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Elliptic Operators and Elliptic Complexes

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Elliptic Operators + Elliptic Complexes

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Ch 1: Sobolev Theory

§ 1.0: Introduction: In these notes we shall discuss Elliptic complexes on compact connected orientable manifolds, the main Examples being the de-Rham and Dolbeault complexes (twisted by a holomorphic vector bundle). First one needs to introduce the analytical machinery of Sobolev Spaces, Pseudodifferential operators and Ellipticity.

§ 1.1: Sobolev Theory on $\mathbb{R}^n$; Distributions ([Nar] Ch III, [Gil] Ch 1., [Hor], Ch I, II, [Rud] Ch 6, Ch 7)

Notation: $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index. $|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \dots \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$

$$d_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}; \quad \partial_j = \frac{\partial}{\partial x_j}, \quad D_j = \frac{1}{\sqrt{-1}}\partial_j = \frac{1}{i}\partial_j,$$

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

We first define some standard function spaces on $\mathbb{R}^n$. For us all functions will be complex-valued.

1.1.1 Def: (i) $C^\infty(\mathbb{R}^n) = \{$ infinitely differentiable (:= smooth) complex valued functions on $\mathbb{R}^n\}$.

With a topology we shall define below, this space is denoted by $\mathcal{E}(\mathbb{R}^n)$, or just $\mathcal{E}$

(ii) $C_0^\infty(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : d_x^\alpha f \to 0 \text{ as } x \to \infty \text{ for all } \alpha \}$

(iii) $C_c^\infty(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \text{Support of } f \text{ is compact} \}$.

With a topology to be defined below, this space is denoted by $\mathcal{D}(\mathbb{R}^n)$, or just $\mathcal{D}$, and its elements are called test-functions.

(iv) $\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha d_x^\beta f| \le \text{some constant } C_{\alpha\beta} \ \forall \alpha, \beta \}$.

This space is known as the Schwartz Space of rapidly decreasing fns.

CAUTION:- Many authors denote $C_c^\infty(\mathbb{R}^n)$ by $C_0^\infty(\mathbb{R}^n)$, which is not to be confused with our $C_0^\infty(\mathbb{R}^n)$!

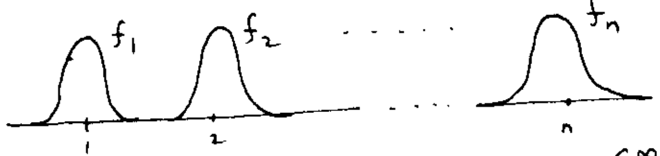
Since all of the function spaces (i) through (iv) are $\mathbb{C}$-vector spaces, it is enough to define the notion of convergence to 0 in order to define convergence and hence a topology.

In (i), the sequence $\{f_n\}$ is defined to converge to 0, i.e. $f_n \to 0$ iff $d_x^\alpha f \to 0$ uniformly on all compact sets, for each α.

In (ii) we put the subspace topology from (i)

In (iii) $f_n \to 0$ if $\exists$ a compact set K such that $\operatorname{supp} f_n \subset K\ \forall n$ and $d_x^\alpha f_n \to 0$ uniformly on K, for each α.

(Note that (iii) does not have the subspace topology from (i), because say in $C_c^\infty(\mathbb{R})$, the sequence



does not converge to 0, whereas it does in $C^\infty(\mathbb{R})$.)

Finally in the Schwartz space (iv), define $f_n \to 0$ if

$$\sup_{x\in\mathbb{R}^n} |x^\alpha d_x^\beta f_n| \to 0 \quad \left(\text{Equivalently } \sup_{\substack{|\alpha|\le N\\ x\in\mathbb{R}^n}} (1+|x|^2)^N |d_x^\alpha f_n(x)| \to 0\ \forall N\right)$$

for each α, β.

From these definitions it is easy to see that (see, e.g. [Rud], 7.10)

$$C_c^\infty \subset \mathcal{S} \subset C_0^\infty \subset C^\infty \qquad (1)$$

are all continuous inclusions, and the first two inclusions are in fact dense. All the inclusions are strict.

For further details on these spaces, see [Rud], Ch 6, 7, [Gil], Ch. 1, [Hor], Ch. I, II. [Nar] § 3

Finally, if one completes $\mathcal{S}$ or C_c^∞ with respect to the norm $\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f|^p\, dx\right)^{1/p}$, we get the space $L^p(\mathbb{R}^n)$ of Lebesgue measurable functions f with $\|f\|_{L^p} < \infty$. (For $p=\infty$ this is false, by looking at the constant functions).

§ 1.2 The Fourier Transform :- ([Rud] Ch.7, [Gil] Lemma 1.1.2, [Nar] § 3.2, [Hor] Ch VII)

For convenience (i.e. to get rid of factors of 2π), introduce the measure dx (or $d\xi$) on $\mathbb{R}^n$ to be $(2\pi)^{-n/2}\, dx_1 \cdots dx_n$ (or $(2\pi)^{-n/2} d\xi_1 \cdots d\xi_n$).

For $f \in \mathcal{S}$, introduce the Fourier transform ~~of~~ f by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x}\, dx \qquad \left(\begin{aligned}\xi\cdot x &= \text{Euclidean inner product}\\ &= \xi_1 x_1 + \cdots \xi_n x_n\end{aligned}\right)$$

(and the inverse Fourier transform by $f^\vee(\xi) = \hat{f}(-\xi) = \int_{\mathbb{R}^n} f(x) e^{i\xi\cdot x}\, dx$)

And for $f, g \in \mathcal{S}$, their convolution by

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}^n} f(y')\, g(x-y')\, dy' = (g*f)(x)$$

1.2.1 Propn It is easily seen that

(a) $f \to \hat{f}$ is an isomorphism of $\mathcal{S}$ with itself, of order 4. In fact $\hat{\hat{f}}(x) = f(-x)$ and $(\hat{f})^{\vee} = f$ $\forall f \in \mathcal{S}$. i.e. $f(x) = \int e^{i\xi \cdot x} \hat{f}(\xi)\, d\xi$ (called the Fourier inversion formula)

(b) $(D_x^\alpha f)^\wedge = \xi^\alpha \hat{f}$ $\left(\Rightarrow (P(D)f)^\wedge = P(\xi)\hat{f}\right.$, for P a polynomial in $\left.\xi_1, \dots, \xi_n\right)$

$(D_\xi^\alpha \hat{f}) = (-1)^{|\alpha|}(x^\alpha f)^\wedge$

(c) $\hat{f}\hat{g} = (f*g)^\wedge$, $\hat{f} * \hat{g} = (fg)^\wedge$

(d) $\wedge : \mathcal{S} \to \mathcal{S}$ extends to an isometry $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (called Plancharel's Theorem)

(e) $(L^1(\mathbb{R}^n))^\wedge \subset C_0(\mathbb{R}^n)$ (continuous functions vanishing at ∞) (called the Riemann–Lebesgue Lemma)

For the proofs, consult the references cited at the beginning. Note that (a) and (b) imply that $\wedge : \mathcal{S} \to \mathcal{S}$ is continuous with respect to the topology introduced in 1.1.1 above. Finally, note if g or $f \in \mathcal{D}$, then $D^\alpha(f*g) = D^\alpha f * g = f * D^\alpha g$. (integration by parts)

§ 1.3 Distributions :- ([Hor.] Ch II, [Rud], Chs. 6, 7, [Sch] Vol 1, Ch I, II)

We will denote the (topological vector space) dual of E by E'. The continuous inclusions in (1) of § 1.1 above imply that

$$(C^\infty(\mathbb{R}^n))' = \mathcal{E}' \subset \mathcal{S}' \subset (C_c^\infty(\mathbb{R}^n))' = \mathcal{D}' \qquad \text{—— } (1)'$$

1.3.1 Def: An element of $\mathcal{D}'$, viz. a continuous linear functional on the space $\mathcal{D}$ of test functions, is called a distribution. An element of $\mathcal{S}'$ is called a tempered distribution, and an element of $\mathcal{E}'$ is called a compactly supported distribution.

1.3.2 : Examples :

(i) If f is a locally L^1 function (i.e. $f \in L^1(K)$ $\forall$ compact $K \subset \mathbb{R}^n$), then f defines a distribution by the formula

$$T_f(g) = \int_{\mathbb{R}^n} f(x) g(x)\, dx \quad \text{for } g \in \mathcal{D}$$

The continuity of T_f follows, e.g. from the Dominated Convergence Theorem. Further if f is a function such that for some N, $(1+|x|)^{-N} f \in L^1$, then f defines a tempered distribution. Thus any polynomial defines a tempered distribution. Similarly, using Hölder's inequality and the fact that $(1+|x|)^{-k}$ is integrable over $\mathbb{R}^n$ for $k > n$, one sees that any L^p function defines a tempered distribution. (Thus, for example, one easily checks that e^x is a distribution on $\mathbb{R}$ which is not tempered, and for example a polynomial is a tempered distribution which is not compactly supported. All inclusions in $(1)'$ are therefore strict.)

(ii) The Dirac – δ-distribution. This (most celebrated) distribution is defined by

$$\delta_a(f) = f(a) \quad \text{for } f \in \mathcal{E} = C^\infty(\mathbb{R}^n)$$

(δ_0 is sometimes denoted δ.). Clearly a compactly supported distribution, with support $= \{a\}$.

Other distributions will arise naturally once we define some basic operations, or calculus on distributions.

1.3.3 Def:- The following operations can be carried out on distributions.

(i) Differentiation: Let $T \in \mathcal{D}'$ be a distribution. Define

$$(D^\alpha T)(f) = (-1)^{|\alpha|} T(D^\alpha f) \quad \text{for } f \in \mathcal{D} = C_c^\infty(\mathbb{R}^n)$$

which is again in $\mathcal{D}'$. The sign is chosen so that if g is a C^k function, then $(D^\alpha T_g) = T_{D^\alpha g}$ from integration by parts (since f above is cptly supported), to keep consistency with the Example 1.3.2 (i) above. It is easy to check that if $T \in \mathcal{S}'$, then so is $D^\alpha T$, and if $T \in \mathcal{E}'$, then so is $D^\alpha T$.

(ii) Multiplication by a C^∞-function. For $\alpha \in C^\infty(\mathbb{R}^n)$, and $T \in \mathcal{D}'$ a distribution, define

$$(\alpha T)(f) = T(\alpha f) \quad \text{for } f \in \mathcal{D}.$$

Note that since multiplication by α takes $\mathcal{E} \to \mathcal{E}$ and $\mathcal{D} \to \mathcal{D}$ but not $\mathcal{S} \to \mathcal{S}$ in general, multiplication by α takes

$\mathcal{E}'$ to $\mathcal{E}'$ and $\mathcal{D}'$ to $\mathcal{D}'$, but needn't take $\mathcal{S}' \to \mathcal{S}'$. If, however, α is of "slow growth", meaning α and all its derivatives have at worst polynomial growth, then α maps $\mathcal{S} \to \mathcal{S}$ continuously, as is easily checked, and thus $\mathcal{S}' \to \mathcal{S}'$, i.e. tempered distributions to tempered distributions.

(iii) Fourier Transform and Inverse Fourier Transform of a tempered distribution.

Since by 1.2.1 (a), (b), (c), (d) $\hat{}: \mathcal{S} \to \mathcal{S}$ and $\check{}: \mathcal{S} \to \mathcal{S}$ are isomorphisms one can define for $T \in \mathcal{S}'$ a tempered distribution, its Fourier transform by

$$\hat{T}(f) = T(\hat{f}) \quad \text{for } f \in \mathcal{S}$$

and its inverse Fourier transform by

$$\check{T}(f) = T(\check{f}) \quad \text{for } f \in \mathcal{S}.$$

This is the key reason for the introduction of the Schwartz Space $\mathcal{S}$ (and tempered distributions $\mathcal{S}'$), since $\hat{}$ does not map $\mathcal{D}$ to $\mathcal{D}$ or $\mathcal{E}$ to $\mathcal{E}$, but maps $\mathcal{S}$ to $\mathcal{S}$ and hence $\mathcal{S}'$ to $\mathcal{S}'$.

(iv) Convolution with functions: Let $\varphi \in \mathcal{S}$, and $T \in \mathcal{S}'$ (a tempered distribution). One defines for $x \in \mathbb{R}^n$, the function φ^x by $\varphi^x(y) = \varphi(x-y)$. Noting that for functions f, $(f * \phi)(x) = \int f(y)\varphi^x(y)\,dy = T_f(\varphi^x)$, we may define the Convolution, which is a function by

$$(\varphi * T)(x) = T(\varphi^x). \quad \text{for } \varphi \in \mathcal{S},\ T \in \mathcal{S}'.$$

It is easy to check that $(\varphi * T)$ is in fact a smooth function with $D^\alpha(\varphi * T) = D^\alpha\varphi * T = \varphi * D^\alpha T$. Also $(\varphi * T)^\wedge = \hat{\varphi}\hat{T}$.

Exercise:- As an application of the definitions (i) through (iv) above show that $\frac{d}{dx}\chi_{[0,\infty)} = \delta$ (on $\mathbb{R}$) where $\chi_{[0,\infty)}$ is the indicator function of $[0,\infty)$.

1.3.4. Propn:- $\hat{}: \mathcal{S}' \to \mathcal{S}'$ defines

(a) A linear 1-1 map (of period 4), and there is the Fourier inversion formula $(\hat{T})^\vee = T$ for $T \in \mathcal{S}'$, a tempered distribution.

(b) $P(D)\hat{T} = P\hat{T}$, $(PT)^\wedge = P(-D)\hat{T}$ for P a polynomial

(c) If $f \in L^1$, then $\hat{T}_f = T_{\hat{f}}$. (d) $(\varphi * T)^\wedge = \hat{\varphi}\hat{T}$ for $\varphi \in \mathcal{S}$.

The Proof is an easy application of the Defns. 1.3.3 and Propn. 1.2.1. We remark that in case $T \in \mathcal{E}'$ is a compactly supported distribution, then $\hat{T}$ is actually a function of slow growth defined by $\hat{T}(\xi) = T(e^{-ix\cdot\xi})$

As an application of the formalism of distributions, we have the

1.3.5 Proposition:– (Cauchy Formula and Green's Function on the Plane)

Let $\bar\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ and $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ on $\mathbb{C} = \mathbb{R}^2$.

Thus $\partial\bar\partial = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta$ where $\Delta =$ the Laplacian on $\mathbb{R}^2$.

We denote by dV_z the volume form $\frac{1}{2\pi}\,dx\,dy$ where $z = x+iy$

Then (i) For $\varphi \in \mathcal{D}(\mathbb{R}^2)$,

$$\varphi(z) = -2\int_{\mathbb{R}^2=\mathbb{C}} \frac{\bar\partial\phi(w)\,dV_w}{(w-z)} \qquad \text{(Cauchy's Integral Formula)}$$

$$= \bar\partial\phi * \left(\frac{2}{z}\right) = \phi * \bar\partial\left(\frac{2}{z}\right)$$

This immediately implies since $\varphi(z) = (\phi * \delta)(z)$ that $\bar\partial\left(\frac{2}{z}\right) = \delta$ i.e. that $\frac{2}{z}$ is a <u>fundamental solution</u> to the Cauchy Problem $\bar\partial F = g$. In other words, the <u>distributional solution</u> to $\bar\partial F = g$ is just $\left(g * \frac{2}{z}\right)$. (This is the reason to look for fundamental solutions). This is because $\bar\partial\left(g * \frac{2}{z}\right) = g * \bar\partial\left(\frac{2}{z}\right)$ $= g * \delta = g$. (by the Def. 1.3.3, (iv))

(ii) For $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\frac{1}{2}\phi(z) = 4\int_{\mathbb{C}} \partial\bar\partial\phi(w)\log|w-z|\,dV_w = \int_{\mathbb{C}} \Delta\phi \log|w-z|\,dV_w$$

which implies $\Delta \log|z| = \delta$, $\log|z|$ is the <u>fundamental solution</u> to the Laplacian. If we want to solve $\Delta F = g$, then $F = (g * \log|z|)$ i.e. $F(z) = \int_{\mathbb{R}^2} g(w)\log|z-w|\,dV_w$

The function $\log|z-w|$ is sometimes denoted $G(z,w)$, the Green's function or Green's <u>kernel</u> for the Laplacian. The corresponding integral operator $\int g(w)G(z,w)\,dV_w = Gg$ is what "inverts" the Laplacian.

Proof of Propn 1.3.5 :

We need the two simple observations below:

Observation 1 :- If δ is the Dirac distribution (on $\mathbb{R}^n$), then

$$\delta^\wedge = 1 .$$

. Proof :-, for $\phi \in C_c^\infty(\mathbb{R}^n) = \mathcal{D}$, (any test function), $\hat{\delta}(\phi) \overset{def}{=} \delta(\hat{\varphi})$

$= \hat{\varphi}(0) = \int e^{ix.0}\phi(x)dx = \int \varphi(x)dx = T_1(\varphi)$ (where T_g as always denotes the distribution defined by the function g). So $\hat{\delta} = 1$ #

Observation 2 :- If $f, g \in C^\infty(U)$ for some open set $U \subsetneq \mathbb{R}^2$, and Ω is any domain contained in U such that $\bar{\Omega} \subset U$ and $\partial\Omega$ is (say) smooth, then by Green's formula on the plane

$$4\pi i \int_\Omega (f\,\bar{\partial}g - g\,\bar{\partial}f)\,dV_z = \int_{\partial\Omega} (fg)\,dz .$$

The Proof follows from the std. Green's Formula $\int_\Omega \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)dx\,dy$ $= \int_{\partial\Omega} (U\,dx + V\,dy)$ by the substitutions $dz = dx + i\,dy$, $dV_z = \frac{1}{2\pi}dx\,dy$ and $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. #.

Now let $f(z) = \frac{1}{z}$. Note that $(1+|z|)^{-2} f \in L^1$ (polar coordinates !! $dx\,dy = r\,dr\,d\theta$) So by 1.3.2 (i) f and $f^\wedge$ are tempered distributions (1.3.3 (iii)), and $T_f(\varphi) = \int_{\mathbb{C}} f(z)\varphi(z)\,dz$ for $\varphi \in C_c^\infty(\mathbb{R}^2)$. Assume $\operatorname{Supp}\varphi \subset \{z : |z| < R\}$, and let $\Omega_{\epsilon,R}$ denote the annulus $\{\epsilon \le |z| \le R\}$. Since $\bar{\partial}f \equiv 0$ (f is holo) on $\Omega_{\epsilon,R}$, we have since $\varphi \equiv 0$ on $|z| = R$ and observation 2,

$$\int_{\Omega_{\epsilon,R}} f\,\bar{\partial}\varphi = \frac{1}{4\pi i}\int_{\theta=0}^{2\pi} \frac{1}{\epsilon e^{i\theta}}\,\phi(\epsilon e^{i\theta})\,i\epsilon e^{i\theta}\,d\theta = \frac{1}{4\pi}\int_0^{2\pi} \phi(\epsilon e^{i\theta})\,d\theta$$

Thus $T_f(\bar{\partial}\varphi) = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_{\Omega_{\epsilon,R}} f\,\bar{\partial}\phi = \frac{1}{4\pi}\cdot 2\pi\,\phi(0) = \frac{1}{2}\phi(0) = \frac{1}{2}\delta(\phi)$

But $T_f(\bar{\partial}\varphi) \underset{def}{=} -\bar{\partial}T_f(\varphi)$ So as distributions $-\bar{\partial}T_f = \frac{1}{2}\delta$.

Fourier Transform both sides, and use 1.3.4 (b) for $P = -\bar{\partial}$ to get $(-\bar{\partial}T_f)^\wedge = \frac{1}{2}(\hat{\delta}) = \frac{1}{2}$ by observation 1, i.e. $\frac{i}{2}\xi\hat{T}_f = \frac{1}{2}$

i.e. $T_{\hat{f}} = -\frac{i}{\xi}$. Thus $\boxed{\left(\frac{1}{z}\right)^\wedge = -\frac{i}{\xi}}$.

Now we can prove our proposition, because for $\varphi \in \mathcal{D}$

$$\frac{1}{2}\phi(z) = \frac{1}{2}(\hat{\varphi})^{\vee}(z) = \left(-\frac{i}{\xi}\cdot\frac{\xi\hat{\varphi}}{-2i}\right)^{\vee}(z)$$

$$= \left(\left(\tfrac{1}{z}\right)^{\wedge}\left(\bar{\partial}\phi\right)^{\wedge}\right)^{\vee}(z) \quad \text{by (3) and (7)}$$

$$= \left(\left(\tfrac{1}{z} * \bar{\partial}\varphi\right)^{\wedge}\right)^{\vee}(z) \quad \text{by 1.2.1 (c) (analogue for tempered dist)}$$

$$= \left(\tfrac{1}{z} * \bar{\partial}\varphi\right)(z) = \int_{\mathbb{C}} \frac{\bar{\partial}\phi(w)}{(z-w)}\, dV_w \quad \text{proving (i) of 1.3.5}$$

Again $\frac{1}{z} = 2\partial(\log|z|^2) = 4\partial(\log|z|)$, so that

$$\frac{1}{2}\varphi(z) = \left(\tfrac{1}{z} * \bar{\partial}\phi\right) = \left(4\partial\log|z| * \bar{\partial}\phi\right) = 4\left(\bar{\partial}\partial\log|z| * \phi\right)$$

$$\text{by (1.3.3) (iv)} = 4\log|z| * \partial\bar{\partial}\varphi$$

$$= 4\int \log|w-z|\,\partial\bar{\partial}\varphi(w)\,dV_w = \int \Delta\varphi(w)\log|w-z|\,dV_w$$

which proves (ii) of 1.3.5. #

§ 1.4. Sobolev Spaces ([Rud], § 8.8, 8.9, [Gil] § 1.1, [Nar] § 3.4, [Hor] 7.9, (Ch VII), [L-M] - Ch III, § 2)

1.4.1 Def: The sobolev space $H_s(\mathbb{R}^n)$ is defined by

$$H_s(\mathbb{R}^n) = \left\{ f \text{ is a tempered distribution} \in \mathcal{S}' : \hat{f} \text{ is a function and } \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty \right\}$$

The quantity $\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$ is defined to be the sobolev s-norm $\|f\|_s$ of f, for $f \in H_s(\mathbb{R}^n)$.

1.4.2 Remarks (i) By the Plancherel Theorem, (Propn 1.2.1 (d),) $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$

(ii) $H_s(\mathbb{R}^n) \cong L^2(\mathbb{R}^n, (1+|\xi|^2)^s d\xi)$. Hence each H_s is a separable Hilbert space with the inner product

$$(f, g)_s = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \hat{f}(\xi)\bar{\hat{g}}(\xi)\, d\xi$$

(iii) Since by the Remark following Propn 1.3.4, if $f \in \mathcal{E}'$ is a compactly supported dist., $\hat{f}$ is a function of slow growth, (which means it's square integrable with respect to some $(1+|\xi|^2)^s d\xi$) it follows that every compactly supported distribution is in $H_s(\mathbb{R}^n)$ for some s.

(iv) Not every tempered distribution is in some H_s. For example the non-zero constant functions are tempered distributions (since they have polynomial growth! See 1.3.2 (i)), but their fourier transforms are multiples of the Dirac distribution, which are not functions).

(iii) + (iv) imply the inclusions (all strict)

$$\mathcal{E}' \subset H_{-\infty} := \bigcup_s H_s \subset \mathcal{S}' \qquad (8)$$

1.4.3 Propn (Basic Properties of the Sobolev Spaces)

(i) $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$; $H_s(\mathbb{R}^n) \hookrightarrow H_t(\mathbb{R}^n)$ is a continuous (bounded) dense inclusion for $s > t$. If $f \in \mathcal{S}$, then $u \to fu$ is a bounded operator $H_s \to H_s$

(ii) If $f \in C^\infty(\mathbb{R}^n) \cap H_m(\mathbb{R}^n)$ (for some positive integer m), then

$$\|f\|_m^2 \sim \sum_{|\alpha| \le m} \int |D^\alpha f|^2 dx \qquad (\sim \text{ denotes equivalent norm})$$

(So for m a positive integer, one can recover $H_m(\mathbb{R}^n)$ as completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm on the right above.)

(iii) If P is a polynomial of degree k, then

$$P(D) : H_s(\mathbb{R}^n) \to H_{s-k}(\mathbb{R}^n)$$

is a bounded operator, where $P(D)$ denotes $P(\underbrace{\tfrac{1}{i}\tfrac{\partial}{\partial x_1}}_{D_{x_1}}, \ldots, \underbrace{\tfrac{1}{i}\tfrac{\partial}{\partial x_n}}_{D_{x_n}})$

(iv) The pairing $\mathcal{S} \times \mathcal{S} \to \mathbb{C}$ defined by

$$f, g \mapsto \int_{\mathbb{R}^n} f\bar{g}\, dx = (f, g)$$

Extends to a pairing $H_s \times H_{-s} \to \mathbb{C}$ which is perfect and identifies H_{-s} with $(H_s)^*$. Furthermore, for $f \in H_s$, $g \in H_{-s}$

$$|(f,g)| \le \|f\|_s \|g\|_{-s} \; ; \quad \|f\|_s = \sup_{\substack{g \in H_{-s} \\ g \neq 0}} \frac{|(f,g)|}{\|g\|_{-s}}$$

The Proofs of (i), (ii) + (iii) are straightforward consequences of the definitions and Prop. 1.2.1. To see (iv) multiply and divide by a factor of $(1+|\xi|^2)^{s/2}$ i.e.

$$(f,g) \underset{\text{Plancherel}}{=} (\hat{f},\hat{g}) = \int \hat{f}(\xi)(1+|\xi|^2)^{s/2}\,\bar{\hat{g}}(\xi)\,(1+|\xi|^2)^{-s/2}d\xi$$

and use Schwartz inequality to get $|(f,g)| \le |f|_s\,|g|_{-s}$. To see that equality is achieved, use g such that $\hat{g} = \hat{f}(1+|\xi|^2)^s \in \mathcal{S}$.
(Compare Lemmas 1.1.3, 1.1.6 in [Gil], e.g.) #

1.4.4 Remark (i) Since $H_s = L^2(\mathbb{R}^n, (1+|\xi|^2)^s d\xi)$, C_c^∞ is dense in H_s and since $C_c^\infty \subset \mathcal{S}$ so is $\mathcal{S}$. Some places in the literature, they define H_s as the completion of $\mathcal{S}$ with respect to the norm $\|f\|_s = \left(\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$. E.g. [Gil] or [L-M]. We have followed the definition in e.g. [Hor].

(ii) The reason for introducing the Sobolev spaces are the parts (ii) + (iii) of 1.4.3 above. One would like to treat differential operators as bounded operators between some natural Hilbert spaces, and these are precisely those which have finite L^2-norm of all derivatives upto order m by (ii), i.e. H_m

1.4.5 Prop: (The Sobolev Embedding Theorem or "Sobolev Lemma")

Let $k \ge 0$ be an integer. If $s > k + n/2$, then

(i) $H_s(\mathbb{R}^n) \subset C_0^k$ (see Def 1.1.1 (ii))

(ii) $\|D^\alpha f\|_\infty \le C_\alpha \|f\|_s$ for all α with $|\alpha| \le k$, $f \in H_s(\mathbb{R}^n)$.

As a consequence there are the inclusions

$$\mathcal{S} \subseteq H_\infty = \bigcap_{s\in\mathbb{R}} H_s \subseteq C_0^\infty(\mathbb{R}^n) \overset{\text{def}}{=} \left\{ \begin{array}{l} C^\infty \text{ fns on } \mathbb{R}^n \text{ vanishing} \\ \text{at } \infty \text{ together with all derivs.} \end{array} \right\}$$

(both of which are strict, e.g.: $f(x) = \frac{1}{1+x^2} \in H_s(\mathbb{R})\ \forall s$, but $\hat{f}(\xi) = e^{-|\xi|}$ which is not smooth, so not in $\mathcal{S}$, thus f is not in $\mathcal{S}$. Again, if f is a smooth function which goes at ∞ like $\frac{1}{x^{1/2}}$, say, then $f \in C_0^\infty(\mathbb{R}^n)$ but $f \notin L^2$ so $f \notin H_0$ so $f \notin H_\infty$.)

Proof:- Since the proof is not too difficult, we quickly run through it.
First there is the

Observation :- If $f \in \mathcal{S}'$ is a tempered distribution such that

$$(D^\alpha f)^\wedge \in L^1(\mathbb{R}^n) \text{ for } |\alpha| \le k, \text{ then } f \in C_0^k$$

(Here $D^\alpha f$ is the distributional derivative of f by Def. 1.3.3 (i).)

First let's see this for $\alpha = 0$. $\hat{f}$ is an L^1-function, and so by the Fourier inversion formula $(\hat{f})^\wedge$ is in $C_0(\mathbb{R}^n)$ by the Riemann-Lebesgue Lemma 1.2.1 (e). But $(\hat{f})^\wedge(x) = f(-x)$ as a distribution and since $^\wedge : \mathcal{S}' \to \mathcal{S}'$ is 1-1 on tempered distributions (Prop 1.3.4 (a)), it follows that $f \in C_0(\mathbb{R}^n)$. The same argument applies to all $(D^\alpha f)$ for $|\alpha| \le k$.

In view of this observation, for proving the Proposition, all we need to do is show that $(D^\alpha f)^\wedge \in L^1$ $\forall$ $|\alpha| \le k$. However, by Prop 1.3.4 (b) we have

$$\int |(D^\alpha f)^\wedge(\xi)|\, d\xi = \int |\xi|^\alpha |\hat{f}(\xi)|\, d\xi = \int |\xi|^\alpha (1+|\xi|^2)^{-s/2} (1+|\xi|^2)^{s/2} |\hat{f}(\xi)|\, d\xi$$

$$\le \left(\int |\xi|^{2\alpha} (1+|\xi|^2)^{-s} d\xi \right)^{1/2} \|f\|_s \quad \text{by Schwartz inequality}$$

Now since $|\alpha| \le k < s - n/2$ the integrand in the 1st integral is $\le C(1+|\xi|^2)^{-\delta}$ with $\delta > n/2$ and hence the 1st integral is finite. Thus

$$\|(D^\alpha f)^\wedge\|_{L^1} \le C \|f\|_s.$$ Taking the inverse Fourier transform

we get $\|D^\alpha f\|_\infty \le \|(D^\alpha f)^\wedge\|_{L^1} \le C_\alpha \|f\|_s$ proving the Proposition

\#

1.4.6 Remark: This result is crucial in proving regularity of distributional solutions of Elliptic equations. By means of "a-priori Estimates" (e.g. Gårding-Friedrichs inequality) one makes the solution lie in H_s for higher and higher s, thus getting smoothness from the Theorem 1.4.5 above.

The other crucial result about Sobolev spaces is the

1.4.7 Rellich's Lemma ([Gil] Lemma 1.1.5, [Nar] Prop 3.5.4, [L-M] Thm. 2.6)

Let $\{f_k\}$ be a sequence in H_s such that

(i) $\exists$ K compact $\subset \mathbb{R}^n$ such that $\mathrm{supp}\, f_k \subset K$ $\forall k$

(ii) $\sup_k \| f_k \|_s < \infty$ (i.e. it is a bounded sequence in H_s)

Then $\exists$ a subsequence $\{f_{k_j}\}$ which converges in H_t $(t < s)$.

Proof :- Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$ and $\phi \equiv 1$ on a nbd. of K. Now there is the elementary (Peetre's) inequality which follows from the Δ-inequality, viz.

$$(1+|\xi|^2)^{s/2} \leq C(1+|\xi-\eta|^2)^{|s|/2}(1+|\eta|^2)^{s/2} \quad \text{— (9)}$$

Thus $(1+|\xi|^2)^{s/2}|\hat{f}_k(\xi)| \leq C\int |\hat{\varphi}(\xi-\eta)|(1+|\xi-\eta|^2)^{|s|/2}|\hat{f}_k(\eta)|(1+|\eta|^2)^{s/2}d\eta$

(by using the fact that $f_k = \phi f_k \Rightarrow \hat{f}_k = \hat{\phi} * \hat{f}_k$ and (9))

$$\Rightarrow (1+|\xi|^2)^{s/2}|\hat{f}_k(\xi)| \leq C\cdot\|\phi\|_{|s|}\,\|f_k\|_s \leq C' \quad \text{— (10)}$$

by Cauchy-Schwartz and hypothesis (ii) above

Likewise for $d_j \hat{f}_k$ since $d_j\hat{f}_k = d_j(\hat{f}_k * \hat{\phi}) = d_j\hat{\varphi} * \hat{f}_k$, i.e.

$$(1+|\xi|^2)^{s/2}|d_j\hat{f}_k(\xi)| \leq C, \text{ for some constant } C \text{ indep. of } \xi, \quad \text{— (11)}$$

Thus $\hat{f}_k$, $d_j\hat{f}_k$ are uniformly bounded on all compact sets. By the Arzela-Ascoli theorem (since the M.V. Thm. $\Rightarrow$ $\{\hat{f}_k\}$ are a bounded equicontinuous family), $\exists$ a subsequence $\{\hat{f}_{k_j}\}$ which converges uniformly on compact sets. For notational convenience, let us denote this subsequence by $\{\hat{f}_k\}$ as well. Now for $t < s$,

$$\|f_j - f_k\|_t^2 \doteq \int_{\mathbb{R}^n} |\hat{f}_j - \hat{f}_k|^2(1+|\xi|^2)^t\, d\xi$$

$$= \int_{|\xi|\geq r}(\ \cdots\ ) + \int_{|\xi|\leq r}(\ \cdots\ )$$

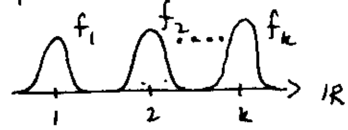
for $r > 0$ to be suitably chosen.

For $|\xi| \geq r$, $(1+|\xi|^2)^t \leq (1+r^2)^{t-s}(1+|\xi|^2)^s$, so that the 1st integral above is majorised (using (10), and hypothesis (ii)) by $(1+r^2)^{t-s}\|f_j - f_k\|_s^2 \leq 2\cdot C\cdot(1+r^2)^{t-s}$

Now given $\epsilon > 0$, choose r so large that $2C((1+r^2)^{t-s} < \epsilon$. The second integral is bounded above by ϵ since $\{\hat{f}_j\}$ converge uniformly on compact sets. Thus $\{f_j\}$ is a Cauchy sequence in H_t and hence convergent.

#

1.4.8 Remark :- Note that the hypothesis (i) cannot be dropped. For example in $H_1(\mathbb{R})$ the sequence



got by translating a fixed bump function will be bounded, but will not contain any convergent subsequence in $H_0(\mathbb{R}) = L^2(\mathbb{R})$.
When we globalise in the next section to compact manifolds, the hypothesis (i) will always be satisfied, and will drop out of the statement of Rellich's lemma.

§1.5. Globalisation to compact manifolds and vector bundles ([Gil] §1.3, Lemma 1.3.4; [Nar], §3.9; [L-M], Thm 2.15)

First, it is clear that by going component by component and using the standard hermitian inner product on $\mathbb{C}^k$, the theorems, and definitions etc. of the preceding §'s generalise from complex valued functions to $(\mathbb{C}^k)$-vector valued functions. So the distributions, tempered distributions, fourier transforms etc. are now $\mathbb{C}^k$-valued, and there are the Sobolev spaces $H_s(\mathbb{R}^n, \mathbb{C}^k)$ obeying the corresponding analogues of Sobolev Embedding, Rellich's Lemma etc., which we will use without further elaboration.

Now and henceforth in these notes let M be a C^∞, compact, Hausdorff, connected, orientable (boundaryless) manifold of dimension n. It therefore possesses a Riemannian metric g, which we fix once and for all. This gives rise to the global volume form attached to g, viz. $dV_g = \sqrt{\det g_{ij}}\, dx_1 \wedge \dots \wedge dx_n$ in local coordinates, where $g = \sum g_{ij} dx_i \otimes dx_j$ in these coordinates. Similarly, if M is a compact complex manifold, with a hermitian metric h, then again there is the associated volume form $dV_h = \left(\frac{i}{2}\right)^n (\det h_{i\bar{j}}) dz_1 \wedge d\bar{z}_1 \dots \wedge dz_n \wedge d\bar{z}_n$ where $h = \sum h_{i\bar{j}}\, dz_i \otimes d\bar{z}_j$ in local coordinates. Thus one can (globally) integrate over M any C^∞-functions on M. (which is all that one really needs to get started.) Also let E be a smooth complex vector bundle on M of rank k.

There is the space $C^\infty(M, E)$ of C^∞-sections of E, which is the same as $C^\infty_c(M, E)$ since M is compact. One way of going about defining Sobolev spaces etc. is to use the covariant differentiation operator ∇^E with respect to a connection on E as the global analogue of differentiation for sections in $C^\infty(M, E)$. This approach maybe found in [Au]. However we'll use partitions of unity to pass from the $\mathbb{R}^n$ case to the global case.

First, let $\{\varphi_\alpha : \overline{U}_\alpha \to \overline{D^n}\}_{\alpha=1}^N$ (where $D^n = \{x \in \mathbb{R}^n, \|x\| < 1\}$) be a system of charts for M such that $E_{|\overline{U}_\alpha} \cong \overline{U}_\alpha \times \mathbb{C}^k$. Such a system clearly exists, M being compact. Letting

$$h : D^n \to \mathbb{R}^n$$
$$x \to \frac{x}{\sqrt{1-|x|^2}}$$

be the standard diffeomorphism of D^n with $\mathbb{R}^n$, there will then be the system of charts $\tilde{\varphi}_\alpha = h \circ \varphi_\alpha : U_\alpha \to \mathbb{R}^n$, $\forall \alpha$. Now if $s \in C^\infty(M, E)$, $s|_{U_\alpha}$ can be thought of as a smooth function in $C^\infty(\mathbb{R}^n, \mathbb{C}^k)$ for each α, (such that $(1+|x|)^{|\alpha|} D^\alpha u(x)$ is bounded $\forall \alpha$.). Let $\{\lambda_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the covering $\{U_\alpha\}_{\alpha=1}^N$. By the above remarks there is an embedding

$$C^\infty(M, E) \hookrightarrow \bigoplus_{\alpha=1}^N C^\infty_c(\mathbb{R}^n, \mathbb{C}^k) \qquad (12)$$
$$s \mapsto \bigoplus_\alpha (\lambda_\alpha s|_{U_\alpha})$$

Since the right hand side has the topology of the space of ($\mathbb{C}^k$ valued) test functions, we get a topology on the left hand side. (The inclusion is in fact automatically continuous if one topologises the left-hand side analogously to 1.1.1 (i) using covariant derivatives instead of usual derivatives). Continuous linear functionals on $C^\infty(M, E)$ define distributional sections. Since M is compact, the notions of distribution, compactly supported distribution and tempered distribution coincide.

Now we can define Sobolev spaces of sections. On $\bigoplus_{\alpha=1}^N H_s(\mathbb{R}^n, \mathbb{C}^k) \cong \bigoplus_{\alpha=1}^N H_s(U_\alpha, E_{|U_\alpha})$ we have the direct-sum Hilbert space structure, and the right hand side of (12) is a dense subspace of this by 1.4.4 Remark (i). Thus on $C^\infty(M,E)$ one has the inner product

$$(f,g)_s = \sum_{\alpha=1}^N (\lambda_\alpha f, \lambda_\alpha g)_s$$

Completing $C^\infty(M,E)$ with respect to this inner product gives $H_s(M,E)$. Alternatively, it is the space of distributional sections f of E whose cut-offs $\lambda_\alpha f$, $\alpha = 1, \dots, N$ belong to $H_s(U_\alpha, E_{|U_\alpha}) \simeq H_s(\mathbb{R}^n, \mathbb{C}^k)$ $\forall\, \alpha$.

One of course needs to show that $H_s(M,E)$ so defined does not depend on the trivialisations, partitions of unity etc. We leave these matters to the reader, (referring to [Gil], Lemmas 1.3.3, 1.3.4 e.g.).

Let us quickly restate the Sobolev Embedding Theorem 1.4.5, Rellich's Lemma 1.4.7 and Propn 1.4.3 (basic properties) for $H_s(M,E)$. These reformulations are immediate from the $\mathbb{R}^n$, $\mathbb{C}^k$ case.

1.5.1 Propn :

(i) $H_0(M,E) = L^2(M,E)$ (where $L^2(M,E)$ is the completion of $C^\infty(M,E)$ with respect to $(f,g) = \int_M \langle f,g\rangle_x \, dV_g$ where $\langle\ \rangle_x$ is the pointwise hermitian inner product with respect to the hermitian metric on E). The inclusion $H_s(M,E) \hookrightarrow H_t(M,E)$ (for $s > t$.) is dense and continuous.

(ii) The pairing

$$C^\infty(M,E) \times C^\infty(M,E) \to \mathbb{C}$$
$$f, \ g \ \longmapsto (f,g) = \int_M \langle f, g\rangle_x \, dV_g$$

Extends to a bilinear ("sesquilinear" = conjugate linear in second variable) pairing $H_s(M,E) \otimes H_{-s}(M,E) \to \mathbb{C}$ and identifies $H_s(M,E)$ with $H_{-s}(M,E^*)$ because it is perfect.

(iii) (Sobolev Embedding): $H_s(M,E) \subset C^k(M,E)$ for $s > k + n/2$

$$\Rightarrow \bigcap_s H_s(M,E) \hookrightarrow C^\infty(M,E)$$
$$\Rightarrow \bigcap_s H_s(M,E) = C^\infty(M,E)$$

(Also distributional sections of E = $\mathcal{D}'(M,E)$ = continuous linear functionals on $C^\infty(M,E)$ are nothing but elements of $H_{-\infty}(M,E) = \bigcup_s H_s(M,E)$)

(iv) (Rellich's Lemma): For $s > t$, the inclusion

$$H_s(M,E) \hookrightarrow H_t(M,E)$$

is a compact operator (i.e. every H_s bounded sequence contains an H_t convergent subsequence)
(Note how the condition (i) in the hypothesis of 1.4.7 is superfluous, since M is compact)

1.5.2 Remarks :-

(i) When M is non-compact, one can use a locally finite partition of unity $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ to define

$$H_s(M,E) = \text{Completion of } C_c^\infty(M,E)$$

with respect to the inner product $(f,g)_s = \sum_{\alpha\in\Lambda} (\lambda_\alpha f, \lambda_\alpha g)_s$, but we will not be needing this.

(ii) $\mathcal{D}'(M, \Lambda^p(T^*(M))) = H_{-\infty}(M, \Lambda^p(T^*(M)))$ (M-compact) is called the space of p-currents. [These are defined, for example in [de R] without use of metrics or partitions of unity as the space of continuous linear functionals on $\Lambda^{n-p}(T^*(M))$, so that a p-form ω defines a p-current T_ω by $T_\omega(\tau) = \int_M \omega\wedge\tau$].

Ch 2 :- Pseudodifferential Operators (ΨDO's)

§ 2.0. Introduction :- When one wants to solve a differential equation on a manifold, one basically wants to "invert" a differential operator. This operator inverse is usually not a differential operator. For example if one solves $\bar\partial f = 0$ in a domain, say like the unit disc $\Delta\subset\mathbb{C}$ with some given boundary values on $\partial\Delta = S^1\subset\mathbb{C}$, the solution (if it exists, which is the holomorphic extension of f to Δ, is given by an integral operator which comes from the Cauchy integral formula). Thus one has to enlarge the class of differential operators to include more general operators. The key to this generalisation is of course the observation that for $P = \sum_{|\alpha|\le d} a_\alpha(x) D^\alpha$ a differential operator of order d, we have by Propn 1.2.1 (a) and (b) that

$$Pf(x) = \sum a_\alpha(x) D^\alpha f(x) = \sum a_\alpha(x) (\widehat{D^\alpha f})^\vee(x) = \int e^{ix\cdot\xi}\, p(x,\xi)\hat f(\xi)\,d\xi$$

where $p(x,\xi) := \sum_{|\alpha|\le d} a_\alpha(x)\xi^\alpha$ (called the "symbol" $\sigma(P)$ of P)

is a polynomial of order d. One has to generalise from polynomials to general functions. The idea is similar in spirit to the "functional calculus" for bounded self-adjoint operators on Hilbert space.

§ 2.1 : Pseudodifferential operators on $\mathbb{R}^n$: A matrix valued function

2.1.1 Def:

$$\mathbb{R}^n\times\mathbb{R}^n \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^m)$$
$$(x,\xi) \mapsto p(x,\xi)$$

is called a symbol of order d if

(i) p is smooth

(ii) For each multiindex α, β, $\exists$ a constant $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|} \qquad (13)$$

(The norm on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^m)$ is the Hilbert-Schmidt norm $\sqrt{tr(AA^*)}$)

The $\mathbb{C}$-vector space of symbols of order d is denoted by S^d. Clearly $S^d \subset S^{d'}$ for $d<d'$. Further denote $S^\infty = \bigcup_d S^d$ and $S^{-\infty} = \bigcap_d S^d$. (Note that this notation is opposite to that for H_∞, $H_{-\infty}$, the reason being that $H_{d'} \subset H_d$ for $d<d'$)

2.1.2 Def: Given a symbol $p \in S^d$ of order d, one defines the pseudodifferential operator (corresponding to p) of order d by the formula

$$Pf = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)\,d\xi$$

which is a linear operator $C^\infty(\mathbb{R}^n, \mathbb{C}^k) \to C^\infty(\mathbb{R}^n, \mathbb{C}^m)$. The space of pseudo-differential operators (ΨDO's for short) of order d is denoted by Ψ^d. $p = \sigma(P)$ is called its symbol

2.1.3 Propn: For $P \in \Psi^d$, P defines a (continuous) linear operator $\mathcal{S} \to \mathcal{S}$ and hence (tempered distributions) $\mathcal{S}' \to \mathcal{S}'$. If the symbol p of P has compact x-support, then P admits a continuous extension $H_{s+d}(\mathbb{R}^n, \mathbb{C}^k) \to H_s(\mathbb{R}^n, \mathbb{C}^m)$.

Proof:- By Δ_x we mean the laplacian $\sum_i \frac{\partial^2}{\partial x_i^2}$ whose symbol is $-|\xi|^2$. Clearly

$$|x|^{2N} Pf(x) = (-1)^N \int (\Delta_\xi^N e^{ix\cdot\xi})\, p(x,\xi)\hat{f}(\xi)\,d\xi$$

$$\left(\begin{matrix}\text{integrating by}\\ \text{parts}\end{matrix}\right) = (-1)^N \int e^{ix\cdot\xi}\, \Delta_\xi^N \left(p(x,\xi)\hat{f}(\xi)\right) d\xi$$

So

$$(1+|x|^2)^N |Pf(x)| \le \int |\Delta_\xi^N p(x,\xi)|\,|\hat{f}(\xi)|\,d\xi \le C\cdot \int (1+|\xi|^2)^{-d} d\xi \le C'$$

(by Def 2.1.1 (i) (13) above and $\hat{f} \in \mathcal{S}$, 1.2.1 (a)) if $d > n/2$. The same reasoning applies for $(1+|x|^2)^N |D^\alpha Pf(x)|$, using (13) above. This proves that P maps $\mathcal{S} \to \mathcal{S}$, and that it is continuous.

To show the second part, let the x-support of $p(x,\xi)$ be the compact set K. ($\Rightarrow p(x,\xi) \equiv 0$ for $x \notin K$). For $f \in \mathcal{S}$,

$$\widehat{Pf}(\zeta) = \int e^{-ix\cdot\zeta}\, e^{ix\cdot\xi}\, p(x,\xi)\hat{f}(\xi)\,d\xi\,dx$$

$$= \int q(\zeta-\xi, \xi)\hat{f}(\xi)d\xi \qquad — \qquad (14)$$

where $$q(\zeta, \xi) = \int e^{-ix\cdot\zeta} p(x,\xi)dx \qquad — \qquad (15)$$

is the partial fourier transform of the symbol p in the x-direction, which makes sense since x-supp p compact. (The interchange of the order of x & ξ integration is also justified since $p(x,\xi)\hat{f}(\xi)$ is compactly x-supported and Schwartz class in the ξ-direction). In fact since p is compactly x-supported, by 1.2.1 (a), its partial x-fourier transform q is rapidly decreasing (Schwartz class) in the ζ-variable, and has the same growth properties as p in the ξ variable. Thus

$$|q(\zeta,\xi)| \le C_k (1+|\xi|^2)^{d/2}(1+|\zeta|^2)^{-k/2} \qquad — \qquad (16)$$

where k will be conveniently chosen large enough later.

By Plancherel 1.2.1 (e) and Schwartz inequality,

$$|(Pf, g)| = |(\hat{Pf}, \hat{g})| = \left|\int q(\zeta-\xi,\xi)\langle \hat{f}(\xi), \hat{g}(\zeta)\rangle d\xi\, d\zeta\right|$$

$$\le \left(\int |K(\zeta,\xi)|(1+|\xi|^2)^s|\hat{f}(\xi)|^2 d\xi\, d\zeta\right)^{1/2}\left(\int |K(\zeta,\xi)|(1+|\zeta|^2)^{d-s} d\xi\, d\zeta\right)^{1/2} \qquad — \qquad (17)$$

where $$K(\zeta,\xi) = q(\zeta-\xi,\xi)(1+|\xi|^2)^{-s/2}(1+|\zeta|^2)^{\frac{s-d}{2}} \qquad — \qquad (18)$$

By (16) and Peetre's inequality (9) viz.

$$(1+|\xi|^2)^{d-s} \le C\cdot(1+|\zeta-\xi|^2)^{\frac{|d-s|}{2}}(1+|\zeta|^2)^{\frac{d-s}{2}}$$

$$|K(\zeta,\xi)| \le C\cdot(1+|\zeta-\xi|^2)^{\frac{|d-s|-k}{2}} \qquad — \qquad (19)$$

which implies

$\int |K(\zeta,\xi)|d\zeta$ and $\int |K(\zeta,\xi)|d\xi$ are both bounded provided we choose

$|d-s|-k > n$.

Then (17) becomes $\quad (Pf, g) \le C\cdot\|f\|_s\|g\|_{d-s}$.

Since by Prop. 1.4.3 (iv), $\displaystyle \|Pf\|_{s-d} = \sup_{\substack{g\in H_{d-s}\\ g\neq 0}} \frac{|(Pf,g)|}{\|g\|_{d-s}}$, we are through #

2.1.4: Remark: If one drops the condition that $p(x,\xi)$ has compact x-support, then $P: H_{s}(\mathbb{R}^n, \mathbb{C}^k) \to H_{s,loc}(\mathbb{R}^n,\mathbb{C}^m)$; where $H_{s,loc}(\mathbb{R}^n,\mathbb{C}^m)$ is defined

as the space of tempered distributions $f \in \mathcal{S}'$ such that $\psi f \in H_s$ for all $\psi \in C_c^\infty(\mathbb{R}^n)$. This follows immediately from the above Propn. 2.1.3 above, because for such a ψ, and P a ΨDO of order d with symbol $p \in S^d$, one notes that the symbol of $P' = \psi P$ is $p' = \psi p$ from the formula defining ΨDO's in Def. 2.1.2. But since p' has compact x-support, $(\subset \operatorname{Supp} \psi)$, by Propn. 2.1.3 above $P'f = (\psi P) f \in H_s$ for $f \in H_{s+d}$. i.e. $\psi(Pf) \in H_s$ for $f \in H_{s+d}$ $\forall\ \psi \in C_c^\infty(\mathbb{R}^n)$
$\Rightarrow$ $Pf \in H_{s,\,loc}$ for $f \in H_{s+d}$.

2.1.5. Cor :- If $P \in \bigcap_d \Psi^d = \Psi^{-\infty}$, then Pf is smooth for any $f \in H_s$, any s. This is because for $f \in H_s$, $Pf \in H_{s-d,\,loc}$ for all d by Remark 2.1.4 above. i.e. $Pf \in \bigcap_t H_{t,\,loc}$. However $\bigcap_t H_{t,\,loc} \subset C^\infty$ by the Sobolev-embedding theorem Prop. 1.4.5, because to prove f is C^∞, it is enough to prove ψf is C^∞ for ψ a cutoff (compactly-supported) function in the neighborhood of a point (and then let the point vary over $\mathbb{R}^n$). But then, by definition, $\psi f \in \bigcap_t H_t \subset C^\infty$.

2.1.6. Remark :- A pseudodifferential operator P can be regarded as an integral operator with a distributional kernel. This is because, by Def. 2.1.2,

$$Pf = \int e^{ix\cdot\xi} p(x,\xi) \hat{f}(\xi)\, d\xi = \int e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi\, f(y)\, dy = \int K(x,y) f(y)\, dy$$

where $K(x,y)$ is the distribution $\int e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi$ by which one means the distribution $p^\vee(x, x-y)$ where $p^\vee$ is the inverse fourier transform of p in the ξ variable. If $p \in S^{-\infty}$, then p is rapidly decreasing (Schwartz class) in the ξ variable by (13), Hence its inverse fourier transform $p^\vee(x,z)$ is smooth, and so is $K(x,y) = p^\vee(x, x-y)$. Thus an infinitely smoothing ΨDO (i.e. a ΨDO in $\Psi^{-\infty}$) is given by an integral operator with a smooth kernel. The converse is false: For example $f(y) = \frac{1}{1+y^2} \in L^2(\mathbb{R}) = H_0(\mathbb{R})$. Taking the smooth kernel $K(x,y) = e^{-ixy}$ shows that $\int K(x,y) f(y)\, dy = \hat{f}(x) = (\text{Const})\, e^{-|x|}$, (as noted in the parenthesis after Propn 1.4.5 (ii)), which is not even C^1, let alone smooth! However an important class of ∞-ly smoothing operators come from

2.1.7 : Convolution Kernels : First let T be a distribution such that $\hat{T}$ is a function of slow growth. (e.g.:- T a compactly supported distribution) Then clearly $\phi \to \hat{T}.\phi$ defines a continuous map $\mathcal{S} \to \mathcal{S}$, so that The map $\phi \to T * \phi$ is also a continuous map $\mathcal{S} \to \mathcal{S}$ (by 1.3.3 (iv)) and 1.3.4 (d), and of course 1.2.1 (a). Then we get a map of tempered distributions by defining $S \to T * S$ (from $\mathcal{S}'$ to $\mathcal{S}'$) by $(T*S)(\varphi) = S(T*\varphi)$

The natural questions which arise about the map $\mathcal{S} \to \mathcal{S}$, given by $\phi \to T * \varphi$ are
(i) When is this a ΨDO and of what order, and what is its symbol?
(ii) When is this ∞-ly smoothing, and in general does it map $H_s \to H_{s'}$?

To simplify matters assume that g is itself a function. Then $\phi \mapsto g * \varphi$ is given by the integral operator $\phi \mapsto \int g(x-y)\varphi(y)\,dy$ with integral kernel $K(x,y) = g(x-y)$. Now, assuming that $\hat{g}$ is of slow growth we have

$$(g * \phi) = \left((g*\varphi)^\wedge\right)^\vee = (\hat{g}\hat{\varphi})^\vee = \int e^{ix\cdot\xi}\, \hat{g}(\xi)\hat{\varphi}(\xi)\,d\xi$$

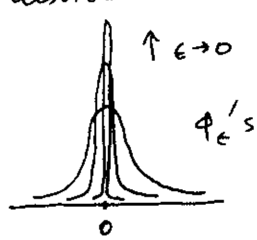
which corresponds to the symbol $p(x,\xi) = \hat{g}(\xi)$

Thus the answer to (i) is $\varphi \to g * \varphi$ is a ΨDO of order d iff

$$|D_\xi^\alpha \hat{g}(\xi)| = |(x^\alpha g)^\wedge| \leq (1+|\xi|)^{d-|\alpha|} \quad \forall \alpha \qquad (20)$$

with symbol $\hat{g}(\xi)$. In particular when $g \in \mathcal{S}$, $\hat{g} \in \mathcal{S}$ as well by 1.2.1 (a) so that $\hat{g} \in S^{-\infty}$, which implies $g \to g * \varphi$ is infinitely smoothing, with $D^\alpha(g*\varphi) = D^\alpha g * \varphi$. Finally, whenever $\hat{g}(\xi)$ obeys $|\hat{g}(\xi)| \leq C(1+|\xi|)^d \leq C'(1+|\xi|^2)^{d/2}$, it follows that if $f \in H_s \Rightarrow \hat{f} \in L^2((1+|\xi|^2)^s)$ then $\hat{g}\hat{f} \in L^2((1+|\xi|^2)^{s-d})$ $\Rightarrow g * f \in H_{s-d}$. This answers (ii). This means when $g \in \mathcal{S}$, $g*(\ )$ maps H_s to $H_{-\infty}$.

2.1.8 : Mollifiers (or Approximate identities). The Schwartz Class $\mathcal{S}$ is clearly closed under the ring operation $\phi, \psi \mapsto \varphi * \psi$. (which is easily checked to be commutative and associative, since it corresponds to multiplication under the Fourier Transform isomorphism $\mathcal{S} \to \mathcal{S}$ by 1.2.1 (a), (c). Since the identity element for the multiplication operation is the constant function 1 (which is not in $\mathcal{S}$, but only in $C^\infty(\mathbb{R}^n)$), the identity element for the convolution product would be the Fourier transform of the constant function 1, which is the Dirac δ-distribution of 1.3.2 (ii), which is as far away from being a function as one can get! However, there are Schwartz class (or even C_c^∞) functions ϕ_ϵ which satisfy $\lim_{\epsilon\to 0}(\phi_\epsilon * f) \to f$ (in suitable norms); which are therefore called approximate identities. Thus the Dirac distribution may be thought of as a "limit" of such ϕ_ϵ's as $\epsilon \to 0$ (see fig.) →



These φ_ϵ's, though only approximate identities (a seeming disadvantage) have the advantage that by 2.1.7 above, they make a pretty irregular function f into a smooth function. More precisely:

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$, $(\operatorname{Supp}\phi) \subset \{|x|<1\}$ = open unit ball around the origin in $\mathbb{R}^n$, and with $\int \phi(y)\,dy = 1$. Now define the regularisers or mollifiers $\varphi_\epsilon := \epsilon^{-n}\phi(x/\epsilon)$, so that $\int \varphi_\epsilon(y)\,dy = 1$ as well, and $\operatorname{Supp}\varphi_\epsilon \subset B(0,\epsilon)$.

Now suppose $f \in C^k(\mathbb{R}^n)$, (i.e. $D^\alpha f$ is continuous on $\mathbb{R}^n$ for $|\alpha| \leq k$).

Claim: $\varphi_\epsilon * f \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(\varphi_\epsilon * f) \to D^\alpha f$ for all α with $|\alpha| \leq k$ uniformly on all compact sets as $\epsilon \to 0$.

Proof:- $\varphi_\epsilon * f \in C^\infty$ by 2.1.7 above, with $D^\beta(\varphi_\epsilon * f) = D^\beta \varphi_\epsilon * f \;\; \forall \beta$. Further

$$|D^\alpha(\varphi_\epsilon * f)(x) - D^\alpha f(x)| = |\varphi_\epsilon * D^\alpha f(x) - D^\alpha f(x)| \quad \text{for } |\alpha| \leq k.$$

Now let L be a compact subset of $\mathbb{R}^n$. Then for $|\alpha| \leq k$, $D^\alpha f$ is uniformly continuous on L_ϵ. This means, given any $\delta > 0$, $\exists\ \epsilon > 0$ so that

$$\sup_{x \in L} |D^\alpha f(x-y) - D^\alpha f(x)| < \delta \quad \text{for } |y| < \epsilon$$

(L_ϵ is the closed ϵ-tubular neighborhood of $L = \{y ; d(y, L) \leq \epsilon\}$)

So for $x \in L$, since $\operatorname{supp}\varphi_\epsilon \subset B(0,\epsilon)$

$$|D^\alpha(\varphi_\epsilon * f)(x) - D^\alpha f(x)| \leq \int_{|y|<\epsilon} \varphi_\epsilon(y)\, |D^\alpha f(x-y) - D^\alpha f(x)|\, dy$$

$$< \delta \int_{|y|<\epsilon} \varphi_\epsilon(y)\,dy = \delta$$

$\Rightarrow \sup_{x \in L} |D^\alpha(\varphi_\epsilon * f)(x) - D^\alpha f(x)| < \delta$ for ϵ small enough, so that $D^\alpha \varphi_\epsilon * f \to D^\alpha f$ uniformly on compacts as $\epsilon \to 0$.

More generally, if $T \in \mathcal{D}'$ is a distribution, then $\varphi_\epsilon * T$ is a C^∞ function which converges to T in $\mathcal{D}'$. We won't elaborate on this point, since we haven't defined a topology on $\mathcal{D}'$, but consult [Sch], Ch VI, §4 for example, for a proof.

2.1.9. Poincare Duality; Currents : Let M be a smooth orientable manifold, not necessarily compact. Following the Remark 1.5.2 (see [de R], §), one can define the space of distributional p-forms or p-currents $(\mathcal{D}')^p(M)$ as the space of continuous linear functionals on $\mathcal{D}^{n-p}(M) = \Lambda_c^{n-p}(M)$. Clearly there is an inclusion $i : \Lambda^p(M) \hookrightarrow (\mathcal{D}')^p(M)$ which is a chain-map of complexes

$$\omega \mapsto T_\omega = \int_M \omega \wedge (\ldots)$$

Similarly there is the space of compactly supported p-currents $(\mathcal{E}')^p(M)$ as the space of continuous linear functionals on $\mathcal{E}^{n-p}(M) = \Lambda^{n-p}(M)$. Again there is an inclusion $i_c : \Lambda_c^p(M) \hookrightarrow (\mathcal{E}')^p(M)$. It turns out (by a patching argument with partitions of unity, that the operators $f \to \varphi_\epsilon * f$ of 2.1.8 above globalise to regularisation operators $R_\epsilon : (\mathcal{D}')^p(M) \to \Lambda^p(M)$ and $\overline{R}_\epsilon : (\mathcal{E}')^p(M) \to \Lambda_c^p(M)$ which are respectively chain-homotopy-inverses of i and i_c respectively defined above. This ends up proving that the cochain complexes of p-currents (resp. compactly supported p-currents) $(\mathcal{D}')^*(M)$ (resp. $(\mathcal{D}')^*_c(M)$) respectively have cohomology $H^*_{dR}(M) = H^*(M)$ (resp. $H^*_c(M)$), i.e. the usual de-Rham (resp. compactly supported de Rham) cohomology.

Note that if we define $\Delta^{BM}_{n-p}(M)$ (resp. $\Delta_{n-p}(M)$) to be the simplicial Borel-Moore chain-complex (resp. simplicial chain complex), one has the inclusions

$$\Delta^{BM}_{n-p}(M) \hookrightarrow (\mathcal{D}')^p(M) \quad (\text{resp. } \Delta_{n-p}(M) \hookrightarrow (\mathcal{E}')^p(M))$$

same map

$$\sigma \mapsto \int_\sigma (\ )$$

which are chain maps by Stoke's theorem. As a consequence one has the natural maps of homology to cohomology $H^{BM}_{n-p}(M) \to H^p(M)$, $H_{n-p}(M) \to H^p_c(M)$ which in fact yield the standard Poincare Duality Isomorphisms. For more on all this, see [G-H], or [de R]

§ 2.2 : Main Properties of Pseudodifferential Operators

If $P = \sum_{|\alpha| \le d} a_\alpha(x) D^\alpha$ is a differential operator of order d, and u is a C^∞ function, then clearly $\operatorname{supp}(Pu) \subset \operatorname{supp} u$. (In fact by a theorem of Peetre, this property characterises differential operators - see Theorem 3.3.8 in [Nar]) This is called the local property for D.O.'s

With pseudodifferential operators this is no longer true because Pf is given by an integral $= \int e^{ix\cdot\xi} p(x,\xi) \hat{f}(\xi) d\xi$ whose x-support has nothing to do with the x-support of f. However ΨDO's have the <u>pseudolocal</u> property which means that $\text{sing supp}(Pu) \subset \text{sing supp}(u)$. So wherever u is smooth, Pu is also smooth. Note that this property is stable under the perturbation $P \to P+S$ where S is an infinitely smoothing ΨDO ($\equiv$ a ΨDO of order $-\infty$). For this, and many other reasons, one introduces an equivalence relation among symbols (and hence ΨDO's).

<u>2.2.1 Def:</u> We say $p, q \in S^d$ are <u>equivalent</u> if $p - q \in S^{-\infty}$, and denoted by $p \sim q$. Correspondingly, if $P, Q \in \Psi^d$ are ΨDO's of order d, then say P and Q are equivalent ($P \sim Q$) if their symbols are equivalent. Finally, if $p_j \in S^{d_j}$, where d_j diminish monotonically with j to $-\infty$, we say $p \sim \sum_j p_j$ ($\sum_j p_j$ is an <u>asymptotic series for p</u>) if given any $k << 0$, $\exists$ an integer j_k such that $p - \sum_{j \le j_k} p_j \in S^k$. i.e. a suitable partial sum of the series differs from p by a symbol of order as negative as we please.

<u>2.2.2 Propn:</u> Let $p_j \in S^{d_j}$ with $d_j \searrow -\infty$ with j. Then there exists a ΨDO p such that $p \sim \sum_j p_j$. This p is uniquely determined upto equivalence.

For the proof, which involves a clever use of cut-off functions with receding supports, we refer to [Gil] Lemma 1.2.8 or [L-M], Prop 3.4. This Lemma is <u>crucial</u> in the construction of parametrices for elliptic operators later. #

The other technical lemma one needs to get into business stems from the following: Let P be a ΨDO given by the symbol p. Substituting $\hat{f}(\xi) = \int e^{-i\xi\cdot y} f(y)\,dy$ in the defining formula for Pf in Def: 2.1.2, we get

$$Pf = \int e^{i(x-y)\cdot\xi}\, p(x,\xi)\, f(y)\, d\xi\, dy$$

which can be viewed as a special case of

$$Kf = \int e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, f(y)\, d\xi\, dy \qquad \text{—} \qquad (21)$$

when $a(x,y,\xi)$ happens to be $a(x,x,\xi) = p(x,\xi)$ for all y. The question is, do we <u>enlarge</u> the class of ΨDO's by defining operators K by (21) above instead of the formula defining ΨDO's in Def. 2.1.2 ?

This is answered by the following definition and lemma.

2.2.3 Def: Let us define a bi-symbol of order d to be a smooth matrix valued function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, denoted $a(x,y,\xi)$, which satisfies

(i) The x-support of a is compact; (the y-support of a is compact) OR

(ii) $$\left| D_y^\alpha D_x^\beta D_\xi^\gamma a(x,y,\xi) \right| \le C_{\alpha,\beta,\gamma} (1+|\xi|)^{d-|\gamma|} \quad (22)$$

By this definition, a symbol $p(x,\xi)$ of order d with compact x-support is a bisymbol, by defining $a(x,y,\xi) = a(x,x,\xi) = p(x,\xi)\ \forall\ y$; of the same order d.

2.2.4 Propn: Let a be a bisymbol of order d, and define the operator K (say on $C_c^\infty(\mathbb{R}^n)$) by the formula (21) above, namely

$$Kf = \int e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)\, d\xi\, dy$$

Then K is a ΨDO of order d whose symbol k has the asymptotic expansion

$$k(x,\xi) \sim \sum_\alpha \frac{(d_\xi^\alpha D_y^\alpha a)}{\alpha!}(x,x,\xi) \quad (23)$$

(Note: In the particular case discussed above when the bisymbol a is defined by $a(x,y,\xi) = a(x,x,\xi) = p(x,\xi)$ where p is a symbol, $D_y^\alpha a = 0$ $\forall\ |\alpha| > 0$ and the series just has one term which is $a(x,x,\xi)$ $= p(x,\xi)$ and this is as it should be!)

The Proof is one big technical mess, whose gory details can be seen in [Gil] Lemma 1.2.2 or [L-M], Theorem 3.5. #

An immediate Consequence is

2.2.5 Propn: If K and a are as in 2.2.4 above, and $a(x,y,\xi)$ vanishes in a neighborhood of the diagonal, then K is infinitely smoothing.

Proof:- $D_y^\alpha a$ is then $\equiv 0$ in said neighborhood of the diagonal, and the right hand side of (23) becomes 0. So the asymptotic series on the right hand side of (23) becomes 0. So k = Symbol of K is ~ 0, i.e. infinitely smoothing. #

2.2.6 Propn : Let $\psi = (\psi_1, \psi_2)$ be a pair of real valued functions in $C_c^\infty(\mathbb{R}^n)$. For $P \in \Psi^d$, define P^ψ by

$$P^\psi(f) = \psi_1 P(\psi_2 f) \qquad (24)$$

for $f \in \mathcal{S}$ (which makes sense). Then P^ψ is also a ΨDO of order d.

Proof:- Clearly by definition

$$P^\psi(f) = \int e^{i(x-y)\cdot\xi} \psi_1(x) p(x,\xi) \psi_2(y) f(y)\, dy\, d\xi$$

which is of the type K of Def. 2.2.3, with the bisymbol

$$a(x,y,\xi) = \psi_1(x) p(x,\xi) \psi_2(y)$$

which is of compact x & y support and of order = order p = d. By Propn. 2.2.4 it is a ΨDO of order d. #

2.2.7 Propn: (The pseudolocal property of ΨDO's). Let $P \in \Psi^d$ be a ΨDO and $f \in H_s$ for some s. Then for an open set $U \subset \mathbb{R}^n$

$$f_{|U} \text{ is } C^\infty \Rightarrow Pf|U \text{ is } C^\infty.$$

Proof:- First note that if $x \in U$, and ψ_1 is a real-valued smooth compactly supported function with $\psi_1 \equiv 1$ in a neighborhood of x, and ψ_2 is a similar function which is $\equiv 1$ on the support of ψ_1, with $\mathrm{Supp}\,\psi_2 \subset U$, then $\psi_2 f \in C_c^\infty(\mathbb{R}^n) \Rightarrow \psi_2 f \in \mathcal{S}$. By Propn 2.1.3, $P(\psi_2 f) \in \mathcal{S}$ as well, so $\psi_1 P \psi_2 f$ is C^∞ with compact support $\subset U$. On the other hand, by Propns 2.2.6, 2.2.5, $\psi_1 P(1-\psi_2)$ is a ΨDO of order d with bisymbol $\psi_1 p(1-\psi_2)$ which $\equiv 0$ for $x \notin \mathrm{Supp}\,\psi_1$ and for $x \in \mathrm{Supp}\,\psi_1$, since $\psi_2 \equiv 1$ on $\mathrm{Supp}\,\psi_1$, this bisymbol vanishes on a neighborhood of the diagonal. Thus $\psi_1 P(1-\psi_2)$ is infinitely smoothing by 2.2.5. Now $\psi_1 Pf = \psi_1 P\psi_2 f + \psi_1 P(1-\psi_2) f$ is smooth in a neighborhood of x. Since $\psi_1 \equiv 1$ in said neighborhood of x, $\psi_1 Pf = Pf$ is smooth on that neighborhood of x. QED. #.

Now we are ready to make ΨDO's into an algebra with adjoints.

The Algebra of ΨDO's

When $\mathcal{H}$ is a Hilbert space, there is the (non-commutative) algebra of bounded operators on $\mathcal{H}$, denoted by $\mathcal{L}(\mathcal{H})$. There is composition $T_1 T_2$ for $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, and the adjoint T^*. Then there is inside $\mathcal{L}(\mathcal{H})$ the closed two-sided ideal of compact operators, and one denotes this ideal by $K(\mathcal{H})$. Finally there is the Calkin algebra $\mathcal{L}(\mathcal{H})/K(\mathcal{H})$, and the Fredholm operators are nothing but (representatives of) invertible elements in $\mathcal{L}(\mathcal{H})/K(\mathcal{H})$ i.e. operators which are invertible mod compacts.

The idea is to mimic all this for ΨDO's. By the Rellich lemma, infinitely smoothing operators are (more or less) compact operators, and the analogue of Fredholm operators will be the forthcoming elliptic operators. The equivalence relation on ΨDO's of Def 2.2.1 corresponds to equality in the Calkin algebras. This is the dictionary by which the Fredholm Theory of operators on Hilbert space (which was invented for this very purpose) will be brought to bear, and issues of regularity, eigenvalues, spectral properties etc. will flow as an outcome.

Let $P \in \Psi^d$, $Q \in \Psi^e$ be pseudodifferential operators. We have seen in Propn. 2.1.3 that $P : \mathcal{S} \to \mathcal{S}$ and $Q : \mathcal{S} \to \mathcal{S}$ are continuous operators, so that $P \circ Q : \mathcal{S} \to \mathcal{S}$ is also continuous. Thus $P \circ Q$ maps tempered distributions ($\in \mathcal{S}'$) to tempered distributions. Natural question: Is $P \circ Q$ a ΨDO, and if so what is its symbol? If we assume for simplicity that the symbols p, q of P, Q resp. have compact x-support, then by 2.1.3 again $Q : H_s \to H_{s-e}$ continuously, and $P : H_{s-e} \to H_{s-e-d}$ continuously, so that seems to suggest that $P \circ Q : H_s \to H_{s-e-d}$ continuously, and so should have order $d+e$ if it is indeed a ΨDO.

Again if $P \in \Psi^d$, by Propn 2.1.3, $P : \mathcal{S} \to \mathcal{S}$ is continuous. Define for $f \in \mathcal{S}$ a tempered distribution by the L^2 adjoint formula ††

$$(P^* f)(\bar{g}) := (f, Pg) \qquad \text{— (25)}$$

where $(\ )$ denotes the L^2-inner product. Another natural question: Is P^* thus defined a ΨDO? Assuming for a minute that $P^* f$ is a function, one uses 1.4.3. (iv) to note that

$$|(P^* f)(\bar{g})| = \left| \int P^* f \, \bar{g} \, dx \right| = |(P^* f, g)| = |(f, Pg)| \leq \|f\|_s \|Pg\|_{-s}$$

$$\leq C \|f\|_s \|g\|_{d-s} \quad \text{by Propn 2.1.3}$$

So that by 1.4.3 (iv)

†† : For us, "adjoints" will always be with respect to the L^2-inner product i.e. $(\ )_0$, and never with respect to the H_s inner product $(\ )_s$

$$\|P^*f\|_{s-d} = \sup_{\substack{g \in H_{d-s} \\ g \neq 0}} \frac{|(P^*f, g)|}{\|g\|_{d-s}} \leq C\|f\|_s$$

which suggests that P^* is of order d, if it is a ΨDO, in view of Propn 2.1.3. Before one answers these two natural questions, there is a definition:

2.2.8 Def: Say $P \in \Psi^d$ is supported in a compact set K if
(i) $\mathrm{Supp}(Pf) \subset K \quad \forall \quad f \in C_c^\infty(\mathbb{R}^n)$
(ii) $Pf \equiv 0 \quad \forall \quad f \in C_c^\infty(\mathbb{R}^n)$ with $(\mathrm{Supp}\, f) \cap K = \phi$

2.2.9 Remark :- If $p(x, \xi)$ is the symbol of P, and P is supported in the compact set K, it follows that x-supp $p \subset K$. For, let ϕ be a positive real valued $\in C_c^\infty(\mathbb{R}^n)$, with $\varphi \equiv 1$ on say $B(0,1)$. Then let $\phi_\epsilon(x) = \phi(\epsilon x)$. Note $\phi_\epsilon \to 1$ uniformly on compact sets as $\epsilon \to 0$. Thus $\hat{\varphi}_\epsilon \to \hat{1} = \delta$ as tempered distributions which implies that for any function g of ξ, $\hat{\varphi}_\epsilon(g) \to \delta(g)$, and in fact $\tau_\xi \hat{\varphi}_\epsilon(g) \to (\tau_\xi \delta)(g)$ where τ_ξ is translation by ξ. However if one takes $g(\eta) = e^{ix\cdot\eta} p(x, \eta)$ Then $\tau_\xi \hat{\varphi}_\epsilon(g) = P(\tau_\xi \hat{\varphi}_\epsilon)$ and $(\tau_\xi \delta)(g) = e^{ix\cdot\xi} p(x, \xi)$. But $\tau_\xi \hat{\varphi}_\epsilon$ are compactly supported, so x-Supp $P(\tau_\xi \hat{\varphi}_\epsilon) \subset K$ by (i) of Def 2.2.8.

$$\Rightarrow \lim_{\epsilon \to 0} x\text{-Supp}\, P(\tau_\xi \hat{\varphi}_\epsilon) \subset K \Rightarrow x\text{-Supp}\,(e^{ix\cdot\xi} p(x,\xi)) = x\text{-supp}\,(p(x,\xi)) \subset K.$$

The converse is false in general, but true for differential operators, as is easily checked.

2.2.10 Propn: Let $P \in \Psi^d$, $Q \in \Psi^e$ be ΨDO's supported in a compact set K. Then, if we denote their symbols by p, q respectively,
(i) P^* is a ΨDO of order d supported in K
(ii) PQ is a ΨDO of order $d+e$ supported in K, and their symbols are respectively given by the asymptotic series expansions (see Def 2.2.1)

$$\sigma(P^*) \sim \sum_\alpha \frac{d_\xi^\alpha D_x^\alpha p^*}{\alpha!} \quad ; \quad \sigma(PQ) \sim \sum_\alpha \frac{d_\xi^\alpha p\, D_x^\alpha q}{\alpha!} \quad \text{— (26)}$$

where p^* is $(\bar{p})^t$, the matrix adjoint of p.

Proof: The idea is to use Propn 2.2.4 about operators defined by bisymbols. (Here, by $A \cdot B$ for row vectors A, B we mean the matrix product AB^* so that $A \cdot \alpha B = \bar{\alpha} A \cdot B$ for $\alpha \in \mathbb{C}$ and $A \cdot TB = T^* A \cdot B$ for a matrix T). Compute

$$(f, Pg) = \int f(y) \cdot Pg(y)\, dy = \int f(y) \cdot e^{iy \cdot \xi} p(y, \xi)\, \hat{g}(\xi)\, d\xi\, dy$$

$$\int f(y) \cdot e^{iy \cdot \xi} e^{-ix \cdot \xi}\, p(y, \xi)\, g(x)\, d\xi\, dy\, dx$$

$$= \int e^{i(x-y)\cdot \xi} p^*(y, \xi) f(y)\, d\xi\, dy \cdot g(x)\, dx \overset{\text{def}}{=} (P^* f, g)$$ which means

$$P^* f = \int e^{i(x-y)\cdot \xi} p^*(y, \xi) f(y)\, d\xi\, dy$$ which is in the form of Eqn (21) with $a(x, y, \xi) = p^*(y, \xi)$

Since by Remark 2.2.9, $p(y, \xi)$ has compact y-support, so does $p^*(y, \xi)$, and the decay properties of p (hence p^*) imply that $a(x, y, \xi)$ fits Def. 2.2.3 for a bisymbol, so Prop. 2.2.4 applies. Thus P^* is a ΨDO with symbol

$$\sigma(P^*) \sim \sum_\alpha \frac{d_\xi^\alpha D_x^\alpha p^*}{\alpha!} \qquad (27)$$

by Formula 23., which implies it is of order d = order of P.

Before we get to PQ, let us note that

$$(\widehat{Qf}, \hat{g}) = (Qf, g) = (f, Q^* g) = \int f(y) (Q^* g)(y)\, dy = \int f(y) e^{iy\cdot \xi} r(y, \xi)\, \hat{g}(\xi)\, d\xi\, dy$$

(Plancherel)

which implies $\widehat{Qf}(\xi) = \int e^{iy \cdot \xi} r(y, \xi) f(y)\, dy$ where $\boxed{r = \text{symbol of } Q^*}$ (28)

Now,

$$PQf(x) = \int e^{ix\cdot \xi} p(x, \xi) (\widehat{Qf})(\xi)\, d\xi = \int e^{i(x-y)\cdot \xi} p(x, \xi) r^*(y, \xi) f(y)\, d\xi\, dy$$

which is in the form of (21) with

$a(x, y, \xi) = p(x, \xi) r^*(y, \xi)$. Thus PQ is a ΨDO with symbol

$$\sigma(PQ) \sim \sum_\alpha \frac{d_\xi^\alpha D_y^\alpha \left(p(x, \xi) r^*(y, \xi)\right)}{\alpha!}\Big|_{y=x} \quad (\text{by } (23)) \qquad (29)$$

But again, by (27) above,

$$r^*(y, \xi) \sim \sum_\beta \frac{d_\xi^\beta D_y^\beta q(y, \xi)}{\alpha!} \qquad (30)$$

Substituting (30) into (29) and using Leibniz formula shows the 2nd formula in (26). The statements on supports are easy to verify:

For Example PQ: If $f \in C_c^\infty(\mathbb{R}^n)$, then $\operatorname{supp}(Qf) \subset K$, since Q is supported in the compact set $K \Rightarrow Qf \in C_c^\infty(\mathbb{R}^n) \Rightarrow \operatorname{supp}(PQf) \subset K$ since P is supported in the compact set K. If $f \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \cap K = \phi$, then $Qf \equiv 0$, so $PQf \equiv 0$. So PQ has compact support $\subset K$.

Similarly for P^*. If $f \in C_c^\infty(\mathbb{R}^n)$, since P^*f is smooth, to show $\operatorname{supp}(P^*f) \subset K$ it is enough to show $(P^*f, g) = 0 \;\forall\; g \in C_c^\infty(\mathbb{R}^n)$ with $(\operatorname{supp} g) \cap K = \phi$. But $(P^*f, g) = (f, Pg) = 0$ since P has cpt support in K (by (ii) of Def 2.2.8). Further, if $f \in C_c^\infty(\mathbb{R}^n)$ with $(\operatorname{supp} f) \cap K = \phi$, then $(P^*f, P^*f) = (f, PP^*f) = 0$ since $\operatorname{supp} PP^*f \subset K$ and $(\operatorname{supp} f) \cap K = \phi \Rightarrow P^*f \equiv 0$. So P^* is supported in K.

#

2.2.11 Corollary: Denote by Ψ_K^d the set of ΨDO's of order d with compact support $\subset K$, $\Psi_K^{-\infty} = \cap_d \Psi_K^d$, $\Psi_K^\infty = \cup_d \Psi_K^d$. Then, by 2.2.10, $\Psi_K^\infty / \Psi_K^{-\infty}$ is a (noncommutative) algebra with adjoints.

§ 2.3 Ellipticity. ([Gil] § 1.3, [Nar] § 3.3, [L-M], § 4, § 5.)

From this point on, "P" will always denote a differential operator of order d,* whose symbol $p(x, \xi)$ will therefore be a polynomial of order d, whose coefficients are C^∞ functions (matrix valued) of $x \in \mathbb{R}^n$.
Thus by Remark 2.2.9, P has compact support $\subset K$ iff the x-support of p is a compact subset of K, or equivalently $p(x, \xi) \equiv 0 \;\forall\; x \notin K$.

2.3.1 Def: A differential operator P of order d is said to be elliptic over U if $\exists\; c > 0$ such that for some open set $V \supset \overline{U}$, $p(x, \xi)$ is invertible for $x \in V$ and $|\xi| \geq c$, and furthermore

$$|p(x, \xi)^{-1}| \leq (\text{Constant})(1 + |\xi|)^{-d} \quad \text{for } x \in V, \; |\xi| \geq c \qquad (31)$$

In this event we say $p(x, \xi)$ is an elliptic symbol (over U).
For Example, for d a positive integer,

$$p(x, \xi) = (1 + |\xi|^2)^d$$

is an elliptic symbol of order $2d$. There is an easy criterion for telling when a differential operator P is elliptic over U: Let's define the leading symbol of $P = \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha$ by $\sigma_L(P) = \sum_{|\alpha| = d} a_\alpha(x) \xi^\alpha$

---

* unless otherwise stated

over U

2.3.2 Lemma: P is elliptic over U iff $\sigma_L(P)$ is elliptic over U.

Proof:- Assume P is elliptic of order d with symbol $p(x,\xi)$, which is an elliptic symbol of order d. By definition for $|\xi|>c$, it is invertible for $x \in V$. Let $q(x,\xi) = (p(x,\xi))^{-1}$ for $|\xi|>c$. For $t>1$, therefore, by (31)

$$|q(x,t\xi)| < C\cdot(1+t|\xi|)^{-d} \quad \text{for } |\xi|>c,\ x\in V \qquad (32)$$

$\Rightarrow \lim_{t\to\infty} |t^d q(x,t\xi)|$ is finite, and $\lim_{t\to\infty} t^d q(x,t\xi)$ exists for $|\xi|>c$. Call it $r(x,\xi)$. In fact (32) implies that for $|\xi|>c$, $|r(x,\xi)| \le C\cdot(1+|\xi|)^{-d}$. We claim that $r(x,\xi)$ is the inverse of $\sigma_L(P)(x,\xi)$ for $|\xi|>c$. This is clear because for $t>1$, $|\xi|>c$,

$$Id = p(x,t\xi)\,q(x,t\xi) = t^{-d} p(x,t\xi)\,.\,t^d q(x,t\xi).$$

But $\lim_{t\to\infty} t^{-d} p(x,t\xi) = \sigma_L P(x,\xi)$ and $\lim_{t\to\infty} t^d q(x,t\xi) = r(x,\xi)$. We've already remarked $|r(x,\xi)| \le C\cdot(1+|\xi|)^{-d}$ for $|\xi|>c$, so by Def. 2.3.1, $\sigma_L(P)$ is elliptic. The converse is also straightforward by writing $p(x,\xi) = \sigma_L(P)(x,\xi)(1-K(x,\xi))$ where $|K|<1$ for $|\xi|$ very large and so $p(x,\xi)^{-1} = \sigma_L(P)(x,\xi)^{-1}(1+K+K^2+\cdots)$ for $|\xi|$ very large. #
(The estimate (31) follows for the corresponding estimate for $\sigma_L(P)$.)

So to check ellipticity for a differential operator, it is enough to prove ellipticity for the leading symbol.

2.3.3. Example: If M is a Riemannian manifold, in a local coordinate system the Laplace-Beltrami operator Δ with respect to this metric is given by
$$\Delta = \sum_{i,j} g^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + (\text{lower order terms})$$
whose leading symbol is $-\sum_{ij} g^{ij}\xi_i\xi_j = -\|\xi\|_g^2$, which shows it is elliptic by Remark 2.3.3 above.

2.3.4 Def: Let $P \in \Psi^d$. We say Q is a parametrix for P if $Q \in \Psi^{-d}$ and $PQ - Id$ and $QP - Id$ are ∞-ly smoothing operators.

2.3.5 Def: Let us say a symbol $s(x,\xi)$ is infinitely smoothing over V if the growth condition (ii) in Def 2.1.1 holds for $x \in V$ and $d = -\infty$. This is equivalent to requiring that $\psi s \in S^{-\infty}$ for all $\psi \in C_c^\infty(V)$. Correspondingly, ΨDO's infinitely smoothing over V are defined as those whose symbols are infinitely smoothing over V.

2.3.6 Lemma: Let p be an elliptic symbol of order d, elliptic over $\mathcal{U}$, and let V, c be as in Def. 2.3.1; with $\bar{\mathcal{U}} \subset V$. Then there exists a symbol $q_0 \in S^{-d}$ such that $pq_0 - Id$ and $q_0 p - Id$ are infinitely smoothing over V_1, for any $V_1 \supset \bar{\mathcal{U}}$, and $\bar{V}_1 \subset V$.

Proof: By hypothesis, $\exists\ c > 0$ and $V \supset \bar{\mathcal{U}}$ such that $p(x,\xi)^{-1}$ exists and satisfies the estimate (31) for $x \in V$. Let ϕ be a +ve real valued C^∞-function such that $\varphi(t) \equiv 0$ $t < c$ and $\varphi(t) \equiv 1$ for $t \geq 2c$.

Define
$$q_0(x,\xi) = \phi(|\xi|)p(x,\xi)^{-1} \quad \text{for } |\xi| > c,\ x \in V$$
$$= 0 \qquad \text{for } |\xi| \leq c, \qquad — (33)$$

By multiplying q_0 with a cut-off function which is $\equiv 1$ on V_1 and with support $\subset V$, we may assume that q_0 is defined all over $\mathbb{R}^n$ and (33) holds over V_1.

Thus $pq_0 - Id$ and $q_0 p - Id$ are equal to $[\varphi(|\xi|) - 1]\, Id\ \forall\ \xi$ and $x \in V_1$. But $(\phi(|\xi| - 1) \equiv 0$ for $|\xi| \geq 2c$, so $(\phi(|\xi|) - I)\, Id$ is infinitely smoothing over V_1. The proof that q_0 obeys the decay conditions for a ΨDO of order $(-d)$ follow from the decay conditions (31) for $p(x,\xi)^{-1}$ and the Leibniz rule, (noting that $\frac{\partial}{\partial x_j}(p^{-1}) = -p^{-1}\frac{\partial p}{\partial x_j}p^{-1}$ and similarly for $\frac{\partial}{\partial \xi_j}$ and Leibniz rule for higher mixed derivatives). #.

2.3.7: Propn (Existence of Parametrices for Elliptic D.O.'s): Let P be a differential operator of order d elliptic over $\mathcal{U}$, and assume that P has compact support $\subset K$. Let $V \supset \bar{\mathcal{U}}$ be as in Def. 2.3.1. (Incidentally, this forces $K \supset \bar{V} \supset U$). Then for any open set V_1 with $\bar{U} \subset V_1 \subset \bar{V}_1 \subset V$, there exists a ΨDO Q of order $-d$ such that $PQ - I$ and $QP - I$ are infinitely smoothing over V_1.

Proof: First define q_0 as in Lemma 2.3.6, so that $pq_0 - Id$ and $q_0 p - Id$ are infinitely smoothing over V_1. q_0 is of order $(-d)$.

For $k > 0$, inductively define a symbol of order $(-d-k)$, viz.

$$q_k = -q_0 \sum_{0<|\alpha|\leq k} \frac{d_\xi^\alpha p \; D_x^\alpha q_{k-|\alpha|}}{\alpha!} \qquad \text{---} \qquad (34)$$

(In fact, if one inductively assumes q_j of order $-d-j$ for $j<k$, then (34) shows that q_k is of order $-d-k$.). By Prop. 2.2.2, $\exists$ a symbol q in S^{-d} with $q \sim \sum_j q_j$. Let $\tilde{Q}$ be the ΨDO of order $(-d)$ corresponding to q. This $\tilde{Q}$ need not have support in a compact set, but if, for example, one defines

$$Q = \lambda \tilde{Q} \psi$$

where $\lambda \equiv 1$ on V_1, $(\operatorname{supp} \lambda) \subset V$ and λ real valued (≥ 0) in $C_c^\infty(\mathbb{R}^n)$ and ψ real valued ≥ 0 in $C_c^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ on V, then

for $x \in V_1$, $\sigma(\tilde{Q})(x,\xi) = \sigma(\lambda\tilde{Q})(x,\xi) = q(x,\xi)$

and also $\lambda\tilde{Q} - \lambda\tilde{Q}\psi = \lambda\tilde{Q}(1-\psi)$, which is infinitely smoothing since $(\operatorname{supp}\lambda) \cap \operatorname{supp}(1-\psi) = \phi$ by Prop 2.2.5. This implies that Q has support in the compact set $(\operatorname{supp}\psi)$; and also that $\sigma(Q) - q(x,\xi)$ is infinitely smoothing over V_1, which we will write as $\sigma(Q) \sim q(x,\xi)$ over V_1.

Now apply the formula (26), since P & Q both have support in the compact $= K \cup (\operatorname{Supp}\psi)$. Over V_1, noting $q \sim \sum_j q_j$

$$\sigma(PQ) \sim \sum_\alpha \frac{d_\xi^\alpha p \; D_x^\alpha q}{\alpha!} = \sum_{k\geq 0} \sum_{0\leq|\alpha|\leq k} \frac{d_\xi^\alpha p \; D_x^\alpha q_{k-|\alpha|}}{\alpha!}$$

$$= pq_0 + \sum_{k>0}\left(pq_k + \sum_{0<|\alpha|\le k} \frac{d_\xi^\alpha p \, D_x^\alpha q_{k-|\alpha|}}{\alpha!}\right)$$

$$\sim pq_0 + (1-pq_0)\left(\sum_{\substack{k>0,\\ 0<|\alpha|\le k}} \frac{d_\xi^\alpha p \, D_x^\alpha q_{k-|\alpha|}}{\alpha!}\right) \quad \text{by } (34)$$

$$\sim \mathrm{Id} + 0\cdot(\ldots) \quad \text{over } V_1$$

$$\sim \mathrm{Id} \quad \text{over } V_1. \quad \Rightarrow \quad PQ \sim \mathrm{Id} \text{ over } V_1$$

By a similar argument, one finds a Q' such that $Q'P \sim \mathrm{Id}$ over V_1. But then $Q' \sim Q'\,\mathrm{Id} \sim Q'PQ \sim Q$ over V_1, so $QP \sim \mathrm{Id}$ over V_1 as well. #

§ 2.4. Globalisation to compact manifolds and vector Bundles

We revert to the set-up of § 1.5, letting E, F, M, U_α, λ_α have the same meanings as they had there. (E & F are cpx. v. bdles of $rk_{\mathbb{C}} = k, l$ resp.)

2.4.1 Def:- Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a $\mathbb{C}$-linear operator. We say P is a ΨDO of order d if for all coordinate charts U_α, U_β of M which are diffeomorphic to $\mathbb{R}^n$, (so that $E_{|U_\alpha}$ and $F_{|U_\beta}$ are trivial bundles), the localised operator

$$\psi P \phi : \underset{\displaystyle C^\infty(\mathbb{R}^n, \mathbb{C}^k)}{\overset{\displaystyle C^\infty(U, E_{|U_\alpha})}{\wr\wr}} \to \underset{\displaystyle C^\infty(\mathbb{R}^n, \mathbb{C}^l)}{\overset{\displaystyle C^\infty(U, F_{|U_\beta})}{\wr\wr}}$$

is a ΨDO of order d $\forall\ \phi \in C_c^\infty(U_\alpha)$, $\psi \in C_c^\infty(U_\beta)$

(Prop 2.2.6 shows that this definition makes sense). Note that this localisation has support in the compact set $(\mathrm{Supp}\,\phi) \cup (\mathrm{Supp}\,\psi)$ (considered as a subset of $\mathbb{R}^n$)

2.4.2 Def: $P : C^\infty(M,E) \to C^\infty(M,F)$ will be called a differential operator of order d if the localisations in Def. 2.4.1 are all differential operators of order d. It will be called an elliptic differential operator of order d if those localisations $\psi P \phi$ are elliptic over each open set U satisfying $\overline{U} \subset \underbrace{\{x : \phi(x) \neq 0\} \cap \{x : \psi(x) \neq 0\}}_{=\{x :\, \phi(x)\psi(x) \neq 0\}} \subset U_\alpha \cap U_\beta$.

The collection of pseudodifferential operators $P : C^\infty(M) \to C^\infty(M)$ of order d is denoted $\Psi^d(M)$.

We now have an analogue of Prop. 2.1.3.

2.4.3 Prop: Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a ΨDO of order d. Then P extends to a continuous (bounded) linear operator of Hilbert spaces

$$P : H_{s+d}(M,E) \to H_s(M,F)$$

where these Hilbert spaces are as in § 1.5.

Proof:- Let $\{\lambda_\alpha\}_{\alpha=1}^N$ be as in § 1.5, the partition of unity subordinate to the covering $\{U_\alpha\}_{\alpha=1}^N$ defined there. By definition, $\forall\ \alpha, \beta$, we have $\lambda_\alpha P \lambda_\beta$ a ΨDO of order d with support in a compact set. Now for $f \in C^\infty(M,E)$, we compute; using $f = \sum \lambda_\beta f_{|U_\beta}$

$$\|Pf\|_s^2 \le \sum_{\beta=1}^N \| P\lambda_\beta f_{|U_\beta}\|_s^2 \overset{\text{def}}{=} \sum_{\alpha,\beta=1}^N \| (\lambda_\alpha P \lambda_\beta) f_{|U_\beta}\|_s^2$$

$$\le \sum_{\alpha,\beta=1}^N C_{\alpha,\beta} \| f_{|U_\beta}\|_{s+d}^2 \quad \text{by Prop 2.1.3 applied to } (\lambda_\alpha P \lambda_\beta)$$

$$\le C \cdot \|f\|_{s+d}^2$$

which implies the proposition. #

Similarly one may easily deduce the analogue of Prop. 2.2.7, (the pseudolocal property), for this setting.

2.4.4 Prop: Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a ΨDO of order d. Using the hermitian metrics define the L^2-adjoint by

$$(P^* f, g)_E = (f, Pg)_F$$

for $f \in C^\infty(M,F)$, $g \in C^\infty(M,E)$. Then P^* is actually a ΨDO of order d. Similarly, if $P : C^\infty(M,E_1) \to C^\infty(M,E_2)$ and $Q : C^\infty(M,E_2) \to C^\infty(M,E_3)$ are ΨDO's of order d and e resp. Then $QP : C^\infty(M,E_1) \to C^\infty(M,E_3)$ is a ΨDO of order $d+e$.

Proof:- Let ϕ, ψ be as in Def. 2.4.1. Then by definition of P^*

$$(\varphi P^* \psi f, g) = (P^* \psi f, \bar\varphi g) = (f, \bar\psi P \bar\varphi g)$$

which implies $\phi P^* \psi = (\bar\psi P \bar\varphi)^*$. Because of Def 2.4.1 and Prop 2.2.10 we're done, and P^* is a ΨDO of order d.

The statement about PQ is left to the reader. #

2.4.5 Propn:- (Parametrices for Elliptic Differential operators on Manifolds)
Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be an elliptic differential operator of order d. Then there exists a ΨDO $Q : C^\infty(M,E) \to C^\infty(M,E)$ of order $(-d)$ such that $PQ - I$ and $QP - I$ are infinitely smoothing operators {i.e. ΨDO's of order $-\infty$).

Proof:- It is enough to construct the left parametrix Q s.t. $QP - I$ is infinitely smoothing, for by the last line of the Proof of 2.3.7, it serves as the right parametrix too.

Let λ_α, U_α be as always. Let us denote

$$\tilde{U}_\alpha = \{x : \lambda_\alpha(x) \neq 0\} \subset U_\alpha$$

Then by the definition of λ_α, $\overline{\tilde{U}_\alpha}$ is a compact subset of U_α $\forall\ \alpha = 1, .., N$.

Let $\psi_\alpha \in C_c^\infty(U_\alpha)$ with $\psi_\alpha \equiv 1$ on $\overline{\tilde{U}_\alpha} = \operatorname{Supp} \lambda_\alpha$, and let $\rho_\alpha \in C_c^\infty(U_\alpha)$ with $\rho_\alpha \equiv 1$ on $\operatorname{Supp} \psi_\alpha$.

Let us consider the localisation $\psi_\alpha P \rho_\alpha$. Since $\tilde{U}_\alpha \subset \{x : \psi_\alpha(x) \neq 0\}$ and $\{x : \psi_\alpha(x) \neq 0\} \subset \{x : \rho_\alpha(x) \neq 0\}$, we have $\tilde{U}_\alpha$ is an open set $\subset \{x : \psi_\alpha(x) \neq 0\} \cap \{x : \rho_\alpha(x) \neq 0\}$.

By the definition 2.4.2 of ellipticity of P, this means $\psi_\alpha P \rho_\alpha$ is elliptic over $\tilde{U}_\alpha$.

Since P is a differential operator and $\rho_\alpha \equiv 1$ on $\operatorname{Supp} \psi_\alpha$, we have $\psi_\alpha P \rho_\alpha \equiv \psi_\alpha P$ for $\alpha = 1, .., N$.

Thus $\psi_\alpha P$ is elliptic over $\tilde{U}_\alpha$. By the fact that $\psi_\alpha \in C_c^\infty(U_\alpha)$, $\psi_\alpha P$ has support in the compact set $K = \operatorname{Supp}(\psi_\alpha)$. Thus, by Prop. 2.3.7, $\exists$ an open set $V_\alpha \supset \overline{\tilde{U}_\alpha}$ and a parametrix Q_α, a ΨDO of order $(-d)$, such that

$$Q_\alpha(\psi_\alpha P) - Id \text{ is infinitely smoothing over } V_\alpha$$

This means $\lambda\left(Q_\alpha(\psi_\alpha P) - Id\right)$ is infinitely smoothing $\forall\ \lambda \in C_c^\infty(V_\alpha)$.

However $\operatorname{supp}\lambda_\alpha = \tilde{\bar{U}}_\alpha$ compact $\subset V_\alpha$, so in particular

$$\lambda_\alpha\left(Q_\alpha(\psi_\alpha P) - Id\right) \text{ is infinitely smoothing}$$

$$\forall \quad \Rightarrow \quad \sum_{\alpha=1}^{N}\left(\lambda_\alpha Q_\alpha \psi_\alpha P - \lambda_\alpha Id\right) \text{ is infinitely smoothing (all over)}$$

But $\{\lambda_\alpha\}_{\alpha=1}^{N}$ is a partition of unity, so $\sum_{\alpha=1}^{N} \lambda_\alpha\, Id = Id$.

Thus $\left(\sum_{\alpha=1}^{N}(\lambda_\alpha Q_\alpha \psi_\alpha)\right)P - Id$ is <u>infinitely</u> smoothing.

Setting $Q = \sum_{\alpha=1}^{N}(\lambda_\alpha Q_\alpha \psi_\alpha)$ does the job. It is of order $-d$ by Prop 2.2.6 since the Q_α are. #

Proposition 2.4.5 above has profound consequences. Let us list some.

<u>2.4.6.</u> <u>Prop</u>. (<u>The Inequality of Garding - Friedrichs</u>). Let $P: C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order d. Then there exists a constant C, (depending only on P, E, and M) such that for $f \in H_{s+d}(M,E)$

$$\|f\|_{s+d} \leq C\left(\|Pf\|_s + \|f\|_s\right) \qquad \text{—} \qquad (35)$$

<u>Proof</u>:- Let Q be the parametrix for P from the previous Propn. 2.4.5. Then by definition

$$f = QPf + Sf \quad \text{where} \quad S \text{ is infinitely smoothing}$$

ψDO (ie. of order $-\infty$). Thus

$$\|f\|_{s+d} \leqq \|QPf\|_{s+d} + \|Sf\|_{s+d} \qquad \text{—} \qquad (36)$$

Since S is of order $-\infty$, it is of order $(-d)$, so by Propn. 2.4.3 $\|Sf\|_{s+d} \leq C\|f\|_s$. By the same proposition, since Q is of order $(-d)$, $\|QPf\|_{s+d} \leq C \cdot \|Pf\|_s$. Plugging into (36), we have (35). #

2.4.7 Corollary :- Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order d. Then the Sobolev norm $\| \, \|_d$ on $H_d(M,E)$ is equivalent to the norm given by the inner product

$$\langle f, g \rangle \stackrel{\text{df}}{=} (Pf, Pg) + (f, g) \qquad (37)$$

where (f,g) on the right side denotes the L^2-~~inner~~ product: $\int_M (f,g)_x \, dV$ $[(f,g)_x$ = pointwise hermitian inner product in $E_x]$, so also (Pf, Pg).

Note : (37) is defined a priori for $f, g \in C^\infty(M,E)$ one has to extend to the completion with respect to the norm arising from $\langle \ \rangle$.

Proof : , If $f \in C^\infty(M,E)$, and one denotes $\langle f, f\rangle^{1/2}$ by $\|f\|'$

Then
$$\|f\|'^2 = \|Pf\|_0^2 + \|f\|_0^2 \qquad \text{Since } (f,f) = \|f\|_0^2$$
$$\leq C\|f\|_d^2 + \|f\|_d^2 \leq C\|f\|_d^2$$

by Propn 2.4.3, whereas
$$\|f\|_d \leq C \cdot (\|Pf\|_0 + \|f\|_0) \leq C(\|f\|' + \|f\|')$$

by (35). This shows that the completion of $C^\infty(M,E)$ with respect to $\| \, \|_d$ and $\| \, \|'$ is the same, and since $\| \, \|'$ is invariantly defined globally without reference to trivialisations or partitions of unity, it shows that $H_d(M,E)$ does not depend on these choices. #

2.4.8 Propn (The Elliptic Regularity Theorem).

Let us call continuous linear functionals on $C^\infty(M,E)$ as in § 1.5 as distributional sections of E. Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be an Elliptic differential operator of order d. Let f be a distributional section of E such that

$$Pf = g \qquad \text{(as distributional sections)}$$

with $g \in H_k(M,E)$. (Such an f is called a weak solution)

Then $f \in H_{k+d}(M,E)$. In particular if $g \in C^\infty(M,E)$, then $g \in H_k(M,E)\ \forall k \Rightarrow f \in H_{k+d}(M,E)\ \forall k \Rightarrow f \in C^\infty(M,E)$ by the Sobolev Embedding Theorem 1.5.1 (iii).

Proof: Since M is compact, as remarked in § 1.5, {distributional sections} $\equiv$ {compactly supported distributional sections} $\equiv$ {tempered distributional sections} $= H_{-\infty}(M,E) = \bigcup_s H_s(M,E)$.

This means $f \in H_s(M,E)$ for some s. Let Q be the parametrix for P from Prop. 2.4.5. Then

$$f = QPf + Sf = Qg + Sf$$

But S is infinitely smoothing, so $Sf \in C^\infty(M,E)$ and hence $H_{k+d}(M,E)$. Also since $g \in H_k(M,E)$, Q is of order $-d$, $Qg \in H_{k+d}(M,E)$ by Prop. 2.4.3. Hence $f \in H_{k+d}(M,E)$ #

§ 2.5: Fredholm Theory, Green's Operator, Spectra:

2.5.0 Introduction:- Most of the operator & Hilbert-space Theory pertaining to ΨDO's and elliptic operators is now in place. We wish to now follow up on the discussion on p. 26 and tie up with Fredholm Theory. This will have important consequences, such as the finite dimensionality of Ker P for P an elliptic differential operator, the behaviour of the spectrum of P, and the construction of a Green's operator, or "inverse" for P. First some definitions:

2.5.1 Def: Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, and let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denote the space of continuous ($\equiv$ bounded) operators $T: \mathcal{H}_1 \to \mathcal{H}_2$. We say $K \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is compact if for every bounded sequence $\{e_n\}$ in $\mathcal{H}_1$, the sequence $\{Ke_n\}$ contains a subsequence converging in $\mathcal{H}_2$. Clearly K compact $\Rightarrow$ SKT is compact $\forall\ S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $T \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$

Examples: (i) By Rellich's Lemma 1.5.1 (iv), the inclusion

$$H_s(M,E) \hookrightarrow H_t(M,E) \quad \text{for } s > t$$

is a compact operator.

(ii) If S is an infinitely smoothing operator, then $S : H_s(M,E) \to H_t(M,E)$ continuously $\forall\, s, t$. We claim <u>S is a compact operator</u>. Just take a $t' > t$, then S can be written as the composite

$$H_s(M,E) \xrightarrow{S} H_{t'}(M,E) \hookrightarrow H_t(M,E)$$

By Example (i), the second map is a compact operator, and precomposing with a bounded operator leaves it compact by the remark in Def. 2.5.1 above.

The space of compact operators $K : \mathcal{H}_1 \to \mathcal{H}_2$ is denoted by $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$. It is not hard to see that it is a norm-closed subspace of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, by a Cantor Diagonal type argument. Thus if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $\mathcal{K}(\mathcal{H})$ is a <u>closed two-sided ideal</u> in $\mathcal{L}(\mathcal{H})$.

<u>2.5.2 Def</u>:- An operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be <u>Fredholm</u> if $\exists$ an operator $S_1 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $S_1 \circ T - \mathrm{Id} \in \mathcal{K}(\mathcal{H}_1)$ and S_2 $(\in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \ni T \circ S_2 - \mathrm{Id} \in \mathcal{K}(\mathcal{H}_2)$. Their space is denoted by $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$

<u>Example</u> (iii) Let P be an elliptic differential operator of order d. Then for any s,

$$P : H_{s+d}(M,E) \to H_s(M,E) \qquad \text{(continuous)}$$

with a parametrix from Prop. 2.4.5 $Q : H_s(M,E) \to H_{s+d}(M,E)$ (of order $(-d)$), also continuous. This means $QP - \mathrm{Id} = S_1$ is an infinitely smoothing operator on $H_{s+d}(M,E)$ and $PQ - \mathrm{Id} = S_2$ is an infinitely smoothing operator on $H_s(M,E)$. By the Example (ii) above, both S_1 & S_2 are compact operators. <u>Thus P is Fredholm</u> (as is Q).

<u>2.5.3 Propn</u>: (<u>The Fredholm Theorem</u>): Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a Fredholm operator. Then $N(T) := \operatorname{Ker} T$ is finite dimensional. Also $R(T) := \operatorname{Im} T$ is closed in $\mathcal{H}_2$ and $(R(T))^\perp = \operatorname{Coker} T$ is also finite dimensional. Finally, T^* is also Fredholm, and $R(T)^\perp = N(T^*)$ and $N(T) = R(T^*)^\perp$.

(Thus the number $\dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T) = \dim(N(T)) - \dim(R(T)^\perp)$ called the <u>Fredholm index</u> is finite.)

Proof: The Proof is not too difficult, so we include it for the sake of completeness. Let $S_1 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and $S_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $S_1 \circ T - Id = K_1 \in \mathcal{L}(\mathcal{H}_1)$ is compact and $T \circ S_2 - Id = K_2 \in \mathcal{L}(\mathcal{H}_2)$ is compact.

Let $x_n \in N(T)$ with $\|x_n\| = 1$. Then $K_1 x_n = S_1 T x_n - x_n = -x_n$ so that $\{K_1 x_n\}$ contains a convergent subsequence. If one chooses $\{x_n\}$ to be an orthonormal basis of the closed subspace $N(T)$, the sequence $\{K_1 x_n\} = \{-x_n\}$ cannot contain an convergent subsequence unless it is a finite set, so dim$(N(T))$ is finite.

To see that $R(T)$ is closed, let $y_n = Tx_n$ and $y_n \to y \in \mathcal{H}_2$ by subtracting out the projections of x_n to $N(T)$ from x_n, we can assume $x_n \perp N(T)$ $\forall n$. There are two cases: Case 1: $\|x_n\| \le C$ $\forall n$. In this case $S_1 y_n = S_1 T x_n = x_n + K_1 x_n$ since $\|x_n\| \le C$, $\{K_1 x_n\}$ contains a cgt. subsequence $\{K_1 x_{n_k}\}$. Then since $y_n \to y$, S_1 is cts., $S_1 y_{n_k} \to S_1 y$, and $K_1 x_{n_k} \to z$ say. Thus $x_{n_k} \to S_1 y - z$. Thus $Tx_{n_k} \to T(S_1 y - z)$. But $Tx_{n_k} \to y$ also, so $y = T(S_1 y - z) \in$ range of T $= R(T)$ and we're through. Case 2: $\|x_n\| \to \infty$ (without loss of generality, by extracting a subsequence). Consider $x_n' = \frac{x_n}{\|x_n\|}$. Then $T(x_n') \to \lim_n \frac{y_n}{\|x_n\|} = 0$. But since $\|x_n'\| = 1$, and $S_1 T(x_n') \to 0$ and K_1 is compact, $K_1 x_n'$ has a convergent subsequence $K_1 x_{n_k}'$, so $x_{n_k}' = S_1 T x_n' - K_1 x_n'$ converges to some x. Since $x_n \perp N(T)$, $x_n' \perp N(T)$ and $N(T)$ is closed so $x \perp N(T)$. However, $T(x) = \lim_k T(x_{n_k}') = 0$ so $x \in N(T)$. Also since $\|x_{n_k}'\| = 1$, $\|x\| = 1$. A contradiction.

Since $R(T)$ is closed, it follows $R(T)^\perp = N(T^*)$ and T^* is immediately Fredholm since T is. (by starring everything and noting K_i^* is cpt for K_i cpt.). Thus dim$(R(T)^\perp) = $ dim$(N(T^*))$ is finite, (and $=$ dim(Coker T)). #

Now we can construct the Green's operator for an Elliptic differential operator.

2.5.4. Proposition : (Existence of Green's Operator)

Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order $d > 0$. Assume further that P is self-adjoint (with respect to the L^2-inner product, i.e. $(Pf, g) = (f, Pg)$ $\forall f$ and $g \in C^\infty(M,E)$). Then, (by Propn. 2.4.3). Consider the continuous operator $P : H_d(M,E) \to H_0(M,E) = L^2(M,E)$

(i) $\operatorname{Ker} P$ is finite dimensional, and also $\operatorname{Ker} P \subset C^\infty(M,E)$. In particular $\operatorname{Ker} P \subset L^2(M,E)$, and $(\operatorname{Ker} P)^\perp = \operatorname{Im} P$.

(ii) There exist a bounded self-adjoint operator (called the Green Operator) $G : L^2(M,E) \to L^2(M,E)$ satisfying

(a) $G \equiv 0$ on $\operatorname{Ker} P \subset L^2(M,E)$; $G = P^{-1}$ on $(\operatorname{Ker} P)^\perp = \operatorname{Im} P \subset L^2(M,E)$.

(b) $G(C^\infty(M,E)) \subset C^\infty(M,E)$, and $GP = PG$ on $C^\infty(M,E)$

(c) $G : L^2(M,E) \to L^2(M,E)$ is a compact operator; $L^2(M,E)$ admits a complete orthonormal basis of eigenfunctions $\{f_i\}$ for G, i.e. $Gf_i = \mu_i f_i$, and μ_i are all real having $0 \in \mathbb{R}$ as the only cluster point.

(d) The eigenfunctions $\{f_i\}$ in (c) are all smooth, and are also eigenfunctions for P, belonging to eigenvalues $\{\lambda_i\}$, where $\{\lambda_i\}$ is a discrete subset of $\mathbb{R}$, $\lim_{i\to\infty} |\lambda_i| = \infty$.

Proof :- By Example (iii) of Def 2.5.2 and Prop 2.5.3, $P : H_d(M,E) \to H_0(M,E) = L^2(M,E)$ is Fredholm. Thus $(\operatorname{Ker} P)$ is finite dimensional by the same Prop. 2.5.3 — i.e. the Fredholm Theorem.

Now if $f \in \operatorname{Ker} P$, $Pf = 0$ with $f \in H_d(M,E)$. By The elliptic regularity Theorem Propn 2.4.8, f is smooth, so in $C^\infty(M,E)$.

Since P is Fredholm, Prop. 2.5.3 tells us that $(\operatorname{Im} P)$ is a <u>closed</u> subspace of $L^2(M,E)$. $(\operatorname{Ker} P)$, being f. dim., is also a closed subspace of $L^2(M,E)$. Thus proving $(\operatorname{Ker} P)^\perp = (\operatorname{Im} P)$ is equivalent to proving that $(\operatorname{Im} P)^\perp = (\operatorname{Ker} P)$. So let $f \in L^2(M,E)$, $f \in (\operatorname{Im} P)^\perp$. Then for any C^∞-function $g \in C^\infty(M,E)$, $(f, Pg) = 0$. <u>We claim</u>: $Pf = 0$. Since $f \in L^2 = H_0$, we know $Pf \in H_{-d}(M,E)$. Find a sequence of smooth functions $f_n \to f$ in $L^2(M,E)$; since $C^\infty(M,E)$ is dense in $L^2(M,E)$. Then $Pf_n \to Pf$ in $H_{-d}(M,E)$ since P is continuous $L^2 \to H_{-d}$. Now $(Pf, g) = \lim_{n\to\infty} (Pf_n, g)$

$= \lim_{n\to\infty} (f_n, Pg)$ (Since P is self-adjt and f_n, g are smooth)

$= (f, Pg) = 0$. So $(Pf, g) = 0 \ \forall\ g \in C^\infty(M,E)$. $\Rightarrow Pf = 0$

Since $C^\infty(M,E)$ is dense in $H_{-d}(M,E)$. This implies $f \in (\operatorname{Ker} P)$ (Note $\operatorname{Ker} P \subset C^\infty \Rightarrow \operatorname{Ker} P$ doesn't depend on the domain $H_s(M,E)$ on which P is considered to be acting). This proves (i), because it is trivial to show that $f \in \operatorname{Ker} P \Rightarrow f \in (\operatorname{Im} P)^\perp$.

Now, to construct G. Decompose $H_d(M,E) = (\operatorname{Ker} P) \oplus E$, where E is a closed subspace of $H_d(M,E)$. Then $P : E \to (\operatorname{Im} P)$ is an isomorphism by the open-mapping theorem. Thus $P^{-1} : (\operatorname{Im} P) \to E$ is a continuous (bounded) operator. We have just seen above that $L^2(M,E) = (\operatorname{Im} P) \oplus (\operatorname{Ker} P)$ as an orthogonal direct sum. Define $G \equiv 0$ on $(\operatorname{Ker} P)$, $G = P^{-1}$ on $\operatorname{Im} P$. This defines $G : L^2(M,E) \to H_d(M,E)$. But by (1.4.3) (i)

$H_d(M,E) \hookrightarrow H_0(M,E) = L^2(M,E)$. This defines $G : L^2(M,E) \to L^2(M,E)$ as a bounded operator. Its self-adjointness follows from that of P, as well as the fact that $PG = GP$ on $C^\infty(M,E)$. Note that by definition, $PGf = f$ for $f \in L^2(M,E)$. Thus if f is C^∞, By the <u>Elliptic regularity</u> theorem (Prop 2.4.8) Gf is also C^∞. This proves (ii) (a) and (b). For (c), note

G is compact, since it is the composite
$$L^2(M,E) \xrightarrow[\text{projection}]{} \operatorname{Im} P \xrightarrow{P^{-1}} H_d(M,E) \hookrightarrow L^2(M,E) = H_0(M,E)$$

and the last inclusion is compact by Rellich's Lemma, and the first two maps are continuous by the above, so G is compact, because pre and post composing compact operators with bounded operators yield compact operators (see Remarks in <u>Def</u>: 2.5.1)

Since $G: L^2(M,E) \to L^2(M,E)$ is a compact self-adjoint operator, the spectral theorem for compact self-adjoint operators implies the rest of (ii) (c).

Finally, since $L^2(M,E) = (\mathrm{Ker}\,P) \oplus (\mathrm{Im}\,P)$, let us write the eigenfunctions $\{f_i\}_i$ so that $\{f_i\}_{i=1}^N$ span $\mathrm{Ker}\,P$ and $\{f_i\}_{i \geq N+1}$ span $(\mathrm{Im}\,P)$.
Note that since $G \equiv 0$ on $(\mathrm{Ker}\,P)$, $Gf_i = 0$ for $i = 1, \dots, N$.
And $Gf_i = \mu_i f_i$, $\mu_i \neq 0$, real for $i \geq N+1$. Thus $Pf_i = 0$ for $1 \leq i \leq N$ and, by (a), $f_i = PGf_i = P(\mu_i f_i) = \mu_i Pf_i$ for $i \geq N+1$
$\Rightarrow Pf_i = \mu_i^{-1} f_i$ for $i \geq N+1$. So let $\lambda_1, \dots, \lambda_N = 0$, and $\lambda_i = \mu_i^{-1}$ for $i \geq N+1$. Since the μ_i only cluster at 0, λ_i cluster only at ∞. Finally since $Pf_i = \lambda_i f_i \Rightarrow (P - \lambda_i) f_i = 0$ and $(P - \lambda_i)$ is also an elliptic operator (since $d > 0$), so by the Elliptic Regularity Theorem Prop. 2.4.8, f_i are smooth. This proves (d) and the Proposition. #.

<u>2.4.5 Remark</u> :- One can show that if P has $(-ve)$-definite leading symbol, then $\{\lambda_i\} \subset [c, \infty)$ for some real no. c.

<u>Ch.3</u> :- <u>Elliptic Complexes, Index Theorem etc.</u>

<u>3.1 Symbols</u> Let M, E, F be as before, and let

$$P : C^\infty(M,E) \to C^\infty(M,F)$$

be a <u>differential</u> operator of order d. We wish to define the <u>leading symbol of P</u>, $\sigma_L(P)$, in an invariant manner. First look at the case of $\mathbb{R}^n$. If we have a C^∞ coordinate change diffeo. $(x_1 \dots x_n) \to (y_1 \dots y_n)$, then

$$\frac{\partial}{\partial x_j} = \sum_i A_{ij} \frac{\partial}{\partial y_i} \quad \text{where } A_{ij} = \left(\frac{\partial y_i}{\partial x_j}\right)$$

= jacobian of coordinate change

$$\text{so} \quad \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j} = D_{x_j} = \sum_i A_{ij} D_{y_i}$$

Now consider the higher order differential operator of order k

$P = D_{x_{j_1}} \cdots D_{x_{j_k}}$ (where repetitions can occur)

$= \left(\sum_{i_1} A_{i_1 j_1} D_{y_{i_1}}\right)\left(\sum_{i_2} A_{i_2 j_2} D_{y_{i_2}}\right) \cdots \left(\sum_{i_k} A_{i_k j_k} D_{y_{i_k}}\right)$

$$= \sum_{i_1 \cdots i_k} (A_{i_1 j_1} \cdot A_{i_2 j_2} \cdots A_{i_k j_k}) D_{y_{i_1}} \cdots D_{y_{i_k}} + (\text{operator of order } (k-1)) \qquad (38)$$

(by Leibniz formula).

If we let ξ be a cotangent vector on $\mathbb{R}^n$, then in the two different coordinate systems:

$$\xi = \sum_{j=1}^{n} \xi_j \, dx_j = \sum_{i=1}^{n} \eta_i \, dy_i$$

where $\xi_j = \sum_i A_{ij} \eta_i$. Thus $(\xi_{j_1} \cdots \xi_{j_k}) = \sum (A_{i_1 j_1} \cdots A_{i_k j_k}) \eta_{i_1} \cdots \eta_{i_k}$ whose left side is the leading symbol of P in the $(x_1 \dots x_n)$ coordinate system, and whose right side is the leading symbol of P in the $(y_1 \dots y_n)$ coordinate system. This shows that for a general manifold M, and $P : C^\infty(M) \to C^\infty(M)$ a differential operator of order d, the leading symbol $\sigma_L(P)$ is a <u>well-defined function on $T^*(M)$</u>. If we choose a coordinate chart $(x_1 \dots x_n)$ around x, and if $\xi_x = \sum_i \xi_{xi} \, dx_i$ is a cotangent vector at x, then $\sigma_L(P)(\xi_x) = \sum_{|\alpha| = d} a_\alpha(x) \, \xi_{x1}^{\alpha_1} \xi_{x2}^{\alpha_2} \cdots \xi_{xn}^{\alpha_n}$ if $P = \sum_{|\alpha| \le d} a_\alpha(x) D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ in that coord. system.

More generally if $P : C^\infty(M, E) \to C^\infty(M, F)$ is a diff. operator of order d, the coefficients $a_\alpha(x)$ are linear maps $E_x \to F_x$ in the local representation of P above, so $\sigma_L(P) : \pi^* E \to \pi^* F$ is a bundle map where $\pi : T^*M \to M$ is the projection of the cotangent bundle of M.

For a more invariant definition of the leading symbol $\sigma_L(P)$, see [Nar] § 3.3.12.

We have the following elementary :-

3.1.1 Lemma :- Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a differential operator of order d. Then P is elliptic iff $\sigma_L(P)$ is a bundle isomorphism outside the zero-section of T^*M. i.e.

$$\sigma_L(P) : \pi^* E_{|T^*M - M} \longrightarrow \pi^* E_{|T^*M - M}$$

is an bundle isomorphism.

Proof :- By the Lemma 2.3.2, the Defn. 2.4.2, and the fact that $\sigma_L(P)$ is a homogeneous polynomial of order d, so the invertibility of $\sigma_L(P)(x, \xi)$ for some $\xi \neq 0$, implies that $\sigma_L(P)(x, \xi') = \left(\frac{\xi'}{\xi}\right)^d \sigma_L(P)(x, \xi)$ is invertible for all $\xi' \neq 0$. #

3.1.2. Lemma :- If $Q : C^\infty(M,E) \to C^\infty(M,F)$ and $P : C^\infty(M,F) \to C^\infty(M,G)$ are differential operators of order d and e respectively, then

$$\sigma_L(PQ) = \sigma_L(P) \cdot \sigma_L(Q) : \pi^*(E) \to \pi^*(G)$$

$$\sigma_L(P^*) = \sigma_L(P)^* : \pi^*(F) \to \pi^*(E)$$

Proof :- Direct computation in the local case. (i.e. of $\mathbb{R}^n$). #.

Now we're ready to define elliptic complexes.

3.1.3 Def : Let $\{E_i\}_{i=0}^r$ be C^∞-complex vector bundles on M equipped with Hermitian metric. We say a sequence of differential operators (all of order d)

$$\cdots \to C^\infty(M, E_i) \xrightarrow{P_i} C^\infty(M, E_{i+1}) \to \cdots$$

is a complex if $P_{i+1} \circ P_i = 0 \ \forall i$. We say it is an elliptic complex if it is a complex and the leading symbol sequence on the complement of the zero-section in T^*M, viz.

$$\cdots \to \pi^*(E_i)_{|T^*M - M} \xrightarrow{\sigma_L(P_i)} \pi^*(E_{i+1})_{|T^*M - M} \longrightarrow \cdots$$

(which is a complex since $0 = \sigma_L(P_{i+1} \circ P_i) = \sigma_L(P_{i+1}) \circ \sigma_L(P_i)$ by 3.1.2 above) is exact.

Before we write down some elliptic complexes, let us record the following ::

3.1.4. Lemma :- Let $\{P_i, E_i\}$ be a complex of differential operators. Define the *generalised Laplacian* $\Delta_{P_i}^i = P_i^* P_i + P_{i-1} P_{i-1}^* : C^\infty(M, E_i) \to C^\infty(M, E_i)$ (which is a differential operator of order 2d.) of the complex. Then the complex is elliptic iff $\Delta_p^i : C^\infty(M, E_i) \to C^\infty(M, E_i)$ is an elliptic differential operator for each i.

Proof :- Let us denote $\sigma_L(P_i) = p_i$, $\sigma_L(P_i^*) = p_i^*$ (by 3.1.2 Lemma) and so $\sigma_L(\Delta_p^i) = p_i^* p_i + p_{i-1} p_{i-1}^*$.

Suppose first that Δ_p^i is elliptic for each i, so that $\sigma_L(\Delta_p^i)(\xi_x)$ is invertible for $\xi_x \neq 0$. Need to show $\{\sigma_L(P_i)\}$ complex is exact off the 0-section. So assume $p_i(\xi_x)e = 0$; for $e \in \pi^*(E_i)$ and $\xi_x \neq 0$. Since $\sigma_L(\Delta_p^i)(\xi_x)$ is invertible, write $e = \sigma_L(\Delta_p^i)(\xi_x)(f)$
$= (p_i^*(\xi_x) p_i(\xi_x) + p_{i-1}(\xi_x) p_{i-1}^*(\xi_x)) f$. Then $p_i(\xi_x)e = 0$, (and $P_i \circ P_{i-1} \equiv 0$)
$\Rightarrow p_i(\xi_x) p_i^*(\xi_x) p_i(\xi_x) f = 0 \Rightarrow (p_i(\xi_x) p_i^*(\xi_x) p_i(\xi_x) f, p_i(\xi_x) f) = 0$
$\Rightarrow (p_i^*(\xi_x) p_i(\xi_x) f, p_i^*(\xi_x) p_i(\xi_x) f) = \| p_i^*(\xi_x) p_i(\xi_x) f \|^2 = 0 \Rightarrow p_i^*(\xi_x) p_i(\xi_x) f = 0$
$\Rightarrow e = p_{i-1}(\xi_x) p_{i-1}^*(\xi_x) f \Rightarrow e \in \operatorname{Im} p_{i-1}(\xi_x) \Rightarrow$ complex is elliptic.

Conversely, suppose the complex is elliptic; and let $\sigma_L(\Delta_p^i)(\xi_x)e = 0$
This implies $(p_i^*(\xi_x) p_i(\xi_x) e + p_{i-1}(\xi_x) p_{i-1}^*(\xi_x) e, e) = 0 \Rightarrow \| p_i(\xi_x) e \|^2 + \| p_{i-1}^*(\xi_x) e \|^2$
$= 0 \Rightarrow p_i(\xi_x) e = 0 = p_{i-1}^*(\xi_x) e \Rightarrow$ either $\xi_x = 0$ or $e = p_{i-1}(\xi_x) f \Rightarrow \xi_x = 0$
or $p_{i-1}^*(\xi_x) p_{i-1}(\xi_x) f = 0 \Rightarrow \xi_x = 0$ or $(p_{i-1}(\xi_x) f, p_{i-1}(\xi_x) f) = 0 \Rightarrow \xi_x = 0$ or $\|e\|^2 = 0$
$\Rightarrow \sigma_L(\Delta_p^i)(\xi_x)$ is injective for $\xi_x \neq 0 \Rightarrow$ (Since $\sigma_L(\Delta_p^i)(\xi_x) : \pi^*(E_i)_{\xi_x} \to \pi^*(E_i)_{\xi_x}$) that $\sigma_L(\Delta_p^i)(\xi_x)$ is invertible for $\xi_x \neq 0$ viz. Δ_p^i is an elliptic D.O. of order 2d.

#

3.1.5. Examples of an Elliptic Complex

(i) The de-Rham Complex. Take $E_i = \Lambda^i(T^*M \otimes_{\mathbb{R}} \mathbb{C})$, the i-th exterior power of the complexified cotangent bundle of a compact oriented Riemannian manifold M. Then one has the de-Rham complex

$$\cdots \to C^\infty(M, \Lambda^i(T^*M \otimes_{\mathbb{R}} \mathbb{C})) \xrightarrow{d_i} C^\infty(M, \Lambda^{i+1}(T^*M \otimes_{\mathbb{R}} \mathbb{C})) \to \cdots$$

$$\text{// def} \qquad\qquad\qquad \text{// def.}$$

$$\{\text{Space of } i\text{-forms}\} = \Lambda^i(M, \mathbb{C}) \qquad\qquad \Lambda^{i+1}(M, \mathbb{C})$$

where d_i is the exterior derivative on $\Lambda^i(M, \mathbb{C})$. We know that this is a complex since $d_{i+1} \circ d_i \equiv 0$. We claim it is an Elliptic Complex. The bundles $\Lambda^i(T^*M \otimes_{\mathbb{R}} \mathbb{C})$ get natural Hermitian bundle metrics from the Riemannian metric on M, so in principle one could try to explicitly write down the corresponding Laplacian and investigate its leading symbol for invertibility at $\xi_x \neq 0$. (using Lemma 3.1.4). However, in this case it is easier to check directly that the leading symbol complex is exact off the 0-section.

In local coordinates, if $\omega \in \Lambda^p(M, \mathbb{C})$ is an i-form, ω has the local coordinate representation

$$\omega = \sum_{|I|=p} \omega_I \, dx_I$$

and
$$d\omega = \left(\sum_{j=1}^n dx_j \frac{\partial}{\partial x_j}\right) \wedge \omega = i\left(\sum_{j=1}^n dx_j D_{x_j}\right) \wedge \omega$$

To get the symbol, replace D_{x_j} by $\xi_{x,j}$ so that for $\xi_x = \sum_j dx_j \xi_{x,j}$

$$\sigma_L(d)(\xi_x) = i\left(\sum_{j=1}^n dx_j \, \xi_{x,j}\right) \wedge \;=\; (i\,\xi_x)\wedge - \;:\; \underbrace{(\pi^*(\Lambda^p(T^*M)\otimes\mathbb{C}))_{\xi_x}}_{\Lambda^p(T^*M_x \otimes \mathbb{C})} \to \underbrace{\pi^*(\Lambda^{p+1}(T^*M\otimes\mathbb{C}))_{\xi_x}}_{\Lambda^{p+1}(T^*M_x\otimes\mathbb{C})}$$

This operator $\xi_x \wedge$ is denoted by $\mathrm{ext}(\xi_x)$, so $\sigma_L(d)(\xi_x) = i\,\mathrm{ext}(\xi_x)$. So we need to show that for $\xi_x \neq 0$, the complex of vector spaces

$$\cdots \to \Lambda^p(T^*M_x \otimes \mathbb{C}) \xrightarrow{i\,\mathrm{ext}(\xi_x)} \Lambda^{p+1}(T^*M_x \otimes \mathbb{C}) \to \cdots$$

is exact. But since $\xi_x \neq 0$, complete ξ_x to a basis $\{\xi_x, e_1, \ldots, e_{n-1}\}$ of $T^*M_x \otimes \mathbb{C}$. Then every elt. $\alpha \in \Lambda^p(T^*M_x \otimes \mathbb{C})$ has a unique decomposition $\alpha = \alpha_1 + \xi_x \wedge \alpha_2$ where α_1 & α_2 don't involve ξ_x. (i.e. can be written as linear combinations of $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots$).

$\mathrm{ext}(\xi_x)(\alpha) = 0 \Rightarrow \xi_x \wedge \alpha_1 = 0$ (since $\xi_x \wedge \xi_x \equiv 0$!). Since $\alpha_1 = \sum_{|J|=p} \alpha_{1J} e_J$

This implies $\sum_{|J|=p} \alpha_{1J}\, \xi_x \wedge e_J = 0 \Rightarrow \alpha_{1J} = 0 \;\forall J \Rightarrow \alpha_1 = 0$

Thus $\alpha = \xi_x \wedge \alpha_2 = i\,ext(\xi_x)(-i\alpha_2) \Rightarrow \alpha \in \operatorname{Im}(i\,ext\,\xi_x)$.
and the sequence is Exact. So the de-Rham Complex is elliptic

3.1.6. Example of Another Elliptic Complex

The Twisted Dolbeault-Complex : Let M be a compact complex manifold of $\dim_{\mathbb{C}} = n$, and let E be a holomorphic vector bundle on M.

- If one takes the underlying real tangent bundle $T_{\mathbb{R}}(M)$ and complexifies it, then

$$T_{\mathbb{R}}(M) \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}(M) \oplus \underbrace{T^{0,1}(M)}_{= \overline{T^{1,0}(M)}}$$

where $T^{1,0}(M)$ is the complex holomorphic tangent bundle (= the tangent bndle of M considered as a complex manifold) and $T^{0,1}(M)$ the antiholo. tangent bundle. (In local holomorphic coordinates $(z_1, \dots, z_n)$ on M, elements of $T^{1,0}(M)$ read like $\sum \alpha_j \frac{\partial}{\partial z_j}$ and those of $T^{0,1}(M)$ like $\sum \beta_j \frac{\partial}{\partial \bar{z}_j}$).

Correspondingly,

$$(T_{\mathbb{R}}(M) \otimes \mathbb{C})^* = T_{\mathbb{R}}^*(M) \otimes \mathbb{C} = (T^{1,0}(M))^* \oplus (T^{0,1}(M))^*$$

So that

$$\Lambda^i(T_{\mathbb{R}}^*(M) \otimes \mathbb{C}) = \bigoplus_{p+q=i} \underbrace{\Lambda^p(T^{1,0}(M))^* \otimes_{\mathbb{C}} \Lambda^q(T^{0,1}(M))^*}_{\text{Sl denote } \Lambda^{p,q}(T^*(M))}$$

and

$$C^\infty(M, \Lambda^i(T_{\mathbb{R}}^*(M) \otimes \mathbb{C})) \overset{def}{=} \Lambda^i(M, \mathbb{C}) = \bigoplus_{p+q=i} \Lambda^{p,q}(M)$$

$$\{\text{space of } (p,q)\text{-forms}\} = C^\infty(M, \Lambda^{p,q}(T^*(M))) \;\; (= \Lambda^{p,q}(M))$$

By similar logic

$$\Lambda^i(M, E) = C^\infty(M, \Lambda^i(T_{\mathbb{R}}^*(M) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E) = \bigoplus_{p+q=i} \Lambda^{p,q}(M, E)$$

where $\Lambda^{p,q}(M,E) := \Lambda^p(T^{1,0}(M))^* \otimes_{\mathbb{C}} \Lambda^q(T^{0,1}(M))^* \otimes_{\mathbb{C}} E$

= space of E-valued (p,q)-forms. In a local holomorphic coordinate neighborhood $\mathcal{U}$, with coords. $(z_1 \dots z_n)$, an element $\alpha \in \Lambda^{p,q}(M,E)|_{\mathcal{U}}$ is expressible as $\sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ} \otimes dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$ where

$\{\alpha_{IJ}\}$ are C^∞-sections of $E_{|U}$. Indeed $\Lambda^{p,q}(M,E)$ is nothing but $\Lambda^{p,q}(M)\otimes C^\infty(E)$ where the "$\otimes$" is over $C^\infty(M)$.

One defines the Dolbeault or $\bar\partial$-operator (and the Dolbeault complex)

$$\bar\partial : \to \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M) \to \cdots$$

by

$$\bar\partial\Big(\sum_{\substack{|I|=p\\|J|=q}} \alpha_{IJ}\, dz_I \wedge d\bar z_J\Big) = (-1)^p \sum_{I,J,j} \frac{\partial \alpha_{IJ}}{\partial \bar z_j}(dz_I \wedge d\bar z_j \wedge d\bar z_J)$$

in local coordinates. As in the case of the de-Rham complex, this may be written as

$$\bar\partial = \Big(\sum d\bar z_j \frac{\partial}{\partial \bar z_j}\Big)\wedge \text{—}$$

One readily checks $\bar\partial\circ\bar\partial = 0$; and the Dolbeault complex is a complex.

Similarly, since E is holomorphic one defines the twisted Dolbeault operator, complex by

$$\begin{array}{ccc}\cdots\to\Lambda^{p,q}(M,E) & \xrightarrow{\quad\bar\partial_E\quad} & \Lambda^{p,q+1}(M,E)\to\cdots\\ \parallel & & \parallel\\ \Lambda^{p,q}(M)\underset{C^\infty(M)}{\otimes} C^\infty(E) & & \Lambda^{p,q+1}(M)\underset{C^\infty(M)}{\otimes} C^\infty(E)\end{array}$$

with $\bar\partial_E\circ\bar\partial_E = 0$. (Locally: if $\{s_i\}$ is a local holo-frame for E, $\bar\partial_E\big(\sum_i \omega_i\otimes s_i\big) \overset{\text{def}}{=} \sum_i \bar\partial\omega_i\otimes s_i$

In this case, the underlying real cotangent bundle $T^*_{\mathbb{R}}(M)$ can be thought of as the underlying real bundle of $(T^{0,1}(M))^*$, and in this identification, for $\xi_x \in (T^{0,1}_x(M))^*$

$$\sigma_L(\bar\partial) = \sigma_L(\bar\partial_E) = \frac{i}{2}\,\mathrm{ext}\,(\xi_x)$$

(since $\frac{\partial}{\partial \bar z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big)$). The symbol complex is exact off the 0-section by the same reasoning as in the Example (i) of the de Rham-complex. viz.

$$\cdots\to \Lambda^p(T^{1,0}_x(M)^*)\otimes\Lambda^q(T^{0,1}_x(M)^*) \xrightarrow{\frac{i}{2}\mathrm{ext}(\xi_x)} \Lambda^p(T^{1,0}_x(M)^*)\otimes\Lambda^{q+1}(T^{0,1}_x(M)^*) \to \cdots$$

is exact for $\xi_x \neq 0$, and tensoring it with E_x leaves it exact.

3.1.7 : Def: Let $\cdots \to C^\infty(M, E_i) \xrightarrow{P_i} C^\infty(M, E_{i+1}) \to \cdots$ be a complex of differential operators of order d., so that $\operatorname{Im} P_{i-1} \subset \operatorname{Ker} P_i$. Define the i-th cohomology of this complex by. $H^i(M, \{E., P.\}) = \operatorname{Ker} P_i / \operatorname{Im} P_{i-1}$

. In the case of Example 3.1.6 (i), the de-Rham Complex, this is nothing but $H^i_{dR}(M, \mathbb{C})$ the i-th (Complex) de-Rham cohomology. and in the case of the Dolbeault (resp. Twisted Dolbeault) Complexes, it is $H^{p,q}_{\bar\partial}(M)$ (resp $H^{p,q}_{\bar\partial}(M, E)$) the corresponding Dolbeault cohomologies.

3.1.8. Propn : Let $\{E., P.\}$ be an Elliptic complex. Then, letting $\Delta^i_p = P_i^* P_i + P_{i-1} P_{i-1}^*$ be the laplacians of the complex as defined in Lemma 3.1.4; we have the following (M is compact, Riemannian oriented, E_i hermitian complex bundles etc. as before)

Hodge Theorem for Elliptic complexes

(i) $H^i(M, \{E. P.\}) \approx (\operatorname{Ker} \Delta^i_p)$ and is finite dimensional

(ii) $H_0(M, E_i) = L^2(M, E_i)$. admits the orthogonal direct sum decomposition.

$$L^2(M, E_i) = (\operatorname{Ker} \Delta^i_p) \oplus \underset{P_i^*(H_d(M, E_{i+1}))}{\overset{\shortparallel}{(\operatorname{Im} P_i^*)}} \oplus \underset{P_{i-1}(H_d(M, E_{i-1}))}{\overset{\shortparallel}{(\operatorname{Im} P_{i-1})}}$$

Proof : Let's first prove (ii), and (i) will follow as a Consequence. The proof is nothing but a Green's operator argument using the Propn. 2.5.4. First, Lemma 3.1.4 implies that Δ^i_p is an Elliptic differential operator $C^\infty(M, E_i) \to C^\infty(M, E_i)$. Thus, by 2.5.4 (i), $\operatorname{Ker}(\Delta^i_p)$ is finite dimensional and also $\operatorname{Ker}(\Delta^i_p) \subset C^\infty(M, E_i). \subset L^2(M, E_i)$.

Let $\pi : L^2(M, E_i) \to \operatorname{Ker}(\Delta^i_p)$ denote the projection (orthogonal projection with respect to L^2 inner product).

Then for $f \in L^2(M, E_i)$, $f - \pi f \in (\mathrm{Ker}\, \Delta_P^i)^\perp$. So if we denote the Greens operator for Δ_P^i guaranteed by 2.5.4 (ii) by G_P^i, (by (a) of Prop 2.5.4, (ii)) we have

$$\Delta_P^i G_P^i (f - \pi f) = f - \pi f \quad \text{for } f \in L^2(M, E_i)$$

But by the same (a) of Prop. 2.5.4 (ii) $G_P^i \equiv 0$ on $\mathrm{Ker}\, \Delta_P^i$ so $G_P^i(\pi f) = 0$. Thus

$$\Delta_P^i G_P^i f = f - \pi f \quad \text{for } f \in L^2(M, E_i) \quad \text{—— (39)}$$

So $f = \pi f + P_i^*(P_i G_P^i f) + P_{i-1}(P_{i-1}^* G_P^i f)$

which shows, (since $G_P^i f \in H_{2d}(M, E_i)$, $P_i G_P^i f \in H_d(M, E_{i+1})$ etc.) that

$$L^2(M, E_i) = \mathrm{Ker}\, \Delta_P^i + P_i^*(H_d(M, E_{i+1})) + P_{i-1}(H_d(M, E_{i-1}))$$
$$= \mathrm{Ker}\, \Delta_P^i + \mathrm{Im}\, P_i^* + \mathrm{Im}\, P_{i-1}.$$

To see that the decomposition <u>is</u> <u>orthogonal</u> is any easy Exercise; [using the fact that $\Delta_P^i f = 0 \Rightarrow P_i^* P_i + P_{i-1} P_{i-1}^* f = 0$ $\Rightarrow (P_i f, P_i f) + (P_{i-1}^* f, P_{i-1}^* f) = 0 \Rightarrow P_i f = 0$ and $P_{i-1}^* f = 0$. Conversely $P_i f = 0 = P_{i-1}^* f \Rightarrow \Delta_P^i f = 0$. Then, for example, if $f \in \mathrm{Ker}(\Delta_P^i)$, $(f, P_i^* g) = (P_i f, g) = 0 \Rightarrow \mathrm{Ker}\, \Delta_P^i \perp \mathrm{Im}\, P_i^*$ etc.].

<u>Remark</u>:- If f in (39) is in $C^\infty(M, E_i)$, then since by 2.5.4 (ii) b, we have $G_P^i(C^\infty(M, E_i)) \subset C^\infty(M, E_i)$, it follows that

$$C^\infty(M, E_i) = \mathrm{Ker}\,(\Delta_P^i) \oplus \Delta_P^i(C^\infty(M, E_i))$$

which is just the <u>elliptic regularity theorem</u> that $\Delta_P^i f = g$ has a unique C^∞- solution f, (for a C^∞ function g which is orthogonal to $\mathrm{Ker}\, \Delta_P^i$) orthogonal to Δ_P^i.

Now we can prove (i). Since we saw above that $\mathrm{Ker}\, \Delta_P^i$ $= \{f \in C^\infty(M, E_i) : P_i f = P_{i-1}^* f = 0\}$, we can define <u>the Hodge map</u>

$$\Phi : \mathrm{Ker}\, \Delta_P^i \longrightarrow \frac{\mathrm{Ker}\,\{P_i : C^\infty(M, E_i) \to C^\infty(M, E_{i+1})\}}{\mathrm{Im}\,\{P_{i-1} : C^\infty(M, E_{i-1}) \to C^\infty(M, E_i)\}} = H^i(M, \{E, P\})$$

We claim that this map Φ is

injective : Suppose $f \in \mathrm{Ker}\,\Delta_P^i$ and $f \in \mathrm{Im}\, P_{i-1}$ i.e. $f = P_{i-1} g$ for some $g \in C^\infty(M, E_{i-1})$. Then since $P_{i-1}^* f = 0$ we have $P_{i-1}^* P_{i-1} g = 0 \Rightarrow (P_{i-1}^* P_{i-1} g, g) = 0$

$$\Rightarrow (P_{i-1}g, P_{i-1}g) = \|P_{i-1} g\|^2 = \|f\|^2 = 0 \Rightarrow f = 0$$

Surjective : Suppose $f \in \mathrm{Ker}(P_i : C^\infty(M, E_i) \to C^\infty(M, E_{i+1}))$. By the Remark above

$$f - \pi f = \Delta_P^i G_P^i f$$
$$= G_P^i \Delta_P^i f \quad (\text{by Prop 2.5.4 (ii), (b)})$$

But $P_i f = 0 \Rightarrow \Delta_P^i f = P_{i-1} P_{i-1}^* f$

Now since (Δ_P^i) commutes with P_i, P_{i-1}^* and G_P^i is either 0 or $(\Delta_P^i)^{-1}$ it also commutes with P_i, P_{i-1}^*. Thus $G_P^i(P_{i-1}P_{i-1}^* f) = P_{i-1} P_{i-1}^* G_P^i f$. Thus $f = \pi(f) + P_{i-1} g$ for some $g \in C^\infty(M, E_{i-1})$

This means $[f] = [\pi(f)] = [\Phi(\pi(f))]$ in $H^i(M, \{E_\cdot, P_\cdot\})$ and Φ is therefore surjective. #. [Note: In [deR], the Green's operator for Δ is constructed by hand, as an integral operator with a smooth integral kernel singular along the diagonal, in the spirit of Prop 1.3.5]

3.1.9 Corollary. (Hodge Theorem for de-Rham Complex). Let M be as in Example 3.1.5 above. The operator $(d_i)^*$ is denoted δ_{i+1}, and the corresponding Laplacian $\Delta_P^i = \delta_{i+1} d_i + d_{i-1}\delta_i$ (called the Laplace-Beltrami operator) has finite-dimensional-kernel $\mathrm{Ker}(\Delta_P^i)$ (denoted by $\mathcal{H}^i$ = the space of harmonic i-forms $= \{ f \in \Lambda^i(M, \mathbb{C}) : df = \delta f = 0 \}$) which is by Propn 3.1.8 (i) above isomorphic to the i-th de-Rham Cohomology $H^i_{dR}(M, \mathbb{C})$. Thus every i-de-Rham cohomology class has a unique harmonic representative.

3.1.10 Corollary :- For the twisted Dolbeault Complex of Example 3.1.6, the Laplacian

$\Box_{\bar\partial, E} = \bar\partial_E^* \bar\partial_E + \bar\partial_E \bar\partial_E^* : \Lambda^{p,q}(M,E) \to \Lambda^{p,q}(M,E)$ has finite dimensional kernel isomorphic to $H^{p,q}_{\bar\partial}(M,E)$ the Dolbeault (p,q)-cohomology (= q-th cohomology of the Dolbeault complex $\{\Lambda^{p,\cdot}(M,E), \bar\partial\}$)

$\Box_{\bar\partial, E}$ is sometimes called the Hodge-Laplacian.

3.1.11 Corollary: In the setting of Example 3.1.6, one can show that the complex of sheaves $\{\Lambda^{p,\cdot}(\ ,E)\}$ is a fine resolution of the sheaf $\Omega^p(E)$, the sheaf of holomorphic p-differentials with coefficients in E, whose complex of global sections therefore computes the Dolbeault cohomology. viz. ; $H^{p,q}_{\bar\partial}(M,E) = H^q(M, \Omega^p(E))$ where the right-side is sheaf-cohomology.

Thus by 3.1.10 we see that the sheaf cohomologies $H^q(M, \Omega^p(E))$ are all finite-dimensional. In particular, the space of global-holomorphic E valued p-forms $H^0(M, \Omega^p(E))$ is finite dimensional $\forall\ p = 0, \dots, n = \dim_{\mathbb{C}} M$

§ 3.2 The Index of an Elliptic Complex

3.2.1. Def:- Let $\{E_\cdot, P_\cdot\}_{i=0}^{r}$, viz.,

$$\cdots\ C^\infty(M, E_i) \xrightarrow{P_i} C^\infty(M, E_{i+1}) \to \cdots$$

be an elliptic complex (of finite length $r+1$). By virtue of the Hodge Theorem Prop. 3.1.8 (i) for elliptic complexes, it makes sense to define the index of this elliptic complex

$$\operatorname{ind}(\{E_\cdot, P_\cdot\}) = \sum_{i=0}^{r} (-1)^i \dim_{\mathbb{C}} H^i(M, \{E_\cdot, P_\cdot\})$$

$$\text{by Prop 3.1.8} \quad = \sum_{i=0}^{r} (-1)^i \dim_{\mathbb{C}}(\operatorname{Ker} \Delta_P^i)$$

For Example, in the case of the de-Rham complex of Example 3.1.5, this index is $\sum_{i=0}^{\dim_{\mathbb{R}} M} (-1)^i H^i_{dR}(M, \mathbb{C}) = \chi(M)$ the Euler characteristic of M. In the case of the Dolbeault Complex with $\boxed{p=0}$

$$\Lambda^{0,0}(E) \xrightarrow{\bar\partial_E} \cdots \quad \Lambda^{0,q}(E) \xrightarrow{\bar\partial_E} \Lambda^{0,q+1}(E) \to \cdots$$

this index is the Arithmetic Euler characteristic of M in E, viz

$$\chi(M,E) = \sum_{q=0}^{\dim_{\mathbb{C}} M} (-1)^q H^{0,q}_{\bar\partial}(M,E)$$

$$= \sum_{q=0}^{\dim_{\mathbb{C}} M} (-1)^q H^q(M, \Omega^0(E))$$

sheaf of holo sections of E (also denoted $\mathcal{O}(E)$)

Many of the theorems of differential geometry and complex geometry such as the Gauss-Bonnet theorem, the Hirzebruch signature Theorem, the Riemann-Roch Theorem are statements to the effect that the index defined in 3.2.1 (which is an <u>analytical</u> object) is in fact a <u>topological</u> invariant. To do this, one exhibits a <u>characteristic class</u>, (arising out of M, E_i), of top dimension, and shows that its <u>Evaluation</u> on the fundamental class $[M]$ of M (i.e. its integral over M, if one represents it as a top-dimensional differential form) is equal to the (analytic) index defined above. The most general such theorem is, of course, the Atiyah-Singer Index Theorem for the Dirac operator, which implies all of the results cited above. One method of proving such a theorem for a general elliptic complex, which we outline below, is the <u>Heat Equation Method</u>.

First suppose that Q is some <u>self-adjoint</u>, <u>Elliptic</u> differential operator

$$Q : C^\infty(M,E) \to C^\infty(M,E)$$

with ~~negative~~ <u>definite</u> leading symbol, so that by <u>Propn. 2.5.4</u>, we have $L^2(M,E) = \bigoplus_{\lambda_i} V_{\lambda_i}$, the L^2-direct sum decomposition into eigenspaces for Q, $Qf_i = \lambda_i f_i$ $\{f_i\}_{i=1}^\infty$ orthonormal, etc. by (ii), (d) of that proposition. Define the <u>heat operator</u> e^{-tQ} <u>associated to Q</u> by

$$e^{-tQ} f_i = e^{-t\lambda_i} f_i \qquad \text{for } t>0 \quad \text{— (40)}$$

on the eigenfunctions f_i. Since $\lambda_i \to \infty$ as $i \to \infty$ (note by Remark 2.4.5, $\lambda_i \in [c,\infty)$) $|e^{-t\lambda_i}| < 1$ for i large enough, so e^{-tQ} extends to a <u>bounded</u> operator $e^{-tQ} : L^2(M,E) \to L^2(M,E)$

$$f = \sum_{i=1}^\infty \alpha_i f_i \mapsto e^{-tQ} f = \sum_{i=1}^\infty e^{-t\lambda_i} \alpha_i f_i$$

It is called the <u>heat operator</u> because by differentiating (40) with respect to t, we see $\frac{\partial}{\partial t}(e^{-tQ} f_i) + Q(e^{-tQ} f_i) = 0$ $\forall i$ so for $f \in L^2(M,E)$, $e^{-tQ} f$ is a solution to $\frac{\partial g}{\partial t} + Qg = 0$, <u>the</u>

heat equation (when $Q = \Delta$ the laplacian, e.g.).

3.2.2 :- Def :- Let Q, λ_i e^{-tQ} be as defined above. Assume that $\lambda_i \sim i^{\alpha}$ for i large,** and α some positive number. Then the series $\sum_{i=1}^{\infty} e^{-t\lambda_i}$ converges for $t > 0$, and we call this number the trace of the heat operator $\mathrm{Tr}\,(e^{-tQ})$.

The behaviour of $\mathrm{Tr}\,(e^{-tQ})$ as $t \to 0$ is of key importance. Before we investigate it, let us view $\mathrm{Tr}\,(e^{-tQ})$ in another way:

3.2.3 Propn : Let Q, λ_i, e^{-tQ}, $\mathrm{Tr}\,(e^{-tQ})$ be as above in Def 3.2.2. Then the operator e^{-tQ} is given by an integral operator with smooth kernel $K(t, x, y)$ ie

(i) $$(e^{-tQ} f)(x) = \int_M K(t, x, y) f(y)\, dy$$

(ii) Furthermore $$\mathrm{Tr}\,(e^{-tQ}) = \int_M K(t, x, x)\, dx.$$

Proof :- Define $K(t, x, y) = \sum_{i=1}^{\infty} e^{-t\lambda_i} f_i(x)\, \bar{f}_i(y)$. (Here $\bar{f}_i$ denotes the basis dual to f_i under the L^2 pairing in $L^2(M, E)$) Since $\lambda_i \to \infty$ as a positive power of i, and f_i, $\bar{f}_i$ are smooth by Prop 2.5.4. (d), this series above converges quite rapidly and uniformly together with its derivatives; so it defines a smooth function.

Also, $$\int_M K(t, x, y) f_k(y)\, dy = \sum_{i=1}^{\infty} e^{-t\lambda_i} f_i(x) \int_M (f_i(y), f_k(y))\, dV$$
$$= \sum_{i=1}^{\infty} e^{-t\lambda_i} f_i(x) . \delta_{ik} = e^{-t\lambda_k} f_k(x) := e^{-tP} f_k(x). \quad \forall\ k$$

So $$\int_M K(t, x, y) f(y)\, dy = (e^{-tP} f)(x) \quad \forall\ f \in L^2(M, E).$$

** It is a fact that if Q is of order d, then $\lambda_i \gtrsim C\, i^{d/n}$ for i large, where $n = \dim M$. See Lemma 1.6.3 (c) in [Gil]

Finally $\int_M K(t,x,x)\,dV_x = \sum_{i=1}^{\infty} e^{-t\lambda_i} \int_M (f_i(x), f_i(x))\,dV_x$

$= \sum_{i=1}^{\infty} e^{-t\lambda_i} = \mathrm{Tr}\,(e^{-tQ})$ by def.

This proves the assertion. #.

Now let us go back to the Elliptic complex $\{E_\cdot, P_\cdot\}_{i=0}^{r}$ on M. There is a beautiful heat-trace interpretation of the index $\{E_\cdot, P_\cdot\}$ viz. As a side remark, note that $\frac{\partial}{\partial t} K(t,x,y) + P_x K(t,x,y) = 0$, i.e. $K_t(x,y)$ is a fundamental solution of the heat equation.

<u>3.2.4 Propn</u>: Let $\{E_\cdot, P_\cdot\}_{i=0}^{r}$ be an <u>elliptic complex</u> on M, and assume that all the associated laplacians Δ_P^i have <u>positive definite leading symbols</u> so that $\mathrm{Tr}(e^{-t\Delta_P^i})$ makes sense for each i by Propn 3.2.3. Then for any $t>0$

$$\sum_{i=0}^{r} (-1)^i \,\mathrm{Tr}\left(e^{-t\Delta_P^i}\right) = \mathrm{ind}\,(E_\cdot, P_\cdot).$$

<u>Proof</u>:- Let us denote by E_λ^i the closed subspace of $L^2(M, E_i)$ corresponding to the eigenvalue λ for Δ_P^i, (i.e. the λ-eigenspace for Δ_P^i). We know by Prop. 2.5.4 (ii), (d) that $E_\lambda^i \subset C^\infty(M, E_i)$. Also $E_0^i = \mathrm{Ker}(\Delta_P^i)$. By definition

$$\mathrm{Tr}\,(e^{-t\Delta_P^i}) = \sum_{\lambda\in\mathbb{R}} e^{-t\lambda} \dim(E_\lambda^i)$$

So

$$\sum_{i=0}^{r} (-1)^i \,\mathrm{Tr}(e^{-t\Delta_P^i}) = \sum_{\lambda\in\mathbb{R}} e^{-t\lambda} \left(\sum_{i=0}^{r} (-1)^i \dim(E_\lambda^i)\right) \quad \text{--- (41)}$$

<u>Claim</u>: $\sum_{i=0}^{r} (-1)^i \dim(E_\lambda^i) = 0$ for $\lambda \neq 0$.

<u>Proof</u>: Note that since P's commute with Δ_P^i,

$$P_i(E_\lambda^i) \subset E_\lambda^{i+1} \quad \forall\, \lambda. \quad \text{Similarly} \quad P_{i-1}^*(E_\lambda^i) \subset E_\lambda^{i-1}$$

for all λ. Thus for each $\lambda \in \mathbb{R}$, we have the subcomplex

$$\begin{array}{ccccccc} \cdots \to & E_\lambda^i & \xrightarrow{P_i} & E_\lambda^{i+1} & \to \cdots \\ & \cap & & \cap & \\ & C^\infty(M, E_i) & & C^\infty(M, E_{i+1}) & \end{array}$$

of the original Elliptic complex.

To prove the claim, it is enough to show that this complex is Exact for $\lambda \neq 0$.

Suppose $f \in E^i_\lambda$, $P_i f = 0$, $\Delta^i_p f = \lambda f$ for $\lambda \neq 0$.

$\Rightarrow (P_i^* P_i + P_{i-1} P_{i-1}^*) f = \lambda f \Rightarrow P_{i-1} (P_{i-1}^* f) = \lambda f$

Since $P_i f = 0 \Rightarrow f = \frac{1}{\lambda} P_{i-1} (P_{i-1}^* f) = P_{i-1} (\frac{1}{\lambda} P_{i-1}^* f)$

(Since $\lambda \neq 0$) $\Rightarrow f \in \operatorname{Im} P_{i-1}$.

This proves the claim.

Thus on the right side of (41), the only term which survives is the one corresponding to $\lambda = 0$, which is

$$e^{-t \cdot 0} \cdot \left(\sum_{i=0}^{r} (-1)^i \dim E_0^i \right) = \sum_{i=0}^{r} (-1)^i \dim \operatorname{Ker}(\Delta^i_p)$$

$= \operatorname{ind}(\{E_\cdot\, P_\cdot\})$ by definition of the index.

This proves the Proposition.

Let us denote the integral kernel corresponding to $e^{-t\Delta^i_p}$ guaranteed by Prop 3.2.3 as $K_i(t, x, y)$ and also $\sum_{i=0}^{r} (-1)^i K_i(t, x, y)$ as $K(t, x, y)$. Then

3.2.5 Corollary :- In the setting of the previous Propn. 3.2.4).

$$\int_M K(t, x, x)\, dV_x = \operatorname{ind}(\{E_\cdot, P_\cdot\}).$$

Proof :- The left side $= \sum_{i=0}^{r} (-1)^i \int_M K_i(t, x, x)\, dV_x$

$= \sum_{i=0}^{r} (-1)^i \operatorname{Tr}(e^{-t\Delta^i_p})$ by Propn. 3.2.3

$= \operatorname{ind}(\{E_\cdot\, P_\cdot\})$ by Propn 3.2.4. #.

In fact since the right side is independent of t, we see that

$$\lim_{t \to \infty} \int_M K(t, x, x)\, dV_x = \operatorname{ind}(\{E_\cdot\, P_\cdot\}) \#$$

$$= \lim_{t \to 0} \int_M K(t, x, x)\, dV_x$$

The last corollary has given us an integral formula for the index. The next task is to identify the integrand $K(t,x,x)$. This is the

3.2.6 Propn Let $Q: C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order d; and let $K(t,x,y)$ be defined as the integral kernel of the corresponding heat operator. (So we are assuming that Q has negative definite leading symbol). as defined in Propn. 3.2.3. Then

(i) $$K(t,x,x) \sim \sum_{m=0}^{\infty} t^{\frac{m-n}{d}} e_m(x) \qquad (42)$$

as $t \to 0^+$ (where $n = \dim_{\mathbb{R}} M$), and $e_m(x)$ depends functorially on the derivatives (of upto some order) of the symbol of Q.

(This means; given $k \geq 0$; $\left| K(t,x,x) - \sum_{n \leq n(k)} t^{\frac{m-n}{d}} e_n(x) \right|_{\infty,k} < C_k t^k$ for $t \in (0,1)$ for some $n(k)$, C_k depending on k)

(ii) $e_m(x) = 0$ for m odd. (For a Proof, see Lemma 1.7.4 [Gil])

3.2.7 Corollary: For $\{E_\cdot, P_\cdot\}$ an elliptic complex, M of dim. n

$$\operatorname{ind}\{E_\cdot, P_\cdot\} = \int_M \alpha_n(\{E_\cdot, P_\cdot\})\, dVol \quad \text{for } n \text{ even}$$
$$= 0 \quad \text{for } n \text{ odd.}$$

where $\alpha_n(\{E_\cdot, P_\cdot\})dVol$ is an n-form on M which depends functorially on derivatives upto a certain order of the symbol of P)

Proof: Let $K(t,x,y) = \sum_{i=0}^{r} (-1)^i K_i(t,x,y)$ as in Cor 3.2.5, where K_i is the integral kernel for $e^{-t\Delta_P^i}$; and let $e_m^i(x)$ denote the coefficients arising in the right side of (42), the asymptotic expansion of $K_i(t,x,x)$. Noting that $\sum_{i=0}^{r}(-1)^i \int K_i(t,x,x)\,dVol$ is the index $\operatorname{ind}\{E_\cdot, P_\cdot\}$, so indep. of t as in Cor 3.2.5, we see that $\int_M \sum_{i=0}^{r} (-1)^i e_n^i(x)\, dVol = \text{index}(\{E_\cdot, P_\cdot\})$

(i.e. only the $m=n$ term survives in (42) after integration, because an (asymptotic) series in powers of t is equal to a constant $=$ index $(\{E_\cdot, P_\cdot\})$ which is independent of t. Because of (ii) of 3.2.6, this is 0 for n odd, (since $e_n^i(x)$ is 0 for n odd), and equal to the claimed quantity for n even by setting $\alpha_n(\{E_\cdot, P_\cdot\}) = \sum_{i=0}^{r} (-1)^i e_n^i(x)$.

The final (and really difficult!) step is to actually compute the form $\alpha_n(\{E_\cdot, P_\cdot\})$. The Atiyah-Singer index Theorem does this in general (see § 13 chapter III, [L,M], or [A-B-P], or [Gil] § 3.9. However, since we've talked only about the de-Rham and Dolbeault complexes (and not about the other classical complexes, such as the Signature Complex, or spin complex), let us at least state what the Theorem says in these cases.

3.2.8 The Gauss-Bonnet Theorem: (Compare [Gil] § 2.4, and [A-B-P].) When M is an even-dimensional manifold (Riemannian, compact, oriented) with the skew symmetric curvature form $[\Omega_{ij}]$. one defines the Pfaffian of Ω as

$$Pf(\Omega) = \sum_{i_1 \dots i_{2n}} \epsilon_{i_1 \dots i_{2n}} (\Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2n-1} i_{2n}})$$

(where Ω is written with respect to a local-orthonormal frame, and the above $Pf(\Omega)$ is independent of the orthonormal frame because $Pf(\Omega) = Pf(A\Omega A^t)$ for A orthogonal, and $\epsilon_{i_1 \dots i_{2n}} = \text{sgn}(i_1 \dots i_{2n})$ permutation). The Gauss-Bonnet Theorem says that a_n (deRham Complex) $dV = C_n Pf(\Omega)$, so that

$$C_n \cdot \int_M Pf(\Omega) = \chi(M) = \text{index (de-Rham Complex)}.$$

Where C_n is a universal constant depending only on $n = \frac{1}{2}(\dim M)$. (which is normalised so that for the sphere S^{2n} with standard metric you get equality).

3.2.9. The Riemann-Roch Theorem: Applying Cor 3.2.7 to the twisted Dolbeault Complex, and defining the Todd class of a complex vector bundle to be the characteristic class coming from the multiplicative formula $\prod_\nu \left(\frac{x_\nu}{1-e^{-x_\nu}}\right)$, the integrand $a_n\left(\begin{smallmatrix}\text{Twisted}\\ \text{Dolbeault}\\ \text{with } p=0\end{smallmatrix}\right)$ is identifiable as

$$a_n(\text{Twisted Dolbeault})\, dV = ch(E) \wedge Td(T^{1,0}(M))$$

so that by the definition of the arithmetic Euler characteristic after Def 3.2.1

$$\chi(M,E) = \text{index}\begin{pmatrix}\text{Twisted Dolbeault}\\ \text{Complex with } p=0\end{pmatrix} = \int_M ch(E)\wedge Td(T^{1,0}(M))$$

which is the Riemann-Roch Theorem.

In the case when M is a Riemann Surface, $T^{1,0}(M)$ is a holomorphic line bundle, and $Td(T^{1,0}(M)) = \dfrac{c_1(M)}{1-e^{-c_1(M)}}$

$$= \left(1 - \frac{c_1(M)}{2} + \frac{c_1^2(M)}{3!} + \cdots\right)^{-1} = 1 + \frac{c_1(M)}{2} \quad \text{Since } c_1^2 = c_1^3 = \cdots = 0$$

If we further assume that E is a line bundle as well, then $ch(E) = 1 + c_1(E)$ so that $ch(E)\wedge Td(T^{1,0}(M))$ has the degree-2 component $= \frac{1}{2}c_1(M) + c_1(E)$. But $\int c_1(M) = 2-2g$ and $\int c_1(E) = \deg([E])$, we get the classical Riemann-Roch formula:

$$\begin{aligned}\chi(M,E) &= \dim H^0(M,\mathcal{O}(E)) - \dim H^1(M,\mathcal{O}(E))\\ &= 1 - g + \deg([E])\end{aligned}$$

where $[E]$ = the divisor corresponding to E.

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