INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.637/28

ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

(31 August - 11 September 1992)

ARITHMETIC CURVES: Algebraic Number Theory interpreted à la Arakelov

Norbert Schappacher
Departement de Mathématiques
Université Louis Pasteur
7, rue René Descartes
F-67084 Strasbourg
France

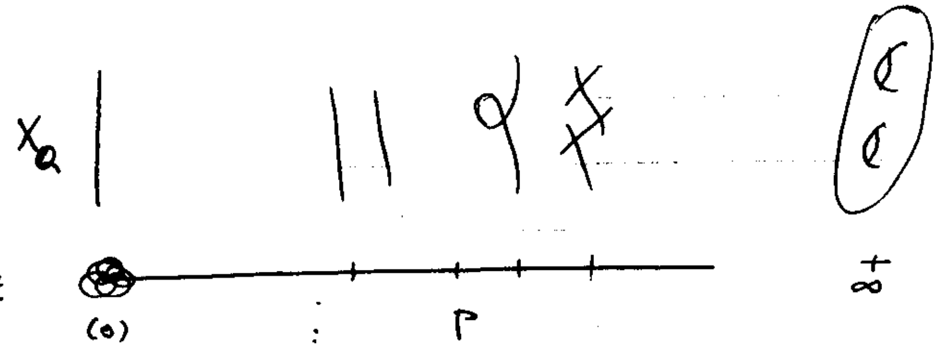
These are preliminary lecture notes, intended only for distribution to participants

ARITHMETIC CURVES: Algebraic Number Theory interpreted à la Arakelov.

N. SCHAPPACHER

[WARNING: Do not overestimate this little theory here; the true Arakelov theory really starts with arithmetic surfaces ...]

Recall Soulé's picture :



Arithmetic curves are of dimension 1 $\Rightarrow$ only the bottom line survives!

$\not\!\!-$

K – algebraic number field

$\mathfrak{o}_K \subset K$ ring of integers — a Dedekind domain: regular of dim. 1.

Spec $\mathfrak{o}_K$: regular integral affine scheme of dimension 1.

"Complete" Spec $\mathfrak{o}_K$ by archimedean places of K, in Arakelov's way!

The (closed) points of the completed Spec $\mathfrak{o}_K$ then are given by:

non-archimedean normalized absolute values on K	archimedean normalized absolute values on K
$\mathfrak{p} \in$ Spec $\mathfrak{o}_K$ closed point:	$\sigma : K \hookrightarrow \mathbb{C}$; $\bar\sigma = c\circ\sigma$, c = cplx. conj.
$\lvert\ \rvert_{\mathfrak{p}} : K \to \mathbb{R}$; $\lvert x\rvert_{\mathfrak{p}} = N\mathfrak{p}^{-v_{\mathfrak{p}}(x)}$	$v = \{\sigma, \bar\sigma\}$ "infinite place" of K
where $N\mathfrak{p} = \#(\mathfrak{o}_K/\mathfrak{p}) = p^{f_{\mathfrak{p}}}$; $\mathfrak{p}\cap\mathbb{Z} = p\mathbb{Z}$	$\lvert x\rvert_v = \lvert\sigma(x)\rvert_{\mathbb{C}} = \sqrt{\sigma(x)\bar\sigma(x)}$
$\lvert p\rvert_{\mathfrak{p}} = (p^{-f_{\mathfrak{p}}})^{e_{\mathfrak{p}}}$.	$f_v = 1$; $e_v = \begin{cases} 1 & \sigma = \bar\sigma \\ 2 & \sigma \neq \bar\sigma \end{cases}$

Completions:

$$K \otimes_{\mathbb{Z}} \mathbb{Q}_p \cong \prod_{\mathfrak{p}\mid p} K_{\mathfrak{p}} \qquad\qquad K \otimes_{\mathbb{Z}} \mathbb{R} \cong \prod_{v\mid\infty} K_v$$

$$[K_{\mathfrak{p}} : \mathbb{Q}_p] = f_{\mathfrak{p}} \cdot e_{\mathfrak{p}} \qquad\qquad [K_v : \mathbb{R}] = e_v$$

$$K_v \cong \begin{cases} \mathbb{R} & e_v = 1 \\ \mathbb{C} & e_v = 2 \end{cases}$$

Now, define analogs of the standard concepts of the theory of algebraic curves for our arithmetic curves: completed $\mathrm{Spec}(\mathcal{O}_K)$.

1. Line bundles

$\mathcal{L}$ line bundle on $\mathrm{Spec}(\mathcal{O}_K)$ $\Leftrightarrow$ invertible coherent module on $\mathrm{Spec}(\mathcal{O}_K)$
$\Leftrightarrow$ projective module L of rank 1 over $\mathcal{O}_K$.

RECALL FROM COMMUTATIVE ALGEBRA:

Fact. Let R be a Dedekind domain, F its field of fractions, M a finitely generated projective R-module. Then there exists a fractional ideal $\mathfrak{a} \subset F$ of R, and a non-negative integer m such that: $M \cong \mathfrak{a} \oplus R^m$.
[See Milnor, Introduction to Algebraic K-theory, Princeton Univ Press]

Consequence: All invertible R-modules L are isomorphic to fractional ideals

Lemma: For fractional ideals: $\mathfrak{a} \cong \mathfrak{b}$ (as R-modules) $\Leftrightarrow \exists \alpha \in F^*: \mathfrak{a} = \alpha \cdot \mathfrak{b}$.

Pf. "$\Leftarrow$": Isomorphism is given by $\mathfrak{b} \xrightarrow{\sim} \mathfrak{a}$, $x \mapsto \alpha x$.
"$\Rightarrow$": $\varphi: \mathfrak{a} \xrightarrow{\sim} \mathfrak{b} \Rightarrow \varphi \otimes_R F : \mathfrak{a} \otimes_R F \cong F \xrightarrow{\sim} F \cong \mathfrak{b} \otimes_R K$ is K-linear
$\Rightarrow \varphi \otimes F$ given by invertible 1×1-matrix : by $\alpha \in F^*$. //

Def. $\mathrm{Pic}(R)$ – group of isomorphism classes of invertible R-modules, group law induced by $(M, L) \mapsto M \otimes_R L$.

Consequence (of lemma and fact above): $\mathrm{Pic}(R) \cong Cl(R)$ ← ideal class group
$[\mathfrak{a}] \longleftarrow [\mathfrak{a}]$

//

Proposition / Definition: (1) A metrized line bundle on o_K is a pair $(L, \{N_v\}_{v|\infty})$, where
- L is an invertible o_K-module,
- for each $v|\infty$, putting $L_v := L \otimes_{o_K} K_v$, $N_v : L_v \to K_v$ is a norm on the 1-dimensional K_v-vector space L_v.

2) $(L, \{N_v\}) \simeq (L', \{N'_v\})$ ("isometric") $:\Longleftrightarrow \exists\, \varphi : L \xrightarrow{\simeq} L'$ isomorphism of o_K-modules, such that $\forall v|\infty\ \forall x \in L_v$:
$$N'_v\big((\varphi \otimes K_v)(x)\big) = N_v(x).$$

3) $\mathrm{Pic}(\bar{o}_K)$ — the group of isometry-classes of metrized line bundles on o_K, with group law induced by the tensor product:
$$(L, \{N_v\}) \otimes (L', \{N'_v\}) = (L \otimes_{o_K} L', \{N''_v\}); \quad N''_v(x \otimes x') = N_v(x) \cdot N'_v(x').$$

Remarks: 1. The neutral element of $\mathrm{Pic}(\bar{o}_K)$ is represented by $(o_K, \{N^0_v\})$, where $N^0_v : K_v \to K_v$; $N^0_v(1) = 1$.

2. Given $v|\infty$ and $\lambda_v \in \mathbb{R}$, define the (in general non-trivial) metric on $(o_K)_v = K_v$:
$$N_v^{\lambda_v}(1) = \exp\left(\frac{\lambda_v}{e_v}\right).$$

3. Define
$$\rho : \bigoplus_{v|\infty} \mathbb{R} \cdot v \longrightarrow \mathrm{Pic}(\bar{o}_K)$$
$$\sum \lambda_v \cdot v \longmapsto \left(o_K, \{N_v^{\lambda_v}\}\right).$$

Lemma: $\ker(\rho) = \mathrm{im}(\log)$, where
$$\log : o_K^* \longrightarrow \bigoplus \mathbb{R} \cdot v$$
$$\varepsilon \longmapsto \sum_{v|\infty} e_v \cdot \log|\varepsilon|_v$$

Proof. "$\subseteq$": $(o_K, \{N_v^{\lambda_v}\}) \simeq (o_K, \{N_v^0\})$ given by $o_K \xrightarrow{\sim} o_K$, $x \longmapsto \varepsilon \cdot x$; $\varepsilon \in o_K^*$
$$\Rightarrow \exp\left(\frac{\lambda_v}{e_v}\right) = N_v^{\lambda_v}(1) = N_v^0(\varepsilon) = |\varepsilon|_v \Rightarrow \lambda_v = e_v \cdot \log|\varepsilon|_v.$$

"$\supseteq$": straightforward! //

Proposition: The following sequence is exact:

$$0 \longrightarrow \bigoplus_{v|\infty} \mathbb{R}\cdot v \Big/ \log(\mathcal{O}_K^*) \xrightarrow[\substack{\text{(induced)}\\ \text{by } \rho}]{\bar{\varsigma}} \operatorname{Pic}(\overline{\mathcal{O}}_K) \xrightarrow{\pi} \operatorname{Pic}(\mathcal{O}_K) \longrightarrow 0$$

$$(L, \{N_v\}) \longmapsto L$$

Pf.: exercise!

Note that, essentially, we know the map "log": it is the logarithmic embedding used in the proof of Dirichlet's unit theorem. So, by this theorem, $\log(\mathcal{O}_K^*)$ is a full lattice in $\log(\mathcal{O}_K^*)\otimes_{\mathbb{Z}}\mathbb{R}$ which itself is a hyperplane in $\bigoplus_{v|\infty}\mathbb{R}\cdot v$. Thus $\dim_{\mathbb{R}}(\log(\mathcal{O}_K^*)\otimes\mathbb{R}) = r_1+r_2-1$.

§2. Divisors

Definition: (1) The group of Arakelov divisors on $\mathcal{O}_K$:

$$\operatorname{Div}(\overline{\mathcal{O}}_K) := \bigoplus_{0\neq \mathfrak{p}\in \operatorname{Spec}(\mathcal{O}_K)} \mathbb{Z}\cdot\mathfrak{p} \;\oplus\; \bigoplus_{v|\infty}\mathbb{R}\cdot v$$

$$\psi$$

$$D = \sum_{\mathfrak{p}} n_{\mathfrak{p}}\cdot\mathfrak{p} + \sum_v \lambda_v\cdot v = D_f + D_\infty .$$

(2) Principal Arakelov divisors: for $f\in K^*$:

$$\overline{\operatorname{div}}(f) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(f)\cdot\mathfrak{p} - \sum_v e_v\cdot\log|f|_v\cdot v .$$

(3) Degree of an Arakelov divisor:

$$\deg\Big(\sum_{\mathfrak{p}} n_{\mathfrak{p}}\cdot\mathfrak{p} + \sum_v\lambda_v\cdot v\Big) = \sum_{\mathfrak{p}} n_{\mathfrak{p}}\cdot\log N\mathfrak{p} + \sum_v \lambda_v \in \mathbb{R}.$$

(4) The Arakelov divisor class group:

$$CH^1(\overline{\mathcal{O}}_K) = \operatorname{Div}(\overline{\mathcal{O}}_K)\big/\overline{\operatorname{div}}(K^*) .$$

Lemma: $\forall f\in K^*$ $\deg(\overline{\operatorname{div}}(f)) = 0$.
(This is just a reformulation of the "product formula" for K.)
Consequence: deg induces a degree map $\deg: CH^1(\overline{\mathcal{O}}_K) \twoheadrightarrow \mathbb{R}$.
(obviously surjective!)

Proposition: The following diagram is commutative with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{v|\infty} \mathbb{R}\cdot v / \log(\mathcal{O}_K^*) \arrow[r, "A"] \arrow[d, equal] & CH^1(\bar{\mathcal{O}}_K) \arrow[r, "B"] \arrow[d, "\cong", "\mathcal{O}"'] & Cl(\mathcal{O}_K) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \bigoplus_{v} \mathbb{R}\cdot v / \log(\mathcal{O}_K^*) \arrow[r, "\bar{\rho}"] & \mathrm{Pic}(\bar{\mathcal{O}}_K) \arrow[r, "\pi"] & \mathrm{Pic}(\mathcal{O}_K) \arrow[r] & 0 \end{tikzcd}$$

<u>Here:</u>
- $A(\sum_v \lambda_v \cdot v) = -\sum_v \lambda_v \cdot v \in CH^1(\bar{\mathcal{O}}_K)$
- $B(D) = [\prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}}]$. (ideal class), for D as above.
- the isomorphism $Cl(\mathcal{O}_K) \to \mathrm{Pic}(\mathcal{O}_K)$ comes from viewing an ideal class as an isom. class of $\mathcal{O}_K$-modules, see above...
- $D \mapsto \mathcal{O}(D) := (\prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}}, \{N_v^{-\lambda_v}\})$ is an isomorphism because the other vertical maps are isomorphisms.

<u>Corollary:</u> via the isomorphism we obtain a degree map on $\mathrm{Pic}(\bar{\mathcal{O}}_K)$

<u>Formulas:</u> (1) $(L, \{N_v\})$ metrized line bundle, $0 \neq s \in L$.
Then $\deg(L, \{N_v\}) = \log \#(L/s\cdot\mathcal{O}_K) - \log \prod_v N_v(s)^{e_v}$

(2) $(\mathfrak{a}, \{N_v\})$ metrized fractional ideal of $\mathcal{O}_K$.
Then $\deg(\mathfrak{a}, \{N_v\}) = -\log(N\mathfrak{a}) - \log \prod_v N_v(1)^{e_v}$.

<u>Proof.</u> (a) The formula in (1) is independent of s.
In fact, if $s' \neq 0$, $s' \in L \Rightarrow s' \otimes 1 = (1 \otimes \alpha)(s \otimes 1)$ in $L \otimes_{\mathcal{O}_K} K$, for some $\alpha \in K^*$. Let $d \in \mathbb{Z}$ such that $d\alpha \in \mathcal{O}_K$, and write $n = [K:\mathbb{Q}]$.
Then
$$d^n \cdot \#(L/s'\cdot\mathcal{O}_K) = \#(L/ds'\cdot\mathcal{O}_K) = \#(L/s\cdot\mathcal{O}_K) \cdot \#\left(\frac{s\cdot\mathcal{O}_K}{d\alpha s\cdot\mathcal{O}_K}\right)$$
$$= \#(L/s\cdot\mathcal{O}_K) \cdot |N_{K/\mathbb{Q}}(d\alpha)| = d^n \cdot |N_{K/\mathbb{Q}}\alpha| \cdot \#(L/s\cdot\mathcal{O}_K).$$
On the other hand, $\prod_v N_v(s')^{e_v} = \prod_v |\alpha|_v^{e_v} \cdot \prod_v N_v(s)^{e_v} = |N_{K/\mathbb{Q}}\alpha| \cdot \prod_v N_v(s)^{e_v}$

This shows (a).

(b) The formula (1) does not change when $(L, \{N_v\})$ is replaced by an isometric metrized line bundle $(L', \{N'_v\})$.

[Clear: take for $s' \in L'$ the image of s under an isometry.]

Using (a) and (b), we are reduced to proving (1) in the case: $L = \mathfrak{a} \supseteq \mathcal{O}_K$ a fractional ideal, and $s = 1$. Then we find

$$\log \#(\mathfrak{a}/\mathcal{O}_K) - \log \prod_v N_v(1)^{e_v} = \log N\mathfrak{a}^{-1} - \log \prod_v N_v(1)^{e_v}.$$

Thus it suffices to prove formula (2). — But writing $(\mathfrak{a}, \{N_v\}) = \mathcal{O}(D)$, for $D \in \mathrm{Div}(\bar{\mathcal{O}}_K)$, we get

$$\deg(\mathfrak{a}, \{N_v\}) = \deg D = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \cdot \log(N\mathfrak{p}) + \sum_v \lambda_v$$

$$= -\log N\underbrace{\left(\prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}}\right)}_{\mathfrak{a}} - \log \prod_v \underbrace{N_v^{-\lambda_v}(1)}_{N_v}{}^{e_v} \qquad //$$

§3. Riemann-Roch

Definition. 1) Given $(L, \{N_v\})$, write $V = L \otimes_{\mathcal{O}_K} \mathbb{R} \cong \prod_v L_v$, and equip this n-dimensional ($n = [K:\mathbb{Q}]$) $\mathbb{R}$-vector space with a volume by treating each factor L_v separately (and then: $\det_{\mathbb{R}}(\prod_v L_v) = \bigotimes_{v,\mathbb{R}} (\det L_v)$):

$v | \infty$ with $e_v = 1$ $\Rightarrow$ get a norm on $\det_{\mathbb{R}} L_v = L_v$ by $|x| := N_v(x)$;

$v | \infty$ " $e_v = 2$ $\Rightarrow$ " " " " $\det_{\mathbb{R}} L_v = \Lambda^2_{\mathbb{R}} L_v$ by $|x \wedge ix| := N_v(x)$

2) $\chi(L, \{N_v\}) = -\log \mathrm{vol}(V/L)$.

[Recall Minkowski's theory!]

Proposition: [Riemann-Roch] $\chi(L, \{N_v\}) = \deg(L, \{N_v\}) + \chi(\mathcal{O}_K, \{N_v^\circ\})$

Lemma: $\chi(\mathcal{O}_K, \{N_v^\circ\}) = \log\left(2^{r_2} \cdot |d_K|^{-1/2}\right)$.

Proof: follows from "$d_K = \det(\sigma_j(\omega_i))^2$".

Proof of proposition: χ and $\deg$ depend only on $(L, \{N_v\})$ mod. isometry. So, reduce to the case $(\mathfrak{a}, \{N_v\})$, with $\mathfrak{a} \supseteq \mathcal{O}_K$ fractional ideal.

$$\Rightarrow \chi(\mathfrak{a}, \{N_v\}) = -\log \mathrm{vol}\left(\frac{\mathfrak{a} \otimes \mathbb{R}}{\mathfrak{a}}\right) = -\log\Big(\prod_v N_v(1)^{e_v} \cdot \underbrace{\mathrm{VOL}}_{\substack{\text{w.r.t.} \\ \{N_v^\circ\}}}\Big(\frac{\mathfrak{a} \otimes \mathbb{R}}{\mathfrak{a}}\Big)\Big)$$

$$= -\log\left(\prod_v N_v(1)^{e_v} \cdot [\mathfrak{a} : o_K] \cdot \mathrm{vol}\left(\frac{o_K \otimes \mathbb{R}}{o_K}\right)\right)$$

$$= \deg(\mathfrak{a}, \{N_v\}) + \chi(o_K, \{N_v^0\}) \quad //$$

<u>Ad hoc substitutes</u> for the cohomology group H^0:

$$H^0(L, \{N_v\}) := \{ s \in L \mid \forall v | \infty \ \ N_v(s) \leq 1 \}.$$

This is just a <u>finite</u> set (not a group, in general).
($L \subset V$ discrete!)

<u>Properties</u>: (1) $H^0(\mathcal{O}(D)) = \{0\} \cup \{f \in K^* \mid \overline{\mathrm{div}}(f) \geq -D\}$.

(2) $H^0(o_K, \{N_v^0\}) = \{0\} \cup \underbrace{\mu_K}_{\text{roots of 1 in } K^*}$.

(3) $\deg(L, \{N_v\}) < 0 \implies H^0(L, \{N_v\}) = \{0\}$

(4) $\deg(L, \{N_v\}) = 0$ and $H^0(L, \{N_v\}) \neq 0 \implies$
$\implies (L, \{N_v\}) \cong (o_K, \{N_v^0\})$.

<u>Lemma</u> [Minkowski]: $\chi(L, \{N_v\}) \geq \log \frac{2^n}{2^{r_1}\pi^{r_2}} \implies H^0(L, \{N_v\}) \neq 0$.

<u>Proofs</u>. (1) $\cdot f \in \mathcal{O}(D) = \prod \mathfrak{y}^{-n_{\mathfrak{y}}} \iff \prod \mathfrak{y}^{v_{\mathfrak{y}}(f)} \subseteq \prod \mathfrak{y}^{-n_{\mathfrak{y}}} \iff v_{\mathfrak{y}}(f) \geq -n_{\mathfrak{y}}$, for all $\mathfrak{y}$

$\cdot\ N_v^{-\lambda_v}(f) \leq 1 \iff 1 \geq N_v(f) \cdot \exp(-\frac{\lambda_v}{e_v}) \iff -e_v \cdot \log N_v(f) \geq -\lambda_v$. //

(2) An integer $\varepsilon \in o_K$ such that $|\sigma(\varepsilon)| \leq 1$, for all $\sigma : K \hookrightarrow \mathbb{C}$, is a root of unity (exercise!).

(3)/(4) Pick $0 \neq s \in H^0(L, \{N_v\})$ (supposing it exist). Then

$$\deg L = \log \underbrace{\#(L/s.o_K)}_{\geq 1} - \log \underbrace{\prod_v N_v(s)^{e_v}}_{\leq 1, \text{ because } s \in H^0} \geq 0.$$

This already shows (3), and if $\deg(L, \{N_v\}) = 0$, then get:

$1 \leq \# \log\left(L/s\cdot o_K\right) = \prod_v N_v(s)^{e_v} \leq 1$ $\Longrightarrow$ equality everywhere
$\Longrightarrow$ $L = s\cdot o_K$ and $N_v(s) = 1$, for all s $\Longrightarrow (L,\{N_v\}) \simeq (o_K, \{N_v^\circ\})$. //

Proof of lemma: $H^0(L, \{N_v\}) = L \cap G_\infty$, where

$$G_\infty = \{ x = (x_v)_v \in \prod L_v \mid N_v(x) \leq 1 \text{ for all } v \} \subset \prod L_v = V.$$

This is a closed, convex, 0-symmetric subset in V, and with respect to the volume defined by $\{N_v\}$ we get:

$$\operatorname{vol} G_\infty = 2^{r_1}\cdot \pi^{r_2}.$$

So by Minkowski's lemma: $\underbrace{\operatorname{vol} G_\infty > 2^n \operatorname{vol}(V/L)}_{\Longleftrightarrow\ \chi(L,\{N_v\}) \geq \log \frac{2^n}{2^{r_1}\pi^{r_2}}} \Longrightarrow \underbrace{L \cap G_\infty}_{= H^0(L,\{N_v\})} \neq \{0\}$.

//

As an illustration of the use of Arakelov theory in proving finiteness results, let us prove with our notations:

Theorem: Every ideal class $[\mathfrak{a}]$ of o_K contains an integral ideal $\mathfrak{b} \subseteq o_K$ with norm $\mathbb{N}\mathfrak{b}$ bounded (indep. of $\mathfrak{a}$).
(Corollary, of course: $Cl(o_K)$ is a finite abelian group.)

Pf. Choose metrics $(\mathfrak{a}^{-1}, \{N_v\})$ such that $\deg(\mathfrak{a}^{-1}, \{N_v\}) =$
$= C_\infty - \chi(o_K, \{N_v^\circ\})$. Here, we write $C_\infty := \log \frac{2^n}{2^{r_1}\pi^{r_2}}$, and we have assumed, without loss of generality, that $\mathfrak{a} \subseteq o_K$.

$\Longrightarrow \chi(\mathfrak{a}^{-1}, \{N_v\}) \underset{RR}{=} \deg(\mathfrak{a}^{-1}, \{N_v\}) + \chi(o_K, \{N_v^\circ\}) = C_\infty$.

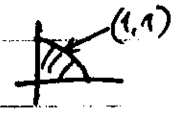
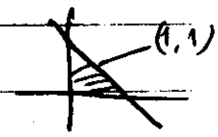
$\Longrightarrow H^0(\mathfrak{a}^{-1}, \{N_v\}) \neq 0$; so choose $0 \neq s \in H^0(\mathfrak{a}^{-1}, \{N_v\})$, and put $\mathfrak{b} := s\cdot\mathfrak{a} \subseteq o_K$ (because $s\cdot o_K \subseteq \mathfrak{a}^{-1}$!).

$\Longrightarrow \deg(\mathfrak{a}^{-1}, \{N_v\}) = \log \#(\mathfrak{a}^{-1}/s\cdot o_K) - \log \underbrace{\prod N_v(s)^{e_v}}_{\leq 1} \geq \log \#\left(\frac{\mathfrak{a}^{-1}}{s\cdot o_K}\right)$

$\Longrightarrow \mathbb{N}\mathfrak{b} = \#(o_K/\mathfrak{b}) = \#\left(\mathfrak{a}^{-1}/s\cdot o_K\right) \leq \underbrace{\exp\left(C_\infty - \chi(o_K, \{N_v^\circ\})\right)}_{\text{indep. bound.}}$ //

Variations of these ad hoc definitions:

$H^0(L,\{N_v\}) := L \cap G_?$, where G may be any of the following

G	vol(G)	picture
$G_\infty = \{ s=(s_v)\in\prod L_v \mid \forall v \ N_v(s_v)\le 1 \}$	$2^{r_1}\pi^{r_2}$	
$G_2 = \{ s=(s_v) \mid \sum_v e_v \cdot N_v(s_v)^2 \le n \}$	$\frac{2\pi^{\frac{n}{2}} n^n}{n\Gamma(\frac{n}{2})2^{r_2}} \ge \frac{2^n}{2^{r_2}}$	
$G_1 = \{ s=(s_v) \mid \sum_v e_v \ N_v(s_v) \le n \}$	$\underbrace{2^{n-r_2}\left(\frac{\pi}{4}\right)^{-r_2}\left(\frac{n^{n}}{n!}\right)^{-1}}_{\ge 2^n}$	

All one needs to know, in fact, is:

(i) $s\in G \implies \prod N_v(s)^{e_v} \le 1$

(ii) $s\in G$ and $\prod_v N_v(s_v)^{e_v} = 1 \implies N_v(s_v)=1$, for each v.

These properties can be easily checked for the choices of G above, by proving: $G_\infty \underset{\text{easy}}{\subseteq} G_2 \underset{\text{C-Schwarz inequ.}}{\subseteq} G_1 \underset{\text{arith-geom-mean inequ.}}{\subseteq} \{ s \mid \prod N_v(s_v)^{e_v} \le 1 \}$

In every case, Minkowski's lemma says: $\chi(L,\{N_v\}) \ge \log \frac{2^n}{\mathrm{vol}\, G} \implies H^0(\dots)\ne 0$

ONE last application:

Variant of Minkowski's lemma: $\deg(L,\{N_v\}) \ge \underbrace{\log\left(\left(\frac{2}{\pi}\right)^{r_2}|d_K|^{1/2}\right)}_{=: A_K}$

$\implies H^0_{(\infty)}(L,\{N_v\}) \ne \{0\}$.

Proof. By Riemann-Roch:

$$\deg(L,\{N_v\}) \geq A_K \iff \chi(L,\{N_v\}) \geq A_K + \chi(\mathcal{O}_K,\{N_v^0\})$$

$$= \log\left(\left(\tfrac{2}{\pi}\right)^{r_2} |d_K|^{1/2} \cdot 2^{r_2} \cdot |d_K|^{1/2}\right)$$

$$= \log\left(\frac{2^n}{2^{r_1}\pi^{r_2}}\right).$$

So we reduced the variant to our original version of Minkowski's lemma.

Corollary: $K \neq \mathbb{Q} \implies |d_K| > 1$ [$\Rightarrow$ at least one p ramifies in K.]

Proof. Assume $d_K = \pm 1$.

Comparing property (3) on page 7 to the variant of Minkowski, we see that $A_K \geq 0$. But for $d_K = \pm 1$, $A_K = \log\left(\left(\frac{2}{\pi}\right)^{r_2}\right)$. So $r_2 = 0$.

Assume $r_1 > 1$, and choose $s = (s_v)_v \in \prod (\mathcal{O}_K)_v = \mathbb{R}^{r_1}$ which is not in the image $\log(\mathcal{O}_K^*)$ [which is discrete, so s certainly exists], and such that $\sum_v s_v = 0$. (in $\{\sum s_v = 0\}$)

Put $\lambda_v = -e_v \cdot \log|s_v|$.

Then $(\mathcal{O}_K, \{N_v^{\lambda_v}\}) \not\cong (\mathcal{O}_K, \{N_v^0\})$, by lemma page 3.

But $\deg(\mathcal{O}_K, \{N_v^{\lambda_v}\}) = \sum \lambda_v = 0$ by construction of s.

Thus, by property (4) on page 7, we see that $H^0(\mathcal{O}_K, \{N_v\}) = \{0\}$.

So, the variant of Minkowski tells us that $\log(|d_K|^{1/2}) > 0$ (remember that $r_2 = 0$); therefore $|d_K| \neq 1$.

qed.