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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Topology and Differential Geometry - Lecture IV

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These are preliminary lecture notes, intended only for distribution to participants

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Topology and Differentialgeometry
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Determinant bundles and Quillen metric

1. A finite dimensional analogy

For a vector bundle or vector space E we denote by

$$\det(E) = \begin{cases} \Lambda^r E & r = rk(E) > 0 \\ I & E = 0 \end{cases}$$

where I is the trivial rank 1 bundle (one 1-dim space $\mathbb{C}$ for vectorspaces)

Exercise: 1) There is a canonical isomorphism for exact sequence $0 \to E' \to E \to E'' \to 0$ we have $\det(E') \otimes \det(E'') \xrightarrow{\sim} \det(E)$ (induced by $\det(E') \otimes \Lambda^{r''} E \xrightarrow{\wedge} \det(E)$, $r'' = rk(E'')$)

2) If $(E^\cdot, D^\cdot) = (\cdots E^{i-1} \xrightarrow{D^i} E^i \to \cdots)$ is a bounded complex of finite dimensional

vector spaces there is a canonical isomorphism

$$DET(D^\bullet) =: Hom(\bigotimes_j \det H^{2j-1}(E), \bigotimes_j \det H^{2j}(E))$$

$$\xrightarrow[\phi]{\sim} Hom(\bigotimes_j \det(E^{2j-1}), \bigotimes_j \det(E^{2j}))$$

Now suppose each E^j has a Hermitian metric and consider the adjoint $(D^j)^* = D_j : E^j \to E^{j-1}$.

Then the analogy of the Laplacian is

$$\Delta_j = D^j \circ D_j + D_{j+1} \circ D^{j+1} : E^j \to E^j$$

and we get an orthogonal decomposition

$$E^j = \mathcal{H}^j \oplus Im(D^j \circ D_j) \oplus Im(D_{j+1} \circ D^{j+1})$$

$$(\mathcal{H}^j = Ker(\Delta_j))$$

$$Ker(D^{j+1}) = \mathcal{H}^j \oplus Im(D^j \circ D_j)$$

hence $\mathcal{H}^j \simeq H^j(E^\bullet, D^\bullet)$

The regularized determinant of Δ_j is defined as

$$\det'(\Delta_j) = \det(Ker(\Delta_j)^\perp \xrightarrow{\Delta_j} Ker(\Delta_j)^\perp)$$

$$= \text{product of non-zero eigenvalues of } \Delta_j.$$

Then $DET(D^\circ) \cong Hom(\bigoplus_j \mathcal{H}^{2j-1}, \oplus \mathcal{H}^{2j})$

but ϕ is not an isometry:

$$\|\phi\|^2 = \prod_j \det{}'(\Delta_{j-1}) \prod_j \det{}'(\Delta_j)^{-1}$$

Hence the induced metric on $DET(D^\bullet)$ multiplied by $\|\phi\|^2$ yields a metric independent on $D^\bullet$

(We leave this as an exercise)

2. Determinant bundle for Fredholm operators

For simplicity consider only complexes of the form $E^1 \xrightarrow{D} E^2$, E^1, E^2 Hilbert spaces, D Fredholm.

We can define ~~DET(D) = Hom(det~~

$DET(D) = \mathrm{Hom}(\det(\mathrm{Ker}(D)), \det(\mathrm{Coker}(D)))$

Proposition There exist a line bundle DET on the space $\mathcal{F} = \mathcal{F}(E^1, E^2)$ of Fredholm operators, with fibre $DET(D)$ in $D \in \mathcal{F}$.

It is characterized by the following

Consider the trivial bundles and the universal map

$$E^1 \times \mathcal{F} \xrightarrow{\mathcal{D}} E^2 \times \mathcal{F}$$

$$\mathcal{D}(v, P) = (P(v), P)$$

and for $\mathcal{U} \subset \mathcal{F}$ open sub bundles of finite rank

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{D}} & F \\ \cap & & \cap \\ E^1 \times \mathcal{U} & \xrightarrow{\mathcal{D}} & E^2 \times \mathcal{U} \end{array}$$

such that $\mathrm{Ker}(\mathcal{D}/\mathcal{U}) \subset E$ and F "transversal to $P \in \mathcal{U}$" which means

$F(P) + \mathrm{Im}(P) = F$ for $P \in \mathcal{U}$

Then $\mathrm{DET}/\mathcal{U} = \mathrm{HOM}(\det(E), \det(F))$

In $\mathcal{F}_0 = \{P \mid \mathrm{index}\, P = 0\} \subset \mathcal{F}$ we have a canonical section $\det(\mathcal{D})$ induced from $E \xrightarrow{\mathcal{D}} F$ over $\mathcal{U} \subset \mathcal{F}_0$ (since $\mathrm{rk}(E) = \mathrm{rk}(F)$ over $\mathcal{F}_0$)

The construction of DET is straightforward an based on

<u>Lemma</u> Suppose $P_0 \in \mathcal{F}$, $W \subset E^2$ a finite dimensional subspace, transversal to P_0 and $V = P_0^{-1}(W)$.

Then

$U_W = \{ P \in \mathcal{L}(E^1, E^2) \mid$ the induced map $P^\perp : V^\perp \to W^\perp$ is an isomorphisme $\}$

is open and contained in $\mathcal{F}$

($\mathcal{L}(E^1, E^2)$ = space of bounded linear operators)

Moreover: $F = W \times U_W$ is transversally to $P \in U_W$ and $E = \mathcal{D}^{-1}(W \times U_W)$ is a subbundle of $E^1 \times U_W$, isomorphic to $V \times U_W$ under the projection

Proof: $P^\perp$ is the block P_{22} in the decomposition

$$E^1 = \begin{matrix} V \\ \oplus \\ V^\perp \end{matrix} \xrightarrow[P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}]{} E^2 = \begin{matrix} W \\ \oplus \\ W^\perp \end{matrix}$$

and $P \mapsto P^\perp$ is a continuous map $\mathcal{L}(E^1, E^2) \to \mathcal{L}(V^\perp, W^\perp)$. U_W is the pre image of the open set $\mathrm{Isom}(V^\perp, W^\perp)$.

Obviously $\mathrm{Ker}(P) \xrightarrow{\cong} V$ under the orthogonal projection of E^1 to V, for

Hence $U_W \subset \mathcal{F}$, $\mathcal{D}^{-1}(W \times U_W) \simeq V \times U_W$.

Thus, DET can be defined by

$$\mathrm{DET}|U_W = \mathrm{HOM}(\det(\mathcal{D}^{-1}(W \times U_W)), \det(W \times U_W))$$

and it is easily checked that this construction is independant on choice of W. ~~(for an~~

3. DET(P) <u>for a family of elliptic operators</u>

We consider a proper submersion $X \xrightarrow{\pi} S$ of smooth manifolds and vector bundles E, F on X. Locally with respect to S:

$$X \simeq X_0 \times S \quad , E = E_0 \times S \quad , F = F_0 \times S$$

where E_0, F_0 are vector bundles on X_0.

We consider operators

$$\mathcal{C}^\infty_X(E) \longrightarrow \mathcal{C}^\infty_X(F)$$

which are $\mathcal{C}^\infty_S$-linear, then
$P_s = P \,|\, \mathcal{C}^\infty_{X_s}(E/X_s) \longrightarrow \mathcal{C}^\infty_{X_s}(F/X_s)$ is well explained. This operators are supposed to be elliptic of order N.

Locally we get therefore a smooth map

$$S \longrightarrow \mathcal{F}(L^2_t(E_0), L^2_{t-N}(F_0)) \qquad (t \in \mathbb{R})$$
$$s \longmapsto (P_s)_t$$

and pulling back DET we get a bundle $DET(P) \to S$ with fibres $\mathrm{Hom}(\det(\mathrm{Ker}(P_s)), \det(\mathrm{Coker}(P_s)))$ in $s \in S$.
If $\mathrm{index}(P_s) = 0$ we also get a section $\det(P)$ of $DET(P)$ vanishing precisely in the locus $\{s, P_s$ not an isomorphism$\}$.

<u>Example</u> For families of ~~complex~~ Riemann ~~manifolds~~ surfaces $X \xrightarrow{\pi} S$ and holomorphic vector bundles E over S, the operator $\bar\partial_{X/S} : \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(T^{01}(X/S) \otimes E)$ is an example.

Take for instance

$$X = \mathbb{P}^1(\mathbb{C}) \times \mathbb{C} \xrightarrow{\pi} S = \mathbb{C}$$

$$E \subset X \times \mathbb{C}^3$$

$$E = \{ (z_0 : z_1, t, y_0, y_1, y_2) \mid z_0^2 y_1 - z_1^2 y_0 = t z_0 z_1 y_2 \}$$

Then $E / \mathbb{P}^1(\mathbb{C}) \times \{t\} = \begin{cases} H^{-1} \oplus H^{-1} & t \neq 0 \\ H^{-\otimes 2} \oplus \underline{1} & t = 0 \end{cases}$

where $H \to \mathbb{P}^1(\mathbb{C})$ is the Hopf bundle

Since $\bar\partial_{X/S}$ on E has index 0 we get a section $\det(\bar\partial_{X/S}) \in DET(\bar\partial_{X/S})$

Since $H^0(E/\mathbb{P}^1(\mathbb{C}) \times \{t\}) = 0$ for $t \neq 0$ we have $\det(\bar\partial_{X/S})~~(t)~~ = 1$ for $t \neq 0$ (generator of $\det(0)$) and $= 0$ for $t = 0$.

This example shows that the L^2-metric on $DET(\bar\partial_{X/S})$ induced from the L^2-metric on $E \subset X \times \mathbb{C}^3$ is not a continuous metric.

4. The Quillen metric

We assume E, F to be Hermitian bundles and X a Riemannian manifold, fibrewise oriented.

Exercise:

Define for self-adjoint ~~elliptic~~ positive elliptic operators

$$\det{}'(P) = \exp\left(-\frac{d}{ds}\Big|_{s=0} (\zeta(s,P))\right)$$

$$\det{}'(\Pi_\Lambda \circ P) = \exp\left(-\frac{d}{ds}\Big|_{s=0} (\zeta(s,P,\Lambda))\right)$$

Then

$$\det{}'(P) = \left(\prod_{0<\lambda_k<\Lambda} \lambda_k\right) \det{}'(\Pi_\Lambda \circ P)$$

where $0 \le \lambda_1 \le \lambda_2 \le \dots$ is the spectrum, counted with multiplicity.

The Quillen metric on $DET(P)(s)$ is defined as

$$\| \ \|_Q^2 = \det{}'(P_s) \| \ \|_{L^2}^2$$

where $\| \, \|_{L^2}$ is the L^2 metric on $\mathrm{Hom}(\det(\mathrm{Ker}(P_s)), \det(\mathrm{Coker}(P_s)))$ induced from the Hermitian metrics on $E|X_s$, $F|X_s$ and the Riemannian metric on X_s

<u>Theorem</u> $\| \, \|_Q$ is a smooth Hermitian metric on $DET(P)$.

The reason is the following:
Consider $s \in S$, $\Lambda > 0$, $\Lambda \notin \mathrm{spec}(P_s^* \circ P_s) = \mathrm{spec}(P_s \circ P_s^*)$.
Then one can show ~~over~~
$\cancel{E = \Gamma_{\leq \Lambda}(E, P^* \circ P), F\ \Gamma_{\leq \Lambda}(F, P^*}$
1) $\{s \in S \mid \Lambda \notin \mathrm{spec}(P_s^* \circ P_s)\} = U_\Lambda(\cancel{s}) = U$
is open in S
2) Over U the families
$s \mapsto \cancel{E_s}\ \Gamma_{\leq\Lambda}(E_s, P_s^* \circ P_s)$
$s \mapsto \Gamma_{\leq\Lambda}(F_s, P_s \circ P_s^*)$

are locally trivial vector bundles $\mathcal{E}_\Lambda, \mathcal{F}_\Lambda$ which can be used to define $DET(P)$ over U as in point 1.

Since the $\mathcal{L}^2$-metric on $HOM(\det(\mathcal{E}_\Lambda), \det(\mathcal{F}_\Lambda)) \simeq DET(P)|U$ differs from the L^2-metric on $DET(P)|U$ by the factor $\prod_{0<\lambda_j<\Lambda} \lambda_j$, where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots = \mathrm{spec}(P_s^* \circ P_s)$ in s, the exercise above shows the ~~independence~~ smoothness of $\| \ \|_Q$.

<u>Remark</u>: If E, F have a Hermitian connection and if there is given a splitting H of the exact sequence

$$0 \to T(X|S) \to T(X) \overset{H}{\leftrightarrows} \pi^* T(S) \to 0$$

one gets a Hermitian connection

on $DET(P)$ (with resp. to $\| \|_Q$)

Sketch of proof:

Consider the relative volume form φ of X/S via the splitting as a form on X and define a 1-form γ by

$$\mathrm{int}_{H(x,v)}(d\varphi) = 2\gamma(H(x,v))\varphi$$

for $x \in X_s$, $v \in T_s(S)$.

Then define

$$^{\mathcal{E}}\nabla_v(\psi) = {}^{\mathcal{E}}\pi\left({}^{E}\nabla_{H(x,v)}\psi\right) + \gamma(H(x,v))\psi$$

(where $^{\mathcal{E}}\pi$ = orthogonal projection of $\Gamma(E, \pi^{-1}U_\lambda)$ onto $\mathcal{E}$)

$$^{\mathcal{F}}\nabla_v(\psi) = {}^{\mathcal{F}}\pi\left({}^{F}\nabla_{H(x,v)}\psi\right) + \gamma(H(x,v))\psi$$

($^{\mathcal{F}}\pi$ analog)

and ~~consider~~ consider the induced connection $^\wedge\nabla$ on $HOM(\det(\mathcal{E}), \det(\mathcal{F})) \simeq DET(P)|U$

Then $\nabla_v = {}^\wedge\nabla_v - \frac{1}{2} v\left(\frac{d}{ds}\Big|_0 \zeta(s, P, \lambda)\right)$