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Jacobian Varieties

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The basic reference for these notes is

J. Milne: Jacobian Varieties. In: Arithmetic Geometry (Proc. Storrs 1984), Eds.: G. Cornell & J. Silverman. Springer 1986, pp. 167-2

There you find all the details left out in these notes, as well as some additional material.

Let k be a field and C a curve (i.e. a onedimensional separated scheme of finite type) over k which we assume to be projective and geometrically smooth and connected.
The aim of these notes is to construct the Jacobian variety J of C as an abelian variety over k, and to discuss some of its properties.

1. Motivation

Recall the construction of the Jacobian of a Riemann surface:
Let X be a compact Riemann surface of genus $g \geq 1$. Then its Jacobian $Jac(X)$ is defined as the analytic torus $\mathbb{C}^g/\Lambda$, where Λ is the period lattice with respect to a fixed basis $\omega_1, \dots, \omega_g$ of the vector space $H^0(X, \Omega_X)$ of global holomorphic differentials on X: $\Lambda = \{(\int_\alpha \omega_1, \dots, \int_\alpha \omega_g) \in \mathbb{C}^g : \alpha \in H_1(X, \mathbb{Z})\}$.
The theory of theta functions shows that this torus is in fact algebraic, hence an abelian variety (and in particular projective).

The Abel-Jacobi theorem states that, as a group, $Jac(X)$ is isomorphic to the group $Pic^\circ(X)$ of invertible sheaves (or line bundles) of degree zero on X.
So the Abel-Jacobi theorem can be viewed as a nice (since functorial!) way of endowing $Pic^\circ(X)$ with the structure of an abelian variety.

In the abstract situation of a curve C (of genus ≥ 1), the Jacobian should do exactly the same job:
"Endow $Pic^\circ(C)$ in a natural way with a structure of abelian variety".

As groups, all abelian varieties of the same dimension (that admit a polarization of a certain fixed type, e.g. a principal polarization) are isomorphic. So the problem of turning $Pic^\circ(C)$ into an abelian variety is not well defined unless we specify what we mean by "in a natural way":
Recall that $Pic^\circ(C)$ classifies line bundles of degree zero on C. We shall define the appropriate moduli problem, and the moduli functor of "families of line bundles of degree zero on C"; then we have to prove that this functor is representable by an abelian variety.
For any k-scheme T let

$$\mathcal{P}ic^\circ_C(T) := \{\mathcal{L} \in Pic(C \times T) : \deg(\mathcal{L}_t) = 0 \text{ for all } t \in T\}/\sim$$

where $C \times T := C \times_{\operatorname{Spec} k} T$, $\mathcal{L}_t$ is the restriction of $\mathcal{L}$ to the fibre C_t ($\cong C$) over t, and $\mathcal{L} \sim \mathcal{L}'$ if there is $\mathcal{M} \in Pic(T)$ such that $\mathcal{L} \otimes (\mathcal{L}')^{-1} \cong q^*\mathcal{M}$, where $q : C \times T \to T$ is the projection onto the second factor (note that $(g^*\mathcal{M})_t \cong \mathcal{O}_{C_t}$ for all $t \in T$, so that these families are in fact trivial.

Conversely, let $\mathcal{L}$ be an invertible sheaf on $C \times T$ such that $\mathcal{L}_t \cong \mathcal{O}_{C_t}$ for all $t \in T$; for an arbitrary $P \in C$, let $\mathcal{M} := \mathcal{L}_P$ (considered as a sheaf on $T \cong T_P$); then if T is integral, by the seesaw principle, $\mathcal{L} \otimes (g^*\mathcal{M})^{-1}$ is trivial on $C \times T$.

This shows that different elements in $\mathcal{P}ic^0_C(T)$ differ in at least one fibre.

Remark: If T is connected, then $\deg(\mathcal{L}_t)$ is independent of $t \in T$.

Obviously, $\mathcal{P}ic^0_C$ is a contravariant functor from the category of k-schemes to the abelian groups, and

$$\mathcal{P}ic^0_C(k) = \mathrm{Pic}^0(C)$$

Theorem: If $C(k)$ is nonempty, the functor $\mathcal{P}ic^0_C$ is representable by an abelian variety J over k.

In the general situation (i.e. if $C(k) = \emptyset$) let k' be a finite Galois extension of k such that $C(k') \neq \emptyset$, and let $G := \mathrm{Gal}(k' : k)$. Then J represents the functor $T \longmapsto \mathcal{P}ic^0_C(T \times_k k')^G$.

Remark: If $C(k) \neq \emptyset$, the functor $\mathcal{P}ic^0_C$ is equivalent to the functor $\mathcal{P}ic'(T) := \{\mathcal{L} \in \mathrm{Pic}(C \times T) : \deg(\mathcal{L}_t) = t \text{ for all } t \in T,$

$$\mathcal{L} | \{P\} \times T \cong \mathcal{O}_{T_P}\}$$

where P is a fixed k-rational point on C.

Proof: For any k-scheme T, the map $t \mapsto (P,t)$ is a section $s: T \to C \times T$ to the projection q. Consequently, $s^*: \mathrm{Pic}(C\times T) \to \mathrm{Pic}\, T$ is a section to q^*, and hence

$$\mathrm{Pic}(C\times T) = \mathrm{Im}(q^*) \oplus \ker(s^*)$$

By definition, $\mathcal{P}ic'(T) = \ker(s^*)$.

We shall always assume that C has a k-rational point P.

The theorem implies that with every $\mathcal{L} \in \mathrm{Pic}^\circ(C) = \mathcal{P}ic^\circ_C(k)$, there corresponds a unique $a \in J(k) = \mathrm{Mor}(\mathrm{Spec}\, k, J)$, and conversely. After translating J by a suitable element of $J(k)$ we may assume that the trivial line bundle $\mathcal{O}_C$ corresponds to the origin $0 \in J(k)$. Then the bijection $\mathrm{Pic}^\circ(C) \to J(k)$ becomes in fact a group isomorphism.

The theorem also implies that there is a "universal" $\mathcal{M} \in \mathrm{Pic}(C\times J)$ corresponding to the identity $id: J \to J$ and normalized by the requirement $\mathcal{M}_0 \cong \mathcal{O}_{C_0}$, with the following property:

(*) Given a connected k-scheme T, a k-rational point $t \in T(k)$, and a line bundle $\mathcal{L} \in \mathrm{Pic}(C\times T)$ such that $\mathcal{L}_t$ and $\mathcal{L}|\{P\}\times T$ are both trivial, there exists a unique morphism $\varphi: T \to J$ such that $\varphi(t) = 0$ and $(1\times\varphi)^*\mathcal{M} \cong \mathcal{L}$.

Proof: The assumption means that we may regard $\mathcal{L}$ as an element of $\mathcal{P}ic'(T)$, which by the theorem is equal to $\mathrm{Mor}(T, J)$.

Conversely, given an abelian variety J and $\mathcal{M} \in \mathrm{Pic}(C\times J)$ satisfying (*), it is clear that J represents $\mathcal{P}ic'$. So in order to prove the theorem, we shall construct an abelian variety J over k, a line bundle $\mathcal{M}$ on $C \times J$, and show that the pair $(J, \mathcal{M})$ satisfies $(*)$.

The following proposition shows that in ~~proving~~ the theorem we may pass to a finite separable extension of k:

Proposition: If the functor $\mathrm{Pic}^0_{C_{k'}}$ is representable for a finite separable extension k' of k, then Pic^0_C is also representable.

Sketch of proof (for details see Milne, Prop. 1.9): Assume k' Galois over k, $G := \mathrm{Gal}(k' : k)$, and let J' be the Jacobian of $C_{k'}$, and $\mathcal{M}$ the universal invertible sheaf on $C_{k'} \times_{k'} J' = C \times_k k' \times_{k'} J' = C \times_k J'$. Hence $\mathcal{M} \in \mathrm{Pic}^0_C(J')$; for $\sigma \in G$, $\sigma\mathcal{M} \in \mathrm{Pic}^0_C(\sigma J')$ induces by $(*)$ a unique morphism $\varphi_\sigma : \sigma J' \to J'$ such that $(1 \times \varphi_\sigma)^* \mathcal{M} = \sigma\mathcal{M}$. These φ_σ form a descent datum for J', (i.e. $\varphi_{\tau\sigma} = \varphi_\tau \circ \tau\varphi_\sigma$), and from descent theory it follows that J' has a model J over k (i.e. $J \times k' = J'$). Now it is not hard to check that J in fact represents Pic^0_C.

Consider the natural map $C(k) \to \mathrm{Pic}^0(C)$, $Q \mapsto \mathcal{O}(Q-P)$, the invertible sheaf associated with the divisor $Q-P$.

Proposition: There is a natural injective morphism $\varphi = \varphi^{(1)} : C \to J$ which on k-rational points induces the above map $Q \mapsto \mathcal{O}(Q-P) \in \mathrm{Pic}^0(C) = J(k)$.

Proof: In view of $(*)$ we have to find a line bundle $\mathcal{L}$ on $C \times C$ such that $\mathcal{L} | C \times \{P\}$ and $\mathcal{L} | \{P\} \times C$ are both trivial and that $\mathcal{L}\,|\,C\times\{Q\} \cong \mathcal{O}(Q-P)$. These properties are clearly satisfied by the invertible sheaf associated with the divisor $D = \Delta - C\times\{P\} - \{P\}\times C$ on $C\times C$, where Δ denotes the diagonal. The injectivity is clear since we have assumed that C has genus $g \geq 1$.

It is considerably harder to show (but also true in general) that $\varphi^{(1)}$ is a closed embedding. For this one first proves that the Zariski tangent space to J at 0 is isomorphic to $H^1(C, \mathcal{O}_C)$, a result of independent interest which implies in particular that J is g-dimensional (see Milne, §2).

2. Construction of the Jacobian

The starting point for the construction of J is the r-fold symmetric product

$$C^{(r)} := C\times \ldots \times C \,/\, S_r$$

of our curve C, where the product has r factors ($r \geq 1$), and where S_r denotes the symmetric group on r letters acting as permutations of the factors.

If C^r is covered by affine symmetric k-varieties $U_i = \operatorname{Spec} A_i$, then $C^{(r)}$ is covered by $U_i^{S_r} := \operatorname{Spec} A_i^{S_r}$, where $A_i^{S_r}$ denotes the subring of S_r-invariants of A_i (finitely generated by Hilbert's invariant theorem). This shows that $C^{(r)}$ is an r-dimensional variety over k. Moreover:

<u>Proposition</u>: $C^{(r)}$ is nonsingular.

Proof: Let $\Delta = \{(P_1, \dots, P_r) \in C^r : \exists\, i \neq j \text{ s.th. } P_i = P_j\}$ be the (generalized) diagonal of C^r, and denote by $\pi : C^r \to C^{(r)}$ the quotient map.

Since $x \in C^r - \Delta$ is not fixed by any nontrivial element of S_r, $\pi(x)$ is a smooth point of $C^{(r)}$. So let $x \in \Delta$; by symmetry we may assume that x is of the form $x = (P_1, \dots, P_1, P_{i+1}, \dots P_r)$ for some $1 \le i \le r$. The completion $\hat{\mathcal{O}}_{C^r, x}$ of the local ring of x in C^r is isomorphic to $k[[X_1, \dots, X_r]]$, and so (assuming for a moment that $\{P_1, P_{i+1}, \dots, P_r\}$ are $r-i+1$ distinct points) the local ring $\hat{\mathcal{O}}_{C^{(r)}, \pi(x)}$ is isomorphic to $k[[X_1, \dots, X_r]]^{S_r}$ which is generated by $X_{i+1}, \dots, X_r$ and the i elementary symmetric functions in $X_1, \dots, X_i$: $s_1 = X_1 + \dots + X_i$, $s_2 = X_1X_2 + \dots + X_{i-1}X_i$, $\dots$, $s_i = X_1 \cdots X_i$. This shows that $\hat{\mathcal{O}}_{C^{(r)}, \pi(x)} \cong k[[s_1, \dots, s_i, X_{i+1}, \dots, X_r]]$, thus $\pi(x)$ is smooth.

If $\bar{k}$ denotes the algebraic closure of k, then $C^{(r)}(\bar{k})$ is the set of unordered r-tupels of elements of $C(\bar{k})$, which can also be interpreted as the set of effective divisors of degree r on $C_{\bar{k}}$. To make this correspondence more conceptual one introduces the notion (and the functor) of relative effective Cartier divisors. We omit the somewhat technical detail (cf. Milne, §3) and merely state the result:

Proposition: $C^{(r)}$ represents the functor $\mathcal{D}iv^r_C$ of relative effective Cartier divisors of degree r.

Instead of giving the general definition of the functor $\mathcal{D}iv^r_C$

we only describe the universal relative effective divisor D_{can} on $C \times T$ over T:
for $i = 1, \dots, r$ let $s_i : C^r \to C \times C^r$ be given by $(P_1, \dots, P_r) \mapsto (P_i, P_1, \dots, P_r)$. Clearly s_i is a section to the projection $p_2 : C \times C^r \to C^r$.
Then $D_i := s_i(C^r)$ is a divisor of $C \times C^r$ which is effective of degree 1 over C^r in the sense that in each fibre of q, D_i induces an effective divisor of degree one.
Now let $D := \sum_{i=1}^{r} D_i$; this is clearly a symmetric divisor, so let $D_{can} := D/S_r$ its image on $C^{(r)}$.
Then D_{can} is the unique relative effective divisor on $C \times C^{(r)}$ over $C^{(r)}$ whose fibre over $D \in C^r(k)$ is exactly D.

With a relative effective Cartier divisor D on $C \times T/T$ for a k-scheme T we can, in the same way as for usual divisors, associate an invertible sheaf $\mathcal{L}(D)$ on $C \times T$ (in particular, the restriction $(\mathcal{L}(D))_t$ to the fibre over $t \in T$ is the sheaf $\mathcal{L}(D_t)$ induced by the effective divisor D_t on C_t). This provides us with a natural transformation of functors $f : \mathcal{D}iv_C^r \to \mathcal{P}ic_C^r$
($\mathcal{P}ic_C^r$ is defined in the same way as $\mathcal{P}ic_C^0$, but with "degree 0" replaced by "degree r").

Remark: For any k-scheme T, the map $\mathcal{L} \mapsto \mathcal{L} \otimes p^* \mathcal{L}(rP)$ is an isomorphism $\mathcal{P}ic_C^0(T) \to \mathcal{P}ic_C^r(T)$ (where $p : C \times T \to C$ is the projection on the first factor and P is our fixed k-rational point on C, see p. 4).

Consequently the functors Pic^0_C and Pic^r_C are equivalent, and in order to prove the theorem it suffices to show that Pic^r_C is representable for some r.

For the proof we shall use the representability of the functor Div^r_C: we shall try to associate in a natural way with every invertible sheaf of degree r an effective divisor of degree r, and this also in the relative situation on $C\times T$ over T. This will work because of the following

<u>**Lemma:**</u> Suppose there is a section $s: \mathrm{Pic}^r_C \to \mathrm{Div}^r_C$ to f. Then Pic^r_C is representable by a closed subscheme of $C^{(r)}$.

<u>Proof</u>: (see Milne, Lemma 4.1): Since $s\circ f$ is a natural transformation $\mathrm{Div}^r_C \to \mathrm{Div}^r_C$, it is represented by a morphism $\varphi: C^{(r)} \to C^{(r)}$. Now it is easy to check that the fibre product $\mathcal{J}' := C^{(r)} \times_{C^{(r)}\times C^{(r)}} C^{(r)}$, where the first morphism is $(1,\varphi)$ and the second the diagonal Δ, satisfies $\mathcal{J}'(T) = \mathrm{Pic}^r_C(T)$ for all k-schemes T. Since Δ is a closed immersion, the projection $\mathcal{J}' \to C^{(r)}$ onto the first factor is also a closed immersion.

Unfortunately the hypothesis of the lemma is not satisfied for any r. But we shall define subfunctors of Pic^r_C over which a section exists and which therefore by the lemma are representable by locally closed subschemes of $C^{(r)}$. Since the images of these sections cover $C^{(r)}$ we arrive at the Jacobian by glueing certain of these subschemes.

Fix $r > 2g$; then for any invertible sheaf $\mathcal{L}$ on C of degree r, the Riemann-Roch formula reads $h^0(\mathcal{L}) := \dim_k H^0(C,\mathcal{L}) = r+1-g$; moreover, for any basis $s_0, \dots, s_{r-g}$ of $H^0(C,\mathcal{L})$, the map $C \to \mathbb{P}^{r-g}$, $Q \mapsto (s_0(Q) : \dots : s_{r-g}(Q))$ is a closed embedding. On the other hand, for any divisor D on C of degree r the linear system $|D|$ of effective divisors linearly equivalent with D, is of dimension $r-g$, which makes a canonical choice of an effective divisors in a given linear equivalence class impossible.

Therefore we fix a set $\gamma = \{P_1, \dots, P_{r-g}\}$ of (not necessarily distinct) k-rational points on C and define $D_\gamma := P_1 + \dots + P_{r-g}$. If we now ask for divisors $\tilde{D} \geq D_\gamma$ in $|D|$ we arrive at the situation described above:

<u>Proposition:</u> a) The functor $\mathcal{D}iv_C^\gamma$ defined by

$$\mathcal{D}iv_C^\gamma(T) := \{D \in \mathcal{D}iv_C^r(T) : h^0(D_t - D_\gamma) = 1 \text{ for all } t \in T\}$$

is representable by an open subvariety C^γ of $C^{(r)}$.

b) If k is separably closed, $C^{(r)} = \bigcup_\gamma C^\gamma$

c) the natural transformation of functors $f: \mathcal{D}iv_C^\gamma \to \mathcal{P}ic_C^\gamma$ has a section, where $\mathcal{P}ic_C^\gamma$ is the subfunctor of $\mathcal{P}ic_C^r$ defined by

$$\mathcal{P}ic_C^\gamma(T) := \{\mathcal{L} \in \mathcal{P}ic_C^r(T) : h^0(\mathcal{L}_t \otimes \mathcal{L}(D_\gamma)^{-1}) = 1 \text{ for all } t \in T\}.$$

<u>Proof</u>: a) Let K be a canonical divisor on C. Then by Riemann-Roch $h^0(D - D_\gamma) := h^0(\mathcal{L}(D - D_\gamma)) = 1 + h^0(K - D + D_\gamma) \geq 1$ for all divisors D of degree r on C, and $h^0(D - D_\gamma) > 1$ if and only if $h^0(K - D + D_\gamma) > 0$. Now $K - D + D_\gamma$ has degree $g-2$, so in the case $g = 1$, $\mathcal{D}iv_C^\gamma = \mathcal{D}iv_C^r$, and part c) gives the Jacobian of C

without any glueing (in fact it follows already from the remark on p. 6 that the Jacobian of an elliptic curve is isomorphic to the curve itself).

So let $g \geq 2$ and $D = Q_1 + \ldots + Q_r$ effective of degree r. Note that $h^0(K + D_\gamma) = r - 1$, and that $H^0(C, \mathcal{L}(K + D_\gamma)) = H^0(C, \Omega(D_\gamma))$ is the vector space of Kähler differentials on C having poles at most at $P_1, \ldots, P_{r-g}$, of order at most the multiplicity of the point in $P_1, \ldots, P_{r-g}$. Then for $i = 1, \ldots, r-1$, $h^0(K + D_\gamma - (Q_1 + \ldots + Q_i)) = h^0(K + D_\gamma - (Q_1 + \ldots + Q_{i-1})) - 1$ unless Q_i is a common zero of all elements of (a basis of) $H^0(C, \mathcal{L}(K + D_\gamma - (Q_1 + \ldots + Q_{i-1})))$; this of course is possible for at most finitely many choices of Q_i which shows that $h^0(K + D_\gamma - D) = 0$ for all D in an open subvariety C^r of $C^{(r)}$. This defines C^r, and it is now routine to check that C^r indeed represents the functor $\mathcal{D}iv^r_C$.

b) Let D be an effective divisor of degree r on C. We have to find $\gamma = \{P_1, \ldots, P_{r-g}\}$ such that $h^0(D - D_\gamma) = 1$.

Let $s_0, \ldots, s_{r-g}$ be a basis of $H^0(C, \mathcal{L}(D))$ and let $\varphi : C \hookrightarrow \mathbb{P}^{r-g}$ the corresponding closed embedding.

Claim: $\varphi(C)$ is not contained in any proper linear subspace of $\mathbb{P}^{r-g}$.

Because, if $\varphi(C)$ were contained in $\sum a_i X^i = 0$ then $\sum a_i s_i$ would be zero on C, a contradiction to the linear independence.

So there exist $P_1, \ldots, P_{r+g}$ in $C(k)$, disjoint from $\mathrm{Supp}(D)$, such that $P_1, \ldots, P_i$ is not contained in any linear subspace of dimension $i-2$ $(i = 2, \ldots, r-g)$. Then $H^0(C, \mathcal{L}(D - (P_1 + \ldots + P_{r-g}))) = \{\sum a_i s_i \in H^0(C, \mathcal{L}(D)) : \sum a_i s_i(P_j) = 0 \text{ for } j = 1, \ldots, r-g\}$ has dimension ≤ 1.

c) in a) we have constructed the section on the level of
$$C^{\gamma} = \mathcal{D}iv_C^{\gamma}(k) \to \mathcal{P}ic_C^{\gamma}(k) = \{\mathcal{L} \in \mathrm{Pic}^r(C) : h^0(\mathcal{L} \otimes \mathcal{L}(D_\gamma)^{-1}) = 1\}.$$
Since we omitted the general definition of $\mathcal{D}iv_C^r$, we have to refer for the proof of c) to Milne, Prop. 4.2 b).

Now by the compactness of $C^{(r)}$ we find finitely many $(r-g)$-tupels $\gamma_1, \ldots, \gamma_n$ of k_s-rational points on C (k_s being the separable closure of k) such that $C^{(r)} = \bigcup_{i=1}^{n} C^{\gamma_i}$. Hence there is a finite separable extension k' of k such that all the points in the γ_i are k'-rational. So by glueing the C^{γ_i} as indicated we obtain a variety J' over k' which represents the functor $\mathcal{P}ic^r_{C_{k'}}$. In view of the proposition on p. 5 this is sufficient to prove the theorem.

It is clear that J is a group variety since it represents a group functor. Moreover we have a surjective morphism $C^{(r)} \to J$ which shows, since $C^{(r)}$ is complete, that J is also complete, thus an abelian variety. This finishes the proof of the theorem.

3. Properties of the Jacobian

(1) For any $r \geq 1$, the natural transformation of functors $\mathcal{D}iv_C^r \to \mathcal{P}ic_C^0$, $D \mapsto \mathcal{L}(D - rP)$ induces a morphism $\varphi^{(r)} : C^{(r)} \longrightarrow J$.

Note that $\varphi^{(1)}$ is the closed embedding of the curve into its Jacobian discussed on p. 5/6. More generally we have:

Theorem: For $1 \le r \le g$, the morphism $\varphi^{(r)}$ is birational onto its image.

(2) The image of $\varphi^{(r)}$ is traditionally denoted by W^r; it can be identified with $\varphi(C) + \ldots + \varphi(C)$ (r summands; addition on the abelian variety J). Of particular importance is W^{g-1} which is a divisor on J:

Theorem: The divisor $\Theta := W^{g-1}$ (also called the theta divisor because over $\mathbb{C}$ it is closely related to the set of zeroes of the Riemann theta function) is ample, and the corresponding line bundle $\mathcal{L}(\Theta)$ defines a principal polarization on J, i.e. an isomorphism between J and its dual abelian variety.

(3) For any $r \geq 1$, the fibre of $\varphi^{(r)}$ over a point in J corresponding to a class $[D]$ of divisors of degree zero consists of all effective divisors of degree r linearly equivalent with $D + rP$, i.e the linear system $|D+rP|$, which is $\mathbb{P}(H^0(C, \mathcal{L}(D+rP)))$. We have proved:

Proposition: For any $r \geq 1$, the fibres of $\varphi^{(r)}$ are projective spaces, hence in particular connected.

(4) The Jacobian has the following universal property which qualifies it as the "Albanese variety" of C:

Theorem: For any morphism $f: C \to A$ into an abelian variety A over k, which maps P to 0, there is a unique homomorphism $g: J \to A$ such that $f = g \circ \varphi^{(1)}$

$$\begin{tikzcd} C \arrow[r, "\varphi^{(1)}"] \arrow[dr, "f"'] & J \arrow[d, dashed, "g"] \\ & A \end{tikzcd}$$

Proof: Consider the morphism $C^g \to A$, $(P_1, \dots, P_g) \mapsto \sum_{i=1}^{g} f(P_i)$; it is clearly symmetric, hence factors through $C^{(g)}$, and since the latter is birational to J, it induces a rational map $g: J \dashrightarrow A$. Since A is complete and J an abelian variety, g is in fact a morphism which clearly makes the diagram commutative. Moreover $g(0) = 0$, so g is a homomorphism. If g' is a second such homomorphism, g and g' agree on $\varphi^{(1)}(C) + \dots + \varphi^{(1)}(C) = J$.

As a corollary we see that the construction of the Jacobian is functorial: If $f: C \to C'$ is a morphism of smooth curves over k sending P to $P' \in C'(k)$, then the composition $\varphi_{C'}^{(1)} \circ f$ is a morphism $C \to J'$, the Jacobian of C', which by the theorem induces a homomorphism $J \to J'$.

(5) Theorem: Assume k is infinite. Then any abelian variety A over k is a quotient of a Jacobian.

Idea of proof: Embed A into projective space $\mathbb{P}_k^n$ and apply Bertini's theorem (several times) to find a linear subspace of $\mathbb{P}^n$ such that its intersection with A is a smooth curve C. Then by (4) the embedding $C \to A$ induces a homomorphism $J \to A$ which can be shown to be surjective.

(6) Over an algebraically closed field it is possible to recover a curve from its Jacobian:

<u>Theorem (Torelli)</u>: Let k be algebraically closed. Then smooth projective curves C, C' over k are isomorphic if and only if their Jacobians J, J' are isomorphic as canonically (principal) polarized abelian varieties.

For the proof we refer to Milne, § 13.

Torelli's theorem can be stated in the context of moduli spaces where it is extremely important:
Let $\mathcal{M}_g$ be the moduli scheme of smooth curves of genus g and $\mathcal{A}_g$ the moduli scheme of principally polarized abelian varieties. Then Torelli's theorem states that the Jacobi map $\mathcal{M}_g \longrightarrow \mathcal{A}_g$, $C \longmapsto J$ is injective.
The famous Schottky problem which has attracted much attention and seen great progress in recent years, ask for the image of the Jacobi map inside $\mathcal{A}_g$.

(7) For the Torelli theorem (and many other purposes!) one wants to construct the Jacobian of a family of curves:
Let $\pi : \mathcal{C} \to S$ be a family of curves over a scheme S, i.e. π is a morphism of schemes which is projective and flat such that the fibre over each $s \in S$ is a curve over the residue field $k(s)$.
In this situation we can define the Picard functor on the category of S-schemes of finite type as before:

$$\mathrm{Pic}^r_{\mathcal{C}}(T) := \{\mathcal{L} \in \mathrm{Pic}(\mathcal{C} \times_S T) : \deg(\mathcal{L}_t) = r \text{ for all } t \in T\} / q^* \mathrm{Pic}\, T$$

Then Grothendieck (Séminaire Bourbaki, exposé 232) proved the following theorem:

<u>Theorem</u>: For a family $\pi : \mathcal{C} \to S$ of integral curves (i.e. all fibres are integral), there exists a group scheme J over S with connected fibres which represents the functor $\mathrm{Pic}^{\circ}_{\mathcal{C}}$, if π has a section (and a somewhat weaker statement holds in general).

Remarks: - if all fibres are smooth, J is proper over S and can be considered as a family of Jacobians.
- if $S = \mathrm{Spec}\, k$, the theorem states the existence of a Jacobian for a singular irreducible curve over k. This construction is carried out explicitly in Serre's book "Groupes algébriques et corps de classes", Chap $\overline{V}$. Roughly speaking the technique consists in lifting the divisors on the singular curve to the normalization.

For the simplest examples, the cubic curve with a node (given by an inhomogeneous equation $y^2 = x^2(x+1)$, e.g.) and the Neil parabola ($y^2 = x^3$) the Jacobians turn out to be the multiplicative group and the additive group, respectively.