb{R}$,

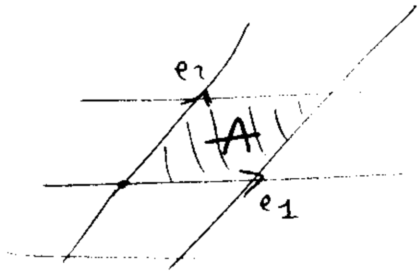
where $\Lambda^r h$ is the metric induced by h on $\Lambda^r(E \otimes_{\mathbb{Z}} \mathbb{C})$.

If $e_1, \dots, e_r$ is a basis of E, ~~except~~ $s_0 = e_1 \wedge \dots \wedge e_r$ is a basis of $\Lambda^r E$, therefore

$$\widehat{\deg}(E,h) = -\log \|s_0\| = -\log \mathrm{vol}(E \otimes_{\mathbb{Z}} \mathbb{R}/E),$$

where the ~~[illegible]~~ measure is the one attached to h (the Lebesgue measure induced by an orthogonal basis of $E \otimes_{\mathbb{Z}} \mathbb{R}$).

When $r = 2$, $\widehat{\deg}(E,h) = -\log\|e_1 \wedge e_2\|$,



$\|e_1 \wedge e_2\|$ = surface of the fundamental domain A.

Proposition 2: If $\widehat{\deg}(E,h) \gg 0$, there exists $s \in E - \{0\}$ such that $\|s\| \leq 1$.

Proof: This follows from Minkowski theorem applied to the unit ball in $E \otimes_{\mathbb{Z}} \mathbb{R}$.

The analogy with Proposition 1 is that $s \in E$ is a section of E on $\mathrm{Spec}(\mathbb{Z})$, when the condition $\|s\| \leq 1$ means that "$v_\infty(s) \geq 0$", i.e. s extends to $\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$.

Assume now M is any finitely generated $\mathbb{Z}$-module and

h is an hermitian metric on $M \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ invariant under F_∞. The definition (1) can be extended by defining

$$(2) \qquad \widehat{\deg}(M, h) = \log \#(M_{\text{torsion}}) - \log \operatorname{vol}(M \underset{\mathbb{Z}}{\otimes} \mathbb{R} / M),$$

where M_{torsion} is the torsion subgroup of M.

2. Ampleness

Let X be a smooth projective variety over $k = \bar{k}$ and L a line bundle on X.

Definition 1: The line bundle L is called ample if one of the following equivalent properties holds:

A 1) For any coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that, if $n > n_0$, $\mathcal{F} \otimes L^{\otimes n}$ is generated by its global section.

$\Updownarrow$

A 2) For any coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that, if $n > n_0$ and $q > 0$, $H^q(X, \mathcal{F} \otimes L^{\otimes n}) = 0$.

$\Updownarrow$

A 3) There exists $n_0 > 0$ and a projective embedding $\varphi : X \hookrightarrow \mathbb{P}^N$ such that $\varphi^*(\mathcal{O}(1)) = L$.

When $k = \mathbb{C}$, these conditions are also equivalent to

A4) There exists an hermitian metric h on L such that $c_1(L, h) > 0$.

In other words, if we write the first Chern form in local holomorphic coordinates,

$$c_1(L,h) = \sum_{i,j} a_{ij}\, dz_i\, d\bar{z}_j$$

the matrix (a_{ij}) should be <u>definite</u> positive at every point.

<u>Theorem 1</u> (Makaĭ - Moishezon) Assume $\dim(X) = 2$.

Then L is ample if and only if $L = \mathcal{O}_X(D)$ with

$$\begin{cases} D^2 > 0 \\ \text{and} \\ D.C > 0 \quad \text{for any curve } C \text{ contained in } X. \end{cases}$$

In other words L is ample if and only if $c_1(L)^2 > 0$ and $\underbrace{\deg(L \mid C) > 0}_{\text{for any } C \text{ on } X}$, where $c_1(L) \in CH^1(X)$ is the first Chern class of L in the first (algebraic) Chow group of X.

3. Arithmetic ampleness:

Let X be a regular projective flat scheme over $\mathbb{Z}$, and (L, h) an hermitian line bundle on X.

The arithmetic first Chern class of (L, h),

$$\hat{c}_1(L,h) \in \widehat{CH}^1(X),$$

is defined to be the class of $(\mathrm{div}(s), -\log\|s\|^2)$, where s is any rational section of L over X, $\mathrm{div}(s)$ its divisor, and $-\log\|s\|^2 \in D^{00}(X(\mathbb{C}))$ the distribution on $X(\mathbb{C})$ defined by the L^1-function $-\log\|s\|^2$ (it is a Green current for $\mathrm{div}(s)$ because of the Poincaré-Lelong equation). Since any other rational section of L may be written $s' = f s$ where f is a rational function on X, the definition of $\hat{c}_1(L,h)$ is independent of the choice of s (but it does depend on h).

If $d = \dim(X)$, the intersection formalism explained in the previous talk gives us a real number

$$\hat{c}_1(\overline{L})^d := \pi_*(\hat{c}_1(L,h)^d) \in \mathbb{R}.$$

We then have the following results

<u>Theorem 2</u> (Gillet-Soulé) (effectivity):

Assume that

a) L is ample

b) the first Chern form $c_1(L_{\mathbb{C}}, h)$ is positive on $X(\mathbb{C})$

c) $\hat{c}_1(\overline{L})^d > 0$.

Then, for any hermitian vector bundle (E, h') on X, there exists $n_0 > 0$ such that, if $n > n_0$, there exists $s \in \Gamma(X, E \otimes L^{\otimes n})$, $s \neq 0$, such that

$$\|s(x)\| \leq 1 \qquad \text{for any } x \in X(\mathbb{C}).$$

Theorem 3 (S. Zhang) (ampleness):

Assume that

a) L is ample on X

b) $c_1(L_{\mathbb{C}}, h) \geqslant 0$ on $X(\mathbb{C})$

c) for any closed irreducible subscheme $Y \subset X$ which is flat over $\mathbb{Z}$

$$\hat{c}_1((L,h)|_Y)^{\dim(Y)} > 0.$$

Then, given any hermitian line bundle (M, h') on X, there exists $n_0 > 0$ such that, if $n > n_0$, the free $\mathbb{Z}$-module $\Gamma(X, M \otimes L^{\otimes n})$ has a <u>basis</u> consisting of sections s such that

$$\|s(x)\| \leq 1 \qquad \text{for any } x \in X(\mathbb{C}).$$

Notice that in the hypotheses of [illegible]
the scheme Y is not necessarily regular. To make sense of $\hat{c}_1((L,h)_{|Y})^{\dim(Y)}$ one can proceed as follows.
Choose any Green current g_Y for Y^{an} and let $y \in \widehat{CH}^p(X)$, $p = d - \dim(Y)$, be the class of (Y, g_Y). Then define

$$\hat{c}_1((L,h)_{|Y})^{\dim(Y)} = \pi_*\left(\hat{c}_1(L,h)^{d-p} \cdot y\right) - \frac{1}{2}\int_{X(\mathbb{C})} g_Y\, c_1(L_{\mathbb{C}}, h)^{d-p}.$$

3. Sketch of the proof of Theorem 2:

3.1. Let X be as above and (E,h) be an hermitian vector bundle on X. For any integer $q \geq 0$ the coherent cohomology group $H^q(X,E)$ is a finitely generated $\mathbb{Z}$-module. It may be equipped as follows with an hermitian metric h_{L^2}.

~~Choose a Kähler metric on $X(\mathbb{C})$ invariant under~~

Denote by

$$A^{0q} = C^\infty(X(\mathbb{C}), E_{\mathbb{C}}) \otimes_{C^\infty(X(\mathbb{C}))} A^{0q}(X(\mathbb{C}))$$

the vector space of smooth forms of type $(0,q)$ with coefficients in $E_{\mathbb{C}}$.

The holomorphic structure on $E_{\mathbb{C}}$ gives rise to a Cauchy-Riemann operator

$$\bar{\partial} : A^{0q} \longrightarrow A^{0,q+1}$$

with $\bar{\partial} \circ \bar{\partial} = 0$, and Dolbeault theorem asserts that $H^q(X(\mathbb{C}), E_{\mathbb{C}})$ is canonically isomorphic to the qth cohomology of the complex

$$\cdots \longrightarrow A^{0,q-1} \xrightarrow{\bar{\partial}} A^{0,q} \xrightarrow{\bar{\partial}} A^{0,q+1} \longrightarrow \cdots .$$

Now choose a Kähler metric h_X on $X(\mathbb{C})$ invariant under F_∞.

The L^2-scalar product on A^{0q} is defined by the formula

$$\langle s, t \rangle_{L^2} = \int_{X(\mathbb{C})} \langle s(x), t(x) \rangle \frac{\omega_0^{d-1}}{(d-1)!} , \quad s, t \in A^{0q},$$

where $\langle s(x), t(x) \rangle$ is the pointwise scalar product coming from our choice of metrics on E and X, and ω_0 is the (normalized) Kähler form.

The operator $\bar{\partial} : A^{0q} \to A^{0,q+1}$ has an adjoint $\bar{\partial}^* : A^{0,q+1} \to A^{0,q}$ for this scalar product.

Let $\Delta_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ be the Laplace operator on A^{0q}. By Hodge theory, $H^q(X(\mathbb{C}), E_{\mathbb{C}})$ is isomorphic to the space of harmonic forms $\mathrm{Ker}(\Delta_q)$.

it follows that

$$H^q(X, E) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \simeq \mathrm{Ker}(\Delta_q)$$

may be equipped with the L^2-scalar product h_{L^2}.

On the other hand, let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the nonzero eigenvalues of Δ_q (~~counted~~ repeated according to their multiplicities) and

$$\zeta_q(s) = \sum_{i \geq 1} \lambda_i^{-s} \quad , s \in \mathbb{C}, \ \mathrm{Re}(s) > d-1,$$

the zeta function of Δ_q. This function extends to a meromorphic function of $s \in \mathbb{C}$ which has no pole at ~~does not vanish~~ at $s = 0$. Therefore it makes sense to talk of the value of its derivative, $\zeta_q'(0)$.

~~The~~ Following Quillen, we define the Euler characteristic of (E, h) to be the real number

$$(3) \quad \chi_Q(E, h) := \sum_{q \geq 0} (-1)^q \, \widehat{\deg}(H^q(X, E), h_{L^2}) + \frac{1}{2} \sum (-1)^q \, q \, \zeta_q'(0),$$

where $\widehat{\deg}$ is defined as in (2).

The arithmetic Riemann-Roch theorem computes this number by a formula of the form

(4) $$\chi_Q(E,h) = \pi_*\big(\widehat{ch}(E,h).\widehat{Td}(X,h_X)\big) + \frac{1}{2}\int_{X(\mathbb{C})} ch(E_{\mathbb{C}})\, Td(X(\mathbb{C}))\, R(X(\mathbb{C}))\,,$$

where $\widehat{ch}$ and $\widehat{Td}$ are characteristic classes with values in $\bigoplus_{p\geq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generalizing $\hat{c}_1$, ch and Td are the usual Chern and Todd classes, and R is the additive characteristic class attached to the power series

$$R(x) = \sum_{\substack{m \text{ odd} \\ m\geq 1}} \left(2\,\zeta'(-m) + \left(1+\tfrac{1}{2}+\dots+\tfrac{1}{m}\right)\zeta(-m)\right)\frac{x^m}{m!}\,,$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta'(s)$ its derivative (statement (4) is due to Gillet and myself; Faltings extended it to higher degrees "à la Grothendieck").

3.2. Let's now make the hypotheses of Theorem 2. Consider the Euler characteristic $\chi_Q(E\otimes L^{\otimes n})$ (for our choice of metrics on E and L and any choice of h_X on X).

Using a) we know that $H^q(X, E\otimes L^{\otimes n}) = 0$ when $q>0$ and $n \gg 0$.

Using b) Bismut and Vasserot proved that, as n goes to infinity,

$$\sum_{q\geq 0} (-1)^{q}\, q\, \zeta_q'(0) = O(n^{d}\log n).$$

Therefore (3) reads

$$(5) \qquad \chi_Q(E\otimes L^{\otimes n}) = \widehat{\deg}\left(H^0(X, E\otimes L^{\otimes n}), h_{L^2}\right) + O(n^{d-1}\log n).$$

On the other hand $\widehat{ch}$ and ch are multiplicative under tensor product, and coincide with $\exp(\hat{c}_1)$ (resp. $\exp(c_1)$) for line bundles. It follows from (4) that

$$(6) \qquad \chi_Q(E\otimes L^{\otimes n}) = n^d \, \frac{\hat{c}_1(\bar{L})^d}{d!} \operatorname{rank}(E) + O(n^{d-1}).$$

Using hypothesis c) together with (5) and (6) we conclude that

$$\widehat{\deg}\left(H^0(X, E\otimes L^{\otimes n}), h_{L^2}\right) \gg 0$$

as $n \to +\infty$. Therefore, by Minkowski (Proposition 2), $H^0(X, E\otimes L^{\otimes n})$ contains $s \neq 0$ with $\|s\|_{L^2} \leq 1$. By comparing the sup norm with the L^2-norm asymptotically in n, one gets Theorem 2.

4. Applications:

A variant of Theorem 2 was used by Vojta in his proof of Mordell (1989). He considered an hermitian line bundle (L, h) on the product of two copies of an arithmetic surface. However, in his case the hypotheses a) and b) of Theorem 2 were not quite satisfied, so that extra work was needed for the argument to work.

Later, Faltings showed that more elementary methods, extending the classical Siegel's lemma from ~~approx~~ diophantine approximation theory, could be used instead of the arithmetic Riemann-Roch theorem to produce small sections (they play the role of the "auxiliary polynomials" of transcendence number theory). Bombieri's proof of Mordell is also in the style of classical diophantine approximation theory, with no use of Arakelov theory.

So the problem remains of using the arithmetic Riemann-Roch theorem more than by the asymptotic study that we gave in section 3 (for which the exact form of the statement (4) is not needed).

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