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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Topology and Differential Geometry - Lecture III

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These are preliminary lecture notes, intended only for distribution to participants

Topology and Differentialgeometry
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Lecture III
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Linear differential operators and basic properties of elliptic operators

I. Generalities, the symbol map

Linear differential operators act as $\mathbb{R}$-linear operators on C^∞-sections of vector bundles.
Locally, if

$$\begin{array}{ccc} E = X \times V & & F = X \times W \\ \downarrow & & \\ \mathbb{R}^n \supset X & & \end{array}$$

with vector spaces V, W, $C^\infty_X(E) = C^\infty_X \otimes V$ and LDO have the form

$$(Ps)(x) = \sum_{|\alpha| \leq N} L_\alpha(x) \frac{\partial^\alpha s}{\partial x^\alpha}(x)$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$

$\frac{\partial^{\alpha}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, L_α smooth functions

$X \longrightarrow \mathrm{Hom}(V, W)$

An intrinsic definition can be given inductively as

$$\mathrm{Diff}^0(E,F) = \mathrm{Hom}_{\mathcal{C}^\infty_X}(\mathcal{C}^\infty_X(E), \mathcal{C}^\infty_X(F))$$
$$= \mathrm{Hom}(E,F) \quad \text{(see lecture II)}$$
$$= \{P : \mathcal{C}^\infty_X(E) \to \mathcal{C}^\infty_X(F),\ \text{for any } f \in \mathcal{C}^\infty_X(U),\ [P,f]=0\}$$

Here, $[P,f](e) = P(fe) - fP(e)$ for operators $P : \mathcal{C}^\infty_X(E) \to \mathcal{C}^\infty_X(F)$.

Suppose $\mathrm{Diff}^p(E,F)$ is defined for $p<N$ we define

$$\mathrm{Diff}^N(E,F) = \{P : \mathcal{C}^\infty_X(E) \to \mathcal{C}^\infty_X(F),\ \mathbb{R} \text{ linear and for all open } U \subset X,\ f \in \mathcal{C}^\infty_X(U) \text{ there holds } [P,f] \in \mathrm{Diff}^{N-1}(E|U, F|U)\}$$

As an exercise, check the following

(1) $P \in \mathrm{Diff}^1(E,F) \Rightarrow f \mapsto [P,f]$ is a derivation $\mathcal{C}^\infty_X \to \mathcal{H}om(E,F) = \mathcal{C}^\infty_X(HOM(E,F))$

(2) $P \in \mathrm{Diff}^N(E,F)$, $f,g \in \mathcal{C}^\infty_X \Rightarrow$
$[[P,f],g] = [[P,g],f]$

As a consequence of (1), (2) we get

There exist a bundle map

$$\sigma_N(P) : S^N T^*X \to HOM(E,F)$$

such that for $f_1, \dots, f_N \in \mathcal{C}^\infty_X$

$$[\cdots[[P,f_1],f_2],\cdots,f_N] = \sigma_N(P)(df_1 \odot \cdots \odot df_N)$$

Here $\odot$ denotes the symmetric product.

Since N-linear symmetric maps can be identified with polynomials maps of degree N (homogeneous) the <u>symbol map</u> $\sigma_N(P)$ is known if we know the map $T^*X \to HOM(E,F)$

$\xi \mapsto \sigma_N(P)(\xi^{\odot N})$

In terms of coordinates $x_1, \dots, x_n$ near a point of X we can describe it as follows:

For $\xi = \xi_1 dx_1 + \cdots + \xi_n dx_n$

$$\sigma_N(P)(\xi^{\odot N}) = [\cdots\underbrace{[[P,f],f],\cdots,f}_{N \text{ times}}]$$

f a function with $df = \xi$, e.g. $f = \xi_1 x_1 + \cdots + \xi_n x_n$

Example: $P = \sum_{|\alpha| \le N} L_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$

Then $\sigma_N(P)(\xi^{\otimes N}) = \sum_{|\alpha|=N} \xi^\alpha L_\alpha(x)$

$\left(\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}\right)$

If P is defined as in the intrinsic way and for a coordinate neighbourhood U and choices of trivializations of E, F the symbol of P has the expression

$$\sigma_N(P) = \sum_{|\alpha|=N} \xi^\alpha L_\alpha(x)$$

the operator $Q = \sum_{|\alpha|=N} L_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ has the same symbol and therefore

$\sigma_N(P-Q) = 0$

By the definition of $\sigma_N(P)$ this means $P-Q \in \mathrm{Diff}^{N-1}(E|U, F|U)$ and by induction on N we can conclude that there exist

$L_\alpha \in \mathcal{C}^\infty_X(\cancel{U}) \otimes \mathrm{Hom}(V, W)$, $|\alpha| \leq N$ and

$$P = \sum_{|\alpha| \leq N} L_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

Note the following formulas

(1) For $P \in Diff^p(E,F)$, $Q \in Diff^q(F,G)$

$Q \circ P \in Diff^{p+q}(E,G)$

$$\sigma_{p+q}(Q \circ P)\left(\frac{\xi^{\odot p+q}}{(p+q)!}\right) = \sigma_q(Q)\left(\frac{\xi^{\odot q}}{q!}\right) \circ \sigma_p(P)\left(\frac{\xi^{\odot p}}{p!}\right)$$

(2) If E, F are Hermitian bundles and if $*1$ is a volume form on X we can define $P^* \in Diff^N(F,E)$ by

$$\int_X \langle P^* f, e\rangle (*1) = \int_X \langle f, Pe\rangle (*1)$$

for smooth sections with compact supports (integration by parts).

Then $\sigma_N(P^*)(\xi^{\odot N}) = -\sigma_N(P)(\xi^{\odot N})^*$

Considering $U \mapsto Diff^N(E|U, F|U)$ we get sheaves of $\mathcal{C}^\infty_X$-modules, denoted by $\mathcal{D}iff^N(E,F)$ and exact sequences

$$0 \to \mathcal{D}iff^{N-1}(E,F) \to \mathcal{D}iff^N(E,F) \xrightarrow{\sigma_N} \mathcal{H}om(S^N TX, HOM(E,F)) \to 0$$

(<u>Symbol sequence</u>)

Thus by induction on N we see that the sheaves $\mathcal{D}iff^N$ are locally free sheaves of $\mathcal{C}^\infty_X$-modules and therefore they are the sheaf of $\mathcal{C}^\infty$-sections of vector bundles, denoted by $DIFF^N(E,F)$

$$\mathcal{D}iff^N(E,F) = \mathcal{C}^\infty_X\,(DIFF^N(E,F))$$

There is a dual language, the dual of the bundle $DIFF^N(E,I)$ (I the trivial rank 1 bundle) has the name "bundle of N-jets of sections of E" denoted by $J_N(E)$.

By the definition we get a operator

$$j_N : \mathcal{C}^\infty_X(E) \longrightarrow \mathcal{C}^\infty_X(J_N(E)) = \mathcal{H}om\,(DIFF^N(E,I),$$

$$j_N(e)\,(P) = P(e)$$

which is a universal differential operator of order N, i.e.

(1) $j_N \in \mathcal{D}iff^N(E, J_N(E))$

(2) For each vector bundle the map

$$Hom\,(J_N(E),F) \longrightarrow \mathcal{D}iff^N(E,F)$$
$$h \longmapsto h \circ j_N$$

is a bijection.

The dual of the symbol sequence is the exact sequence (symbol sequence)

$$0 \to S^N T^*X \otimes E \to J_N(E) \to J_{N-1}(E) \to 0$$

and the symbol of $P = h \circ j_N$ is the restriction of h to $S^N T^*X \otimes E$,

$$\sigma_N(h \circ j_N) = h \mid S^N T^*X \otimes E \, .$$

Remark: $j_N(e)$ is an intrinsic version of "Taylor expansion of e up to order N".
To be precise: choose connections ∇, ∇_0 on T^*X and E, define the tensor product connections

$$\nabla_k : \mathcal{C}^\infty_X(\overset{k}{\otimes} T^*X \otimes E) \to \mathcal{C}^\infty_X(\overset{k+1}{\otimes} T^*X \otimes E)$$

and operators

$$\begin{array}{ccc} D_N : \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(S^N T^*X \otimes E) \\ & {\scriptstyle \nabla_N \circ \cdots \circ \nabla_0} \searrow \quad \nearrow {\scriptstyle \frac{1}{N!}\text{ symmetrization}} & \\ & \mathcal{C}^\infty(\overset{N}{\otimes} T^*X \otimes E) & \end{array}$$

with $\sigma_N(D_N)(\xi^{\otimes N})e = \xi^{\otimes N} \otimes e$, i.e. a splitting of the symbol sequence. Hence

$$J_N(E) = E \oplus (T^*X \otimes E) \oplus (S^2 T^*X \otimes E) \oplus \cdots \oplus (S^N T^*X \otimes E)$$

<u>Exercise</u>: Check that in case $X \subset \mathbb{R}^n$ open, ∇ the trivial connection on $T^*X = X \times (\mathbb{R}^n)^*$, $E = X \times V$, D_0 the trivial connection this yields the homogeneous components of Taylor expansions of sections.

The decomposition yields intrinsic metrics on the ~~S~~Sobolov-spaces $L^2_N(E)$, if E and T^*X are metric bundles, by

$$(e, e')_{L^2_N(E)} = (j_N(e), j_N(e'))_{L^2(J_N(E))}$$

($L^2_S(E) = W_s(E)$ in Denningers lecture)

<u>Remark</u> We get extensions $\mathrm{Diff}^N(E,F) \to \mathcal{L}(L^2_s(E), L^2_{s-N}(F))$ (bounded lineare operators) $P \mapsto P_s$

II. <u>Basic properties of elliptic operators</u>

The operator $P \in \mathrm{Diff}^N(E,F)$ is called elliptic of order N, if for all $x \in X$ and all $\xi \in T^*_x(X) \setminus (0)$ the symbol $\sigma_N(P)(\xi): E_x \to F_x$ is an isomorphism

Examples:

1. $E = F = \Lambda^* T^* X$, X oriented Riemannian manifold

$d : C^\infty_X(E) \to C^\infty_X(E)$ has symbol

$\sigma_1(d)(\xi) = \xi\wedge$

so it is not elliptic.

$\sigma_1(d^*)(\xi) = -\xi\lrcorner$

(where $\xi\lrcorner$ denotes contraction with ξ

$\xi\lrcorner\,\alpha_0\wedge\dots\wedge\alpha_p = \sum(-1)^j\langle\xi,\alpha_j\rangle\,\alpha_0\wedge\dots\wedge\check{\alpha}_j\wedge\dots\wedge\alpha_p$)

hence $d+d^*$ is elliptic of order 1

$$\sigma_1(d+d^*)(\xi)(\alpha) = \xi\wedge\alpha - \xi\lrcorner\alpha$$

and $\Delta = (d+d^*)^2 = d\circ d^* + d^*\circ d$

is elliptic of order 2 and

$$\sigma_2(\Delta)(\xi^{\otimes 2}) = -2\,\|\xi\|^2\,\mathrm{Id}$$

2. If ∇ is a connection on E, then $\nabla\in\mathrm{Diff}^1(E, T^*X\otimes E)$ and

$\sigma_1(\nabla)(\xi) = \xi\otimes$

i.e. a connection is the same as a splitting of the symbol sequence (for $N=1$) or the jet sequence for $N=1$

For oriented Riemannian manifolds and metric bundles E we get an elliptic operator of order 2

$D^* \circ D$, $\sigma_2(D^* \circ D) = -2\|\xi\|^2 Id$.

The basic property of an elliptic operator, invertibility of the symbol, implies that the operator P is almost invertible, i.e. there exist a so called "parametrix

<u>Theorem</u> Assume X a closed manifold, E, F vector bundles, $P \in Diff^N(E,F)$ elliptic of order N.
There exist an operator Q of order $-N$ (as explained in Deningers lecture) such that

$$Id - P \circ Q = S$$
$$Id - Q \circ P = S'$$

both are "infinitely smoothing" operators, i.e. defined by smooth sections $S \in \Gamma(HOM(pr_2^*F, pr_1^*F))$ and

$$(Sf)(x) = \int_X S(x,y) f(y) (*1)_y$$

$S' \in \Gamma(HOM(pr_2^*E, pr_1^*E))$ and

$$(S'e)(x) = \int_X S'(x,y) e(y) (*1)_y$$

References: Consult for example one of the books:

1) P.B. Gilkey, Invariant theory. The Heat Equation And the Atiyah Singer Index Theorem. (Publ. or Perish, Inc. 1984)

2) Lawson, Michelsohn, Spin geometry, (Princeton Univ. press 1989)

Consequences: In the following X is compact $P \in \mathrm{Diff}^N(E,F)$ elliptic of order.

(1) <u>Regularity</u>

$P \in \mathrm{Diff}^N(E,F)$ elliptic, $e \in L^2_s(E)$ then

$Pe \in \Gamma(E) \implies e \in \Gamma(E)$

(remember: $\Gamma(E)$ denotes smooth $= C^\infty$-sections of E).

Proof: Use $e = Q \circ Pe + Se$

(2) <u>Fredholm property</u>

The extensions $P_s : L^2_s(E) \to L^2_{s-N}(F)$

are Fredholm (of index independant of s)

Proof: By Rellich's lemma (see Denninger's lecture)

$Id - Q_{s-N} \circ P_s = S$, $Id - P_{s+N} \circ Q_s = S'$

are compact operators.

$S | Ker(P) = Id | Ker(P) \Rightarrow \dim Ker(P) < \infty$
$S^{\prime *} | Ker(P^*) = Id | Ker(P^*) \Rightarrow \dim Ker(P^*) < \infty$

(Remember: If Id is a compact operator $\Rightarrow$ the space is finite-dimensional)

Since $Ker(P^*) = Im(P)^\perp$ it remains to show that $Im(P)$ is closed.

We have to show:

If $e_k \in \Gamma_s(E)$ and $P_s e_k \to f$ in F then $f \in Im(P_s)$

We can assume $e_k \in Ker(P_s)^\perp$

Claim: $\{e_k\}$ is bounded

To prove this we may assume that $S\left(\frac{e_k}{\|e_k\|}\right)$ is convergent (Rellich's lemma)

~~hence~~ $\cancel{S\left(\frac{e_k}{\|e_k\|}\right) = \frac{e_k}{\|e_k\|} - \frac{1}{\|e_k\|} \mathrm{QoP}(e_k)}$

~~and $\|e_k\| \to \infty$ would~~

If $\|e_k\| \to \infty$ we could infere that

$$\frac{e_k}{\|e_k\|} = S\left(\frac{e_k}{\|e_k\|}\right) + Q_{s-N}\left(\frac{P_s(e_k)}{\|e_k\|}\right)$$

is convergent to a vector e, $\|e\|_s = 1$, in contradiction to $P_s e = \lim_{k \to \infty} P_s\left(\frac{e_k}{\|e_k\|}\right) = 0$ and $e \perp Ker(P)$.

Thus we can again use Rellich's lemma and assume that $Se_k \to e'$ in $L^2_s(E)$.
Then $Q_{s-N} P_s e_k = e_k - Se_k \to Q_{s-N}(f) - e'$
and $e_k \to e =: Q_{s-N}(f) + e'$, $P(e) = f$.

③ <u>Hodge decomposition</u>

$\Gamma(E) = \mathrm{Ker}(P) \oplus \mathrm{Im}(P^*)$
orthogonal with respect to $L^2_0(E)$

Proof: Clearly $\mathrm{Im}(P^*) \perp \mathrm{Ker}(P)$ and
$\mathrm{Ker}(P) = \mathrm{Ker}(P^* \circ P)$, $\mathrm{Im}(P^*) \supseteq \mathrm{Im}(P^* \circ P)$,
so we can replace P by $P^* \circ P$.
Hence for the proof we can assume
$P = P^*$.
Write $L^2(E) = \mathrm{Ker}(P_0) \oplus \mathrm{Ker}(P_0)^\perp$ then
$\Gamma(E) = \mathrm{Ker}(P) \oplus (\Gamma(E) \cap \mathrm{Ker}(P_0)^\perp)$
(by Regularity)
Since $L^2_{-N}(E)^* = L^2_N(E)$ we get $P_0^* = P_N$
we have $\mathrm{Ker}(P_0)^\perp = \mathrm{Im}(P^*) = P_N(L^2_N(E))$
and by Regularity $\mathrm{Ker}(P_0)^\perp \cap \Gamma(E) = P(\Gamma(E))$ q.e.d.

④ Spectral properties

$P \in \mathrm{Diff}^N(E,E)$, $P^* = P$, $N > 0$.

Since $P - \lambda \mathrm{Id}$ is elliptic we get finite dimensional subspaces

$$\Gamma_\lambda(E) = \mathrm{Ker}(P - \lambda \mathrm{Id}) \subset \Gamma(E)$$

There holds

(i) $\exists c > 0$ such that for each $\Lambda > 0$,

$$\Gamma_{\leq \Lambda}(E,P) = \sum_{|\lambda| \leq \Lambda} \Gamma_\lambda(E) \text{ satisfies}$$

$$\dim \Gamma_{\leq \Lambda}(E,P) \leq c\, \Lambda^{n(n+2N+2)/2N}$$

($n = \dim X$)

i.e. if we write the sequence of eigenvalues, counted with multiplicity as

$$0 \leq |\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_k| \leq \dots$$

then $|\lambda_k| \geq \mathrm{const}\; k^{2N/(n+2N+2)n}$

(ii) $L^2_0(E) = \hat{\bigoplus} \Gamma_\lambda(E)$

We will sketch the proof of (i).

Write $v \in \Gamma_{\leq \Lambda}$ as $v = \sum_\lambda a_\lambda e_\lambda$,

$Pe_\lambda = \lambda e_\lambda$, $\|e_\lambda\|_0 = 1$, then

$$\|P^k v\|_0^2 \leq \sum |a_\lambda|^2 \lambda^{2k} \leq \|v\|_0^2 \Lambda^{2k}$$

Using a parametrix Q_k for P^k we find

$$v = Q_k \circ P^k v + Sv$$

$$\|v\|_s \leq \|Q_k \circ P^k v\|_s + \|Sv\|_s$$

$$\leq C_k \left(\|P^k v\|_{s-kN} + \|v\|_{s-kN} \right)$$

(C_k a constant)

For $s = kN$ we get

$$\|v\|_{kN} \leq C_k \|v\|_0 (1 + \Lambda^k)$$

Using Sobolev's theorem (see Demingers lecture) we get for ~~s~~ $kN > \frac{n}{2} + 1$

$L^2_{kN}(E) \subset \Gamma^{(1)}(E)$ (C^1-sections with uniform C^1-norm)

$$\sup_X \|\nabla v\| \leq C_k' (1 + \Lambda^k) \|v\|_0$$

If $B_\varepsilon(a)$ denotes the ε-ball in X with ~~radius~~ center a and if $v(a) = 0$ we therefore get

$$\sup_{B_\varepsilon(a)} \|v\| \leq \|v(a)\| + C_k'' \varepsilon (1 + \Lambda^k) \|v\|_0$$

(C_k'' a constant depending on k only)

If we fix $\varepsilon > 0$ and cover X by say m ε-balls and if $\dim \Gamma_{\leq \Lambda} > m$ we find a section $v \in \Gamma_{\leq\Lambda}$ vanishing in the centers of the balls ~~with~~ $\|v\|_0 = 1$.

Then $\sup_X \|v\| \leq \varepsilon \cdot C_k'' (1+\Lambda^k)$

$$1 = \|v\|_0^2 = \int_X \|v\|^2(x) \leq \varepsilon^2 C_k''^2 (1+\Lambda^k)^2 \operatorname{vol}(X)$$

$$\varepsilon \geq \frac{1}{C_k''(1+\Lambda^k)\sqrt{\operatorname{vol}(X)}}$$

If $m(\varepsilon)$ denotes the minimal number of ε-balls covering X and

$$\varepsilon_\Lambda = \frac{1}{2} \frac{1}{C_k''(1+\Lambda^k)\sqrt{\operatorname{vol}(X)}}$$

it follows

$$\dim \Gamma_{\leq\Lambda}(E) \leq m(\varepsilon_\Lambda)$$

<u>Exercise</u> If $n = \dim X$ then $\varepsilon^n m(\varepsilon)$ is bounded for $0 < \varepsilon \ll 1$.

Using the exercise we get

$$\frac{\dim \Gamma_{\leq\Lambda}(E)}{(1+\Lambda^k)^n} \text{ is bounded.}$$

Choosing $k = \left[\frac{n+2N+2}{2N}\right]$ the assumption

$kN \geq \frac{n}{2}+1$ is satisfied and we get the estimate (i).

III. The ζ-function

Suppose $P \in \mathrm{Diff}^N(E,E)$ elliptic of order $N > 0$, $P^* = P$ and P positive (i.e. $(Pe, e)_{L^2_0(E)} \geq 0$ for alle e)

Write the spectrum (with multiplicity) as $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ and fix $\Lambda > 0$.

Then $\lambda_k \geq \text{const.} \cdot k^{\frac{2N}{n(n+2N+2)}}$ and therefore by

$$\zeta(s,P) = \sum_{\lambda_k \neq 0} \lambda_k^{-s}$$

$$\zeta(s,P,\Lambda) = \sum_{\Lambda \leq \lambda_k} \lambda_k^{-s}$$

we get holomorphic functions on the half-plane $\mathrm{Re}(s) > \frac{n(n+2N+2)}{2N}$

I quote the following

<u>Theorem</u> The functions $\zeta(s,P,\Lambda)$ admit meromorphic continuation to $\mathbb{C}$, with simple poles only. The poles are

contained in the set

$$\left\{ s = \frac{n-\nu}{N} ,\ \nu = 0,1,2,\dots \right\} \setminus \{0,-1,-2,\dots\}$$

Remark: The meromorphic continuation comes via Mellin transformation from the asymptotic of the heat kernel for $t \to 0$.

Remember

$$\int_0^\infty t^{s-1} e^{-\lambda t}\, dt = \lambda^{-s}\Gamma(s)$$

hence

$$\zeta(s, P, \Lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(\Pi_\Lambda \circ \exp(-tP))\, dt$$

where Π_Λ denotes the orthogonal projection onto $\Gamma_{\mathcal{L}\Lambda}(E)^\perp$

The heat operator $\exp(-tP)$ is defined by the smooth kernel

$$K_t(x,y) = \sum_{k=1}^{\infty} e^{-t\lambda_k} u_k(x) \otimes \langle u_k(y), - \rangle$$

$$\in \Gamma(X \times X \times (0,\infty), HOM(pr_2^* E, pr_1^* E))$$

as $\exp(-tP)u = \int_X K_t(x,y)u(y)\,(x)_y$

There exist $e_\nu \in End(E)$ such that

$$K_t(x,x) \sim \sum_{\nu=0}^{\infty} t^{\frac{\nu-n}{N}} e_\nu(x)$$

i.e. $\forall r\ \exists\ \nu(r)$ and constant c_r s.th.

$$\left\| K_t(x,x) - \sum_{\nu \leq \nu(r)} t^{\frac{\nu-n}{N}} e_\nu(x) \right\|_{C^r, X} < c_r t^r$$

(Reference: Gilkey, loc cit. p 54).