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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Topology and Differential Geometry - Lecture I

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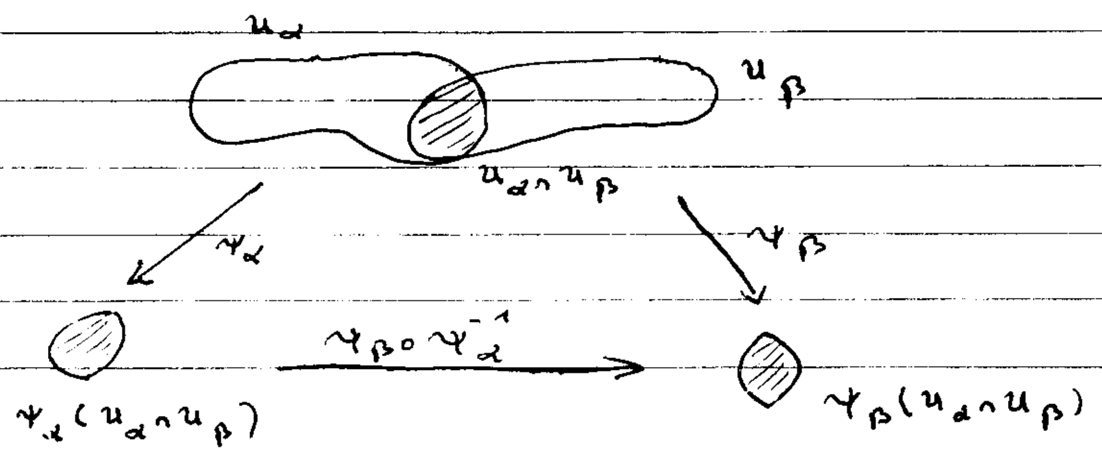
TOPOLOGY AND DIFFERENTIAL GEOMETRY

LECTURE I

(P. SLODOWY, 2.9.92)

1. BUNDLES

In the following, M will denote an n-dimensional C^∞-manifold. Recall that this involves the existence of an atlas $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n)_{\alpha \in I}\}$ of manifold charts, U_α open in M, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ a homeomorphism of U_α onto an open subset of $\mathbb{R}^n$, such that the transition $\varphi_\beta \circ \varphi_\alpha^{-1}$ from $\varphi_\alpha(U_\alpha \cap U_\beta)$ to $\varphi_\beta(U_\alpha \cap U_\beta)$ is a C^∞-diffeomorphism:



We will assume that M is paracompact, i.e. Hausdorff and any open covering $\{V_\alpha\}$ of M admits a locally finite refinement $\{W_\beta\}$. In view of the applications in mind (M is derived from an affine or projective variety), this is no restriction, and it provides us with partitions of unity, the basic differential topologist's tool to extend local to global constructions. More precisely, if $\{\mathcal{U}_\alpha\}$ is an open covering of M, then exists a subordinate partition of unity, i.e. a family $\{\lambda_\beta\}$ of C^∞-functions $\lambda_\beta : M \longrightarrow \mathbb{R}$ s.t.

- $\forall_\beta \quad \operatorname{supp}(\lambda_\beta) \subset \mathcal{U}_\alpha$ for some α
- $\forall_{x \in M} \quad x \in \operatorname{supp}(\lambda_\beta)$ for only finitely many β
- $\sum_\beta \lambda_\beta = 1$

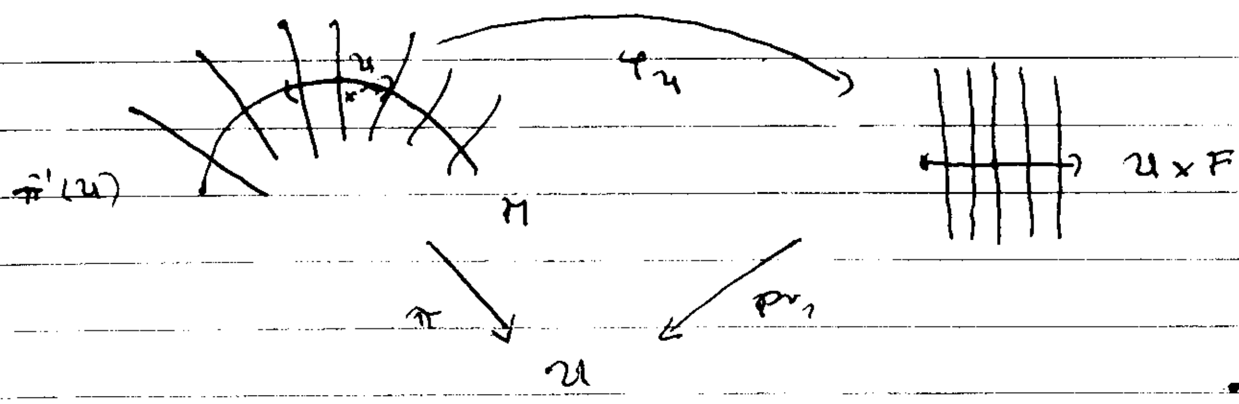
Let F be another C^∞-manifold (from now on we will suppress the prefix C^∞, all manifolds and maps being understood to be of this class).

<u>Definitions</u>:

A <u>fibre bundle</u> over M with typical fibre F is a manifold E together with a map $\pi : E \longrightarrow M$ such that the following condition (<u>L</u>ocal <u>T</u>riviality) holds:

(LT) For any $x \in M$ there is an open neighborhood $\mathcal{U}$ of x in M and a diffeomorphism

$$\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times F \quad \text{s.t.} \quad pr_1 \circ \varphi_{\mathcal{U}} = \pi$$



We call M the <u>base</u>, E the <u>total space</u>, and π the <u>bundle map</u> (projection) of the bundle (E, π, M).

Very often, the shorthand notation

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \\ & & M \end{array}$$

is used. Fibres $\pi^{-1}(x)$ are also denoted E_x.

A diffeomorphism φ_U as in (2.7) is called a <u>bundle chart</u>, a collection $\{(U_\alpha, \varphi_\alpha)\}$ of such charts is called a <u>bundle atlas</u> if $\bigcup_\alpha U_\alpha = M$.

Two bundles (E, π, M) and (E', π', M) are called <u>isomorphic</u> if there is a diffeomorphism $\Phi : E \to E'$ such that $\pi' \circ \Phi = \pi$.

In that case, the fibres of E and E' are diffeomorphic.

A bundle is <u>trivial</u> if it is isomorphic to $M \times F$.

A section of the bundle (E, π, M) is a C^∞-map $s : M \to E$ such that $\pi \circ s = id_M$ (i.e. $s(x) \in \pi^{-1}(x) = E_x$). We shall write $C^\infty(M, E)$ or $C^\infty(E)$ for the set of all sections of E.

The transition of one bundle chart $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ to another one $\varphi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times F$ takes a more specific form, now. We have a commutative tetrahedron

$$\begin{tikzcd}
& \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] \arrow[dd, "\pi"] & \\
U_\alpha \cap U_\beta \times F \arrow[rr, "\varphi_\beta \circ \varphi_\alpha^{-1}"] \arrow[dr, "pr_1"'] & & U_\alpha \cap U_\beta \times F \arrow[dl, "pr_1"] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

hence $\varphi_\beta \circ \varphi_\alpha^{-1}(x, f) = (x, g_{\alpha\beta}(x)(f))$ where $g_{\alpha\beta}(x)$ is an element of group of diffeomorphisms $\mathrm{Diff}(F)$, depending "differentiably" on x (in the

sense that the map $(x,f) \mapsto g_{\alpha\beta}(x)(f)$ is differentiable).
The map $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Diff}(F)$ is called a
<u>transition function</u>. Transition functions obviously
satisfy the following rules:

- $g_{\alpha\alpha}(x) = id_F$ (all $x \in U_\alpha$)
- $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$ (all $x \in U_\alpha \cap U_\beta$)
- $g_{\beta\gamma}(x) \circ g_{\alpha\beta}(x) = g_{\alpha\gamma}(x)$ (all $x \in U_\alpha \cap U_\beta \cap U_\gamma$).

(It is easily seen that the first two conditions can be derived from the third one, the so-called cocycle condition.)

In the same way as a manifold may be obtained from an atlas $\{(U_\alpha, \varphi_\alpha)\}$ as the result of glueing all chart domains U_α along their intersections $U_\alpha \cap U_\beta$ (by means of $\varphi_\beta \circ \varphi_\alpha^{-1}$), a fibre bundle may be viewed as the result of glueing of the trivial bundles $\mathcal{U}_\alpha \times F$, essentially being given by the family of transition functions $\{ g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{Diff}(F) \}$, a 1-cocycle on the open covering $\{\mathcal{U}_\alpha\}$ of M with values in $\mathrm{Diff}(F)$. Two such cocycles $\{g_{\alpha\beta}\}$, $\{g'_{\alpha\beta}\}$ give rise to isomorphic bundles exactly when they are cobordant, i.e. when there exists a family $\{h_\alpha\}$ of "differentiable" maps $h_\alpha : \mathcal{U}_\alpha \longrightarrow \mathrm{Diff}(F)$ such that

$$g'_{\alpha\beta}(x) = h_\beta(x) \circ g_{\alpha\beta}(x) \circ h_\alpha^{-1}(x)$$

for all α, β and $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

2. STRUCTURES AND OPERATION ON BUNDLES

The description of bundles by means of transition functions allows an easy, though non-intrinsic definition of structures and operations on bundles. Assume (E, π, M) being given by a cocycle $\{ g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{Diff}(F) \}$ and that the images of all $g_{\alpha\beta}$ lie in a subgroup G of $\mathrm{Diff}(F)$. Then any structure or operation on F which is respected by G can be transplanted to the fibers of E in a well-defined way: just use one of the bundle charts $\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times F$ to transport the structure from F to the fibers E_x, $x \in \mathcal{U}_\alpha$; since $g_{\alpha\beta}(x) \in G$, this result is independent of the choice of the chart.

Here are the most important examples:

a) $F \cong \mathbb{R}^m$ and G is $GL_m(\mathbb{R})$. Then E carries the structure of a real vector bundle (of rank m).

For $x \in M$, $u, v \in E_x$, $\lambda \in \mathbb{R}$, one defines

$$u+v \in E_x \qquad \text{resp.} \qquad \lambda u \in E_x$$

by choosing a bundle chart $\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ with $x \in U_\alpha$ and putting

$$u+v := \varphi_\alpha^{-1}\big((x, pr_2 \circ \varphi_\alpha(u) + pr_2 \circ \varphi_\alpha(v))\big)$$

$$\lambda u := \varphi_\alpha^{-1}\big((x, \lambda\, pr_2 \circ \varphi_\alpha(u))\big) .$$

b) $F \cong \mathbb{C}^m$, $G = GL_m(\mathbb{C})$. Then E carries the structure of a complex vector bundle (of rank m).

c) $F \cong \mathbb{R}^m$, $G = O_m(\mathbb{R})$ preserving the euclidean inner product $\langle\, ,\, \rangle : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $\langle u, v\rangle = \sum_{i=1}^m u_i v_i$.

In this case, the vector bundle E carries a Riemannian metric, i.e. $\langle\, ,\, \rangle : E_x \times E_x \longrightarrow \mathbb{R}$

is well-defined and depends differentiably on x.

d) $F \cong \mathbb{C}^m$, $G = U_m(\mathbb{C})$ preserving the hermitian inner product $\langle\ ,\ \rangle : \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}$, $\langle u, v\rangle = \sum_{i=1}^{m} \bar{u}_i v_i$.

Now, F is a complex vector bundle carrying a hermitian metric.

e) Consider your own favorite structures: orientation, symplectic, etc. ... .

As an example, let us look at the tangent bundle $E = TM$ of the manifold M. Let us choose an atlas $\{(U_\alpha, \varphi_\alpha : U_\alpha \hookrightarrow \mathbb{R}^n)\}$ on M. Then TM can be obtained by glueing the local tangent bundles $TU_\alpha = U_\alpha \times \mathbb{R}^n$ by means of the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$,

$$g_{\alpha\beta}(x) = D_{\varphi_\alpha(x)}(\varphi_\beta \circ \varphi_\alpha^{-1}) .$$

Additional structures on E induce additional structures on the set $C^\infty(M, E)$ of sections. Thus, by defining addition and multiplication pointwise, we obtain a $C^\infty(M, \mathbb{R})$ - module structure on $C^\infty(M, E)$ in the above case (a) of a real vector bundle, or a $C^\infty(M, \mathbb{R})$ - bilinear form $C^\infty(M, E) \times C^\infty(M, E) \longrightarrow C^\infty(M, \mathbb{R})$ in case (c). (The complex cases (b), (d) are completely analogous.)

By using transition functions, we can also describe a number of operations with vector bundles.

Let (E, π, M) be a fibre bundle and $f : M' \to M$ a map from another manifold M' into M. Then the <u>pull-back</u> of π (or E) by f is

$$f^*(E) = \{ (x, e) \in M' \times E \mid f(x) = \pi(e) \}$$

together with the first projection onto M'. If E is given by the cocycle $\{g_{\alpha\beta}\}$ with respect to the covering $\{U_\alpha\}$ of M, then f^*E is given by $\{g_{\alpha\beta}\circ f\}$ with respect to the covering $\{f^{-1}(U_\alpha)\}$ of M'. In particular, f^*E carries the same additional structures as E.

For simplicity, let us concentrate now on the case of real vector bundles $E, E_1, E_2, \ldots$ over M.

i) The Whitney sum $E_1 \oplus E_2 \xrightarrow{\pi} M$ is defined by $(E_1 \oplus E_2)_x = E_{1,x} \oplus E_{2,x}$ and transition functions $g_{\alpha\beta} = g_{1,\alpha\beta} \oplus g_{2,\alpha\beta} = \left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array}\right)$.

ii) The tensor product $E_1 \otimes E_2 \xrightarrow{\pi} M$ is defined by $(E_1 \otimes E_2)_x = E_{1,x} \otimes E_{2,x}$ and transition functions $g_{\alpha\beta} = g_{1,\alpha\beta} \otimes g_{2,\alpha\beta}$.

iii) The homomorphism bundle $\underline{\mathrm{Hom}}(E_1, E_2)$ is defined by $\underline{\mathrm{Hom}}(E_1, E_2)_x = \mathrm{Hom}(E_{1,x}, E_{2,x})$ and

transition functions $g_{\alpha\beta} = L(g_{2,\alpha\beta}) \cdot R(g_{1,\alpha\beta}^{-1})$,
where R and L denote right and left multiplication on $\mathrm{Hom}(\mathbb{R}^{m_1}, \mathbb{R}^{m_2})$. In particular, we have the dual bundle $E^* = \underline{\mathrm{Hom}}(E, \mathbb{R})$ with transition functions $g^*_{\alpha\beta} = {}^t g_{\alpha\beta}^{-1}$.

The global sections $C^\infty(M, \underline{\mathrm{Hom}}(E_1, E_2))$ are also called vector bundle homomorphisms from E_1 to E_2 (and denoted by $\mathrm{Hom}(E_1, E_2)$). A vector bundle homomorphism is an isomorphism, if it is an isomorphism of fibres at each point $x \in M$. For example, $\underline{\mathrm{Hom}}(E_1, E_2)$ and $E_1^* \otimes E_2$ are isomorphic (E_i of finite rank).

Iteration, combination, and generalisation of the above constructions lead to the most interesting bundles in geometry, i.e. starting from the tangent bundle TM of M we get its dual T^*M and the various tensor powers (symmetric, alternating) $\otimes^r TM$, $\otimes^r T^*M$, $S^r TM$, $S^r T^*M$, $\Lambda^r TM$, $\Lambda^r T^*M$.

Note that $C^\infty(M, S^r T^*M)$ is the "home" of symbols of r-th order differential operators, and $C^\infty(M, \Lambda^r T^*M)$ is the $C^\infty(M, \mathbb{R})$-module of alternating differential forms of degree r, also denoted by $A^r(M)$ or $\Omega^r(M)$.

Let us conclude this section by mentioning some useful facts:

i) Any real vector bundle admits a Riemannian metric.

ii) Any complex vector bundle admits a Hermitian metric.

iii) The association

$$E \longmapsto C^\infty(M, E)$$

$$\Phi \in \mathrm{Hom}(E_1, E_2) \longmapsto \Phi_* : C^\infty(M, E_1) \to C^\infty(M, E_2)$$
$$s \mapsto \Phi \circ s$$

induces an equivalence between the category of real vector bundles over M and the category of projective $C^\infty(M, \mathbb{R})$-modules of finite rank.

(An analogous statement holds for complex bundles.)

To prove i) and ii) one makes use of a partition of unity $\underbrace{}_{}$ $\sum \lambda_\alpha = 1$ subordinate to some bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$.

On $\pi^{-1}(U_\alpha)$ one obtains a metric by pull-back from $U_\alpha \times F$ (via φ_α). Multiplication by λ_α provides an extension of this metric to E, and summing over all α gives the required positivity properties.

Note that the association in (iii) is dual to Grothendieck's scheme theoretic approach, which, in the above notation, corresponds to $E \longmapsto C^\infty(M, E^*)$.

3. CONNECTIONS

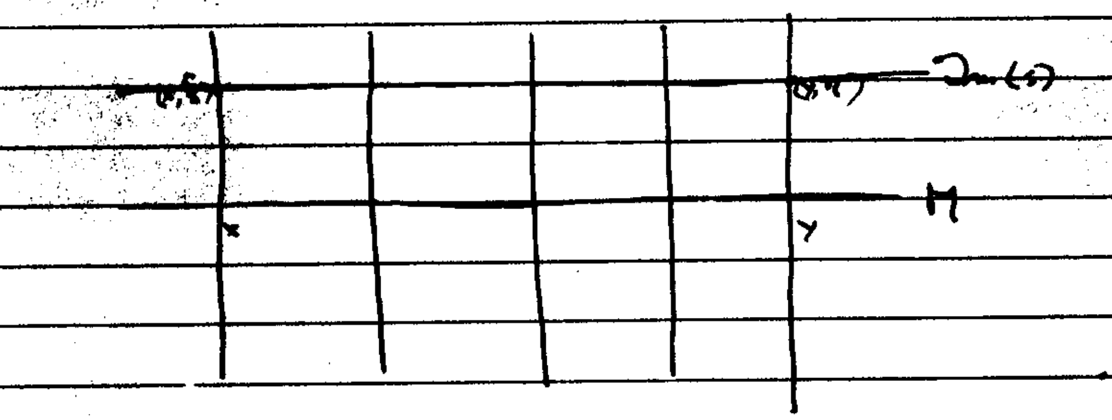
As indicated by its name, a connection on a vector bundle (E, π, M) provides a way of identifying different fibres of E in a "natural" way. However, in the case of a general E, there certainly can't be any natural global identification of all fibres, since this would result in the triviality $E \cong M \times F$ of E. Despite local triviality of E, there is also no way of natural local identification for general bundles (cf. the relations between curvature and characteristic classes to be developped later; Chern-Weil-theory). The only solution that remains, is the identification of "infinitesimally" near fibres. This can be given by means of a "covariant" derivative on sections of E.

As a model, let us first look at the trivial bundle

$E = M \times F$, $F \cong \mathbb{R}^m$. Two points $(x, \xi) \in E_x$ and $(y, \eta) \in E_y$ are identified if $\xi = \eta$, or, in more complicated terms, if (x, ξ) and (y, η) lie in the image of some "horizontal", i.e. constant, section

$s : M \to E$, $s(z) = (z, s_1(z), \dots, s_m(z)) \in M \times \mathbb{R}^m$,

$s_i(z) = \xi_i$ constant:



Infinitesimally, horizontal sections can be characterised by the vanishing of their (total) derivative

$$ds = (ds_1, \dots, ds_m) = 0.$$

For a general $s : M \to E = M \times \mathbb{R}^m$, ds may be viewed as a global section of the bundle <u>Hom</u>(TM, E), attaching to a tangent vector $X \in T_x M$ the infinitesimal deviation $ds(X) \in E_x = \mathbb{R}^m$ at x from a horizontal section.

This leads to the following definition for a general real vector bundle.

<u>Def.:</u> A <u>connection</u> on (E, π, M) is an $\mathbb{R}$-linear map

$$\nabla : C^\infty(M, E) \longrightarrow C^\infty(M, \underline{\mathrm{Hom}}(TM, E)) = C^\infty(M, T^*M \otimes E)$$

satisfying the Leibniz-rule:

$$\nabla(fs) = df \otimes s + f \nabla s$$

for all $s \in C^\infty(M, E)$, $f \in C^\infty(M, \mathbb{R})$.

We call ∇s the (total) <u>covariant derivative</u> of s, and for any $X_x \in T_x M$, we call $\nabla_{X_x} s := (\nabla s)(x)(X_x) \in E_x$ the <u>partial covariant derivative</u> of s <u>at</u> x <u>in the direction of</u> X_x. For any vector field X on M we call $\nabla_X s \in C^\infty(M, E)$, given by $(\nabla_X s)(x) = \nabla_{X(x)} s$, the <u>partial covariant derivative</u> of s <u>along</u> X. It satisfies:

- $\nabla_X : C^\infty(M, E) \longrightarrow C^\infty(M, E)$ is $\mathbb{R}$-linear
- $X \longmapsto \nabla_X$ is $C^\infty(M, \mathbb{R})$-linear
- $\nabla_X(fs) = df(X) . s + f \nabla_X s$ for all $f \in C^\infty(M, \mathbb{R})$, $s \in C^\infty(M, E)$

Examples: 1) "Ordinary" derivation on $M = \mathbb{R}^n$, $E = \mathbb{R}^n \times \mathbb{R}$,

i.e.

$$\begin{array}{ccc} \nabla = d : C^\infty(M, E) & \longrightarrow & C^\infty(M, \underline{Hom}(TM, E)) \\ \| & & \| \\ C^\infty(M, \mathbb{R}) & & \Omega^1(M) \\ \cup & & \\ f & \longmapsto & df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \end{array}$$

The Leibniz-rule is the "classical" one $d(f \cdot g) = df \cdot g + f dg$.

If X is the constant vector field $X(x) = e_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$,

∇_X is the i-th partial derivative

$$\begin{array}{ccc} \nabla_X : C^\infty(M, \mathbb{R}) & \longrightarrow & C^\infty(M, \mathbb{R}) \\ f & \longmapsto & df(X) = \frac{\partial f}{\partial x_i} \end{array}$$

2) $\nabla = d$ on a trivial bundle $E = M \times \mathbb{R}^m$ over a general manifold M (cf. our motivation).

3) Let (E, ∇) be a bundle over M with connection, and let $D \overset{i}{\hookrightarrow} E$ be a sub-bundle. Assume there is a projection $p \in Hom(E, D)$ such that $p \circ i = id_D$ (such a p can be constructed, for example, by using the orthogonal projection in E with respect to a Riemannian metric $\langle , \rangle$ on E).

An easy verification shows that the composite operator $\nabla' := p_* \circ \nabla \circ i_*$

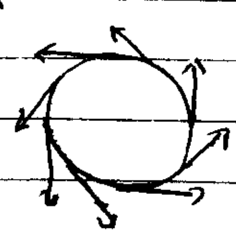
$$C^\infty(M,D) \xrightarrow{i_*} C^\infty(M,E) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes E) \xrightarrow{p_*} C^\infty(M, T^*M \otimes D)$$

is a connection (i_* is induced by the inclusion $D \to E$ and p_* by the projection $p: E \to D$).

This example lies at the origin of the theory of connections (Levi-Civita) since it is studied in the theory of submanifolds S of $\mathbb{R}^n$. In that case, D is the tangent bundle TS of S and E is the restriction of the tangent bundle $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ to S. Furthermore, E is equipped with the trivial connection d and the standard euclidean metric.

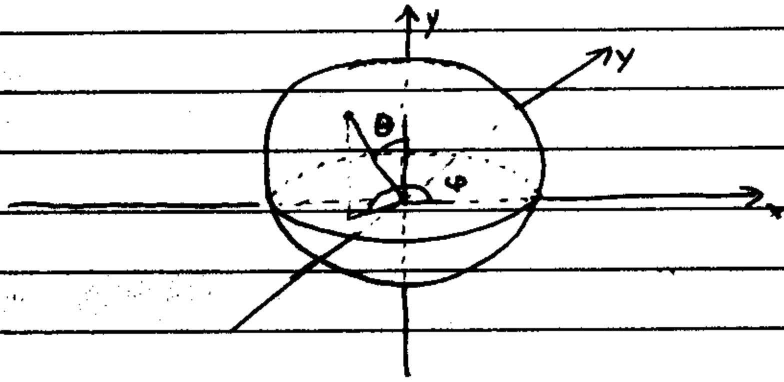
As a rather trivial example, consider $S^1 \subset \mathbb{R}^2$, the unit circle with unit tangent vector field

$$X(\theta) = (-\sin\theta, \cos\theta)$$



One easily checks: $\nabla' X = 0$.

Now look at $S^2 \subset \mathbb{R}^3$, the unit sphere parametrised by polar coordinates $\Pi : (\theta, \varphi) \longmapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$



For later use, we consider the two orthonormal vector fields on $S^2 \setminus \{\text{poles}\} =: \mathring{S}^2$:

$$X(\theta,\varphi) = (-\sin\varphi, \cos\varphi, 0)$$

$$Y(\theta,\varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

As sections of [illegible] $T^*\mathring{S}^2 \otimes E$ we have

$$dX = d\varphi \otimes (-\cos\varphi, -\sin\varphi, 0)$$

$$dY = d\varphi \otimes (-\cos\theta\sin\varphi, \cos\theta\cos\varphi, 0) + \\ + d\theta \otimes (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, -\cos\theta)$$

leading to

$$\nabla X = -\cos\theta \, d\varphi \otimes Y$$

$$\nabla Y = \cos\theta \, d\varphi \otimes X$$

after orthogonal projection $T^*\mathring{S}^2 \otimes T\mathbb{R}^3|_{S^2} \longrightarrow T^*\mathring{S}^2 \otimes T\mathring{S}^2$.

To work effectively with connections, one has to know their local form.

<u>Proposition</u>: $\nabla : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E)$ is a "local" operator, i.e. $\operatorname{supp}(\nabla s) \subset \operatorname{supp}(s)$ for all $s \in C^\infty(M,E)$.

<u>Proof</u>: Assume ~~[illegible]~~.

$x \notin \operatorname{supp}(s)$, i.e. there is an open neighborhood U of x s.th. $s(y) = 0$ for all $y \in U$. Choose an $f \in C^\infty(M,\mathbb{R})$ s.t. $f(y) = 1$ for all y in a neighborhood U' of x and $f(y) = 0$ for all $y \in \operatorname{supp}(s)$. Then we have $f.s = 0$, thus $0 = \nabla(fs) = df \otimes s + f\nabla s = f \nabla s$, and it follows that $\nabla s(y) = 0$ for all $y \in U'$, i.e. $\operatorname{supp}(\nabla s) \subset \operatorname{supp} s$.

<u>Corollary</u>: i) For any open $U \subset M$, the restriction $\nabla|_U : C^\infty(U, E) \to C^\infty(U, T^*U \otimes E|_U)$ is a connection.

ii) Let $\{U_\alpha\}$ be an open covering of M, then ∇

is uniquely determined by the family of its restrictions $\{\nabla|_{U_\alpha}\}$.

Note that the set of connections on E does not form a vector space (due to the Leibniz rule).

However, one has

<u>**Lemma**</u>: Let ∇_1, ∇_2 be connections on E and $\lambda \in C^\infty(M,\mathbb{R})$. Then $\lambda\nabla_1 + (1-\lambda)\nabla_2$ is a connection on E.

<u>**Proof**</u>: $(\lambda\nabla_1 + (1-\lambda)\nabla_2)(fs) = \lambda\nabla_1(fs) + (1-\lambda)\nabla_2(fs)$

$= \lambda\, df \otimes s + (1-\lambda)\, df \otimes s + \lambda f \nabla_1 s + (1-\lambda) f \nabla_2 s$

$= df \otimes s + f(\lambda\nabla_1 + (1-\lambda)\nabla_2)(s)$.

<u>Corollary</u>: Any vector bundle $E \xrightarrow{\pi} M$ admits a connection.

<u>**Proof**</u>: Let $\{U_\alpha\}$ be an open covering of M with subordinate partition of unity $\sum_\alpha \lambda_\alpha = 1$ and such that $E|_{U_\alpha}$ is trivial, $\cong U_\alpha \times \mathbb{R}^m$. Then we can choose some connection ∇_α on $E|_{U_\alpha}$ (e.g. the "trivial" one given by d). Using the preceding

lemma, one derives that $\nabla = \sum_\alpha \lambda_\alpha \nabla_\alpha$ is a connection on E.

Now consider a trivial bundle $E = \mathcal{U} \times \mathbb{R}^m$ over (an open piece of) some manifold. Then $C^\infty(\mathcal{U}, E)$ is a free $C^\infty(\mathcal{U}, \mathbb{R})$-module of rank m

$$C^\infty(\mathcal{U}, E) = \bigoplus_{i=1}^{m} C^\infty(\mathcal{U}, \mathbb{R}).s_i \quad ,$$

where $s_1, \dots, s_m$ are generating sections (e.g. $s_i(x) = (x, e_i) \in \mathcal{U} \times \mathbb{R}^m$). Similarly, we have

$$C^\infty(\mathcal{U}, T^*\mathcal{U} \otimes E) = \bigoplus_{i=1}^{m} \Omega^1(\mathcal{U}) \otimes s_i \quad .$$

It follows that

$$\nabla s_i = \sum_{j=1}^{m} \omega_i{}^j s_j \qquad \text{(we drop the } \otimes\text{-sign!)}$$

for all $i = 1, \dots, m$ and some matrix $\omega = ((\omega_i{}^j))$ of 1-forms, $\omega \in \Omega^1(\mathcal{U}) \otimes M_m(\mathbb{R})$. If we write $\vec{s}$ for the column vector ${}^t(s_1, \dots, s_m)$, we have

$$\nabla \vec{s} = \omega \vec{s} \quad .$$

The matrix ω is called the <u>connection matrix</u> of ∇

relative to the basis $\vec{s}$ of $C^\infty(U,E)$.

Obviously, ω and $\vec{s}$ determine ∇ due to the Leibniz-rule: If $s = \sum_{i=1}^{m} f^i s_i$ is an arbitrary section, we have

$$\nabla s = \nabla \sum_i f^i s_i = \sum_i df^i \otimes s_i + f^i \nabla s_i$$

$$= \sum_i \left(df^i \otimes s_i + f^i \sum_j \omega_i^{\,j} s_j \right)$$

$$= \sum_i \left(df^i + \sum_j f^j \omega_j^{\,i} \right) s_i \qquad \left(\text{after re-indexing}\right),$$

or, in matrix notation (identifying s with the row $(f^1,\dots,f^m)$)

$$\nabla (f^1,\dots,f^m) = (f^1,\dots,f^m)\cdot\omega + (df^1,\dots,df^m) \,.$$

Assume in addition that U is open in $\mathbb{R}^n$ (or diffeomorphic to such a set). Then we can choose standard vector fields $e_1,\dots,e_n$ in $C^\infty(U,TU)$, generating this $C^\infty(U,\mathbb{R})$ module freely. The connection ∇ is then completely determined by the Christoffel-symbols:

$$\Gamma^j_{ki} = \text{[redacted]}\ \omega_i^{\,j}(e_k) \in C^\infty(U,\mathbb{R})$$

where $i, j = 1, \dots, m$, $k = 1, \dots, n$. In particular, we have

$$\nabla_{e_k}(s) = \nabla_{e_k}\left(\sum f^i s_i\right) = \sum_i \left(\frac{\partial f^i}{\partial x_k} + \sum_j f^j \Gamma^i_{kj} \right) s_i .$$

Since the connection form ω depends on the basis $\vec{s}$, let us look at the transformation behaviour of ω w.r.to a change of basis $\vec{s} = {}^t(s_1, \dots, s_m) \longrightarrow \vec{s}' = {}^t(s'_1, \dots, s'_m)$. We have $\vec{s}' = g\,\vec{s}$, where g is an element of $C^\infty(\mathcal{U}, GL_m(\mathbb{R}))$, i.e. an $m \times m$-matrix of C^∞-functions with nowhere vanishing determinant. (In physicist's language, $\vec{s}$ is a fixed gauge and g is an element of the gauge group.) Let ω' denote the connection matrix of ∇ w.r.to $\vec{s}'$, i.e. $\nabla \vec{s}' = \omega' \vec{s}'$. Then

$$\omega' \vec{s}' = \nabla \vec{s}' = \nabla g \vec{s} = dg \otimes \vec{s} + g \nabla \vec{s} = dg \otimes \vec{s} + g\omega \otimes \vec{s} =$$
$$= dg \otimes g^{-1} \vec{s}' + g \omega g^{-1} \otimes \vec{s}' = (dg \cdot g^{-1} + g\omega g^{-1})\, \vec{s}'$$

or

$$\omega' = dg \cdot g^{-1} + g\omega g^{-1}$$

in obvious short-hand notation.

4. PARALLEL TRANSPORT AND CURVATURE

Let us return to our original problem of identifying different fibres of our (real) vector bundle (E, π, M) which we now assume to be equipped with a connection ∇. In the case of a trivial bundle we would effect such an identification by means of horizontal sections. We now define:

__Def.:__ A section $s \in C^\infty(M, E)$ is called __horizontal__ if $\nabla s = 0$.

Locally, this condition can be expressed by a system of partial differential equations:

If $s = \sum_{i=1}^m f^i s_i$ in $C^\infty(U, E)$, we have

$\nabla s = 0 \iff df^i + \sum_j f^j \omega_j^i = 0$ for all $i = 1, \dots, m$

$$\iff \frac{\partial f^i}{\partial x_k} + \sum_{j=1}^m \Gamma^i_{kj} f^j = 0 \quad \text{for all } i = 1, \dots, m,\ k = 1, \dots, n.$$

This system cannot be integrated, in general, as is shown already by the following example:

Ex.: Let $E = \mathbb{R}^n \times \mathbb{R}$, i.e. $m = 1$. Then $\nabla s = 0$ is expressed by $df + f\omega = 0$. If we assume some positive initial value for f (e.g. at $0 \in \mathbb{R}^n$), we must have $\omega = -d(\log f)$, i.e. ω has to be closed, $d\omega = 0$, to admit a solution f of the original equation.

However, if M is 1-dimensional, the PDF-system degenerates to an ODE-system:

$$\nabla s = 0 \iff \frac{df^i}{dt} + \sum_{j=1}^{m} \Gamma^i_j f^j = 0 \qquad (i=1,\dots,m)$$

$$\iff \frac{d\vec{f}}{dt} + \vec{f}\cdot\Gamma = 0$$

for some local parameter t on M. By the theory of linear ODE there is a uniquely determined solution $\vec{f}: t \mapsto \vec{f}(t) \in \mathbb{R}^m$ for any given initial value $\vec{f}(0) \in \mathbb{R}^m$. The map $\vec{f}(0) \mapsto \vec{f}(t)$ induces ~~an iso~~ a linear isomorphism

$$E_0 \xrightarrow{\sim} E_t$$

for any value of t in the relevant local domain. It is called the <u>parallel transport</u> from ~~~~ E_0 to E_t.

Let M be an arbitrary connected manifold, now. Then any two points x, y can be joined by a differentiable curve $\gamma: [0,1] \longrightarrow M$, $\gamma(0) = x$, $\gamma(1) = y$.

Carrying over the above arguments, either to $\gamma([0,1])$, if this is a submanifold (with boundary), or, more generally, to the pull-back $\gamma^* E$ of E to $[0,1]$ (the connection can also be pulled back), one sees that there is a linear isomorphism

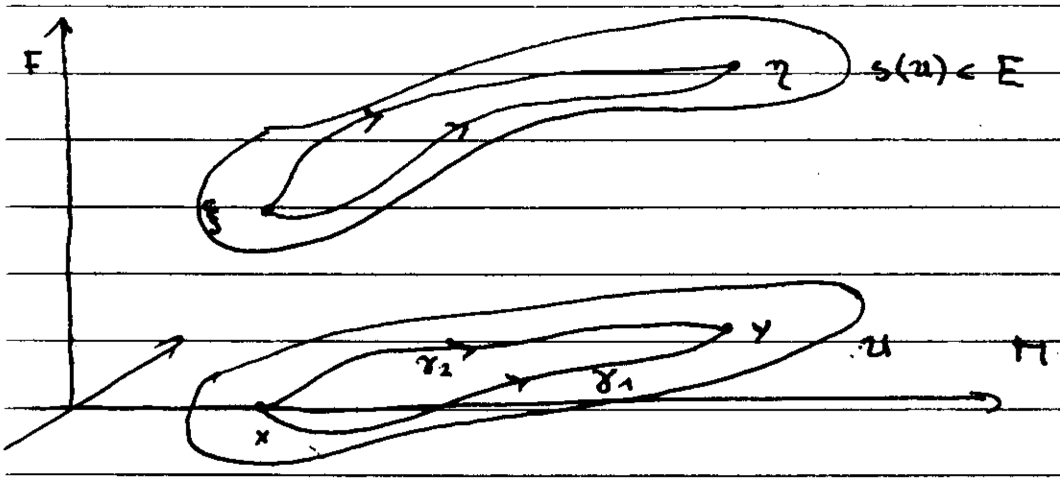
$$\tau_\gamma : E_x = E_{\gamma(0)} \longrightarrow E_{\gamma(1)} = E_y$$

obtained by parallel transport along γ.

Of course, τ_γ depends on γ, in general. In particular, if γ is a loop, $\gamma(0) = \gamma(1) = x$, we get a linear automorphism $\tau_\gamma : E_x \to E_x$. It can be shown (cf. Kobayashi–Nomizu) that the set of all τ_γ, γ a loop at x, forms a Lie subgroup of $GL(E_x) \cong GL_m(\mathbb{R})$. It is called the <u>holonomy group</u> of ∇, and is independent of x up to conjugacy. The dependence of τ_γ on γ is somewhat weaker for the following type of ~~[illegible]~~ connections (for which we are giving a preliminary definition):

<u>Def.</u>: A connection ∇ on E is called <u>flat</u> if every point $x \in M$ has a neighborhood U in M such that horizontal sections $s \in C^\infty(U, E)$ exist.

In this case, the parallel transport within U is independent of the path, since it is induced by the local horizontal sections:



$\xi \in E_x\,,\ \eta \in E_y \qquad \eta = \tau_{\gamma_1}(\xi) = \tau_{\gamma_2}(\xi)$

On the global scale, this local uniqueness implies independence under homotopies, i.e. τ_γ depends only on the homotopy class of γ. In particular, the holonomy group at a point x collapses to the <u>monodromy group</u> of the flat bundle (E,∇):

$$\{\tau_\gamma \mid \gamma \text{ loop at } x\} = \{\tau_{[\gamma]} \mid [\gamma] \in \pi_1(M,x)\}.$$

The representation $\begin{array}{ccc}\pi_1(M,x) & \longrightarrow & GL_m(\mathbb{R}) \\ [\gamma] & \longmapsto & \tau_{[\gamma]}\end{array}$ is the monodromy representation of (E,∇) and determines (E,∇) up to isomorphism (respecting ∇!).

The question wether a connection is flat can be answered in terms of the curvature of this connection.

(Due to lack of time, in the lecture as well as in the completion of this notes, we have to proceed in a more abstract way than before.)

To define the curvature tensor $K \in C^\infty(M, \Lambda^2 T^*M \otimes \underline{\mathrm{Hom}}(E,E))$ we extend the definition of ∇ to a map

$$\nabla : C^\infty(M, T^*M \otimes E) \longrightarrow C^\infty(M, \Lambda^2 T^*M \otimes E)$$

which is $\mathbb{R}$-linear and satisfies $\nabla(\alpha \otimes s) = d\alpha \otimes s - \alpha \wedge \nabla s$ for all $\alpha \in C^\infty(M, T^*M) = \Omega^1(M)$ and all $s \in C^\infty(M, E)$.

For that, view $C^\infty(M, T^*M \otimes E)$ as $\mathrm{Hom}(TM, E)$ and $C^\infty(M, \Lambda^2 T^*M \otimes E)$ as $\mathrm{Hom}(\Lambda^2 TM, E)$.

Now define $\nabla\varphi(X \wedge Y) := \nabla_X \varphi(Y) - \nabla_Y \varphi(X) - \varphi([X,Y])$ for $\varphi \in \mathrm{Hom}(TM, E)$, $X, Y \in C^\infty(M, TM)$ (note that $\varphi(Y), \varphi(X)$ are sections of E).

<u>Lemma:</u> Let $f \in C^\infty(M,\mathbb{R})$, $s \in C^\infty(M,E)$, $\alpha \in \Omega^1(M) = C^\infty(M,T^*M)$

$\beta \in C^\infty(M, T^*M \otimes E)$. Then we have:

i) $\nabla(f\beta) = df \wedge \beta + f\nabla\beta$

ii) $\nabla(\alpha \otimes s) = d\alpha \otimes s - \alpha \wedge \nabla s$

iii) $\nabla \circ \nabla(fs) = f\,\nabla\circ\nabla(s)$

<u>Proof:</u> For simplicity of notation we shall denote the derivation of a function f by a vector field X by $X.f$ $(= df(X))$.

i) $$\nabla(f\beta)(X \wedge Y) = \nabla_X f\beta(Y) - \nabla_Y f\beta(X) - f\beta([X,Y])$$

$$\begin{aligned} &= X.f\,\beta(Y) + f\,\nabla_X\beta(Y) \\ &\quad - Y.f\,\beta(X) - f\,\nabla_Y\beta(X) \\ &\quad - f\,\beta([X,Y]) \end{aligned}$$

$$= df \wedge \beta(X,Y) + f\,\nabla\beta(X \wedge Y)$$

ii) $$\nabla(\alpha\otimes s)(X \wedge Y) = \nabla_X \alpha(Y)s - \nabla_Y\alpha(X)s - \alpha([X,Y])s$$

$$\begin{aligned} &= (X.\alpha(Y))s + \alpha(Y)\nabla_X s \\ &\quad - (Y.\alpha(X))s - \alpha(X)\nabla_Y s \\ &\quad - \alpha([X,Y])s \end{aligned}$$

$$= d\alpha(X,Y)s - (\alpha \wedge \nabla s)(X,Y)$$

iii) We use i) and ii):

$$\nabla\circ\nabla(fs) = \nabla(df\otimes s + f\nabla s) = ddf\otimes s - df\wedge\nabla s + df\wedge\nabla s + f\,\nabla\circ\nabla s = f\,\nabla\circ\nabla(s).$$

Due to property iii) $\nabla\circ\nabla$ corresponds to a homomorphism of bundles $E \longrightarrow \Lambda^2T^*M\otimes E$, i.e. to an element K in $C^\infty(M, \Lambda^2T^*M\otimes \underline{\mathrm{Hom}}(E,E))$.

<u>Def.</u>: K is called the <u>curvature tensor</u> of (E,∇).

<u>Theorem</u>: The connection ∇ is flat iff $K=0$.

In other words, the vanishing of the curvature is the integrability condition for the PDE-system $\nabla s = 0$.

For the proof of this theorem we refer to the references (Spivak, Kobayashi-Nomizu). However, let us derive the local expression of K in terms of the connection matrix ω attached to a basis $s_1,\dots,s_m$ of sections of E (on some trivialising set $U\subset M$), cf. p. 24.

Assume $\nabla s_i = \sum_{j=1}^{m} \omega_i^j \otimes s_j$, $\omega = ((\omega_i^j)) \in \Omega^1(U) \otimes M_m(\mathbb{R})$,

then
$$\nabla \circ \nabla s_i = \sum_{j=1}^{m} d\omega_i^j \otimes s_j - \omega_i^j \wedge \nabla s_j$$
$$= \sum_{j=1}^{m} d\omega_i^j \otimes s_j - \omega_i^j \wedge \left(\sum_{k=1}^{m} \omega_j^k \otimes s_k \right)$$
$$= \sum_{j=1}^{m} \left(d\omega_i^j - (\omega \wedge \omega)_i^j \right) \otimes s_j$$
$$= \sum_{j=1}^{m} \Omega_i^j \otimes s_j$$

where $\Omega = ((\Omega_i^j)) \in \Omega^2(M) \otimes M_m(\mathbb{R})$ is a matrix of 2-forms; $\omega \wedge \omega$ is the matrix product of ω with itself, the ~~compon~~ matrix entries being multiplied by the wedge product "$\wedge$".

<u>Definition</u>: $\Omega = d\omega - \omega \wedge \omega \in \Omega^2(M) \otimes M_m(\mathbb{R})$ is the <u>curvature matrix of</u> ∇ relative to the basis $\xi = {}^t(s_1, \dots, s_m)$ of $C^\infty(U, E)$.

<u>Example</u>: We continue the study of the example on p. 21, where $E = TS^2$. If we denote the vector fields X and Y

by s_1 and s_2, we obtain the connection matrix

$$\omega = \begin{pmatrix} 0 & -\cos\theta\, d\varphi \\ \cos\theta\, d\varphi & 0 \end{pmatrix}$$

and the curvature matrix

$$\Omega = d\omega - \omega\wedge\omega = \begin{pmatrix} 0 & \sin\theta\, d\theta\wedge d\varphi \\ -\sin\theta\, d\theta\wedge d\varphi & 0 \end{pmatrix}.$$

Note that $\sin\theta\, d\theta\wedge d\varphi$ is the standard volume (= area) form on the unit sphere, hence

$$\Omega = d\mathrm{vol}_{S^2(1)} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In case of the sphere $S^2(R)$ of radius R all computations would be the same apart from the volume form being $R^2 \sin\theta\, d\theta\wedge d\varphi$. Thus, in this case

$$\Omega = d\mathrm{vol}_{S^2(R)} \cdot \begin{pmatrix} 0 & 1/R^2 \\ -1/R^2 & 0 \end{pmatrix}.$$

(Hence you see the curvature of $S^2(R)$ creeping in !)

The transformation behaviour of Ω under gauge transformations $g \in C^\infty(\mathcal{U}, GL_m(\mathbb{R}))$ is simpler than that for ω since Ω is a section of the bundle $\Lambda^2 T^*M \otimes \underline{\mathrm{Hom}}(E,E)\big|_{\mathcal{U}}$, i.e. we must have

$$\Omega' = g\,\Omega\, g^{-1}.$$

This can also be derived directly from the relation

$$\omega' g = dg + g\omega \qquad \text{(cf. p. 36)}$$

which gives $d\omega' g - \omega' \wedge dg = dg \wedge \omega + g\, d\omega$ after exterior derivation. Substituting $dg = \omega' g - g\omega$ and multiplying on the right by g^{-1} gives

$$d\omega' - \omega' \wedge \omega' = g\, d\omega\, g^{-1} - g\,\omega \wedge \omega\, g^{-1}.$$

Applying d to $\Omega = d\omega - \omega \wedge \omega$ we obtain the

<u>Bianchi - identity</u> $\qquad d\Omega = \omega \wedge \Omega - \Omega \wedge \omega =: [\omega, \Omega]$

useful in the Chern - Weil theory of characteristic classes.

Up to now we have discussed only real vector bundles (E,π,M). One can of course look at vector bundles with additional structure given by some Lie subgroup G of $GL_m(\mathbb{R})$, like complex bundles, $G = GL_\ell(\mathbb{C}) \subset GL_{2\ell}(\mathbb{R})$, Riemannian bundles, $G = O_m(\mathbb{R}) \subset GL_m(\mathbb{R})$, Hermitian bundles, $G = U_\ell(\mathbb{C}) \subset GL_{2\ell}(\mathbb{R})$, and others.

Def.: Let (E,π,M) be a real vector bundle with additional structure (given by $G \subset GL_m(\mathbb{R})$), and let ∇ be a connection on E. Then ∇ is called admissible if the parallel transport respects this structure.

The theory of admissible connections can be best understood by the theory of principal fibre bundles and their connections (cf. Spivak or Kobayashi-Nomizu).

We only want to add that admissible connections exist for Riemannian and Hermitian bundles. This can be easily derived from the form of their connection matrices with respect to orthonormal bases of local section $C^\infty(\mathcal{U}, E)$, which have to lie in $\Omega^1(\mathcal{U}) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $O_m(\mathbb{R})$ resp. $U_r(\mathbb{C})$, i.e.

$$\mathfrak{g} = \mathrm{Lie}\, O_m(\mathbb{R}) = \{ A \in M_m(\mathbb{R}) \mid {}^tA = -A \}$$

resp. $\mathfrak{g} = \mathrm{Lie}\, U_r(\mathbb{C}) = \{ A \in M_r(\mathbb{C}) \mid {}^t\bar{A} = -A \}$.

The curvature matrix lies in $\Omega^2(\mathcal{U}) \otimes \mathfrak{g}$. In the Riemannian case we have seen that in our discussion of TS^2 (pp. 21, 36).

As indicated in this example, curvature goes back to the curvature of surfaces in $\mathbb{R}^3$ or, more generally, to curvature of Riemannian manifolds. Because of lack of time, we leave it to the reader to explore these relations (cf. the references).

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