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ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

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Grothendieck-Riemann-Roch

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Grothendieck - Riemann - Roch

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Introduction.

Let k be an algebraically closed field. Let X be a connected smooth projective scheme over k of dimension d.

Then for any vectorbundle $\mathcal{E}$ on X and more generally for any coherent sheaf of $\mathcal{O}_X$-modules one has the following two facts concerning the cohomology of $\mathcal{E}$:

i) For all $q \geq 0$ the cohomology groups $H^q(X, \mathcal{E})$ are k-vectorspaces of finite dimension.

ii) For all $q > d$ one has $H^q(X, \mathcal{E}) = 0$.

So we can form the so called <u>Euler - Poincaré - Characteristic</u>

$$\chi(X, \mathcal{E}) := \sum_{q \geq 0} (-1)^q \dim_k H^q(X, \mathcal{E})$$

of $\mathcal{E}$. The Riemann - Roch problem is to compute $\chi(X, \mathcal{E})$.

We recall the Riemann-Roch theorem in the lower dimensions:

If X is a curve ($d=1$) and $\mathcal{L}$ a line bundle on X, then

$$\chi(X, \mathcal{L}) = \deg \mathcal{L} + \chi(X, \mathcal{O}_X)$$

with $\chi(X, \mathcal{O}_X) = 1 - g$, where $g = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$ is the genus of X.

If X is a surface ($d=2$) and $\mathcal{L}$ a line bundle on X, then

$$\chi(X, \mathcal{L}) = \frac{1}{2}(\mathcal{L} \cdot \mathcal{L} - \omega_X \cdot \mathcal{L}) + \chi(X, \mathcal{O}_X)$$

with $\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2)$, where the Chern classes c_1, c_2 will be explained later.

The solution of the Riemann-Roch problem in arbitrary dimensions was given by Hirzebruch (in the case $k = \mathbb{C}$), namely:

Hirzebruch-Riemann-Roch

For any vector bundle $\mathcal{E}$ on X one has

$$\chi(X, \mathcal{E}) = \int \mathrm{Td}(X/k)\,\mathrm{ch}(\mathcal{E}).$$

Here

$$\mathrm{Td}(X/E) = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) + \cdots$$

is the Toddclass of X built up with the Chernclasses $c_i := (-1)^i c_i(\Omega^1_{X/E})$;

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rank}\,\mathcal{E} + c_1(\mathcal{E}) + \tfrac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \cdots$$

is the Chern character of $\mathcal{E}$ built up with the Chernclasses of $\mathcal{E}$;

the Chernclasses are elements of the graded ring $CH^{\cdot}(X)\otimes\mathbb{Q}$, where $CH^{\cdot}(X)$ is the Chow-ring defined by the cycles on X modulo rational equivalence and the intersection product as multiplication;

and finally $\int \mathrm{Td}(X/E)\,\mathrm{ch}(\mathcal{E})$ means to take the degree of the top component $(\mathrm{Td}(X/E)\,\mathrm{ch}(\mathcal{E}))_d \in CH^d(X)\otimes\mathbb{Q}$. —

The structure morphism $f: X \to \mathrm{Spec}(E)$ of the E-scheme X is a projective morphism. Regarding Riemann-Roch as a theorem for the morphism $f: X \to \mathrm{Spec}(k)$ one may ask for a generalization of Riemann-Roch for any projective morphism

$$f: X \to Y$$

of schemes, say noetherian. For instance think of a k-morphism between k-schemes of the type above or think of projective arithmetic schemes $X \to \mathrm{Spec}(\mathbb{Z})$.

The question of generalizing Riemann-Roch for projective morphisms $f: X \to Y$ of noetherian schemes makes sense because of the following cohomological facts for vector bundles E on X and more generally for coherent $\mathcal{O}_X$-modules:

i) For all $q \geq 0$ the $\mathcal{O}_Y$-modules $R^q f_* E$ are coherent.

ii) For all q big enough the $R^q f_* E$ vanishes.

So the following things are to be done:

– Define the Euler-Poincaré characteristic for vector bundles E on X with respect to the morphism $f: X \to Y$,

- define a theory of Chern classes on the considered schemes,

- formulate and prove Riemann-Roch.

In order to do this Grothendieck established the theory of K- and K'-groups of schemes.

<u>I. The K'-groups and the Euler-Poincaré-characteristic.</u>

In the following let X be a noetherian scheme.

Let $F(X)$ be the free abelian group of the isomorphism classes $\{\mathcal{F}\}$ of coherent $\mathcal{O}_X$-modules $\mathcal{F}$ on X. In $F(X)$ we consider the subgroup $R(X)$ generated by all elements

$$\{\mathcal{F}\} - \{\mathcal{F}'\} - \{\mathcal{F}''\}$$

for any exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent $\mathcal{O}_X$-modules on X.

<u>Definition</u>: The quotient

$$K'(X) := F(X)/R(X)$$

is called the Grothendieck group of the coherent $\mathcal{O}_X$-modules or the K'-group of X. –

For any coherent $\mathcal{O}_X$-module $\mathcal{F}$ on X we denote by $[\mathcal{F}]$ the element in $K'(X)$ which is defined by $\mathcal{F}$, i.e. $[\mathcal{F}] = \{\mathcal{F}\} \bmod R(X)$, and we call $[\mathcal{F}]$ the class of $\mathcal{F}$ in $K'(X)$. These classes $[\mathcal{F}]$ generate $K'(X)$, and in fact any element of $K'(X)$ can be written as a difference $[\mathcal{F}] - [\mathcal{F}']$ of such classes.

The group $K'(X)$ has an obvious universal property, namely:

Any additive map

$$\begin{pmatrix}\text{Coherent} \\ \mathcal{O}_X\text{-modules}\end{pmatrix} \xrightarrow{\varphi} G$$

of the category of coherent $\mathcal{O}_X$-modules into an abelian group G has a unique factorization

$$\begin{tikzcd}
\begin{pmatrix}\text{Coherent} \\ \mathcal{O}_X\text{-modules}\end{pmatrix} \arrow[r, "\varphi"] \arrow[d] & G \\
K^{\cdot}(X) \arrow[ur, "\Phi"] &
\end{tikzcd}$$

with a group homomorphism Φ, and where the vertical map is given by $\mathcal{F} \mapsto (\mathcal{F})$.

Now let

$$f : X \to Y$$

be a proper morphism of noetherian schemes.

For each coherent $\mathcal{O}_X$-module $\mathcal{F}$ and for $q \geq 0$ we consider the q-th direct image

$R^q f_* \mathcal{F} :=$ associated sheaf to the presheaf $\mathcal{U} \mapsto H^q(f^{-1}(\mathcal{U}), \mathcal{F})$

on Y. Then we have the two fundamental facts:

i) For all q the $\mathcal{O}_Y$-module $R^q f_* \mathcal{F}$ is coherent.

ii) For all q big enough the $R^q f_* \mathcal{F}$ vanish.

Hence we can form the element

$$\sum_{q \geq 0} (-1)^q [R^q f_* \mathcal{F}] \quad \text{in } K(Y)$$

and call it the <u>Euler-Poincaré-characteristic</u> of $\mathcal{F}$ relative to $f : X \to Y$.

<u>Prop.</u>: The Euler-Poincaré-characteristic is additive in $\mathcal{F}$ and hence induces a homomorphism

$$f_* : K(X) \to K(Y)$$
$$[\mathcal{F}] \mapsto \sum_{q \geq 0} (-1)^q [R^q f_* \mathcal{F}].$$

Proof:

This comes out from the long exact cohomology sequence associated to a short exact sequence $0 \to F' \to F \to F'' \to 0$ of coherent $\mathcal{O}_X$-modules.

Prop.: If $f: X \to Y$ and $g: Y \to Z$ are proper morphisms of noetherian schemes, then

$$(g \circ f)_* = g_* \circ f_* ,$$

i.e. the K^0 groups are covariant for proper morphisms.

Proof:

This comes out from the Leray spectral sequence

$$E_2^{pq} = R^p_{g_*} \circ R^q_{f_*}(F) \Rightarrow E^{p+q} = R^{p+q}_{(g \circ f)_*}(F). —$$

The Riemann-Roch problem is now to "compute" the Euler-Poincaré characteristic $f_*: K^0(X) \to K^0(Y)$ for proper morphisms $f: X \to Y$. —

II. The K-group and the regular case.

Let again X be a noetherian scheme.

We consider now the free abelian group of the isomorphism classes $\{\mathcal{E}\}$ of vector bundles $\mathcal{E}$ on X and its subgroup generated by the elements $\{\mathcal{E}\} - \{\mathcal{E}'\} - \{\mathcal{E}''\}$ for any exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of vector bundles on X.

<u>Definition</u>: The quotient $K(X)$ is called the Grothendieck group of the vector bundles on X or the K-group of X. –

Again we have the notion of the class $[\mathcal{E}]$ in $K(X)$ of a vector bundle $\mathcal{E}$ on X, and again we have the obvious universal property of the group $K(X)$. –

If $f: X \to Y$ is a morphism of noetherian schemes the pullback of an exact sequence of vector bundles on Y remains exact, and hence we get a homomorphism

$$f^*: K(Y) \to K(X).$$

So the formation of the K-groups is contravariant for arbitrary morphisms. —

As any vector bundle on a noetherian scheme is coherent, we have the canonical homomorphism

$$K(X) \longrightarrow K'(X)$$

which sends the class of a vector bundle in $K(X)$ to its class in $K'(X)$. This so called Poincaré homomorphism is in general neither injective nor surjective. But one has the following result going back to Hilbert:

Recall that a noetherian scheme is called regular if all its local rings are regular.

Theorem: If X is regular and separated then the Poincaré map

$$K(X) \longrightarrow K'(X)$$

is an isomorphism.

Proof:

Under the assumptions on the scheme X every coherent $\mathcal{O}_X$-module $\mathcal{F}$ has a finite locally free resolution, i.e. there exists an exact sequence

$$0 \to \mathcal{E}_n \to \dots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

with vector bundles $\mathcal{E}_i$ on X.

Then one shows that

$$K'(X) \longrightarrow K(X)$$
$$[\mathcal{F}] \longmapsto \sum_{q \geq 0} (-1)^q [\mathcal{E}_q]$$

is a well defined homomorphism which is inverse to $K(X) \to K'(X)$. —

We shall make use of this theorem later.

III. Universal Chern theory

A Chern theory on a scheme X associates to any vector bundle $\mathcal{E}$ on X elements $c_1(\mathcal{E}), c_2(\mathcal{E}), \dots$ in a graded ring with certain properties with respect to the formation of

- short exact sequences
- tensor products
- exterior powers
- rank

of vector bundles. We shall now develop such a Chern theory by means of $K(X)$ and the formations above. This Chern theory is universal for all Chern theories on X.

In the following let X be a noetherian scheme. We begin with the tensorproduct; the short exact sequences are already involved in the definition of $K(X)$.

1. The ringstructure on $K(X)$

If E is a vector bundle and if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of vector bundles (or coherent $\mathcal{O}_X$-modules) on X, then the tensored sequence $0 \to F' \otimes_{\mathcal{O}_X} E \to F \otimes_{\mathcal{O}_X} E \to F'' \otimes_{\mathcal{O}_X} E \to 0$ remains exact. Hence we get on $K(X)$ via

$$[\mathcal{E}]\cdot[\mathcal{F}] := [\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}]$$

a multiplication with respect to which $K(X)$ becomes a commutative ring with unity ($=[\mathcal{O}_X]$), and $K'(X)$ becomes a module over $K(X)$.

2. The λ-structure on $K(X)$.

For every vector bundle $\mathcal{E}$ on X and every $p\geq 0$ there are defined the exterior power $\Lambda^p\mathcal{E}$; these are vector bundles of rank $\binom{n}{p}$ if $\mathcal{E}$ is of rank n; in particular one has $\Lambda^0\mathcal{E}=\mathcal{O}_X$, $\Lambda^1\mathcal{E}=\mathcal{E}$ and $\Lambda^p\mathcal{E}=0$ for $p>n$.

Prop.: The exterior powers of vector bundles on X defines on $K(X)$ a family $(\lambda^p)_{p\geq 0}$ of maps

$$\lambda^p: K(X)\to K(X)$$
$$[\mathcal{E}]\mapsto[\Lambda^p\mathcal{E}]$$

with the following properties:

i) $\lambda^0(x) = 1$

ii) $\lambda^1(x) = x$

iii) $\lambda^p(x+y) = \sum_{i+j=p} \lambda^i(x)\lambda^j(y)$.

This family $(\lambda^p)_{p\geq 0}$ is called a <u>pre-λ-structure</u> on the ring $K(X)$.

<u>Proof:</u>

For any vectorbundle E on X we put

$$\lambda_t(E) := \sum_{p\geq 0} [\Lambda^p E]\, t^p \in 1 + t\,K(X)[[t]].$$

Then one has to show: For any short exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles on X one has

$$\lambda_t(E) = \lambda_t(E') \cdot \lambda_t(E'').$$

This comes out with a suitable filtration of $\Lambda^p E$ due to Koszul. —

Now we come to the first important theorem, namely:

Theorem: The pre-λ-structure $(\lambda^p)_{p\geq 0}$ on $K(X)$ is a λ-structure, that means: For elements $x, y \in K(X)$ one has:

iv) $\lambda^n(xy) = P_n(\lambda^1 x, \dots, \lambda^n x; \lambda^1 y, \dots, \lambda^n y)$

v) $\lambda^m(\lambda^n x) = P_{m,n}(\lambda^1 x, \dots, \lambda^{mn} x)$,

where $P_n(X_1, \dots, X_n; Y_1, \dots, Y_n)$ and $P_{m,n}(X_1, \dots, X_{mn})$ are certain universal polynomial with coefficients in the ring $\mathbb{Z}$.

Remark:

The universal polynomials in the theorem are defined as follows:

Let $U_1, \dots, U_n$ and $V_1, \dots, V_n$ be variables and let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be their elementary symmetric functions respectively. Then

$$P_n(X_1, \dots, X_n; Y_1, \dots, Y_n) := \text{coefficient of } t^n \text{ in } \prod_{1\leq i,j\leq n} (1 + U_i V_j t) ;$$

For example:

$$P_0 = 1$$
$$P_1 = X_1 Y_1$$
$$P_2 = X_1^2 Y_2 + X_2 Y_1^2 - 2 X_2 Y_2 .$$

Furthermore let $\xi_1, \dots, \xi_{nm}$ be variables and $X_1, \dots, X_{nm}$ their elementary symmetric functions. Then

$$P_{n,m}(X_1, \dots, X_{nm}) := \text{coefficient of } t^n \text{ in } \prod_{1 \le i_1 < \dots < i_m \le nm} (1 + \xi_{i_1} \cdots \xi_{i_m} t)$$

<u>Remark:</u>

On the ring $\mathbb{Z}$ there are plenty of pre-λ-structures $(\lambda^p)_{p \ge 0}$. But there is one and only one λ-structure, namely

$$\lambda^p(x) = \binom{x}{p} \quad \text{for } x \in \mathbb{Z}.$$

The theorem above is by far not trivial. One needs a suitable splitting principle. This splitting principle is given by the structure theorem of the K-ring of projective fibre bundles, which is fundamental also in the proof of Grothendieck-Riemann-Roch.

Let $\mathcal{E}$ be a vectorbundle of rank r on X and let $p: \mathbb{P}(\mathcal{E}) \to X$ be the associated projective fibre bundle. We look at $K(\mathbb{P}(\mathcal{E}))$ as an algebra over $K(X)$ via $p^*: K(X) \to K(\mathbb{P}(\mathcal{E}))$.

<u>Theorem</u>: The assignment $t \mapsto [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)]$ induces an isomorphism

$$K(X)[t]/(\lambda_t(\mathcal{E})) \xrightarrow{\cong} K(\mathbb{P}(\mathcal{E}))$$

of $K(X)$-algebras, where $\lambda_t(\mathcal{E}) = \sum_{i=0}^{r} (-1)^i [\Lambda^i \mathcal{E}] t^i$.

<u>But proof</u>: see Quillen, Algebraic K-theory. —

From the theorem, it follows in particular that the map $p^*: K(X) \to K(\mathbb{P}(\mathcal{E}))$ is injective. Now on $\mathbb{P}(\mathcal{E})$ we have the universal exact sequence

$$0 \to \mathcal{Z} \to p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to 0$$

with the canonical line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$

and hence in $K(\mathbb{P}(E))$:

$$[p^*E] = [\mathcal{H}] + [\mathcal{O}_{\mathbb{P}(E)}(1)]$$

in $K(\mathbb{P}(E))$. By repeating this consideration we get:

<u>Corollary</u> (splitting principle): For any vectorbundle E on X there exists a projective morphism $f: X' \to X$ such that

i) $f^*: K(X) \to K(X')$ is injective

ii) $[f^*E] = [L_1] + \dots + [L_r]$
with line bundles $L_1, \dots, L_r$ on X'.

With the help of this splitting principle it is now easy to prove the theorem about the λ-structure on $K(X)$. —

3. The Grothendieck filtration on $K(X)$

For a vectorbundle E on X the rank of E is constant on every connected component of X, and so $\operatorname{rank}(E)$ is an element of the ring $H^0(X, \mathbb{Z}) = \prod_{\pi_0(X)} \mathbb{Z}$, where $\pi_0(X)$

denote the set of the connected components of X.

Obviously the rank of vector bundles defines a ringhomomorphism

$$\mathrm{rank}: K(X) \longrightarrow H^0(X, \mathbb{Z})$$
$$[E] \longmapsto \mathrm{rank}(E) \quad ,$$

which is an augmentation of the canonical homomorphism $H^0(X, \mathbb{Z}) \to K(X)$.

The λ-structure $(\lambda^p)_{p\geq 0}$ and the rank augmentation on $K(X)$ define an ideal filtration on $K(X)$ as follows:

First we define the Grothendieck-operations

$$\gamma^p : K(X) \to K(X)$$

for $p \geq 0$ by $\gamma^p(x) := \lambda^p(x+p-1)$ for $x \in K(X)$.
One has

$$\begin{cases} \gamma^0(x) = 1 \\ \gamma^1(x) = x \\ \gamma^p(x+y) = \sum\limits_{i+j} \gamma^i(x)\gamma^j(y) \end{cases} .$$

Then we put:

__Definition:__ For $n \geq 0$ let $F^n K(X)$ be the $f^q(X,\mathbb{Z})$-submodule of $K(X)$ generated by all elements of the form

$$\gamma^{n_1}(x_1)\cdots\gamma^{n_r}(x_r) \quad \text{with} \quad \begin{cases} \operatorname{rank}(x_i) = 0 \\ \sum n_i \geq n \end{cases}.$$

__Prop.:__ One has

i) $K(X) = F^0 K(X) \supseteq F^1 K(X) \supseteq \cdots$

ii) $F^n K(X) \cdot F^m K(X) \subseteq F^{n+m} K(X)$.

__Proof:__
From the definition.

It follows that the family $(F^n K(X))_{n \geq 0}$ is an ideal filtration on $K(X)$. It is called the Grothendieck filtration on $K(X)$.

__Remark:__
For $n=1$ and $n=2$ one has the following interpretation of the ideals $F^n K(X)$:

$$F^1K(X) = \mathrm{Ker}(K(X) \xrightarrow{\mathrm{rank}} H^0(X,\mathbb{Z}))$$

$$F^2K(X) = \mathrm{Ker}(F^1K(X) \xrightarrow{\det} \mathrm{Pic}(X)).$$

Here we use the determinant homomorphism

$$\det : K(X) \to \mathrm{Pic}(X)$$

which exists since for every short exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles one has a (canonical) isomorphism $\det E \xrightarrow{\sim} \det E' \otimes \det E''$. —

In order to prove finiteness conditions for the Grothendieck filtration $(F^\nu K(X))$ on $K(X)$ one needs an assumption of projectivity. – Recall that a line bundle L on X is called ample, if for any coherent $\mathcal{O}_X$-module F on X there exists a n_0 such that $F \otimes_{\mathcal{O}_X} L^{\otimes n}$ for $n \geq n_0$ is generated by its global sections.

Then one has the following theorem which is hard to prove.

Theorem:

i) If X has an ample line bundle, then the filtration $(F^nK(X))$ is locally nilpotent, that means: for every $x \in F^1K(X)$ there exists an integer $n \geq 0$ such that $\gamma^{n_1}(x)\cdots\gamma^{n_r}(x) = 0$ whenever $n_1 + \dots + n_r \geq n$.

ii) If moreover X is of finite dimension d, then

$$F^nK(X) = 0 \text{ for all } n > d.$$

The second result is very beautiful.

4. The graded ring $Gr^{\cdot}K(X)$ and Chern classes.

We have the Grothendieck filtration $(F^nK(X))_{n \geq 0}$ on $K(X)$, and we now take the associated graded object

$$Gr^{\cdot}K(X) := \bigoplus_{n \geq 0} F^nK(X)/F^{n+1}K(X).$$

This is a graded ring.

<u>Remark:</u>
In the lower degrees we have the interpretations:

$$Gr^0 K(X) \xrightarrow{\cong} H^0(X, \mathbb{Z})$$

$$Gr^1 K(X) \xrightarrow{\cong} Pic(X)$$

where the isomorphisms are induced by the rank and by the determinant respectively. If the scheme is of finite dimension and has an ample line bundle, then we have

$$Gr^n K(X) = 0 \quad \text{for } n > d.$$

We now define the chern classes of elements of $K(X)$ with values in the graded ring $Gr^{\cdot} K(X)$ as follows:

Let $x \in K(X)$ be given. Then by definition for all $n \geq 0$ we have

$$\gamma^n(x - \mathrm{rank}(x)) \in F^n K(X)$$

and so we can form the element

$$c_n(x) := \gamma^n(x - \operatorname{rank} \varepsilon(x)) \bmod F^{n+1}K(X)$$

in $Gr^n K(X)$.

<u>Definition</u>: $c_n(x)$ is called the <u>n-th Chern class of x</u>.

The Chern classes are contravariant for arbitrary morphisms $f: X \to Y$ of noetherian schemes, i.e. the diagram

$$\begin{tikzcd} K(X) \arrow[r, "c_n"] & Gr^n K(X) \\ K(Y) \arrow[u, "f^*"] \arrow[r, "c_n"] & Gr^n K(Y) \arrow[u, "f^*"'] \end{tikzcd}$$

commutes. —

<u>Remark</u>:
If E is a vectorbundle on X the Chern classes $c_n(E) \in Gr^n K(X)$ are defined to be those of the element $[E]$ in $K(X)$. These Chern classes have all the usual properties mentioned (but not explicitly

given) at the beginning. —

Our Chern theory $(c_n)_{n\geq 0}$ on X with values in $Gr^{\cdot} K(X)$ is universal in the sense that every Chern theory on X with values in a graded $\mathbb{Q}$-algebra $A^{\cdot}$ comes from $(c_n)_{n\geq 0}$ via a unique homomorphism $Gr^{\cdot} K(X) \to A^{\cdot}$. We won't go into the details.

Having now a Chern theory on a noetherian scheme X we can now define

IV The Chern character and the Todd-homomorphism.

For $n \geq 1$ let $N_n(X_1, \dots, X_n)$ be the n-th Newton polynomial defined to be

$$N_n(X_1, \dots, X_n) = t_1^n + \dots + t_n^n$$

where $X_1, \dots, X_n$ are the elementary symmetric functions of variables $t_1, \dots, t_n$. One has

$$N_1(X_1) = X_1$$

$$N_2(X_1, X_2) = X_1^2 - 2X_2.$$

For $x \in K(X)$ we have

$$N_n(c_1(x), \dots, c_n(x)) \in Gr^n K(X).$$

Definition: For $x \in K(X)$ the element

$$ch(x) := \operatorname{rank}(x) + \sum_{n \geq 1} \frac{1}{n!} N_n(c_1(x), \dots, c_n(x))$$

in the ring $\prod_{n \geq 0} Gr^n K(X) \otimes \mathbb{Q}$ is called the Chern character of x.

Prop: The Chern character

$$ch : K(X) \to \prod_{n \geq 0} Gr^n K(X) \otimes \mathbb{Q}$$

is a ring homomorphism.

Proof
Calculations.

The fundamental property of the Chern character comes out if we assume the Grothendieck filtration $(F^n K(X))$ to be locally nilpotent. Remember, this is the case, if X has an ample line bundle. In case of a locally nilpotent Grothendieck filtration, the Chern character obviously has its values already in $Gr^{\cdot} K(X) \otimes \mathbb{Q}$.

<u>Theorem</u>: If the filtration $(F^{n} K(X))$ is locally nilpotent, then the Chern character

$$ch : K(X) \otimes \mathbb{Q} \longrightarrow Gr^{\cdot} K(X) \otimes \mathbb{Q}$$

is an <u>isomorphism</u>.

<u>Proof</u>:
See Adams.

<u>Comments</u>:

1) By the theorem an element $x \in K(X)$ is known modulo torsion, if the value $ch(x) = \mathrm{rank}(x) + c_1(x) + \cdots$ is known.

2) The theorem gives $K(X) \otimes \mathbb{Q}$ the structure of a <u>graded</u> ring. What's about it?

We now come to the Todd homomorphism. To any series $f(t) \in 1 + t\,\mathbb{Q}[[t]]$ there is associated a group homomorphism

$$td_f : K(X) \longrightarrow 1 + \prod_{n \geq 0} H^{2n}(X) \otimes \mathbb{Q}$$

$$x \longmapsto \sum_{n \geq 0} K_n(c_1(x), \ldots, c_n(x)),$$

where the so called Hirzebruch polynomials $K_n(X_1, \ldots, X_n)$ are defined by

$$K_n(X_1, \ldots, X_n) := \text{coefficient of } t^n \text{ in } \prod_{i=1}^{n} f(x_i t).$$

In the following we take the special series

$$f(t) := B(t) e^{-t}$$

with the Bernoulli series

$$B(t) = \frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n}{n!} t^n,$$

B_n the Bernoulli numbers. The associated

homomorphism td is called the Todd homomorphism and simply denoted by td.
One has for $x \in K(X)$:

$$td(x) = 1 + \frac{1}{2} c_1(x) + \frac{1}{12}\left(c_1(x)^2 + c_2(x)\right) + \dots$$

V. The Adams operations.

For $k \geq 1$ the Adams operations

$$\psi^k : K(X) \to K(X)$$

are defined by the series

$$\sum_{k \geq 1} (-1)^{k-1} \psi^k(x) t^{k-1} = \frac{d}{dt} \log \lambda_t(x)$$

for $x \in K(X)$, and they are also given by the formulas

$$\psi^k = N_k(\lambda^1, \dots, \lambda^k)$$

where N_k is the k-th Newton polynomial.

It is not difficult to prove but a wonderful fact that the Adams-operations $\psi^k : K(X) \to K(X)$ are nilpotent endomorphisms.

Now we look at the induced homomorphism

$$\psi^k : K(X)\otimes\mathbb{Q} \longrightarrow K(X)\otimes\mathbb{Q}.$$

For $k=1$ this is the identity. For $k>1$ we let

$$(K(X)\otimes\mathbb{Q})^{(n)}$$

be the k^n-eigenspace of ψ^k, i.e.

$$(K(X)\otimes\mathbb{Q})^{(n)} = \{x \in K(X)\otimes\mathbb{Q} \mid \psi^k(x) = k^n x\}.$$

Then we have the following theorem:

Theorem: Let the filtration $(F^n K(X))_{n\geq 0}$ be locally nilpotent. Then:

i) For every $n \geq 0$ the k^n-eigenspace $(K(X)\otimes\mathbb{Q})^{(n)}$ is independent of $k>1$.

ii) For the induced Grothendieck filtration $(F^n K(X)\otimes\mathbb{Q})$ on $K(X)\otimes\mathbb{Q}$ one has

$$F^n K(X)\otimes\mathbb{Q} = \bigoplus_{j\geq n} (K(X)\otimes\mathbb{Q})^{(j)}.$$

iii) On the eigenspace $(K(X)\otimes\mathbb{Q})^{(n)}$ the Chern character is given simply by

$$ch(x) = x \bmod F^{n+1}K(X)\otimes\mathbb{Q}.$$

Comments:

1) Taking $n=0$ in ii) we get

$$K(X)\otimes\mathbb{Q} = \bigoplus_{j\geq 0}(K(X)\otimes\mathbb{Q})^{(j)}$$

and this obviously makes $K(X)\otimes\mathbb{Q}$ into a graded ring.

2) In $K(X)\otimes\mathbb{Q}$ the Grothendieck filtration is rediscovered as the natural filtration of the graded ring.

3) From iii) it follows that the Chern character is simply the map

$$K(X)\otimes\mathbb{Q} \longrightarrow Gr\, K(X)\otimes\mathbb{Q}$$

$$x = \sum_{n\geq 0} x_n \mapsto \sum_{n\geq 0} x_n \bmod F^{n+1}K(X)\otimes\mathbb{Q}$$

which by ii) is an isomorphism of graded rings. —

We won't prove this theorem here.

VI. Grothendieck-Riemann-Roch

Let $f: X \to Y$ be a projective morphism of regular noetherian schemes which have ample line bundles. *)

Then the K-groups of the schemes identify with their K'-groups, and the Euler-Poincaré-characteristic is defined:

$$f_*: K(X) \to K(Y)$$
$$[E] \mapsto \sum_{q \geq 0} (-1)^q [R^q f_* E].$$

We want to compute $f_*(x)$ for an element $x \in K(X)$. Now the Chern character

$$ch: K(Y) \otimes \mathbb{Q} \to Gr\, K(Y) \otimes \mathbb{Q}$$

is an isomorphism. Hence we know $f_*(x)$ in $K(Y)$ at least up to torsion if we know the value $ch(f_*(x)) = \mathrm{rank}(f_*(x)) + c_1(f_*(x)) + \dots$
The computation will be done by completing the diagram

*) It is possible to consider a more general situation.

$$\begin{tikzcd} K(X) \arrow[d, "f_*"'] & \mathrm{Gr}\, K(X) \otimes \mathbb{Q} \\ K(Y) \arrow[r, "ch"] & \mathrm{Gr}\, K(Y) \otimes \mathbb{Q} \end{tikzcd}$$

is a commutative square.

We first define the Todd class and the virtual relative dimension of $f: X \to Y$.

The morphism $f: X \to Y$ has a factorization of the form

$$f: X \xrightarrow{i} \mathbb{P}^r \xrightarrow{p} Y$$

with a closed immersion i and the projection p. Let N be the conormal sheaf of i and Ω be the sheaf of 1-differentials of p. Both are vector bundles.

<u>Prop:</u> The element

$$T_f := [i^* \check{\Omega}] - [\check{N}] \text{ in } K(X)$$

is independent of the factorization above.

Definition: The element

$$Td(X|Y) := td(T_f) \in Gr^{\cdot} K(X) \otimes \mathbb{Q}$$

is called the Todd class of $f: X \to Y$ and

$$\mathrm{rank}\,\Omega(T_f) \in K^0(X, \mathbb{Z})$$

is called the virtual relative dimension of $f: X \to Y$.

Now we formulate

Grothendieck - Riemann - Roch.

Let $f: X \to Y$ be of constant virtual relative dimension d. Then

(i) The Gysin-Poincaré morphism $f_*: K(X) \to K(Y)$ is of degree $-d$ modulo torsion, i.e.

$$f_*(F^n K(X) \otimes \mathbb{Q}) \subseteq F^{n-d} K(Y) \otimes \mathbb{Q}$$

for all n, and hence induces a homomorphism

$$f_*: Gr^{\cdot} K(X) \otimes \mathbb{Q} \longrightarrow Gr^{\cdot} K(Y) \otimes \mathbb{Q}$$

of degree $-d$, the so called Gysin.

ii) The diagram

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{Todd}(T_f)\,ch"] \arrow[d, "f_*"'] & \mathrm{Gr}^{\cdot} K(X)\otimes\mathbb{Q} \arrow[d, "f_*\ (\mathrm{Gysin})"] \\ K(Y) \arrow[r, "ch"] & \mathrm{Gr}^{\cdot} K(Y)\otimes\mathbb{Q} \end{tikzcd}$$

commutes. —

Remark: The Gysin morphism can be described by means of the push forward for cycles.

Example (Hirzebruch-Riemann-Roch)

Let X be a projective smooth scheme over a field k of pure dimension d. If $f: X \to \mathrm{Spec}(k)$ is the structure morphism then by Grothendieck-Riemann-Roch the diagram

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{Todd}(X/k)\,ch"] \arrow[d, "f_*"'] & \mathrm{Gr}^{\cdot} K(X)\otimes\mathbb{Q} \arrow[d, "f_*"] \\ K(\mathrm{Spec}(k)) \arrow[r, "ch"] & \mathrm{Gr}^{\cdot} K(\mathrm{Spec}(k))\otimes\mathbb{Q} \end{tikzcd}$$

commute.

As rank: $K(\mathrm{Spec}(S)) \to \mathbb{Z}$ is an isomorphism one has $\mathrm{Gr}\, K(\mathrm{Spec}(S)) = \mathbb{Z}$ and the chern character reduces to the rank and hence we have

$$\mathrm{ch}(f_*[\mathcal{F}]) = \sum_{q \geq 0} (-1)^q \operatorname{rank} H^q(X, \mathcal{F})$$
$$= \chi(X, \mathcal{F}).$$

The Todd class of X/S is given by

$$\mathrm{Td}(X/S) = 1 + \frac{1}{2} c_1 + \frac{1}{12}(c_1^2 + c_2) + \dots$$

with $c_i = (-1)^i c_i(\Omega_{X/k})$ as $X \to \mathrm{Spec}(S)$ is smooth.

The Gysin $f_* : \mathrm{Gr}^* K(X) \otimes \mathbb{Q} \to \mathbb{Q}$ being of degree $-d$, it annihilates all components $\mathrm{Gr}^q K(X) \otimes \mathbb{Q}$ for $q < d$, and on $\mathrm{Gr}^d K(X) \otimes \mathbb{Q}$ it is given by

$$f_* : \mathrm{Gr}^d K(X) \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$
$$[\mathcal{O}_{\{x\}}] \longmapsto [k(x) : k]$$

where the classes $[\mathcal{O}_{\{x\}}]$ of the closed points x on X form a system of generators of the $\mathbb{Q}$-vectorspace $G^d_X(\ldots)\otimes\mathbb{Q}$.

So we get

$$\chi(X,E) = \int \mathrm{Td}(T(X))\,\mathrm{ch}(E)$$

(with the explanation above). —

Coming now to the proof of Grothendieck-Riemann-Roch we remark first the following easy observation:

If $f: X \to Y$ and $g: Y \to Z$ are two morphisms as above and if G.R.R. is true for both of them then G.R.R. is true for the composition $g\circ f: X \to Z$.

So, if we factorize the given morphism $f: X \to Y$ in the form

$$f: X \xrightarrow{i} \mathbb{P}^r_Y \xrightarrow{p} Y$$

with a closed immersion i and the projection p, it suffices to prove G.R.R. for p and for i.

The proof for $p: \mathbb{P}^r_Y \to Y$ is rather easy because of the very good knowledge of the ring $K(\mathbb{P}^r_{\mathbb{C}})$ as an algebra over $K(Y)$, see the structure theorem above.

The proof for the closed immersion $i : X \to \mathbb{P}^r_{\mathbb{C}}$ is difficult.

VII. Riemann-Roch without denominators

Let $i : Y \to X$ be a closed immersion of regular noetherian schemes with ample line bundles. Let i be of constant codimension d and let $N^\vee$ be the conormal sheaf. $N^\vee$ is a vector bundle of rank d on Y.

We look at the Euler-Poincaré-characteristic

$$i_* : K(Y) \longrightarrow K(X)$$

and on its effect on the Adams operations ψ^k acting on both rings. The key theorem is:

Riemann-Roch without denominators

For all $\ell \geq 1$ the square

$$\begin{tikzcd} K(Y) \arrow[r, "\theta^{\ell}(N^{\vee})\,\psi^{\ell}"] \arrow[d, "i_*"'] & K(Y) \arrow[d, "i_*"] \\ K(X) \arrow[r, "\psi^{\ell}"] & K(X) \end{tikzcd}$$

commutes, where $\theta^{\ell}(N^{\vee})$ is the ℓ-th cannibalistic <u>Bott class</u> of N. —

The Bott classes are defined for vector bundles E on Y as elements in $K(Y)$ by the following properties:

i) $\theta^{\ell}(E \oplus E') = \theta^{\ell}(E) \cdot \theta^{\ell}(E')$

ii) $\theta^{\ell}(L) = 1 + [L] + \dots + [L]^{\ell-1}$
for a line bundle L

iii) functoriality. —

The theorem above is hard to prove. One

proves it first for the zero section
$i : Y \to \mathbb{P}(W \oplus \mathcal{O}_Y)$ given by $W \oplus \mathcal{O}_Y \to \mathcal{O}_Y$
and then in the general case $i : Y \to X$
by the so called deformation of i
into the zero section of $\mathbb{P}(W \oplus \mathcal{O}_Y)$. —

In order to deduce Grothendieck-Riemann-Roch from Riemann-Roch without denominator we use a lemma bringing all the names together: Adams, Bott, Chern, Todd:

<u>Lemma</u>: If W is a vector bundle of rank d on Y then for all $k \geq 1$

$$\theta^k(W)\,\psi^k(ch^{-1}(td(W^\vee))) = k^d\, ch^{-1}(td(W^\vee)).$$

<u>Proof</u>:
One reduces to a line bundle and then it's easy. —

We put
$$u := ch^{-1}(td(W^\vee)).$$

This is an element in $K(Y)\otimes\mathbb{Q}$, and in fact a unit with augmentation 1, so the lemma we have

$$\theta^\xi(N^\vee)\,\psi^\xi(u) = \xi^d u.$$

We now deduce Grothendieck–Riemann–Roch.

1. Step: The map

$$K(Y)\otimes\mathbb{Q} \longrightarrow K(X)\otimes\mathbb{Q}$$
$$y \longmapsto i_*(uy)$$

maps the eigenspace $(K(Y)\otimes\mathbb{Q})^{(j)}$ into the eigenspace $(K(X)\otimes\mathbb{Q})^{(d+j)}$.

In fact, let $y \in (K(Y)\otimes\mathbb{Q})^{(j)}$ be given. Then by Riemann–Roch without denominators and the lemma we have:

$$\begin{aligned}
\psi^\xi(i_*(uy)) &= i_*(\theta^\xi(N^\vee)\,\psi^\xi(uy)) \\
&= i_*(\theta^\xi(N^\vee)\,\psi^\xi(u)\,\psi^\xi y) \\
&= i_*(\xi^d u\, \xi^j y) \\
&= \xi^{d+j}\, i_*(uy)
\end{aligned}$$

hence $i_*(uy) \in (K(X)\otimes\mathbb{Q})^{(d+j)}$.

2. Step. The map $i_*: K(Y) \to K(X)$ is compatible with filtrations of degree d, i.e.

$$i_*(F^n K(Y)\otimes\mathbb{Q}) \subseteq F^{d+n} K(X)\otimes\mathbb{Q}.$$

In fact, let $y \in F^n K(Y)\otimes\mathbb{Q}$ be given and write $y = u y'$ with $y' = u^{-1} y$. With respect to the decomposition

$$F^n K(Y)\otimes\mathbb{Q} = \bigoplus_{j\geq n} (K(Y)\otimes\mathbb{Q})^{(j)}$$

write

$$y' = \sum_{j\geq n} y'_j.$$

Then

$$i_*(y) = i_*(uy') = \sum_{j\geq n} i_*(u y'_j).$$

By the first step we have

$$i_*(u y'_j) \in (K(X)\otimes\mathbb{Q})^{(d+j)}$$

and hence

$$i_*(y) \in \bigoplus_{j\geq n} (K(X)\otimes\mathbb{Q})^{(d+j)} = F^{d+n} K(X)\otimes\mathbb{Q}.$$

3. Step. The diagram

$$\begin{tikzcd}
K(Y) \arrow[r, "td(N)^{-1} ch"] \arrow[d, "i_*"'] & Gr\, K(Y)\otimes\mathbb{Q} \arrow[d, "i_*"] \\
K(X) \arrow[r, "ch"] & Gr\, K(X)\otimes\mathbb{Q}
\end{tikzcd}$$

commutes.

In fact, let $y \in K(Y)$ be given and write again $y = u y'$ with $y' \in K(Y)\otimes\mathbb{Q}$. We have to show

$$ch(i_*(u y')) = i_*(td(N)^{-1} ch(u y'))$$

Because of $u = ch^{-1}(td(N))$ the right hand side is equal to $i_*(ch(y'))$. So we have to prove

$$ch(i_*(u y')) = i_*(ch(y'))$$

for elements $y' \in K(Y)\otimes\mathbb{Q}$. This way assume $y \in (K(Y)\otimes\mathbb{Q})^{(n)}$.

Then we have

$$ch(y) = y \bmod F^{n+1} K(Y)\otimes\mathbb{Q}$$

and hence by Step 2:

$(*)\quad i_*(ch(y)) = i_*(y) \bmod F^{d+n+1} K(M)\otimes\mathbb{Q}$.

By step 1 we have $i_*(uy) \in F^{d+n}(K(X)\otimes\mathbb{Q})$ and hence

$(**)\quad ch(i_*(uy)) = i_*(uy) \bmod F^{d+n+1} K(X)\otimes\mathbb{Q}$.

But u now being of augmentation 1 we have $uy \equiv y \bmod F^{n+1} K(Y)\otimes\mathbb{Q}$ and hence by step 2 $i_*(uy) = i_*(y) \bmod F^{d+n+1} K(X)\otimes\mathbb{Q}$. So from $(*)$ and $(**)$ we get

$$ch(i_*(uy)) = i_*(ch(y)).$$

<u>Some literature.</u>

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