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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Complex Geometry - Lecture I

R.R. Simha
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay - 400 005
India

1 Differentiable Manifolds. de Rham groups.

Let M be a differentiable (C^∞) manifold, of dimension m. For any open set $U \subset M$ and any r, $1 \leq r \leq m$, let $A^r(U)$ denote the space of C^∞ differential forms of degree r on U. Recall that if $(U, \underline{x} = (x_1, \dots, x_r))$ is a chart for M, then the elements of $A^r(U)$ are uniquely representable as

$$\sum_{1 \leq i_1 < \dots < i_r \leq m} a_{i_1, \dots, i_r} \, dx_{i_1} \wedge \dots \wedge dx_{i_r}$$

where the $a_{i_1, \dots, i_r}$ are C^∞ functions on U. ~~($A^0(U)$ is just the space of C^∞ functions on U)~~. $A^0(U)$ will by convention be the space of C^∞ functions on U.

Recall that we have the operation of exterior differentiation on the $A^r(U)$:

$$d : A^r(U) \to A^{r+1}(U).$$

This operation is determined by the requirement (in a chart $(U, \underline{x})$):

$$d(f \, dx_{i_1} \wedge \dots \wedge dx_{i_r}) = \sum_s \frac{\partial f}{\partial x_s} dx_s \wedge dx_{i_1} \wedge \dots \wedge d$$

It is easy to see that $d^2 = 0$ and that $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$ if

$\omega_1 \in A^r$.

The r-th de Rham cohomology group (or vector space) $H^r_{dR}(M)$ is the quotient

$$H^r_{dR}(M) = \{\omega \in A^r(M) : d\omega = 0\} / \{d\psi : \psi \in A^{r-1}(M)\}$$

0) $H^0_{dR}(M)$ is 1-dimensional (M connected).

Examples. 1) If $M = S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ then $H^1_{dR}(M)$ is one-dimensional.

2) The classical Poincaré lemma for d states that, if U is a convex open set in $\mathbb{R}^n$, then $H^r_{dR}(U) = 0$ for $r \geq 1$.

From this, it follows fairly easily that the de Rham cohomology groups are topological invariants of M, being the corresponding sheaf cohomology groups of the "constant sheaf" ($\mathbb{R}$ or $\mathbb{C}$) on M.

However, it turns out that the study of the de Rham groups gives very important ~~str~~ information about the complex structure of M, when M is a compact complex manifold admitting a certain kind of metric, as we shall explain now.

2. Complex Manifolds. Dolbeault Cohomology

Thus let M now be a complex manifold. This means that we have charts $(U, \underline{z} = (z_1, \cdots, z_m))$ for M, where the U's form an open cover of M, and $\underline{z}$ is a homeomorphism of U onto an open set in $\mathbb{C}^m$ (M is supposed to be a Hausdorff topological space). If $(U, \underline{z})$ and $(V, \underline{\zeta})$ are two such charts (with $U \cap V \neq \emptyset$), then the map $\underline{z} \cdot \underline{\zeta}^{-1} : \zeta(U \cap V) \to z(U \cap V)$ is a holomorphic map of open sets in $\mathbb{C}^m$. Thus M is in particular also a C^∞ manifold.

We shall now consider complex-valued differential forms: for a complex-valued C^∞ function $f = u + iv$, $df = du + i\,dv$ etc. Now let $(U, \underline{z})$ be a chart of the complex structure on M; write $z_k = x_k + i y_k$. Then, for any $f \in A^0(U)$,

$$df = \sum \frac{\partial f}{\partial x_k} dx_k + \sum \frac{\partial f}{\partial y_k} dy_k$$

$$= \sum \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - i \frac{\partial f}{\partial y_k}\right)(dx_k + i\,dy_k)$$

$$+ \sum \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i \frac{\partial f}{\partial y_k}\right)(dx_k - i\,dy_k)$$

$$=: \sum \frac{\partial f}{\partial z_k} dz_k + \sum \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k$$

The Cauchy-Riemann equations for f are equivalent to the equations $\frac{\partial f}{\partial \bar{z}_k} = 0$.

Thus the 1-forms expressible as $\sum \varphi_k dz_k$ in U are intrinsically characterised (i.e without reference to the coordinates $\underline{z}$) as forms of the type $\sum \varphi_j dh_j$, $h_j \in \mathcal{O}(U)$:= space of <u>holomorphic</u> functions in U. In other words, we have the direct sum decomposition

$$A^1(U) = A^{1,0}(U) \oplus A^{0,1}(U)$$

where $A^{1,0}(U)$ consists 1-forms representable as $\sum \varphi_k dh_k$, and $A^{0,1}(U)$ consists of those representable as $\sum \varphi_k d\bar{h}_k$, $h_k \in \mathcal{O}(U)$. This decomposition is now clearly valid for <u>all</u> open subsets U of M. Moreover, the map $d : A^0(U) \to A^1(U)$ breaks as $d = \partial \oplus \bar{\partial}$: $df = \partial f \oplus \bar{\partial} f$, $\partial f \in A^{1,0}(U)$, $\bar{\partial} f \in A^{0,1}(U)$.

It easily follows that, for forms of higher degree r, we have a decomposition

$$A^r(U) = \bigoplus_{p+q=r} A^{p,q}(U),$$

where $A^{p,q}(U)$ is charactised by the condition that, if $(U, \underline{z})$ is a complex chart, $\omega \in A^{p,q}(U)$ is a linear combination (over $A^0(U)$) of the forms $dz_I \wedge d\overline{z}_J$ where $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$ are subsets of $\{1, \ldots, m\}$ of cardinalities p and q respectively, and

$$dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}.$$

Using the local representation of the operator d, it follows moreover that the map $d : A^r(U) \longrightarrow A^{r+1}(U)$ maps $A^{p,q}(U)$ into $A^{p+1,q}(U) \oplus A^{p,q+1}(U)$. Thus the operators $\partial : A^0 \to A^{1,0}$, $\overline{\partial} : A^0 \to A^{0,1}$ introduced above generalise to operators

$$\partial : A^{p,q} \longrightarrow A^{p+1,q}, \quad \overline{\partial} : A^{p,q} \longrightarrow A^{p,q+1}.$$

From $d^2 = 0$, it follows that $\partial^2 = \overline{\partial}^2 = 0$, and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

In particular, in analogy with the de Rham groups, we can can consider the Dolbeault groups $H^{p,q}(M)$:

$$H^{p,q}(M) = \{\omega \in A^{p,q}(M) : \overline{\partial}\omega = 0\} \big/ \{\overline{\partial}\omega : \omega \in A^{p,q-1}(M)\}$$

<u>Examples</u> <u>and</u> <u>Remarks</u>. (1) For $\omega \in A^{p,0}(U)$, $\bar\partial\omega = 0$ if and only if ω is a holomorphic p-form, that ω is locally of the form

$$\sum f_{i_1,\dots,i_p}\, dz_{i_1} \wedge \dots \wedge dz_{i_p}$$

with the $f_{i_1,\dots,i_p} \in \mathcal{O}(U)$

(2) Unlike in the case of complex dimension one, a holomorphic form ω <u>need</u> <u>not</u> be d-closed (i.e $d\omega$ need not be zero).

(3) The $H^{p,q}(M)$ in general depend ~~very strongly~~ on the complex structure on M (unlike the $H^r_{dR}(M)$ which are topological invariants). The analogue of Poincaré's lemma is true for $\bar\partial$: for convex open sets U in $\mathbb{C}^n$, $H^{p,q}(U) = 0$ for all $q \geq 1$. Sheaf theory then says that, for example, the $H^{0,q}(M)$ are the cohomology groups of the sheaf $\mathcal{O}_M$ of germs of holomorphic functions on M. Generally, the $H^{p,q}(M)$ are the cohomology groups of the sheaf (or holomorphic vector bundle) Ω^p of holomorphic p-forms on M

4) For any complex manifold M, there are natural homomorphisms $H^1_{dR}(M,\mathbb{R}) \to H^{0,1}(M)$, $H^1(M,\mathbb{C}) \to H^{0,1}(M)$ got by sending

$\omega = \overset{1,0}{\omega} + \overset{0,1}{\omega} \in A^1(M)$ to $\overset{0,1}{\omega}$. Note that $d\omega = 0 \Rightarrow \bar\partial \overset{0,1}{\omega} = 0$ and $\omega = df \Rightarrow \overset{0,1}{\omega} = \bar\partial f$. These maps are in general neither injective nor surjective. For example, for any open subset U of $\mathbb{C}$, $H^{0,1}(U) = 0$, while $H^1_{dR}(U) = 0$ iff U is simply connected. On the other hand, if $U = \mathbb{C}^2 - \{0\}$, then $H^{0,1}(U)$ is infinite dimensional, while $H^1_{dR}(U) = 0$. However, for a compact complex manifold, the map $H^1_{dR}(X, \mathbb{R}) \to H^{0,1}(X)$ is injective. (Proof: suppose $\omega = \overset{1,0}{\omega} + \bar\partial f$ is a closed 1-form which is real; then $\omega = \bar\omega$ $\Rightarrow \omega = \partial \bar f + \bar\partial f$. Hence $\omega - df = \partial(\bar f - f)$, and now $d\omega = 0 \Rightarrow \bar\partial\partial(\bar f - f) = 0$. Thus $i(\bar f - f)$ is a plurisubharmonic function, hence satisfies the maximum principle, and is therefore constant (X is compact). QED).

One of the consequences of the theory of compact Kähler manifolds is that, for such a manifold M, the above map $H^1_{dR}(M, \mathbb{R}) \to H^{0,1}(M)$ is also surjective. This result is applicable to compact Riemann surfaces, and implies the uniformisation theorem in the compact case: a simply connected compact Riemann surface is bi-holomorphic to the Riemann sphere.

Hermitian and Kähler metrics

Let M be a complex manifold of complex dimension m. For each $\xi \in M$, we have the tangent space $T_\xi(M)$ to M at ξ (M considered as a C^∞ manifold of dimension $2m$), thus $T_\xi(M)$ is an $\mathbb{R}$-vector space of dimension. On the other hand, we also the complexified cotangent space $\mathrm{Hom}_{\mathbb{R}}(T_\xi(M), \mathbb{C}) = T^*_{\xi,\mathbb{C}}$, which is

a complex vector space of dimension $2m$. As we have already seen, $T^*_{\xi,\mathbb{C}} = \Omega^{1,0}_\xi \oplus \Omega^{0,1}_\xi$, where $\Omega^{1,0}_\xi$ is the $\mathbb{C}$-span of the differentials $df(\xi)$, f holomorphic, and $\Omega^{0,1}_\xi$ is the span of the $d\bar{f}(\xi)$, f holomorphic. $T^*_{\xi,\mathbb{C}}$ has the obvious complex conjugation map: $\langle u, \bar{\omega}\rangle = \overline{\langle u, \omega\rangle}$ for $u \in T_\xi$ and $\omega \in T^*_{\xi,\mathbb{C}}$.

The natural map $T_\xi \to \mathrm{Hom}_{\mathbb{C}}(\Omega^{1,0}_\xi, \mathbb{C})$

is an $\mathbb{R}$-isomorphism: if $u \in T_\xi$ satisfies $\langle u, dz_k\rangle = \langle u, dx_k\rangle + i\langle u, dy_k\rangle = 0$ for all k,

(where $(U, \underline{z})$ is a complex chart with $\xi \in U$), then $\langle u, dx_k\rangle = 0 = \langle u, dy_k\rangle$ for all k, i.e. $u = 0$. Thus the map in question is injective, hence also surjective for dimension reasons. We can therefore make T_ξ a $\mathbb{C}$-vector-space via this isomorphism. If we introduce the notation $J : T_\xi \to T_\xi$ for the $\mathbb{R}$-endomorphism of T_ξ corresponding to multiplication by i in this $\mathbb{C}$-vector-space structure, an easy calculation shows

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}, \quad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}$$

(recall that $\frac{\partial}{\partial x_k}(dz_l) = \delta_{kl}$, $\frac{\partial}{\partial y_k}(dz_l) = i\delta_{kl}$).

We can now define a Hermitian metric on M as a Riemannian metric, i.e. a positive definite symmetric bilinear form $\langle\, ,\, \rangle_\xi$ on each T_ξ such that

$\langle Ju, Jv\rangle_\xi = \langle u, v\rangle$ (and of course the

functions $\left\langle \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}\right\rangle$ are C^∞ in U for any

chart (U, ξ) of M. Such a scalar product is also characterised as the real part of a unique positive definite Hermitian form H on the complex vector space $T_\xi(M)$ for the J-structure (Hermitian means: $H(\lambda u, v) = \lambda H(u,v)$ for all $\lambda \in \mathbb{C}$, and $H(v,u) = \overline{H(u,v)}$):

$$H(u,v) = \langle u, v\rangle - i\langle Ju, v\rangle$$

(Observe that a Hermitian form on a

complex vector space is uniquely determined by either its real part S (which is symmetric or its imaginary part A (which is skew-symmetric); conversely any such S (resp. A) satisfying $S(iu, iv) = S(u, v)$ (resp. $A(iu, iv) = A(u, v)$) determines a unique Hermitian form.

Hermitian metrics always exist on any complex manifold: this can be proved as in the Riemannian case by a partition of unity, or one may observe that, if $\langle\ ,\ \rangle$ is a Riemannian metric, then $\langle u, v\rangle' = \langle u, v\rangle + \langle Ju, Jv\rangle$

is a Hermitian metric. A Hermitian metric on a complex manifold is said to be a Kähler metric if the two-form $\langle u, v\rangle \to \langle Ju, v\rangle$ is closed

Examples 1) When M is a Riemann surface, i.e. $\dim_{\mathbb{C}} M = 1$, then every Hermitian metric is Kähler (because every C^∞ 2-form is closed on M).

2) The restriction of a Kähler metric to a ~~X~~ complex submanifold is Kähler. Since complex projective space carries a Kähler metric, all projective manifolds are Kähler manifolds

3) It is easy to show that, on a

compact Kähler manifold, the 2-form determined by the metric is never zero in H^2_{dR}. Thus the non-vanishing of H^2_{dR} is a necessary condition for a compact complex manifold to carry a Kähler metric. For example, the Hopf surface, defined as the quotient of $\mathbb{C}^2 \setminus \{0\}$ by $\mathbb{Z}$ under the action $z \to 2^n z$, cannot carry a Kähler metric.

We conclude by stating the Hodge Decomposition theorem for compact Kähler manifolds in a preliminary form:

Theorem. If M is a compact Kähler manifold, then we have, for every r, an isomorphism

$$H^r_{dR}(M, \mathbb{C}) \approx \bigoplus_{p+q=r} H^{pq}(M)$$

Remark On $H^r_{dR}(M, \mathbb{C})$, we have the operation of complex conjugation $\omega \to \overline{\omega}$ (since d is a real operator). Under the isomorphism of the above theorem, $H^{pq}(M)$ gets conjugated to H^{qp}. The above isomorphism has additional important properties which take more preparation to state.

4. Vector Bundles

Although we have avoided the terminology of vector bundles in the preceding discussion, the spaces of f mo we have been considering, like $A^r(U)$ and $A^{p,q}(U)$ etc, are actually the spaces of C^∞ sections over U of appropriate vector bundles over the manifold in question. We make these notions precise in what follows.

Let M be (for instance) a C^∞ manifold. Suppose we have attached to each $\xi \in M$ an r-dimensional vector space V_ξ (over $\mathbb{R}$ or $\mathbb{C}$). Consider $V = \coprod_{\xi \in M} V_\xi$; let $\pi : V \to M$ be the map such that $\pi(V_\xi) = M$.

Notation. A section of V over $U \subset M$ is a map $\sigma : U \to V$ such that $\pi \circ \sigma = Id_U$. A frame for V over $U \subset M$ is an (ordered) r-tuple $(\sigma_1, \ldots, \sigma_r) =: \underline{\sigma}$ of sections of V over U such that, for each $\xi \in U$, $\underline{\sigma}(\xi)$ is a basis for the vector space V_ξ.

Definition: A structure of C^∞ vector bundle on $\pi : V \to M$ consists of a set $(U, \underline{\sigma})$ of frames for V such that
(i) the U form an open cover of M,

(ii) if $(U, \underline{\sigma})$ and $(V, \underline{\tau})$ are two charts of the set such that $U \cap V \neq \emptyset$, then the map $U \cap V \to GL(n)$ ($\mathbb{R}$ or $\mathbb{C}$) defined by the matrices expressing $\underline{\sigma}(\xi)$ in terms of $\underline{\tau}(\xi)$, $\xi \in U \cap V$, is C^∞.

Examples. 1) $V = M \times \mathbb{R}^r$, $\pi : V \to M$ the natural projection. This bundle is called the trivial bundle over M of rank r

2) The tangent and cotangent bundles of a differentiable manifold M correspond to the cases when the V_ξ above are $T_\xi(M) = T_\xi$, the tangent space to M at ξ, or its dual T_ξ^* respectively. For instance, for the cotangent bundle, the defining frames can be taken as $\{(U, \underline{dx}) : (U, \underline{x})$ a coordinate chart for $M\}$.

3) The exterior powers $\overset{p}{\Lambda} V$ of a vector bundle V, or the tensor product ~~of~~ $V_1 \otimes V_2$ of two vector bundles, are defined in the obvious way. E.g. the fibres $(\overset{p}{\Lambda} V)_\xi$ of $\overset{p}{\Lambda} V$ are just the $\overset{r}{\Lambda} V_\xi$, and the frames are $(U, (\sigma_{i_1} \wedge \sigma_{i_2} \wedge \ldots \wedge \sigma_{i_r};$ $1 \leq i_1 < \ldots < i_r \leq s := \operatorname{rank} V))$, where $(U, \underline{\sigma})$ is a frame for V.

Let $V \to M$ be a C^∞ vector bundle. A section s of V over an open subset W of M is said to be C^∞ if, for each frame $(U, \underline{\sigma})$ of V, the coefficients of the relation expressing s in terms of the σ_i are C^∞ on $W \cap U$.

Remark. The space $V(W)$ of C^∞ sections of V over W is a module over $A^0(U)$. If $(U, \underline{\sigma})$ is a frame for V, $V(U)$ is a free $A^0(U)$-module with basis $\underline{\sigma}$.

We have defined C^∞ vector bundles on a C^∞ manifold in the above discussion. But it is clear that holomorphic vector bundles over a complex manifold M, or topological vector bundles on a topological space, holomorphic sections of a holomorphic bundle etc. can be defined in an analogous way.

5. Dolbeault Cohomology of a holomorphic vector bundle.

To get some familiarity with the above notions, we consider a holomorphic vector bundle V on a complex manifold M, and indicate how operators

$$\bar{\partial} : V \otimes A^{0,q} \to V \otimes A^{0,q+1}$$

can be defined such that $\bar\partial^2 = 0$.

To do this, we consider a complex chart $(U, \underline{z})$ for M, such that V has a holomorphic frame $(U, \underline{\sigma})$ over U; clearly such U cover M. If we think of V as a C^∞ vector bundle, we can form the C^∞ vector bundle $V \otimes A^{0,q}$. By definition, $V \otimes A^{0,q}$ has the frame $(d\bar z_{i_1} \wedge \ldots \wedge d\bar z_{i_q} \otimes \sigma_j,\ 1 \leq i_1 < \ldots < i_q \leq m = \dim M$

$1 \leq j \leq r := \operatorname{rank} V)$ over U. Then $\bar\partial : V \otimes A^{0,q} \to V \otimes A^{0,q+1}$ is uniquely determined the the requirement (for $f \in A^0(U)$):

$$\bar\partial (f\, d\bar z_{i_1} \wedge \ldots \wedge d\bar z_{i_q} \otimes \sigma_j)$$

$$= \left\{ \bar\partial (f\, d\bar z_{i_1} \wedge \ldots \wedge d\bar z_{i_q}) \right\} \otimes \sigma_j .$$

Using the fact that $\bar\partial f = 0$ if f is holomorphic, it is easy to see that $\bar\partial$ is then well-defined. It is obvious that $\bar\partial^2 = 0$. One can now define the Dolbeault group $H^q(V)$ as the quotient of

$$\ker \{ \bar\partial : V \otimes A^{0,q} \to V \otimes A^{0,q+1} \}$$

by

$$\operatorname{Im} \{ \bar\partial : V \otimes A^{0,q-1} \to V \otimes A^{0,q} \}.$$

Since the $\bar\partial$-Poincaré lemma is still valid in the present situation, sheaf theory again tells us that $H^q(V)$ is the q-th cohomology of the "sheaf of holomorphic sections of V".