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Notes on Chern Weil Theory

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This is a brief account of Chern - Weil theory for characteristic classes. We assume some familiarity with basic concepts and facts about smooth and complex manifolds such as tangent spaces, the operators $d, d_z, d_{\bar{z}}$ etc.

§ 1. Vector Bundles

Let M be a smooth manifold. A complex vector bundle ξ of rank n is a smooth manifold $E(\xi)$ together with a smooth map $\pi(\xi) : E(\xi) \to M$ and a vector space structure on $E_m(\xi) = \pi(\xi)^{-1}(m)$ for each $m \epsilon M$ with the following property $(*)$.

For each $m \epsilon M$, there is a neighbourhood U of m and a diffeomorphism $\Phi : U \times \mathbf{C}^n \to \pi(\xi)^{-1}(U)$ such that $\pi(\xi)\Phi(m, v) = m$ for all $(m, v) \epsilon U \times \mathbf{C}^n$ and for each $m \epsilon U, v \mapsto \Phi(m, v)$ is a vector space isomorphism of $\mathbf{C}^n$ on $E_m(\xi)$. $E(\xi)$ is called the *total space of* ξ, M the *base of* ξ and for $m \epsilon M, E_m(\xi)$ the *fibre over* ξ. For a fixed Φ as above and $m \epsilon U$, we denote by Φ_m the isomorphism $v \mapsto \Phi(m, v)$ of $\mathbf{C}^n$ on $E_m(\xi)$. Also ϕ will be called a *"trivialisation* of ξ over U".

(The property $(*)$ is a "local triviality" assumption and may be replaced by an apparently weaker but equivalent condition. Such a modification of the definition is of no great interest in the differential geometric context. However, one can make analogous definitions in the algebraic geometric category and there the Zariski topology being a some what coarse topology, the local triviality assumption with U replaced by a Zariski open set in the variety M would appear too strong; and indeed to take care of many naturally occuring objects, it is necessary to modify the assumption to a weaker one. As it turns out the modified assumption that naturally suggests itself turns out to be equivalent to the more stringent condition but unlike in the differential geometric cateogry, this equivalence lies at a subtler level).

A morphism of a vector bundle ξ on M in another vector bundle η also on M is a smooth map $\Phi : E(\xi) \to E(\eta)$ such that $\pi(\eta) \circ \Phi = \pi(\xi)$ and ϕ induces for each $m \epsilon M$ a vector space homomorphism of $E_m(\xi)$ on $E_m(\eta)$. Evidently

the class of all vector bundles with these morphisms form a category.
Exercise show that this is an abelian category.

The simplest example of a vector bundle on M is of course the product $M \times \mathbf{C}^n$ (as the total space) and each $\{m\} \times \mathbf{C}^n$ given the vector space structure on the second factor. A vector bundle ξ of rank n on M is trivial if it is isomorphic to the above bundle.

A basic notion in the theory is that of induced bundles. Let ξ be a vector bundle on M and $f : M' \to M$ be a smooth map of a smooth manifold M' in M. Then we define a vector bundle ξ' to be denoted $f^*(\xi)$ on M' as follows. The total space $E(\xi')$ of ξ' is the fibre product of M' and $E(\xi)$ over M.

$$E(\xi') = \{(m', x) \epsilon M' \times E(\xi) \mid f(m') = \pi(\xi)(x)\}$$

$E(\xi')$ is a closed smooth submanifold and the carterian projection of $M' \times E(\xi)$ restricted to $E(\xi')$ is the smooth map $\pi(\xi')$. $E_{m'}(\xi') = \pi(\xi')^{-1}(m') = \{(m', v) \mid v \epsilon E_m(\xi)\}$ and acquires a vector space structure under this identification with $E_m(\xi)$. We leave it to the reader to check that ξ' is indeed a vector bundle on M'. A special case of this construction is the case when M' is an open set in M and f is the inclusion. In this situation $E(\xi')$ is naturally identified with $\pi(\xi)^{-1}(M')$. Property $(*)$ says then that for each $m \epsilon M$, there is an open subset U such that the induced bundle on U (also called restriction to U) by ξ is trivial. Over such an open set U, the inverse image under $\pi(\xi)$ is a product of U and a fixed vector space $\mathbf{C}^n$. A vector bundle on M may be viewed as a continuously varying family of vector spaces (parametrized by M).

Examples 1. Let M be a smooth manifold and for each $m \epsilon M$, let T_m denote the tangent space at m to M. We define the tangent vector bundle $\tau = \tau(M)$ of M as follows. The underlying set of the total space $E(\tau)$ of τ is the disjoint union $\cup_{m \epsilon M} T_m$ and $\pi(\tau)$ is defined by setting $\pi(\xi)(T_m) = m$. We make $E(\tau)$ into a smooth manifold in the following manner. Let $U_i, i \epsilon I$ be open sets in M admitting coordinate charts $\varphi_i : U_i \to \Omega_i (=$ open set in $\mathbf{R}^n)$. The coordinates $\{x^i_\alpha \mid 1 \leq \alpha \leq n\}$ define vector fields $\partial/\partial x^i_\alpha, 1 \leq \alpha \leq n$ in the open set U_i for each $i \epsilon I$. We then obtain a bijection $\Phi_i : \Omega_i \times \mathbf{R}^n \to \pi(\xi)^{-1}(U_i)$ by setting

$$\Phi_i(x, t_1, \cdots, t_n) = \sum_{1 \leq i \leq n} t_\alpha \partial/\partial x^i_\alpha \,|_{\varphi_i^{-1}(x)} .$$

$E(\tau)$ is topologised by requiring that a set U in $E(\tau)$ is open iff $\Phi_i^{-1}(U)$ is open for all $i\epsilon I$ and then made into a smooth manifold by taking $(\pi(\xi)^{-1}(U_i), \Phi_i^{-1})_{i\epsilon I}$ as an atlas. Observe that the Φ_i play the role of the Φ in our definition of a vector bundles.

2. Let M denote the Grassmann manifold of r dimensional subspaces of $\mathbf{C}^n (r \leq n)$. We define a vector bundle u of rank r on M as follows. The total space $E(u)$ of u is the smooth submanifold $\{(m,v)\epsilon M \times \mathbf{C}^n \mid v\epsilon \text{ vector subspace } m\}$. $E(u)$ is called the canonical vector bundle on the Grassmannian

3. Let S^n be the $n-$sphere $\{(x_1 \cdots x_{n+1})\epsilon \mathbf{R}^{n+1} \mid \sum_{1\leq i \leq n+1} x_i^2 = 1\}$. Let $E(\tau) = \{(x,v)\epsilon S^n \times \mathbf{R}^{n+1} \mid < x, v = 0\}$. Then $E(\tau)$ is the total space of a rank $n-$vector bundle τ on S^n with $\pi(\tau)(x,v) = x$ and the obvious vector space structure on $E_x(\tau) = \{(x,v) \mid v\epsilon \mathbf{R}^n, < x, v = 0\}$.

Exercise: show that $E(\tau)$ is isomorphic to the tangent bundle of S^n.

A host of other examples can be given starting with one or more vector bundles. Thus if ξ, τ are vector bundles on M one can construct the direct sum $\xi \oplus \tau$ of ξ and τ. The total space $E(\xi \oplus \tau)$ is the fibre product of $E(\xi)$ and $E(\tau)$ over M and $\pi(\xi \oplus \tau)$ is the natural map of $E(\xi \oplus \tau)$ in M. For $m\epsilon M$, then $E_m(\xi \oplus \tau) = \{(m, v, w) \mid v\epsilon E_m(\xi), w\epsilon E_m(\xi) \oplus E_m(\tau)\}$ and hence acquires a vector space structure. Slightly more involved is the construction of the tensor product $\xi \otimes \eta$ of ξ and η. The underlying set of the total space $E(\xi \otimes \eta)$ of $\xi \otimes \eta$ is $\cup_{m\epsilon M} E_m(\xi) \otimes E_m(\eta)$. Let $U_i, i\epsilon I$, be a covering of M by open subsets and $\Phi_i : U_i \times \mathbf{C}^n \to \pi(\xi)^{-1}(U_i)$ and $\psi_i : U_i \times \mathbf{C}^m \to \pi(\eta)^{-1}(U_i)$ be "trivialisations" of ξ and η over U_i. (such U_i, Φ_i, ψ_i exist). Let $\wedge_i : U_i \times (\mathbf{C}^m \otimes \mathbf{C}^n) \to \pi(\xi \otimes \eta)^{-1}(U_i)$ (here $\pi(\xi \otimes \eta) : E(\xi \otimes \eta) \to M$ is the map $\pi(\xi \otimes \eta)(E_m(\xi) \otimes E_m(\eta)) = m$) be the bijection $\wedge_i(m, v, w) = \Phi_i(m,v) \otimes \psi_i(m,w)$. We make $E(\xi \otimes \eta)$ into a smooth manifold by the requirement that a set $\Omega \subset E(\xi \otimes \eta)$ is open if and only if $\wedge_i^{-1}(\Omega)$ is open for all $i\epsilon I$ and the smooth structure is such that $\wedge_i$ is a diffeomorphism. The details to be checked are left to the reader. One can similarly define for any vector bundle ξ and an integer $m \geq 0$, its m^{th} symmetric or exterior power vector bundles which are denoted $S^m(\xi)$ and $\wedge^m(\xi)$ respectively. The dual ξ^* of a vector bundle ξ is again defined in a similar way: the fibres $E_m(\xi^*)$ of ξ^* are the duals of $E_m(\xi)$.

A smooth section of a vector bundle ξ over an open subset $U \subset M$ is a smooth map $\sigma : U \to E(\xi)$ such that $\pi(\xi)\sigma(m) = m$ for all $m\epsilon U$. We

denote by $\Gamma(U,\xi)$ the set of all smooth sections of ξ over U. If σ, σ' are two elements of $\Gamma(U,\xi)$ and f is complex valued C^∞ function on U, then we define $\sigma + \sigma' \epsilon \Gamma(U,\xi)$ and $f\sigma\epsilon\Gamma(U,\xi)$ by setting $(\sigma + \sigma')(m) = \sigma(m) + \sigma'(m)-$ note that both $\sigma(m)$ $\sigma'(m)$ are in $E_m(\xi)$ and $(f\sigma)(m) = f(m).\sigma(m)$. Thus $\Gamma(U,\xi)$ form a module over $C^\infty(U) = \mathbf{C}-$ valued C^∞ functions on U.

A smooth exterior $p-$ form α on M with values in a vector bundle ξ is an assignment to each $m\epsilon M$, an alternating $p-$ form α_m on $T_m(=$ tangent space at $m)$ with values in $E_m(\xi)$ satisfying the following conditions. Let U be any open set in M admitting a coordinate system and Φ a trivialisation of ξ over U. Let $x_1 \cdots x_n$ be the coordinates in U and $\partial/\partial x_1 \cdots \partial/\partial x_n$ the vector fields in U corresponding to these coordinates. Then the function $m \mapsto \Phi_m^{-1}\alpha_m(\partial/\partial x_{i_1} \cdots \partial/\partial x_{i_p})$ is a smooth (vector valued) function $\alpha_{i_1} \cdots i_p$ on U for every $p-$tuple $i_1 \cdots i_p$ with $1 \leq i_r \leq n$. In U, then one has

$$\alpha = \sum_{i_1 < \cdots < i_p} \alpha_{i_1} \cdots i_p dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where for $1 \leq i \leq n, dx_i$ is the differential 1- form $dx_i(\partial/\partial x_j) = \delta_{ij}$ and the exterior product is taken in the usual sense. In fact we define below the exterior multiplication of vector bundle valued forms in a more general context. The vector space of exterior $p-$forms on M with values in the vector bundle ξ will be denoted $\Omega^p(\xi)$ in the sequel. Observe that if $\rho : M' \to M$ is a smooth map and ξ is a vector bundle on M, then for each $m'\epsilon M$, there is a natural identification of $E_{m'}(\xi')$ with $E_{f(m)}(\xi)$ where ξ' is the bundle on M' induced by ξ through f. Consequently one can define for each $\alpha\epsilon\Omega^p(\xi)$ its inverse image $f^*(\alpha)\epsilon\Omega^p(\xi')$: for tangent vectors $v_1' \cdots v_p'$ to M' at m', let $v_1 \cdots v_p$ be their images in M; then $f^*(\alpha)(v_1' \cdots v_p') = \alpha(v_1 \cdots v_p)$ (note that $\alpha(v_1 \cdots v_p)\epsilon E_{f(m')}(\xi) = E_{m'}(\xi')$. $f^*(\alpha)$ is the "inverse image" of α under f.

We start with vector bundles ξ, η, ζ on M together with a vector bundle morphism

$$\mu : \xi \otimes \eta \to \zeta.$$

Let α (resp β) be a smooth exterior $p-$ (resp $q-$) form on M with values in ξ (resp. η). Then $\alpha \wedge \beta$ is the $(p+q)-$ form with values in ζ defined by the following formula: let $X_1 \cdots, X_{p+q}$ be vector fields (or tangent vectors at a point of m); then

$$(\alpha \wedge \beta)(X_1 \cdots X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma\epsilon S_{p+q}} \mu(\alpha(X_{\sigma(1)} \cdots X_{\sigma(p)}) \odot \beta(X_{\sigma(p+1)} \cdots X_{\sigma(p+q)})).$$

The case of Scalar forms corresponds to the case when ξ, η and ζ are given trivialisations and the map μ is the usual multiplication of functions). Observe that for ξ and η one has the natural (identity) map $id : \xi \otimes \eta \to \xi \otimes \eta$ so that for any α, β above one has a $\xi \otimes \eta$ valued form $\alpha \wedge \beta$ (with respect id) and the general case is obtained from this exterior product by composing with μ).

We also remark that $\xi-$ valued $p-$ forms on M are the same as sections of the bundle $\wedge^p \tau^* \otimes \xi$ where τ is the tangent bundle of M and τ^* is its dual.

§2. de Rham's theorem.

As in §1, let M be a smooth parecompact manifold of dimension n. For an open set U of M, let $\Omega^p(U)$ denote the vector space of all smooth complex valued exterior differential forms of degree p in U.

Recall that one has a differential operator

$$d : \Omega^p(U) \to \Omega^{p+1}(U)$$

defined as follows: let $X_1 \cdots X_{p+1}$ be smooth vector fields on U. Then for $\alpha \epsilon \Omega^p(U)$,

$$\begin{aligned} d\alpha(X_1 \cdots X_{p+1}) &= \sum_{1 \le i \le p+1} (-1)^{i+1} X_i \alpha(X_i \alpha(X_1 \cdots \widehat{X}_i \cdots X_{p+1}) \\ &\quad + \sum_{i<j} (-1)\alpha([X_1, X_j], X_1 \cdots \widehat{X}_i \cdots \widehat{X}_j \cdots X_{p+1}) \end{aligned}$$

One has of course to check that $d\alpha(X_1 \cdots X_{p+1})$ at point $m \epsilon U$ depends only $X_1(m) \cdots X_{p+1}(m)$ and not on the choice of line vector fields themselves. The operator d is local i.e., if α is zero on an open set $V \subset U$ so is $d\alpha$. It is clear from the definition that d is compatible with restrictions. For an open subset $U' \subset U$, the diagram

$$\begin{array}{ccc} \Omega^p(U) & \xrightarrow{d} & \Omega^{p+1}(U) \\ \downarrow & & \downarrow \\ \Omega^p(U') & \xrightarrow{d} & \Omega^{p+1}(U') \end{array}$$

is commutative. More generally if $f : M' \to M'$ is a smooth map one has a natural vector space homomorphism $f^* : \Omega^p(M^*) \to \Omega^p(M')$ given by $f^*(\alpha)(v_1 \cdots v_p) = \alpha(df(v_1) \cdots df(v_p))$ where $v_1 \cdots v_p$ are tangent vectors at a

point $m' \epsilon M'$ and df denotes the differential of f at m'. The diagram

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \\ f^* \downarrow & & \downarrow f^* \\ \Omega^p(M') & \xrightarrow{d} & \Omega^{p+1}(M') \end{array} \qquad (*)$$

is commutative.

It is well known and easy to deduce from the definitions made above that we have the following two properties for d:

$$\begin{aligned} d(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \\ d^2\alpha &= 0 \end{aligned}$$

where α (resp β) is a p form (resp. q form). The second condition shows that $\Omega^\circ(M) = (\prod_p \Omega^p(M), d)$ is a cochain complex and the second one shows that the exterior multiplication of forms gives rise to a product again denoted $\wedge$ on the total cohomology

$$H^*(\Omega^\circ(M)) = \prod_{p \ge 0} H^p(\Omega^\circ(M))$$

of the complex $\Omega^\circ(M)$; further one has for cohomology classes $\alpha \epsilon H^p(\Omega^\circ(M)), \beta \epsilon H^q(\Omega^*(M))$ one has $\alpha \wedge \beta = (-1)^{p+q} \beta \wedge \alpha$ (since such a formula holds for forms themselves). Thus $H^*(\Omega^\circ(M))$ is a graded (anti-) commutative ring for any manifold. The commutative diagram $(*)$ shows that $M \to \Omega^\circ(M)$ is a contravariant functor from the category of smooth manifolds into the category of (cochain) complexes of vector spaces over $\mathbf{C}$ and hence that $M \mapsto H^*(\Omega^\circ(M))$ is a functor from the category of smooth manifolds into the category of graded commutative algebras.

de Rham's theorem asserts that $H^*(\Omega^\circ(M))$ is a topological invariant of M ie. it depends only on the underlying topology of M and not on the C^∞ structure. (although $\Omega^\circ(M)$ depends heavily on the C^∞ structure for its definition). The theorem carries in fact more precise information and we outline below the necessary back-ground from algebraic topology.

Let $\triangle^p$ denote the compact convex subset $\{\underline{x} = (x_1 \cdots x_{p+1}) \epsilon \mathbf{R}^{p+1} \mid x_i \ge 0, \sum_{1 \le i \le n+1} x_i = 1\}$ of $\mathbf{R}^{p+1}$. It is in fact contained in the linear subspace $L_p = \{\underline{x} = (x_1 \cdots x_{p+1}) \epsilon \mathbf{R}^{p+1} \mid \sum_{1 \le i \le n+1} x_q = 1\}$. A map $f : \triangle^p \to M, (M$ a smooth manifold) is smooth, if there is a open neighbourhood U of Δ^p in L and a smooth function $\tilde{f} : U \to M$ such that $\tilde{f}$ restricts to f on Δ^p. A *singular p-simplex* (resp smooth singular simplex) in M is a continuous (resp smooth) map of Δ^p in M. Let $S_p(M)$ (resp. $S_p^{diff}(M)$) denote the set of singular (resp smooth singular) simplices in M. Clearly we have an inclusion.

$$S_p^{diff}(M) \xrightarrow{i} S_p(M).$$

Now, let $C^p(M)$ resp $C^p_{diff}(M)$ denote the vector space of all complex valued continuous functions on the set $S_p(M)$ (resp $S_p^{diff}(M)$). The above inclusion clearly induces a *surjective* vector-space homomorphism

$$i_p^* : C^p(M) \to C^p_{diff}(M).$$

Next we will introduce a "coboundary" homomorphisms $\partial : C^p(M) \to C^{p+1}(M)$ and $\partial : Cdiff^p(M) \to C^{p+1}_{diff}(M)$ compatible with i_p^* and i_{p+1}^* and such that $\partial^2 = 0$ in both cases. To define ∂ we first observe that we have for each $1 \le i \le p+1$,inclusions $u_i : \Delta^p \to \Delta^{p+1}$ defined by

$$u_i(x_1 \cdots x_{p+1}) = (x_1 \cdots x_{i-1}, 0, x_i \cdots x_{p+1}).$$

If $f : \Delta^{p+1} \to M$ is a singular (resp smooth singular) $(p+1)-$ simple x in M evidently $f \circ u_i$ is a singular (resp smooth singular) $p-$simplex in M. For $\varphi \epsilon C^p(M)$ (resp. $C^p_{diff}(M)$), and $1 \le i \le p+1$, let $\partial_i \varphi \epsilon C^{p+1}(M)$ (resp. $C^{p+1}_{diff}(M)$) is defined by setting $\partial_i \varphi(f) = \varphi(f \circ u_i)$ and let

$$\partial \varphi = \sum_{1 \le i \le p+1} (-1)^{i+1} \partial_i \varphi.$$

Then one has $\partial^2 = 0$ in both $C^\circ(M) = \prod_{p \ge 0} C^p(M) C^\circ_{diff}(M) = \prod_{p \ge 0} C^p_{diff}(M)$. One sees easily too that $M \mapsto C^\circ(M)$ and $\mapsto C^\circ_{diff}(M)$ are both functors on the category of smooth manifolds. Also the diagram (for each M)

$$\begin{array}{ccc} C^p(M) & \xrightarrow{\partial} & C^{p+1}(M) \\ \downarrow & & \downarrow \\ C^p_{diff}(M) & \xrightarrow[\partial]{} & C^{p+1}_{diff}(M) \end{array}$$

is commutative.

Less obvious, though standard topology techniques yield it, is the following

Theorem. The natural homomorphism $C^\circ(M) \to C^\circ_{diff}(M)$ induces isomorphism in the cohomology groups. (which will be denoted $H^p(M)$ in the sequel).

Our next step is to define a homomorphism of complexes of $\Omega^\circ(M)$ in $C^\circ_{diff}(M)$. Let then $\alpha \epsilon \Omega^p(M)$. Suppose that $f : \Delta^p \to M$ is a smooth map. Then the pull back $f^*(\alpha)$ is a $C^\infty p-$ form defined in a neighbourhood of Δ^p in L_P. Now $(x_1 \cdots x_p) \mapsto (x_1 \cdots x_p, 1 - \sum_{1 \le i \le p} x_p)$ gives a standard identification of $\mathbf{R}^p$ with L_p and under this identification Δ^p corresponds to the compact set $\Delta_1^p = \{\underline{x} = (x_1 \cdots x_p) \epsilon \mathbf{R}^p \mid x_i \ge 0, \sum x_i \le 1\}$. Now $f^*(\alpha)$, a $p-$form on $\mathbf{R}^p$ can be expressed uniquely as

$$f^*(\alpha) = \underline{\alpha} dx_1 \wedge \cdots \wedge dx_p$$

where $\underline{\alpha}$ is a smooth function in a neighbourhood of Δ_1^p in $\mathbf{R}^p$. We then set

$$deR(\alpha)(f) = \int_{\Delta_1^p} \underline{\alpha} d\mu_p$$

where $d\mu_p$ is the standard Lebesgue measure on $\mathbf{R}^p$.

Theorem. $deR : \Omega^p(M) \to C^p_{diff}(M)$ defines a homomorphism of complexes $\Omega^\circ \to C^\circ$ inducing an isomorphism in the cohomology groups.

The first assertion is immediate from Stoke's theorem. The second in conjunction with the earlier theorem shows that $H^p(\Omega(M))$ is a topological invariant of M. A proof of the theorem is beyond the scope of these notes.

We need one important fact, viz. that f and g are smooth maps of a manifold M' into a manifold M which are homotopic by a smooth homotopy then the map induced by f and g on $H^*(\Omega(M'))$ are the same.

§3. Principal Bundles

The notion of a principal G-bundle, G a Lie group is closely related to the concept of vector bundles and serves to clarify many ideas connected with vector bundles. We begin with the definition.

A principal G-bundle or simply a G-bundle ξ on M is a smooth manifold $P(\xi)$, a smooth map $\pi(\xi) : P(\xi) \to M$ and a smooth right action $P(\xi) \times G \to P(\xi)$ of G on $P(\xi)$ satisfying the following conditions

(i) $\pi(\xi)(xg) = \pi(\xi)(x)$ for $x \in P, g \in G$

(ii) For every $m \in M$, there is an open set U with $m \in U$ and a diffeomorphism $\Phi : U \times G \to \pi(\xi)^{-1}(U)$ such that

$$\Phi(u, gh) = \Phi(u, g).h$$

(for $x \in P$ and $g \in G, xg$ is the image of x under the action of g).Φ is called a local trivialisation of ξ over U. For $m \in M, P_m(\xi) = \pi(\xi)^{-1}(m)$ is the *fibre* over m. Also $P(\xi)$ is the *total space of* ξ. For Φ as above, the map $g \mapsto \Phi(m, g)$ is a diffeomorphism of G on $\pi(\xi)^{-1}(m)$ compatable with the right translation action of G on G and the given action on $\pi(\xi)^{-1}(m)$. We see thus that G acts *simply trnasitively on the fibres of* $\pi(\xi)$. (As with vector bundles the local triviality property of a principal bundle can be replaced with an equivalent condition that would at first glance appear weaker. The interest in such a weakening actually stems from attempting a definition of principal bundles in the algebraic geometric category where local triviality for the Zariski topology would be an unsatisfactory assumption as it would exclude even a very natural situation like the morphism $H \to H/G$ where H is an algebraic group and G is an algebraic subgroup).

The property (*) can be replaced by the following condition : $\pi(\xi)$ is of maximal rank everywhere in $P(\xi)$ and for each $m \in M, G$ acts simply transitively on $\pi(\xi)^{-1}(m)$. This is essentially a consequence of the implicit function theorem and Saard's theorem and is left as an exercise to the reader.

If ξ and η are G-bundles, a morphism of ξ in η is a smooth map $f : P(\xi) \to P(\eta)$ such that $\pi(\eta)f(x) = \pi(\xi)(x)$ for all $x \in P(\xi)$ and $f(xg) = f(x).g$ for all $g \in G$ and $x \in P(\xi)$. Evidently under these morphisms the G-bundles on M form a category. Note that *all morphisms* in this category are *isomorphisms*.

An obvious example of a principal bundle is the product $M \times G$ with G acting on the second factor through right translations: $(m, g)h = (m, gh)$ for $m \in M$ and $g, h \in G$. A G-bundle ξ is *trivial* if it is isomorphic to this bundle.

If $f : M' \to M$ is a smooth map and ξ is a principal G-bundle on M, one defines the induced G-bundles on $M', f^*(\xi)$ as follows: set $f^*(\xi) = \eta$; then $P(\eta)$ is the fibre product of M' and $P(\xi)$ over M - it is easy to see that $P(\eta)$ is a smooth submanifold of $M' \times P(\xi)$ and is stable under the action of G (acting on $M' \times P(\xi)$ through the second factor).

The local triviality condition (*) says that for each $m \in M$ there is a neighbourhood U of m such that the G-bundle induced by the inclusion $U \hookrightarrow M$ from ξ is trivial. The following lemma is often useful

Lemma. A principal G-bundle ξ over M is trivial if and only if it admits a smooth section if there is a smooth map $\sigma : M \to P$ such that $\pi(\xi).\sigma(m) = m$ for all $m \in M$.

Proof. If ξ is trivial, one has an isomorphism Φ of $M \times G$ with $\xi - \Phi : M \times G \to P(\xi)$ and we need only define σ by $\sigma(m) = \Phi(m, 1)$ ($1 \equiv G$ the identity element). Conversely if σ is a section defined $\Phi : M \times G \to P(\xi)$ by setting $\Phi(m, g) = \sigma(\xi).g$. One checks that Φ is an isomorphism.

Examples 1. Let $S^{m+1} = \{z \in^{n+1} \mid \mid z \mid^2 = \sum_{1 \le i \le n+1} \mid z_i^2 \mid = 1\}$. The group $S^1 = \{z \in^* \mid \mid z \mid = 1\}$ acts on S^{2n+1} by $v, z \mapsto v.z$, $v \in^{n+1}, z \in S^1$ and the quotient is the complex projective space $P^n()$ of (complex) dimension n. We have then the natural map $S^{2n+1} \to P^n()$ and this with the action of S^1 gives a principal S^1 -bundle. The reader may check the details.

2. Let H be a Lie group and G a *closed* subgroup. Then $H \xrightarrow{\pi} H/G$ with the right translation action of G on H is a G-bundle. The local triviality condition is a consequence of a well known fact from Lie theory viz. that for any point $m_0 \in H/G$, there is an open neighbourhood U of m_0 and a smooth map $\sigma : U \to H$ such that $\pi.\sigma(m) = m$ for all $m \in U$.

3. More generally - and the first example is a special case of this, if H is a Lie group and H', H'' closed subgroups and H'' is a *normal* subgroup of H' and we set $G = H'/H''$, then one has a natural action of G on G/H'' on the right : for $g \in G$ and a coset xH'' we define $xH''.g = x\tilde{g}H''$ where $\tilde{g}$ is any lift of g to H'. This action of G on H/H'' together with the smooth map $H/H'' \to H/H'$ is a principal G-bundle on G/H'.

4. This example establishes the connection between vector bundles and principal bundles. Let ξ be a complex vector -bundle of rank n over M. We consider the set $\{(m, \varphi) \mid m \in M, \varphi$ is a vector space isomorphism of ^{n} on $E_m(\xi)\}$. Then $P_m(\xi)$ can be identified with an *open* set in $E(\xi \underbrace{\oplus \cdots \oplus}_{n \text{ copies}} \xi)$ (exercise) and is thus a smooth manifold. One defines an action of $GL(n,)$ on $P(\xi)$ by setting $(m, \varphi).g = (m, \varphi.g)$ for $(m, \varphi) \in P(\xi)$ and $g \in GL(n,)$. Thus to each vector bundle ξ of rank n on M. Conversely if ξ is principal $GL(n,)$ bundle on M we construct a vector bundle ξ_0 on M as follows: Consider the product $P(\xi) \times^n .GL(n,)$ acts on this space by the rule

$$(x,v)g = (x.g, g^{-1}(v))$$

$x \in P(\xi), v \in^n$ and $g \in GL(n,)$. Let $E(\xi_0)$ be the quotient of $P(\xi)\times^n$ for this action and $\pi(\xi_0) : E(\xi_0) \to M$ the map induced by the map $P(\xi)\times^n \to M$ given by $(x,v) \mapsto \pi(\xi)(x)$. Then for each $x \in P(\xi)$ with $\pi(\xi)(x) = m$, the composite map $^n \to E_m(\xi_0)$ given by $v \mapsto$ image of (x,v) is a bijection and we may transport the vector space structure on ^{n} by this bijection to $E_m(\xi_0)$: the structure so obtained is independent of the choice of $x \in \pi(\xi)^{-1}(m)$ since any two such x are in the same $GL(n,)$ orbit and $GL(n,)$ acts *linearly* on ^{n}. One checks easily that if ξ is a $GL(n,)$-bundle, $(\xi_0)^* \simeq \xi$ and similarly if ξ is a vector bundle of rank n, $(\xi^*)_0 \simeq \xi$. Thus there is a categorical equivalence between principal $GL(n,)$ bundles and rank n vector bundles on M (with morphisms as isomorphisms).

The last construction which associates to each $GL(n,)$-bundle a rank n vector bundle is a special case of a more general situation. We start with a G-bundle ξ on M and a continuous, hence smooth, representation ρ of G on a vector space $V(\rho)$. To this data we can associate a vector bundle ξ_ρ on M as follows: One forms the quotient $E(\xi_\rho) = (P(\xi) \times V(\rho))/G$ for the diagonal action $(x,v).g = (xg, \rho(g)^{-1}(v))x \in P(\xi), v \in V(\rho), g \in G$ of G on $P(\xi)\times V(\rho)$. The map $\pi(\xi) : P \to M$ enables one to define a map $E(\xi_\rho) \to M$ and the composite map $v \mapsto$ Image (x,v) in $E(\xi_\rho)$ gives an identification of $V(\rho)$ with $E_{\pi_\xi(x)}(\xi_\rho)$ and hence a vector space structure (that depends only on $\pi_\xi(x)$) making $E(\xi_\rho)$ into a vector bundle.

Observe that $\rho \mapsto \xi_\rho$ (for a fixed principal bundle ξ) is a functor from the category of G-modules into the category of vector bundles on M. (Define the morphism $\xi_f : \xi_\rho \to \xi_{\rho'}$ corresponding to each G-module morphism $f : V(\rho) \to V(\rho')$).

§4. **Connections**

Let ξ be a G-bundle on M. Then for each $X \in$ Lie G, the Lie algebra of G, we have a 1-parameter group of diffeomorphisms of $P(\xi)$ viz

$$x \mapsto x.\exp tX, -\infty < t < \infty$$

where exp : Lie $G \to G$ is the exponential map. It follows that each $X \in$ Lie G defines a vector field on $P(\xi)$. Observe that any such is tangential to the

fibres of $\pi(\xi)$ so that its projects to zero under $d\pi(\xi)$. The map $X \mapsto$ is an injective Lie algebra homomorphism of Lie G into the Lie algebra of vector fields on $P(\xi)$ (tangential to the fibres of $\pi(\xi)$). Since the map $g \mapsto xg$ for fixed $x \in P(\xi)$ is a diffeomorphism of G on $\pi(\xi)^{-1}(\pi(\xi)(x))$, one sees that for $\alpha \in P(\xi), \mapsto (x)$ is an isomorphism of Lie G on the tangent space $V(x)$ to the fibre of $\pi(\xi)$ through x. We denote the inverse $V(x) \mapsto$ Lie G of this isomorphism by $\lambda(x) : \lambda(x)((x)) = X$. For a tangent vector v to $P(\xi)$ at $x \in P(\xi)$ and $g \in G$ we denote by $v.g$ the image of v under the map $T_x \to T_{xg}$ of tangent spaces (T_x and T_{xg} at x and xg) induced by the smooth map $y \mapsto yg$ of $P(\xi)$ into itself. With this notation one checks easily that

$$(x).g = (Ad_g^{-1}X)(xg) \qquad (*)$$

where $Ad : G \to GL($ Lie $G)$ is the adjoint representation. Equivalently one has

$$\lambda(xg)(vg) - Adg^{-1}\lambda(x)(v) \qquad (*')$$

for any vector $v \in V(x)$ (note that $vg \in V(xg)$).

A *connection* on ξ is a smooth Lie G- valued 1 form ω on $P(\xi)$ such that

(i) $\omega(vg) = Adg^{-1}\omega(v)$ for $v \in T_x, x \in P(\xi)$ and $g \in G$.

(ii) $\omega \mid_{V(x)} = \lambda(x)$.

(In view of $(*')$ (ii) is consistent with (i)). Since ω maps $V(x)$ isomorphically on to Lie G and $V(x)$ is precisely the kernel of the tangent map $d\pi(\xi)_x$: $T_x \to T_{\pi(\xi)(x)}$, we conclude that $T_x \simeq V(x) \oplus H(x)$ where

$$H(x) = \text{Kernel } (\omega : T_x \to \text{Lie } G).$$

$H(x)$ is called the *horizontal space* of the connection at x. For a tangent vector $v \in T_x$ denote by $H(v)$ its component in $H(x)$ for the above direct sum decomposition and call it the horizontal projection. In view of (i), one has evidently $H(x).g = H(xg)$ for all $g \in G$ and $x \in P(\xi)$. For $v \in T_m, m \in M$, and $x \in P(\xi)$ with $\pi(\xi)(x) = m$, the unique vector $\tilde{v} \in H(x)$ that maps to v under the tangent map of $\pi(\xi)$ is the *horizontal* lift of v at x. Any two horizontal lifts of v (at two points $P(\xi)$) are mutual transforms under an element of G. If $X_1 \cdots X_n$ are vector fields on an open set $U \subset M$, their horizontal lifts $\widetilde{X}_1 \cdots \widetilde{X}_n$ to $P(\xi)$ (defined by $\widetilde{X}_i(x) = (X_i(\pi(\xi)(x))^\sim)$ are such that for each $x \in P(\xi), \{\widetilde{X}_i(x) \mid 1 \leq i \leq n\}$ is a basis of $H(x)$. This shows that the subset $\cup_{x \in P(\xi)} H(x)$ is indeed a sub-bundle of $T(P(\xi))$ the tangent bundle of $P(\xi)$. The map $X \mapsto$ of Lie G into vector fields on $P(\xi)$ on the other hand gives us an isomorphism of the trivial bundle on (the sub bundle) $\cup_{x \in P(\xi)} V(x)$. Thus a connection gives a direct-sum decomposition of the tangent bundle $T(P(\xi))$ of $P(\xi)$:

$$T(P(\xi)) = V(\xi) \oplus H(\xi).$$

Moreover the two sub-bundles are stable under the tangent action of G on $T(P(\xi))$.

A connection on ξ enables one to define differentiation of sections of associated vector bundles to ξ. Thus let ρ be a representation of G on a vector space $V(\rho)$ and let ξ_ρ denote the associated vector bundle. We will first identify smooth sections of ξ_ρ over an open set $U \subset M$ with certain $V(\rho)$-valued smooth functions on $\pi(\xi)^{-1}(U)$. Recall that $E(\xi_\rho)$ is defined as the quotient of $P(\xi) \times V(\rho)$ under the diagonal action of G : for $x \in P(\xi), v \in V(\rho)$ and $g \in G, (x, v).g = (xg, \rho(g)^{-1}v)$. Let $\Theta : P(\xi) \times V(\rho) \to E(\xi_\rho)(= P(\xi) \times V(\rho)/G)$ be the natural map. For $x \in P(\xi)$ let $\Theta(x)$ be the isomorphism of $V(\rho)$ on $E_{\pi(\xi)(x)}(\xi(\rho))$ defined by $\Theta(x)(v) = \Theta(x, v), v \in V(\rho)$. Suppose now $\sigma : U \to E(\xi_\rho)$ is a smooth section so that $\sigma(m) \in E_m(\xi_\rho)$ for all $m \in U$, we can define $\tilde{\sigma} : \pi(\xi)^{-1}(U) \to V(\rho)$ as the smooth function

$$\tilde{\sigma}(x) = \Theta(x)^{-1}\sigma(\pi(\xi)(x)).$$

One then checks easily that $\tilde{\sigma}$ satisfies the condition

$$\tilde{\sigma}(xg) = \rho(g)^{-1}\tilde{\sigma}(x). \qquad (*)$$

for all $x \in \pi(\xi)^{-1}(U)$ and $g \in G$. Conversely if $\tilde{\sigma}$ is $V(\rho)$ valued smooth function on $\pi(\xi)^{-1}(U)$ satisfying (*) above, $\sigma(\pi(\xi)(x)) = \Theta(x)\tilde{\sigma}(x)$ is well defined and gives a section of ξ_ρ over U. Thus we see that there is a natural isomorphism of the vector space $\Gamma(U, E(\xi_\rho))$ of smooth sections of ξ_ρ over U on the vector space of smooth $V(\rho)$-valued functions $\tilde{\sigma}$ on $\pi(\xi)^{-1}(V)$ satisfying (*).

More generally suppose α is an exterior differential p-form on an open set U of M with values in $E(\xi_\rho)$, we define a $V(\rho)$-valued exterior differential form $\tilde{\alpha}$ on $\pi(\xi)^{-1}(U)$ as follows: let $v_1 \cdots v_p$ be tangent vectors at a point $x \in p(\xi)^{-1}(U)$. Let $\overline{v}_i$ be the image of v_i under the tangent map of $\pi(\xi)$. Then $\tilde{\alpha}(v_1, \cdots, v_p) = \Theta(x)^{-1}\alpha(\overline{v}_1, \cdots \overline{v}_p)$. One checks easily the following:

(i) $\tilde{\alpha}(v_1 \cdots v_p) = 0$ if some $v_i \in V(x)$.

(ii) $\tilde{\alpha}(v_1 g \cdots v_p g) = \rho(g)^{-1}\tilde{\alpha}(v_1 \cdots v_p)$.

The map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism of the vector space $\Omega^p(\xi_\rho)$ of smooth ξ_ρ valued exterior p-forms on the space of $V(\rho)$-valued smooth exterior p-forms $\tilde{\alpha}$ on $\pi(\xi)^{-1}(V)$ satisfying (i) and (ii) above.

The identification of sections of ξ_ρ over U with suitable functions on $\pi(\xi)^{-1}(U)$ enables one to define differentiation of these sections by tangent vectors to U. Let $v \in T_m, m \in U$. Let σ be a smooth section of ξ_ρ over U. Then we define $D_v\sigma$ to be the element of $E_m(\xi_\rho)$ given by $\Theta(x)(\tilde{v}.\tilde{\sigma})$ where x is any point of $\pi(\xi)^{-1}(m), \tilde{v}$ is the horizontal lift of v at x and $\tilde{\sigma}$ is the $V(\rho)$-valued function associated to σ. It is easily checked that this definition is independent of the choices involved. If X is a smooth vector field on U, one has a vector space homomorphism

$$D_X : \Gamma(U, \xi_\rho) \to \Gamma(U, \xi_\rho)$$

given by $(D_X\sigma)(m) = D_{X(m)}\sigma$. This differential operator has the following properties:

$$D_X f.\sigma = Xf.\sigma + fD_X\sigma$$

for $\sigma \in \Gamma(U, \xi_\rho)$ and f a smooth complex valued function. Secondly one has

$$D_{fX}\sigma = f.D_X\sigma$$

(Note that one has for $v \in T_m$ itself $D_u(f.\sigma) = vf.\sigma + fD_v\sigma$). The operation that assigns to each section σ of ξ_ρ defined in a neighbourhood of a point $m \in M$, the element $D_v\sigma \in E_m(\xi_\rho)$ is called covariant differentiation of σ with respect to v associated to the connection ω on ξ. If ρ is the trivial representation on (the vector space $V(\rho)$), ξ_ρ is naturally isomorphic to $M \times V(\rho)$ and sections σ of ξ_ρ are $V(\rho)$-valued functions on M while $\tilde{\sigma}$ are simply $G-$ *invariant* $V(\rho)$-valued functions $P(\xi)$. All this leads to the conclusion that $D_X\sigma = X\sigma$ where σ is treated as a $V(\rho)$-valued function on M through the identifications above.

Suppose now ξ is a G-bundle on M and ω a connection on ξ. Let $f : M' \to M$ be a smooth map. Then one has the induced bundle $f^*(\xi)$ on M whose total space $P(f^*(\xi)) = \{(m', x) \in M' \times P(\xi) \mid f(m') = \pi(\xi)(x)\}$. The map $\tilde{f}$ defined by $(m', x) \mapsto x$ of $P(f^*(\xi))$ in $P(\xi)$ is evidently compactible with the G action on both the spaces. It is then clear that $\tilde{f}^*(\omega)$ is a connection of $f^*(\xi)$. This is called the *induced connection.*

§ 5. Exterior differentiation and the Curvature Form

In §2 we saw the definition of a differential operator $d : \Omega^p(M) \to \Omega^{p+1}(M)$ on ordinary exterior differential forms. If now we have a principal G-bundle ξ with a connection ω on it, we can extend this notion to exterior forms with values in the vector bundle ξ_ρ associated to ξ and a representation ρ of G. The extension is straightforward once we have the covariant differentiation with respect to the connection introduced in §4. If $\alpha \in \Omega^p(\xi_\rho)$ is a p-form with values in ξ_ρ, $d_\omega\alpha$ is the element of $\Omega^{p+1}(\xi_\rho)$ defined by the formula

$$\begin{aligned} d_\omega\alpha(X_1\cdots X_{p+1}) &= \sum_{1\le i\le p+1}(-1)^{i+1}D_{X_i}\omega(X_1\cdots\widehat{X_i}\cdots X_{p+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1\cdots\widehat{X_i}\cdots\widehat{X_j}\cdots X_{p+1}). \end{aligned}$$

That this definition depends only on the values of the X_i at a point m and not on the vector fields themselves is checked easily - as in the case of the trivial representation when we have to deal with ordinary forms. The definition of d_ω is natural to ensure some properties that one expects. If $u : V(\rho) \to V(\rho')$ is a G-module homomorphism, one has a corresponding vector bundle homomorphism $\xi_u : \xi_\rho \mapsto \xi_{\rho'}$ leading to a homomorphism $\Omega^p(\xi_\rho) \to \Omega^p(\xi_{\rho'})$ denoted ξ_u as well. One then has

$$d_\omega(\xi_u.\alpha) = \xi_u.(d_\omega\alpha).$$

Next suppose that one has a G-module morphism $\mu : V(\rho)\otimes V(\rho') \to V(\tau)$ for some representations ρ, ρ' and τ and $\alpha \in \Omega^p(\xi_\rho), \beta \in \Omega^q(\xi_{\rho'})$, then

$$d_\omega(\alpha\wedge\beta) = d_\omega\alpha\wedge\beta + (-1)^p\alpha\wedge d_\omega\beta$$

where the exterior product is with respect to μ (so that all the forms in the above equation are ξ_τ-valued). When ρ is the trivial representation one has an identification of $\Omega^p(\xi_\rho)$ with $\Omega^p(M)$ under which d_ω gets identified with d. However it should be noted that there can be situations in which ξ_ρ is trivial as a vector bundle while ρ may be non-trivial. In such cases there is no "natural" identification of ξ_ρ is with the trivial bundle and hence d_ω need not coincide with d.

In general d_ω^2 (unliked d^2) need not be the zero operator. We will now examine the operator d_ω^2 more closely. It will turn out that $d_\omega^2 : \Omega^p(\xi_\rho) \to \Omega^{p+2}(\xi_\rho)$ is in fact linear over $\Omega^0(M)$-the algebra of complex valued C^∞ functions on M. Towards this end we will first prove the following formula

Proposition. Let $\alpha \in \Omega^p(\xi_\rho)$ and $\tilde\alpha$ the $V(\rho)$-valued form on $P(\xi)$ corresponding to α. Then

$$(d_\omega\alpha)^\sim = d\tilde\alpha + \omega\wedge\tilde\alpha.$$

Equivalently

$$(d_\omega\alpha)^\sim(X_1\cdots X_{p+1}) = d\tilde\alpha(H(X_1)\cdots H(X_{p+1})).$$

Here the exterior multiplication of the $V(\rho)$-valued from α and the $L(G)$ valued form ω is with respect to the bilinear pairing

$$\text{Lie } G\otimes V(\rho) \to V(\rho)$$

given by $(X, v) \mapsto \rho(X)v$.

Proof. Locally on M we can write α as a linear combiantion of forms of the form $\sigma\wedge\alpha_0$ where σ is a section of ξ_ρ and α_0 is an ordinary p-form. One then has $d_\omega(\sigma\wedge\alpha_o) = d_\omega\sigma\wedge\alpha_0 + \sigma\wedge d_\omega\alpha_0 = d_\omega\sigma\wedge\alpha_0 + \sigma\wedge d\alpha_0$. Since $\omega\wedge\beta = 0$ for any scalar forms (which are identified with elements of $\Omega^p(\xi_1)$, 1 being the trivial representation, one sees that for proving the first assertion it suffices to consider the case when $p = 0$. In this case one has

$$\begin{aligned} (d_\omega\alpha)^\sim(X) &= d\tilde\alpha(H(X)) = H(X)\tilde\alpha \\ &= (X - \omega(\mathbf{X}))\tilde\alpha = X\tilde\alpha + \rho(\omega(X))\tilde\alpha \end{aligned}$$

since $\tilde\alpha(xg) = \rho(g)^{-1}\tilde\alpha(x)$ for $x \in P(\xi)$ and $g \in G$; and $d\tilde\alpha(X) = X\tilde\alpha$. Thus we have

$$(d_\omega(\alpha))^\sim(X) = d\tilde\alpha(X) + \omega\wedge\tilde\alpha(X)$$

proving the first assertion. For proving the second we note that since $H(H(X_i)) = H(X_i)$ and $(d_\omega\alpha)^\sim(X_1\cdots X_{p+1}) = (d_\omega\alpha)^\sim(H(X_1)\cdots H(X_{p+1}))$ we may assume that $X_i = H(X_i)$. In that case $\omega(X_i) = 0$ for all i so that $(\omega\wedge\tilde\alpha)(X_1\cdots X_{p+1}) = 0$. Thus

$$(d\tilde\alpha + \omega\wedge\tilde\alpha)(X_1\cdots X_{p+1}) = d\alpha(H(X_1)\cdots H(X_{p+1})$$

proving the proposition.

Assume $p = 0$ and we will now calculate $(d^2_\omega\alpha)^\sim(X,Y)$ for X, Y vector fields on $P(\xi)$. Let $\Theta = d_\omega\alpha$; then one has.

$$\begin{aligned}
(d\Theta)^\sim(X,Y) &= d\tilde{\Theta}(X,Y) + (\omega\wedge\tilde{\Theta})(X,Y)\\
&= X.\tilde{\Theta}(Y) - Y\tilde{\Theta}(X) - \tilde{\Theta}([X,Y]\\
&+ \dot{\rho}\,(\omega(X))\tilde{\Theta}(Y) - \dot{\rho}\,(\omega(Y))\tilde{\Theta}(X)\\
&= XY\tilde{\alpha} + X(\dot{\rho}\,(\omega(Y)\tilde{\alpha}) - YX\tilde{\alpha} - Y(\rho(\omega(X))\tilde{\alpha})\\
&- [X,Y]\tilde{\alpha} - \dot{\rho}\,\omega([X,Y])\tilde{\alpha}\\
&+ \dot{\rho}\,(\omega(X))Y\tilde{\alpha} + \dot{\rho}\,(\omega(X))\,\dot{\rho}\,(\omega(X))\tilde{\alpha}\\
&= (X.\,\dot{\rho}\,(\omega\,(Y)))\tilde{\alpha} - (Y.\,\dot{\rho}\,(\omega(X))\tilde{\alpha}\\
&- \dot{\rho}\,(\omega([X,Y])\tilde{\alpha} + \dot{\rho}\,([\omega(X),\omega(Y)])\tilde{\alpha}\\
&= dw(X,Y)\tilde{\alpha} + (\omega\wedge\omega)/2(X,Y)\tilde{\alpha}
\end{aligned}$$

where the exterior multiplication of Lie G valued forms is with reference to the bilinear product Lie $G\times$ Lie $G \to$ Lie G given by $(A,B)\mapsto[A,B]$, $A,B\in$ Lie G.

We have thus proved

Theorem. We define K^*_ω to be the Lie G valued 2-form on $P(\xi)$ by

$$K^*_\omega(X,Y) = d\omega(X,Y) + (\omega\wedge\omega/2)(X,Y),$$

then one has for any $\sigma\in\Gamma(U,\xi_\rho)$,

$$(d^2_\omega\alpha)^\sim = K^*_\omega\wedge\tilde{\alpha}.$$

Now if X is of the form with $A\in$ Lie G, we have

$$d\omega(X,Y) = d\omega(,Y) = \omega(Y) - \omega([A,Y]).$$

If in addition we assume that Y is horizontal, we find that $d\omega(X,Y) = 0$. Note that since

$$\omega([\mathbf{A},Y]) = \omega(d/dt(X.exptA\mid_{t=0}) = d/dt\mid_{t=0}\omega(X.exptA) = 0.$$

On the other hand $(\omega\wedge\omega)(,) = 0$ if is horizontal. If $=$ with $B\in$ Lie G one has

$$\begin{aligned}
d\omega(,) &= \omega() - \omega(\overline{A}) - \omega([,])\\
&= \omega[A,B]
\end{aligned}$$

while $(\omega\wedge\omega)/2(A,B) = [A,B]$ as well. Thus one has $K^*_\omega(,Y) = 0$ if Y is horizontal or $Y =$ with $B\in$ Lie G. The space $H(x)$ and $\{(x)\mid B\in$ Lie $G\}$ span all of T_x. Moreover since $\omega(v.g) = Adg^{-1}\omega(v)$ for any tangent vector v to $P(\xi)$ and $g\in G$, we conclude that

$$K^*_\omega(vg,wg) = \rho(g)^{-1}K^*_\omega(v,w)$$

for tangent vectors v, w to $P(\xi)$ and $g\in G$. We conclude that $K^*_\omega = \widetilde{K}_\omega$ for a suitable ξ_{Ad} valued 2-form K_ω on M so that we may write

$$d^2_\omega\alpha = K_\omega\wedge\alpha$$

for $\alpha\in\Omega^0(\xi_\rho)$.

Corollary. $d^2_\omega\alpha = K_\omega\wedge\alpha$ for $\alpha\in\Omega^p(\xi_\rho)$

Proof. Locally we may write α as a linear combination of forms of the type $\sigma\wedge\alpha_0$, α_0 is a scalar p-form and $\sigma\in\Omega^0(\xi_\rho)$. One has then $d^2_\omega(\sigma\wedge\alpha_0) = d_\omega\sigma\wedge\alpha_0 + \sigma\wedge d\alpha_0 = d^2_\omega\sigma\wedge\alpha_0 - d_\omega\sigma\wedge d_\omega\alpha_0 + d_\omega\sigma\wedge d_\omega\wedge\sigma\wedge\alpha$ since $d^2_\omega\alpha_0 = d^2\alpha_0 = 0$. Hence the corollary.

The corollary clearly shows that one has for a smooth complex valued function f,

$$d^2_\omega(f.\alpha) = f.d^2_\omega\alpha.$$

One has also the following

Proposition. $\dot{\rho}\,(K_\omega(X,Y))(\sigma) = D_X D_Y\sigma - D_Y D_X\sigma - D_{[X,Y]}\sigma$ for $\sigma\in\Omega(\xi_\rho)$ and X, Y vector fields on M.

Proof.

$$\begin{aligned}
(d^2_\omega\sigma)(X,Y) &= D_X d_\omega\sigma(Y) - D_Y d_\omega\sigma(X) - d_\omega\sigma([X,Y])\\
&= D_X D_Y\sigma - D_Y D_X\sigma - D_{[X,Y]}\sigma
\end{aligned}$$

and we have already seen that $d^2_\omega\sigma = K\wedge\sigma$. Hence the proposition. This leads to

Proposition. $K(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$.

The right hand side which is evidently a differential operator which takes sections of ξ_ρ into itself is in fact a vector bundle endomorphism of ξ_ρ.

We close this section with one final remark. Let ξ be a G-bundle on M and $f : M' \to M$ a smooth map. Let $\xi' = f^*(\xi)$ be the induced G-bundle on M'. Now if ω is a connection on ξ and ω' is the connection on ξ' induced by ω (through f), it is easily seen from our definitions that $K_{\omega'} = f^*(K_\omega)$. (Note that the bundle induced by ξ_{Ad} has a natural identification with ξ'_{Ad}.

§ 6. The Bianchi identity and Chern-Weil Theory

We begin with the following

Theorem. (Bianchi identity) $d_\omega K_\omega = 0$.

Proof. $(d_\omega K_\omega)^{\sim} = d\, \tilde{K}_\omega + \omega \Lambda\, \tilde{K}_\omega$. Since $(d_\omega K_\omega)^{\sim}$ is determined by the value it takes on triples X, Y, Z of tangent vectors which are all horizontal it suffices to show that $(d_\omega K_\omega)^{\sim}(X, Y, Z) = 0$ for X, Y, Z *horizontal*. Now

$$\tilde{K}_\omega = d\omega + (\omega\Lambda\omega)/2$$

so that since $d^2 = 0$

$$(d_\omega K_\omega)^{\sim} = d(\omega\Lambda\omega)/2 + \omega\Lambda d\omega + \omega\Lambda(\omega\Lambda\omega)/2.$$

Moreover, since $d(\omega\Lambda\omega) = d\omega\Lambda\omega - \omega\Lambda d\omega$, we have

$$(d_\omega K_\omega)^{\sim} = (d\omega\Lambda\omega)/2 + (\omega\Lambda d\omega)/2 + \omega\Lambda(\omega\Lambda\omega)/2.$$

Now if X, Y, Z are all horizontal

$$(d\omega\Lambda\omega)(X, Y, Z) = [d\omega(X, Y) \cdot \omega(Z)] - [d\omega(X, Z), \omega(Y)] + [d\omega(Y, Z), \omega(X)]$$

vanishes and similarly so does $(\omega\Lambda d\omega)(X, Y, Z)$ since $\omega(X) = \omega(Y) = \omega(Z) = 0$. Clearly $\omega\Lambda(\omega\Lambda\omega)(X, Y, Z) = 0$ as well. This proves the Binachi identity.

Consider now the symmetric algebra $S(Lie\ G)$ of $Lie\ G$. For any two finite dimensional subrepresentation ρ, ρ' of G contained in $S(Lie\ G)$ we have a natural G-homomorphism of $V(\rho) \otimes V(\rho') \to S(Lie\ G)$ (defined by the commutative multiplication in $S(Lie\ G)$ whose image is a finite dimensional G-stable submodule of $S(Lie\ G)$. Thus one sees that if α, β are forms on M of degree p and q respectively with values in ξ_ρ and $\xi_{\rho'}$ where ρ, ρ' are subrepresentations of $S(Lie\ G), \alpha\Lambda\beta$ is defined as a form again with values

in ξ_τ, where τ is a subrepresentation $S(Lie\ G)$ of finite dimension. In the sequel we will denote by σ the natural (infinite dimensional) representation of G on $S(Lie\ G)$ and define a ξ_σ valued form on M to be a p-form with values in ξ_ρ, where ρ is a finite dimensional subrepresentation of $S(Lie\ G)$. We thus obtain a multiplication $(\alpha, \beta) \mapsto \alpha\Lambda\beta$ on the vector space of all ξ_σ valued forms. Since the bilinear map $S(Lie\ G) \otimes S(Lie\ G) \to S(Lie\ G)$ with reference to which our exterior product is defined is *associative* and *commutative* we have

$$\alpha\Lambda\beta = (-1)^{pq}\beta\Lambda\alpha$$

for ξ_σ valued forms α and β of degree p and q respectively. In particular even degree ξ_σ-valued forms commute with all other ξ_σ valued forms. Note that if m is an integer $\geq 0, S^m(Lie\ G)$ is a G-submodule of $S(Lie\ G)$. We denote the representation as $S^m(Lie\ G)$ by σ_n. If α (resp. β) is a p-form (resp. q form) on M with values in σ_ℓ (resp. σ_m) then $\alpha\Lambda\beta \in \Omega^{p+q}\left(\xi_{\sigma_{\ell+m}}\right)$.

Now the subalgebra $\Omega_{even}(\xi_\sigma)$ of even degree forms on M with values in ξ_σ is a commutative algebra. It follows that one has a unique homomorphism

$$\mathcal{A} : \mathbf{C}[T] \to \Omega_{even}(\xi_\sigma)$$

of the polynomial ring $\mathbf{C}[T]$ in 1 variable T over $\mathbf{C}$ into $\Omega_{even}(\xi_\sigma)$ such that $\mathcal{A}(T) = K_\omega$. Since $K_\omega \in \Omega^2(\xi_{\sigma_1}), K^q_\omega$ (= image T^q under $\mathcal{A}$) belongs to $\Omega^{2q}(\xi_{\sigma_q})$. (Also note that $K^q_\omega = 0$ if $2q dim\ M$ so that all the elements in the image of $\mathcal{A}$ take values in $\coprod_{0 \leq q \leq dim\ M/2} \xi_{\sigma_q}$). Suppose now $I : S^q(Lie\ G) \to \mathbf{C}$ is a G-invariant linear form. Then one can define a (scalar) $2q$ form $I(\omega)$ by setting

$$I(\omega) = I \circ K^q_\omega$$

i.e. by setting

$$I(\omega)(X_1 \cdots X_q) = \xi_I(K^q_\omega(X_1 \cdots X_q))$$

(recall that ξ_I is the vector bundle morphism induced by I and is a homomorphism of ξ_{σ_q} (in which K^q_ω takes values) in the trivial bundle. Now since $d_\omega(K_\omega) = 0$ (Bianchi identity) one has $d_\omega(K^q_\omega) = 0$ and hence

$$d_\omega(\xi_I \circ K^q_\omega) = \xi_I \circ (d_\omega(K^q_\omega)) = 0.$$

Thus for any invariant linear form I on $S^q(Lie\ G)$. $I(\omega)$-which is obtained by "substituting" the curvature form in I is a closed q-form on M.

Theorem (Chern-Weil). The deRham cohomology class of $I(\omega)$ depends only on the isomorphism class of ξ and not on the connection ω (and hence may be designated $I(\xi)$).

Proof. Let ω, ω' be two connections on ξ. Consider now the manifold $M \times \mathbb{R}$. We have on $M \times \mathbb{R}$ a natural bundle $\tilde{\xi}$ deduced from ξ : $P(\tilde{\xi}) = P(\xi) \times \mathbb{R}, \pi(\tilde{\xi})$ $(x,t) = (\pi(\xi)(x), t)$ and the action of G on $P(\tilde{\xi})$ is through the first factor. We can then define on $P(\tilde{\xi})$ a connection $\tilde{\omega}$ as follows : let $\left(v, \alpha\frac{d}{dt}\right)$ be a tangent vector to $P(\tilde{\xi})$ at the point (x,t); then

$$\tilde{\omega}\,(v, \alpha\frac{d}{dt}) \;= t\omega(v) + (1-t)\omega'(v).$$

One checks easily that this is a connection on $\tilde{\xi}$. If $r_i, i = 0, 1$ are the smooth inclusions $x \mapsto (x, i)$, one sees that one has canonical identifications of ξ with $r_i^*(\tilde{\xi})$ and of $r_0^*(\tilde{\omega})$ with ω' and $r_1^*(\tilde{\omega})$ with ω. It follows that $K_{\omega'} = r_0^*(K_{\tilde{\omega}})$ and $K_\omega = r_1^*(K_{\tilde{\omega}})$ so that $r_0^*(I(\tilde{\omega})$ and $I(\omega')$ are induced by the *closed* form $I(\tilde{\omega})$ on $M \times \mathbb{R}$ through *homotopic maps* r_1 and r_0. It follows that $I(\omega)$ and $I(\tilde{\omega})$ define the same deRham cohomology class in $H^*(\Omega(M))$.

The cohomology class $I(\xi)$ determined by I is called the characteristic class of ξ associated to I. The kth Chern class of a vector bundle ξ is defined as follows. Let ξ^* be the $G = GL(n, \mathbf{C})$-bundle associated to ξ. The Lie algebra Lie G of G has a natural identification with $M(n, \mathbf{C})$. Let C_k' denote the homogeneous polynomial on Lie G which associates to each $A \in Lie\ G (= M(n, \mathbf{C}))$ the coefficient T^{n-k} in the characteristic polynomial $det(T \cdot Id - A)$ of A (Then C_k' may be regarded as a G-invariant linear form on $S^k(Lie\ G)$ as well). The kth *Chern class* of ξ denoted $c_k(\xi)$ is defined as the $2k$th cohomology class $c_k(\xi^*)$. Where $c_k = c_k'/(2\pi i)^k$.

We will calculate C_1 in a sample special case. Let ξ be the Hopf-bundle i.e. the $GL(1, \mathbf{C}) = \mathbf{C}^*$-bundle whose total space is $\mathbf{C}^{n+1} \setminus \{0\}$, the action of $\mathbf{C}^*$ on $\mathbf{C}^{n+1} \setminus \{0\}$ is given by $\underline{v} \mapsto z\underline{v}$ for $\underline{v} \in \mathbf{C}^{n+1} \setminus \{0\}$ and $z \in \mathbf{C}^*$ (since $\mathbf{C}^*$ is abelian no distinction need be made between right and left actions) and the base space is the quotient $(\mathbf{C}^{n+1} \setminus \{0\})/\mathbf{C}^*$, the complex projective space of complex dimension n denoted $\mathbf{C}P^n$ in the sequel. As is well known the hyperplane $z_{n+1} = 1$ maps complex analytic isomorphically onto a dense

open subset of $\mathbf{C}P^n$ whose complement is infact a hyperplane in $\mathbf{C}P^n$. The Lie algebra of $GL(1, \mathbf{C})$ is $M(1, \mathbf{C}) = \mathbf{C}$. If $\alpha \in \mathbf{C}$, a simple calculation shows that the vector field $\underline{\alpha}$ on $P(\xi) = \mathbf{C}^{n+1} \setminus \{0\}$ corresponding to α is $\alpha/2 \sum_{1 \le i \le n+1} z_i \frac{\partial}{\partial z_i} + \overline{\alpha}/2 \sum_{1 \le i \le m+1} \overline{z}_i \partial/\partial \overline{z}_i$. Let ω $\mathbf{C} = M(1, \mathbf{C}) = Lie\ GL(1, \mathbf{C})$ valued 1-form on $\mathbf{C}^{n+1} \setminus \{0\}$ defined by

$$\omega = \left(\sum_{1 \le i \le n+1} \overline{z}_i dz_i\right) / r^2 = d_z(log\ r^2)$$

where $r^2 = \sum_{1 \le i \le n+1} |\ z_i\ |^2$. Then $\omega(\underline{\alpha}) = \alpha$ and that $\omega(vg) = \omega(v)(= Adg^{-1}(\omega(v))$ for any tangent vector v to $\mathbf{C}^{n+1} \setminus \{0\}$ and $g \in \mathbf{C}^*$. It follows that ω is a connection on ξ. The curvature from K_ω of this connection is dw since Lie $GL(1, \mathbf{C})$ is abelian and as ω is a 1 form $\omega \Lambda \omega = 0$. Thus $K_\omega = d\omega = (d_z + d_{\overline{z}})d_z(log\ r^2) = d_{\overline{z}}d_z(log\ r^2) = d_{\overline{z}}(d_z r^2)/r^2 = \frac{1}{r^2} d_{\overline{z}} d_z r^2 - \frac{1}{r^4}(d_{\overline{z}} r^2 \Lambda d_z r^2)$. Now as already remarked $z_{n+1} = 1$ maps analytically isomorphically onto an open dense set in $\mathbf{C}P^n$ and on this open set $(z_1 \cdots z_n)$ serve as a complex coordinate system. With respect to this coordinate system then, setting $\rho = \sum_{1 \le i \le n} z_i^2$, we have

$$K_\omega = \frac{1}{1+\rho^2}(d_{\overline{z}} d_z \rho^2) - \frac{1}{(1+\rho^2)^2}(d_{\overline{z}} \rho \Lambda d_z \rho).$$

In the case $n = 1$, so that $\mathbf{C}P^1$ is the sphere one has $\rho =|\ z_1\ |^2$ so that $d\overline{z}\rho \Lambda d_z \rho =|\ z_1^2\ |\ dz_1 \Lambda d\overline{z}_1$ and hence

$$\begin{aligned} K_\omega &= \tfrac{1}{1+|z_1|^2} dz_1 \Lambda d\overline{z}_1 - |\ z_1\ |^2 / (1+ |\ z_1\ |^2)^2 dz_1 \Lambda d\overline{z}_1 \\ &= 1/(1+ |\ z_1\ |^2)^2 \cdot dz_1 \Lambda d\overline{z}_1 \end{aligned}$$

so that $2K_\omega/i$ is the standard volume form on the unit sphere in $\mathbb{R}^3$ with its induced Riemanian metric. Thus we see that $\frac{1}{2\pi i}\int_{\mathbf{C}P^1} K_\omega = 1$. In other words the chern class $c_1(\xi)$ of ξ in this case is just the fundamental class of $\mathbf{C}P^1 = S^2$ with its orientation as a complex manifold.

From our definitions it is not difficult to see that the following holds :

(i) $c_0(\xi) = 1$ for any vector bundle ξ,

(ii) if ξ, η are two vector bundles,

$$c_k(\xi \oplus \eta) = \sum_{i+j=k} c_1(\xi)c_j(\eta)$$

where we have set $c_i(\xi) = 0$ if $i \geq$ rank ξ; in particular if ξ is trivial, $c_i(\xi) = 0$ for $i0$.

(iii) $c_i(f^*(\xi)) = f^*(c_i(\xi))$ for any smooth map f.

In fact these three properties along with the fact that the first Chern class of the Hopf bundle over S^2 is the fundamental class of S^2 characterise Chern classes.

§ 7. Complements.

In § 6 we defined characteristic classes for a principal $G-$ bundle corresponding to a homogeneous invariant polynomial I on Lie G. Evidently the smaller the G, the more the invariants available to us. In this context it is clear that the following definition will be useful:

Let $f : H \to G$ be an (analytic) homomorphism of Lie groups and ξ a $G-$bundle on M. *A reduction* of ξ to H (following f) is a H bundle ξ_H together with a smooth map $\Phi : P(\xi_H) \to P(\xi)$ such that $\pi(\xi_H) = \pi(\xi) \circ \Phi$ and $\Phi(x,h) = \Phi(x)f(h)$ for all $h \in H$. The following lemma though elementary is very useful.

Lemma. Let $H \hookrightarrow G$ be a closed subgroup. A $G-$bundle ξ admits a reduction to H if and only if the following holds. Consider the quotient $P(\xi)/H$ of $P(\xi)$ for the action of H and let $\pi(\xi)' : P(\xi)/H \to M$ be the natural projection. Then there is a smooth map $\sigma : M \to P(\xi)/H$ such that $\pi(\xi)'\sigma(m) = m$ for all $m \in$.

Proof. Note that $P(\xi)/H$ is in a natural fashion a smooth manifold (this is easily seen using the local triviality property of ξ). If ξ_H is a reduction of ξ to H, and $P(\xi_H) \to P(\xi)$ the smooth map compatible with the H-action. We obtain by passage to the quotient a smooth map σ of $M = P(\xi_H)/H$ to $P(\xi)/H$. Evidently $\pi(\xi)' \circ \sigma$ is the identity on M. Conversely suppose a smooth section σ to $\pi(\xi)'$ is given. We define a $H-$ bundle ξ_H as follows. The total space $P(\xi_H)$ will be $\{x \in P(\xi) \mid \pi(\xi)'(x) = \sigma\pi(\xi)(x)\}$. It is evident

that $P(\xi_H)$ defined above is $H-$ stable. Using local triviality it is not difficult to see that it is a smooth submanifold of $P(\xi)$ and (consequently) that H acts smoothly on it; the restriction of $\pi(\xi)$ to $P(\xi_H)$ is defined to be $\pi(\xi_H)$. It is now easily seen that ξ_H is a $H-$bundle and $P(\xi_H) \hookrightarrow P(\xi)$ gives us the required reduction.

Of particular interest is the case when G is a Lie group with finitely many connected components and H is taken as a maximal compact subgroup. (It is well known that all maximal compact subgroups of such a G are all mutually conjugate in G). The homogeneous space G/H is diffeomorphic to euclidean space (of dimension dim $G-$dim H). Further there is a smooth map $\tau : G/H \to G$ such that $\beta \circ \tau$ is the identity (where $\beta : G \Rightarrow G/H$ is the natural projection) and $\tau(xk) = k^{-1}\tau(x)k$ for $x \in G/H$ and $k \in K$. One has then the following.

Proposition. Let G, H be as in last paragraph. Let ξ be a $G-$bundle on a (paracompact) manifold. Then ξ admits a reduction ξ_H to H. Moreover if ξ_H and ξ'_H are two reductions of ξ to H, ξ_H and ξ'_H are isomorphic (as $H-$ bundles).

Proof. To prove the first assertion we need only construct a smooth section $M \to P(\xi)/H$. Let U_r, $1 \leq r < \infty$ be a locally finite covering of M by open subsets such that for each r there is a diffeomorphism Φ_r of $U_r \times ' /H$ on $\pi(\xi)'^{-1}(U_r)$ with $\pi(\xi)'\Phi_r(m,x) = x$ for all $m \in U_r$. Let V_r, $1 \leq \cdot \infty$ be a shrinking of U_r, $1 \leq r < \infty$ and let $F_r = \cup_{1 \leq i \leq r} V_i$. Then one shows inductively that there is a smooth section σ_r over an open neighbourhood of F_r for the map $\pi(\xi)'$. The passage from F_r to F_{r+1} is taken care of by the smooth analogue of the Tietze extension theorem (note that this passage from F_r to F_{r+1} is essentially the same as passing from $F_r \cap V_{r+1}$ to V_{r+1} and over $V_{r+1}, P(\xi)/H$ looks like a product $V_{r+1} \times G/H$ so that sections are same as G/H-valued functions and G/H is a euclidean space).

To prove the second assertion we will make use of the following fact (which we will not prove):

Let ξ be a principal $H-$ bundle on a smooth manifold Y and $f, g : X \to Y$ be two smooth maps of a manifold X in Y which are homotopic to each other. Then $f^*(\xi)$ is isomorphic to $g^*(\xi)$.

To use this result we make the following observation first. If ξ_H is a reduction of ξ to H, we have as already remarked a section $\sigma : M \to P(\xi)/H$

obtained from the $H-$ morphism $f : P(\xi_H) \to P(\xi)$ by passage to the quotient by H. Thus the diagram

$$\begin{array}{ccc} P(\xi_H) & \to & P(\xi) \\ \pi(\xi_H)\downarrow & & \downarrow \pi(\xi)' \\ M & \xrightarrow{\sigma} & P(\xi)/H \end{array}$$

is commutative. In other words ξ_H is isomorphic to the bundle induced through σ by the $H-$ bundle $\eta = (P(\xi), \pi(\xi)' : P(\xi) \to P(\xi)/H)$ on $P(\xi)/H)$. We see therefore that the two reductions ξ_H, ξ'_H of ξ to H are obtained as induced bundles from the bundle η through maps σ and σ' of M in $P(\xi)/H$. Thus it suffices to show that σ and σ' are *homotopic*. This is again done inductively by constructing homotopies between σ and σ' in a neighbourhood of the closed sets F_r for $r = 1, 2 \cdots$ as was done for proving the existence of the reduction: once more one uses the fact that G/H is a euclidean space.

The proposition enables one to define characteristic cohomology classes for a $G-$ bundle other than the ones defined by the invariant polynomials on Lie G. If ξ is a $G-$bundle, with G a Lie group having only finitely many connected components, ξ admits a reduction ξ_H to a maximal compact subgroup H, the reduction being unique upto isomorphism of $H-$bundles. Thus to ξ we can associate the classes of ξ_H; and here we use invariant polynomials on Lie H; and one may find some which are not restrictions of invariant polynomials on Lie G.

The comments of the last paragraph are best illustrated by the following example. Exactly as in the case of $GL(q, \mathbf{C})$ one can set up a bijective correspondence between isomorphism classes of $GL(q, \mathbf{R})-$ bundles and isomorphism classes *real* vector bundles on M. We say that $GL(q, \mathbf{R})$-bundle ξ is orientable if it admits a reduction to $SL(q, \mathbf{R})$; equivalently if ξ_o denotes the associated real vector bundle of rank q, $q \wedge^q \xi_o$ is *trivial*. Now $SO(q)$ is a maximal compact subgroup and the Lie algebra $SO(q)$ of $SO(q)$ is the Lie algebra of $(q \times q)$ skew symmetric matrices. Now if q is even, there is a unique polynomial the Pfaffian, Pfaff:$SO(q) \to \mathbf{R}$ such that for $A \epsilon SO(q)$, det $A =$(Pfaff $A)^2$ and Pfaff $\left(\begin{smallmatrix} 0 & 1_q \\ -1_q & 0 \end{smallmatrix}\right) = 1$ where 1_q is the $(q \times q)$ identity matrix. Moreover Pfaff is a $SO(q)$ (and *not* $O(q)$) invariant polynomial of degree $q/2$. Nor is the Pfaffian invariant under $SL(n, \mathbf{R})$. The substitution of the curvature form K_ω of a connection in the invariant polynomial Pfaff)

$/(2\pi)^{q/2}$ thus gives rise to a $2q$ dimensional cohomology class. This is called the Euler class of the $SO(q)-$ bundle (q even); and for the tangent bundle of a compact oriented manifold M this class is precisely $\chi(M).[M]$, where $[M]$ is the fundamental class of M and $\chi(M)$ is the Euler characteristic of M. This last assertion is the celebrated Gauss Bonnet Theorem.