INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.637/6

ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

(31 August - 11 September 1992)

Sheaves and Schemes

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Advanced Workshop on Arithmetic Algebraic Geometry
International Centre for Theoretical Physics
Trieste, Italy, 31 August - 11 September 1992

SHEAVES AND SCHEMES.

Philippe NUSS.

INTRODUCTION.

The starting point of the algebraic geometry is the study of the solutions of polynomial equations in several indeterminates

$$f_\alpha(x_1, \dots, x_n) = 0 \tag{1}$$

where f_α belongs to $k[x_1, \dots, x_n]$, α runs through a certain set, and k is in the beginning the field of real numbers $\mathbf{R}$.

This is however too restrictive. In fact, if we consider the equation

$$x_1^2 + x_2^2 + 1 = 0.$$

There are of course no solutions in $\mathbf{R}$. So one has to enlarge the field, and to look for solutions in $\mathbf{C}$ (for example, $(0, i)$ is such a solution).

On the other hand the diophantine problems suggest to restrict ourselves to the rational solutions, that is to take $k = \mathbf{Q}$. To simplify the problem, one may restrict it by considering only solutions modulo p (p prime in $\mathbf{Z}$), that is by taking $k = \mathbf{Z}/p\mathbf{Z}$, or even modulo p^n, that is $k = \mathbf{Z}/p^n\mathbf{Z}$ (remark that for $n > 1$, k is no longer a field!), and then to pass to the ring of p-adic integers $\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z}$, or to its field of fractions $\mathbf{Q}_p$.

The above remarks show that one has to take the coefficient ring k as general as possible. The study of the solution-sets of the equations of the type (1) (the so-called *algebraic sets*) leads naturally to the notion of schemes.

We do not give here the proofs. They may be found in the book of R. Hartshorne (see the bibliography).

VARIETIES.

Affine algebraic sets and affine varieties

Let k be a field which we shall assume, for simplicity, *algebraically closed.* As usually, denote the space k^n by $\mathbb{A}^n$: it is the *affine space of dimension n*. Denote by $\mathcal{V}(f_\alpha)$ the set of zeros in $\mathbb{A}^n$ of the equations (1):

$$\mathcal{V}(f_\alpha) = \{x = (x_1, \dots, x_n); \forall \alpha, f_\alpha(x) = 0\}.$$

This set $\mathcal{V}(f_\alpha)$ is called the *(affine) algebraic (sub)set* defined by the (f_α)'s.

If S is any subset of the polynomial ring $k[x_1, \dots, x_n]$, denote by $\mathcal{V}(S)$ the subset of $\mathbb{A}^n$ of those x such that $f(x) = 0$ for each f in S.

If $S = (f_\alpha)$ is the ideal of $k[x_1, \dots, x_n]$ generated by the (f_α)'s, it is easy to see that

$$\mathcal{V}(f_\alpha) = \mathcal{V}((f_\alpha)).$$

Let, on the other hand, be X a subset of $\mathbb{A}^n$. Set

$$\mathcal{I}(X) = \{f \in k[x_1, \dots, x_n]; \forall x \in X, f(x) = 0\}.$$

Clearly, $\mathcal{I}(X)$ is an ideal of $k[x_1, \dots, x_n]$. If $X = \mathcal{V}(I)$, then it is clear that $\mathcal{I}(\mathcal{V}(I))$ contains I. The relationship between $\mathcal{I}(\mathcal{V}(I))$ and I is given by the

Theorem (*Hilbert Nullstellensatz*). *Let I be an ideal of $k[x_1, \ldots, x_n]$, then*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$$

(recall that $\sqrt{I}$ is the radical of the ideal I, consisting of the elements of $k[x_1, \ldots, x_n]$ such that some power lies in I.)

One can prove the general results:

Proposition. *Let I, J, I_α be ideals of $k[x_1, \ldots, x_n]$ and X, Y be algebraic sets of $\mathbb{A}^n$. Then:*

$$\begin{array}{ccc} I \subset J \Rightarrow \mathcal{V}(I) \supset \mathcal{V}(J) & X \subset Y \Rightarrow \mathcal{I}(X) \supset \mathcal{I}(Y) & \\ \mathcal{I}(X \cup Y) = \mathcal{I}(X) \cap \mathcal{I}(Y) & \mathcal{I}(\emptyset) = k[x_1, \ldots, x_n] & \mathcal{I}(\mathbb{A}^n) = \{0\} \\ \mathcal{V}(I \cap J) = \mathcal{V}(IJ) = \mathcal{V}(I) \cup \mathcal{V}(J) & \mathcal{V}(k[x_1, \ldots, x_n]) = \emptyset & \mathcal{V}(0) = \mathbb{A}^n \\ \mathcal{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathcal{V}(I_\alpha) & & \end{array}$$

The two last lines prove that one can take the $\mathcal{V}(I)$ (with I ideal of $k[x_1, \ldots, x_n]$) as the closed sets of a topology on $\mathbb{A}^n$, called the *Zariski topology*. One endows each algebraic set X with the topology induced by the Zariski topology of $\mathbb{A}^n$.

Proposition. *Let X be any subset of $\mathbb{A}^n$, then*

$$\mathcal{V}(\mathcal{I}(X)) = \bar{X} \text{ (the closure of } X).$$

Remark: There is a one-to-one correspondance between radical ideals of $k[x_1, \ldots, x_n]$ (that is, ideals I such that $I = \sqrt{I}$) and algebraic subsets of $\mathbb{A}^n$. One can easily prove that $X = \mathcal{V}(I)$ is irreducible if and only if I is a prime ideal.

An *affine variety* is an irreducible closed subset of $\mathbb{A}^n$ with the induced topology. A *quasi-affine variety* is an open subset of an affine variety. For example $\mathbb{A}^n$ is an affine variety.

Let $X = \mathcal{V}(I)$ be an algebraic subset of $\mathbb{A}^n$. Each polynomial f in $k[x_1, \ldots, x_n]$ determines a function on X by

$$\begin{array}{rccc} f: & X & \longrightarrow & k \\ & x = (x_1, \ldots, x_n) & \longmapsto & f(x_1, \ldots, x_n) \end{array}$$

The restriction of f to X is called a ***regular function*** on X. The set of regular functions on X is an algebra isomorphic to $k[x_1, \ldots, x_n]/\mathcal{I}(\mathcal{V}(I)) = k[x_1, \ldots, x_n]/\sqrt{I}$ and is called *coordinate ring of* X. This algebra is denoted by $\mathcal{O}(X)$. It is a finitely generated noetherian k-algebra, which is reduced (that is, there are no non-zero nilpotent elements). Moreover X is irreducible if and only if $\mathcal{O}(X)$ is an integral domain. Conversely, any finitely generated k-algebra which is a domain is the coordinate ring of some affine variety.

Proposition. *An algebraic set is determined by its coordinate ring.*

Proof: The points of X are in one-to-one correspondance with the k-homomorphisms from $\mathcal{O}(X)$ to k. For, to $x \in X$ one associates the evaluation $f \longmapsto f(x)$. Conversely, to a k-homomorphism $\varphi : \mathcal{O}(X) \longrightarrow k$, one associates the point $(x_1, \ldots, x_n)$ in X, where $x_1, \ldots, x_n$ are the images by φ of a fixed set of generators $u_1, \ldots, u_n$ of $\mathcal{O}(X)$. ◇

Let X be an algebraic subset of $\mathbb{A}^n$, and x a point of X. A rational function $f \in k(x_1, \ldots, x_n)$ is called ***regular*** or ***defined*** at x, if $f = \frac{u}{v}$, where u and v are polynomials, and $v(x) \neq 0$. So $f(x) = \frac{u(x)}{v(x)}$ is well defined. By definition, a ***rational function*** f' ***on*** X is the restriction of a rational function on $\mathbb{A}^n$. Hence f' is not defined everywhere on X, but its domain is an open set in X.

If U is an open set in X, one defines the ring $\Gamma(U, \mathcal{O}_X)$ consisting of rational functions on X regular at every point of U. The assignment $U \longmapsto \Gamma(U, \mathcal{O}_X)$ has all the properties of what we shall later call a *sheaf*.

Proposition. *Let $f \in \mathcal{O}(X)$. Denote by X_f the open subset*

$$X_f = \{x \in X; f(x) \neq 0\}.$$

Then, the following isomorphism holds:

$$\Gamma(X_f, \mathcal{O}_X) \cong \mathcal{O}(X)_f.$$

(here $\mathcal{O}(X)_f$ is the localization of $\mathcal{O}(X)$ with respect to f.)

As a corollary, one gets:

$$\Gamma(X, \mathcal{O}_X) \cong \mathcal{O}(X).$$

Let $X = \mathcal{V}(I)$ be an algebraic subset in $\mathbb{A}^n$. The Hilbert Nullstellensatz proves that the set $\mathrm{Max}\mathcal{O}(X)$ of maximal ideals of $\mathcal{O}(X)$ (the so-called *maximal spectrum*) is in bijection with the set X.

Hence one has canonical bijections:

$$Hom_{k-\mathfrak{Alg}}(\mathcal{O}(X), k) \simeq \mathrm{Max}\mathcal{O}(X) \simeq X.$$

Morphisms of affine varieties.

Let X and Y be two affine varieties. A *morphism* $\varphi : X \longrightarrow Y$ is a continuous map such that, for every open subset $V \subseteq Y$ and every regular function $f : V \longrightarrow k$, the function $f \circ \varphi : \varphi^{-1}(V) \longrightarrow k$ is regular. We get now the category of affine varieties over k, $\mathfrak{Var}/k$. One can prove that there is a bijective mapping of sets

$$Mor_{\mathfrak{Var}/k}(X, Y) \cong Hom_{k-\mathfrak{Alg}}(\mathcal{O}(Y), \mathcal{O}(X))$$

As a corollary, two affine varieties are isomorphic in $\mathfrak{Var}/k$ if and only if the k-algebras are isomorphic. This means that the category $\mathfrak{Var}/k$ is equivalent to the opposite category of finitely generated integral domains over k.

We are now in the following situation: We have a subset X defined by an ideal I of the ring $k[x_1, \dots, x_n]$. It is completely determined by its coordinate ring $\mathcal{O}(X)$. On X there is a topology, and to each open subset U of X, one associates a ring of functions, this assignment having nice properties (it is a sheaf). In particular, if we take for U the whole space X, we recover $\mathcal{O}(X)$.

This situation is pretty usual in mathematics. Instead of dealing with polynomial functions, if one takes continuous (resp. differentiable, resp. holomorphic) functions, we are in the framework of topology (resp. of differential geometry, resp. of analytical geometry).

The next step is to generalize the above construction for $k[x_1, \dots, x_n]$ to arbitrary rings, for example $\mathbb{Z}$, and to give a general setting of the algebraic geometry. This program was achieved by Alexander GROTHENDIECK with the theory of *schemes*. Before doing this, let us consider the projective case.

Projective varieties.

Instead of the affine space $\mathbb{A}^n$, we work with the projective space $\mathbf{P}^n$ over k, and instead of rings, we use graded rings. Recall that a ring S is *graded* when there exists a decomposition $S = \bigoplus_{d \geq 0} S_d$ of S into a direct sum of abelian groups S_d, such that $S_d.S_e \subseteq S_{d+e}$ for any $d, e \geq 0$. An element in S_d is called *homogeneous of degree d*. An ideal I in S is called *homogeneous* if $I = \bigoplus_{d \geq 0} (I \cap S_d)$. The ring of polynomials $S = k[x_1, \dots, x_n]$ is naturally graded.

Now we can paraphrase the affine case by replacing rings by graded rings and ideals by homogeneous ideals, the only thing to take care of is that a polynomial (even homogeneous) in $S = k[x_1, \dots, x_n]$ *does not* define a function on $\mathbf{P}^n$, but nevertheless it makes sense to see whether a homogeneous polynomial is zero or not on an equivalence class defining a point in $\mathbf{P}^n$. Hence the *zeros* of a *homogeneous* polynomial are defined and we can form

$$\mathcal{V}(T) = \{P \in \mathbf{P}^n; \forall f \in T, f(P) = 0\}.$$

Here T is any set of homogeneous elements of S. This set $\mathcal{V}(T)$ is called the *(projective) algebraic (sub)set* defined by the T.

If Y is a subset of $\mathbf{P}^n$, one can define the *homogeneous ideal $\mathcal{I}(Y)$ of Y in $S = k[x_1, \ldots, x_n]$* by

$$\mathcal{I}(Y) = \{f \in S; f \text{ is homogeneous and } \forall P \in Y, f(P) = 0\}.$$

An algebraic set Y is irreducible if and only if $\mathcal{I}(Y)$ is prime. For example $\mathbf{P}^n$ is irreducible.

The homogeneous Hilbert Nullstellensatz insures that if I is an homogeneous ideal such that $\mathcal{V}(I) \neq \emptyset$, then

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

Moreover there is a one-to-one correspondance between algebraic sets in $\mathbf{P}^n$, and homogeneous radical ideals of $S = k[x_1, \ldots, x_n]$ not equal to $S_+ = \bigoplus_{d>0} S_d$.

If Y is an algebraic subset of $\mathbf{P}^n$, the ***homogeneous coordinate ring of** Y* is the graded ring

$$\mathcal{S}(Y) = k[x_1, \ldots, x_n]/\mathcal{I}(Y).$$

As in the affine case, one can define the ***Zariski topology*** on $\mathbf{P}^n$ by taking the closed sets to be the algebraic sets.

A ***projective variety*** is an irreducible algebraic set in $\mathbf{P}^n$ together with the topology induced by the Zariski topology of $\mathbf{P}^n$. An open subset of a projective variety is called a ***quasi-projective variety.***

Proposition. *A projective (resp. quasi-projective) variety has a covering by open sets which are homeomorphic to affine (resp. quasi-affine) varieties.*

One has the analogue of regular functions: they are locally defined as quotients $\frac{u}{v}$, where u and v are homogeneous polynomials of ***same*** degree.

SHEAVES.

Presheaves.

Let X be a topological space, and $\mathfrak{C}$ a category, which will be either the category $\mathfrak{Ab}$ of abelian goups, the category $\mathfrak{Rings}$ of commutative rings, or the category $k - \mathfrak{Mod}$ of modules over a fixed ring k.

A ***presheaf $\mathcal{F}$ on X with values in $\mathfrak{C}$*** is the following data:

-to every open subset U in X, one assigns an object $\mathcal{F}(U)$ in $\mathfrak{C}$

-to every inclusion $V \subseteq U$ of open subsets in X, one assigns a morphism $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ in $\mathfrak{C}$ called ***restriction morphism***

subject to the conditions:

$\rho_{UU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is the identity map

if $W \subseteq V \subseteq U$ are open subsets in X, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

A nice way to see that, is using the categorical language. Let $\mathfrak{Top}_X$ be the category whose objects are the open subsets U of X. The morphisms between U and V are

$$Hom_{\mathfrak{Top}_X}(U, V) = \begin{cases} \text{the inclusion map } U \rightarrow V \text{ if } U \subseteq V \\ \emptyset \quad \text{otherwise} \end{cases}$$

A presheaf $\mathcal{F}$ on X with values in $\mathfrak{C}$ is then nothing but a contravariant functor from $\mathfrak{Top}_X$ to $\mathfrak{C}$.

If U is an open subset in X, an element in $\mathcal{F}(U)$ is called ***section of $\mathcal{F}$ over U***. An element in $\mathcal{F}(X)$ is called ***global section.*** One usually denotes $\mathcal{F}(U)$ by $\Gamma(U, \mathcal{F})$.

Example: Let X be an affine variety. The assignment $U \longmapsto \Gamma(U, \mathcal{O}_X)$ is a presheaf of rings.

A *morphism of presheaves* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of functors. This means that for each open set U in X, there is a morphism in $\mathfrak{C}$, $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$, compatible with the restrictions, *ie* if $V \subseteq U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes.

Denote by $\mathfrak{P}(X)$ the category of presheaves over X.

Presheaves on a base of open sets.

Let $\mathfrak{B}$ be a base of open sets in X. This means that $\mathfrak{B}$ consits of open subsets of X such that, for each open subset A in X, there exists a covering of A by elements in $\mathfrak{B}$. We see $\mathfrak{B}$ as a full subcategory of $\mathfrak{Top}_X$.

A *presheaf* $\mathcal{F}$ *on* $\mathfrak{B}$ is a contravariant functor from $\mathfrak{B}$ to $\mathfrak{C}$. (One can replace this definition by the "down-to-earth" one, by replacing in the first definition of a presheaf "open subset" by "open subset belonging to $\mathfrak{B}$").

A *morphism* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ *of presheaves on* $\mathfrak{B}$ is a morphism of functors. The category of presheaves on $\mathfrak{B}$ is denoted by $\mathfrak{P}(\mathfrak{B})$.

The presheaf on X associated to a presheaf on $\mathfrak{B}$.

We keep the same notations as above. There are adjoint functors

$$res_{\mathfrak{B}} : \ \mathfrak{P}(X) \longrightarrow \mathfrak{P}(\mathfrak{B})$$

$$ext^X : \ \mathfrak{P}(\mathfrak{B}) \longrightarrow \mathfrak{P}(X)$$

The definition of the restriction $res_{\mathfrak{B}}$ is obvious: just forget about the open subsets in X which do not belong to $\mathfrak{B}$.

Define now the extension ext^X. Let $\mathcal{F}$ be a presheaf on $\mathfrak{B}$. Then $\mathcal{F}' = ext^X(\mathcal{F})$ is defined on an open subset U in X by

$$\mathcal{F}'(U) = \varprojlim \mathcal{F}(V)$$

where the inverse limit is taken over all the $V \in \mathfrak{B}$ contained in U. So, an element $s' \in \mathcal{F}'(U)$ is a family $(s_V)_{V \in \mathfrak{B}, V \subseteq U}$ such that, if $W \subseteq V$ with $V, W \in \mathfrak{B}$, then $\rho_{VW}(s_V) = s_W$.

If already U belongs to $\mathfrak{B}$, then $\mathcal{F}'(U) \cong \mathcal{F}(U)$.

Stalks.

Let $\mathcal{F}$ be a presheaf on X and x a point of X. The *stalk of* $\mathcal{F}$ *at* x is the object of $\mathfrak{C}$ denoted by $\mathcal{F}_x$ and defined by

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U)$$

where the direct limit is taken over all open subsets U in X containing x. For each U containing x, one has a canonical homomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}_x$. If s is a section of $\mathcal{F}$ over U, the image of s under this homomorphism is denoted by s_x.

Sheaves.

A presheaf $\mathcal{F}$ on X is a *sheaf* if the two following conditions hold:

1) For each open subset U in X and for each open covering $(U_\alpha)_{\alpha\in A}$ of U, if for each sections s and t over U, all the restrictions $s_\alpha := \rho_{UU_\alpha}(s)$ and $t_\alpha := \rho_{UU_\alpha}(t)$ agree, then $s = t$.

2) For each open subset U in X and for each open covering $(U_\alpha)_{\alpha\in A}$ of U, if for each family $\{s_\alpha \in \mathcal{F}(U_\alpha), \alpha \in A\}$ such that $\rho_{U_\alpha V}(s_\alpha) = \rho_{U_\beta V}(s_\beta)$ for each open $V \subseteq U_\alpha \cap U_\beta$, then there exists a unique $s \in \mathcal{F}(U)$ such that $\rho_{UU_\alpha}(s) = s_\alpha$ for all α.

A more compact way to express that is saying that a presheaf $\mathcal{F}$ on X is a sheaf if and only if, for each open subset U in X and for each open covering $(U_\alpha)_{\alpha\in A}$ of U, the diagram

$$\mathcal{F}(U) \xrightarrow{u} \prod_\alpha \mathcal{F}(U_\alpha) \overset{v_1}{\underset{v_2}{\rightrightarrows}} \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact. The map u is a restriction map and the maps v_1 and v_2 are products of restriction maps. (Recall that the above diagram is exact if $Imu = Ker(v_1, v_2) := \{x \in \prod_\alpha \mathcal{F}(U_\alpha); v_1(x) = v_2(x)\}$.)

A *morphism of sheaves* is a morphism of presheaves. So one gets the category of sheaves on X, denoted by $\mathfrak{S}(X)$ which is a full subcategory of $\mathfrak{P}(X)$. One can define in a similar way the category $\mathfrak{S}(\mathfrak{B})$ of sheaves on a base $\mathfrak{B}$.

Example: Let X be an affine variety. The assignment $U \longmapsto \Gamma(U, \mathcal{O}_X)$ is a sheaf of rings.

The sheaf associated to a presheaf.

Let $\mathcal{F}$ be a presheaf on X. There is a sheaf $\tilde{\mathcal{F}}$, and a morphism $\theta : \mathcal{F} \longrightarrow \tilde{\mathcal{F}}$ with the universal property: *for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$, there is a unique morphism $\psi : \tilde{\mathcal{F}} \longrightarrow \mathcal{G}$ such that $\varphi = \psi \circ \theta$. This pair $(\tilde{\mathcal{F}}, \theta)$ is unique up to isomorphism. $\tilde{\mathcal{F}}$ is called the sheaf associated to the presheaf $\mathcal{F}$.*

The construction of $\tilde{\mathcal{F}}$ is the following. Denote by E the disjoint union of the stalks $\mathcal{F}_x$. There is a canonical projection $p : E = \coprod \mathcal{F}_x \longrightarrow X$ with $p^{-1}(x) = \mathcal{F}_x$. Let U be an open set in X and $s \in \mathcal{F}(U)$. Denote by $\tilde{s} : U \longrightarrow E$ the map defined by $\tilde{s}(x) = s_x$. Then $p \circ \tilde{s} = id$. One makes E into a topological space by giving E the coarsest topology for which all the maps $\tilde{s}$ are continuous. Denote then by $\tilde{\mathcal{F}}(U)$ the set of continuous functions from U to E. Thus an element of $\tilde{\mathcal{F}}(U)$ is a family $(s'_x)_{x\in U}$, where $(s'_x) \in \mathcal{F}_x, \forall x \in U$ and for each $x \in U$, there exists an open neighbourhood V of x contained in U and $s \in \mathcal{F}(V)$ such that $s'_y = s_y$ for all $y \in V$.

One can easily prove that $\tilde{\mathcal{F}}$ with the natural restriction maps is a sheaf satisfying the universal property.

Remark 1: For each point $x \in X$, we have an isomorphism $\tilde{\mathcal{F}}_x \cong \mathcal{F}_x$.
Remark 2: If $\mathcal{F}$ is already a sheaf, then $\tilde{\mathcal{F}} \cong \mathcal{F}$.

We have thus a functor

$$\begin{array}{rccc} ass: & \mathfrak{P}(X) & \longrightarrow & \mathfrak{S}(X) \\ & \mathcal{F} & \longmapsto & \tilde{\mathcal{F}} \end{array}$$

which is left-adjoint to the forgetfull functor $\mathfrak{S}(X) \longrightarrow \mathfrak{P}(X)$

The direct image of a sheaf.

Let $f : X \longrightarrow Y$ be a continuous map between topological spaces and $\mathcal{F}$ be a sheaf on X. One defines a presheaf $f_*\mathcal{F}$ on Y by setting

$$\Gamma(V, f_*\mathcal{F}) = \Gamma(f^{-1}(V), \mathcal{F})$$

for V an open set in Y. One can show that $f_*\mathcal{F}$ is actually a sheaf, which is called the *direct image of $\mathcal{F}$*. We have thus two functors

$$f_* : \mathfrak{P}(X) \longrightarrow \mathfrak{P}(Y)$$

$$f_* : \mathfrak{S}(X) \longrightarrow \mathfrak{S}(Y)$$

Restriction of a (pre)sheaf to an open subset.

Let $\mathcal{F}$ be a presheaf on X and U an open subset in X. Then the $\mathcal{F}(V)$ for which $V \subseteq U$ form a presheaf on U, called the *restriction of $\mathcal{F}$ to U* and denoted by $\mathcal{F}|_U$. If $\mathcal{F}$ is a sheaf, then $\mathcal{F}|_U$ is also a sheaf.

Ringed spaces.

A *ringed space* (in french *espace annelé*) is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings (called the *structure sheaf*).

A *geometrical space* is a ringed space $(X, \mathcal{O}_X)$ such that, for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Example: Let X be an affine variety in $\mathbb{A}^n$. Then $(X, \mathcal{O}_X)$ is a geometrical space. Indeed, let $a = (a_1, \ldots, a_n)$ be a point of X. Let $\mathfrak{m}$ be the maximal ideal corresponding to a : $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$. Then

$$\mathcal{O}_{X,a} = (\mathcal{O}(X))_{\mathfrak{m}}$$

($(\mathcal{O}(X))_{\mathfrak{m}}$ is the localization of the coordinate ring $\mathcal{O}(X)$ at $\mathfrak{m}$.)

A *morphism of ringed spaces* $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair $(\psi : X \longrightarrow Y, \psi^! : \mathcal{O}_Y \longrightarrow \psi_*\mathcal{O}_X)$, where ψ is a continuous map and $\psi^!$ is a morphism of sheaves. Denote by $\mathfrak{RingSp}$ the category of ringed spaces.

For each $x \in X$, $\psi^!$ induces a homomorphism of rings

$$\psi^!_x : \mathcal{O}_{Y,\psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

given by the composition

$$\mathcal{O}_{Y,\psi(x)} = \varinjlim_{V \ni \psi(x)} \mathcal{O}_Y(V) \longrightarrow \varinjlim_{V \ni \psi(x)} \psi_*\mathcal{O}_X(V) = \varinjlim_{V \ni \psi(x)} \mathcal{O}_X(\psi^{-1}(V)) \longrightarrow \varinjlim_{U \ni x} \mathcal{O}_X(U) = \mathcal{O}_{X,x}.$$

A *morphism of geometrical spaces* is a morphism of ringed spaces, such that the above homomorphism induced on the stalks is local. Denote by $\mathfrak{GeomSp}$ the category of geometrical spaces.

SCHEMES.

The spectrum of a ring.

Let A be a ring (ring means always commutative ring with 1). As a set, let $X = \mathrm{Spec}A$ be the set of all prime ideals of A. (Recall that $\mathfrak{p} \subset A$ is *prime* if and only if $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, which is equivalent to the fact that $A/\mathfrak{p}$ is an integral domain.) Let $\mathfrak{a}$ be any ideal of A. Set $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}A; \mathfrak{p} \supset \mathfrak{a}\}$.

Lemma. *The family $(V(\mathfrak{a}))$ can be taken as the closed sets of a topology on* Spec$A$, *called the Zariski topology.*

Proof: If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.
$V(A) = \emptyset$ and $V(0) = \mathrm{Spec}A$. ◇

Remark: If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \Longleftrightarrow \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.

Define now a sheaf of rings $\mathcal{O}_X$ on $X = \mathrm{Spec}A$, called the *structure sheaf*. Let U be an open subset in X. Define

$$\mathcal{O}_X(U) = \{(x_\mathfrak{p})_{\mathfrak{p} \in U} \in \prod_{\mathfrak{p} \in U} A_\mathfrak{p}; \forall \mathfrak{p} \in U, \exists V \text{ neighbourhood of } \mathfrak{p} \text{ contained in } U,$$

$$\exists a, b \in A \text{ such that } \forall \mathfrak{q} \in V, b \notin \mathfrak{q} \text{ and } x_\mathfrak{q} = \frac{a}{b} \text{ in } A_\mathfrak{q}\}.$$

Note that this definition is very similar to those of regular functions on algebraic sets, except that here the functions take their values in the different localizations of A, instead of the fixed field k.

Lemma. *$\mathcal{O}_X$ is a sheaf on X.*

The ringed space $(X = \mathrm{Spec}A, \mathcal{O}_X)$ is called the *spectrum of the ring A*

Let $f \in A$. Denote by $D(f)$ the complement of $V((f))$. It is by definition an open subset in X. We have $D(f) = \{\mathfrak{p} \in \mathrm{Spec}A; \mathfrak{p} \not\ni f\}$. It is easy to see that $(D(f))_{f\in A}$ is a base of the Zariski topology of $\mathrm{Spec}A$.

Theorem. *Let A be a ring. One has the following results:*

1) For any $\mathfrak{p} \in \mathrm{Spec}A$, $\mathcal{O}_{X,\mathfrak{p}} \cong A_\mathfrak{p}$
2) For any element $f \in A$, $\Gamma(D(f), \mathcal{O}_X) \cong A_f$
3) In particular, $\Gamma(\mathrm{Spec}A, \mathcal{O}_X) \cong A$.

As a corollary, $(X = \mathrm{Spec}A, \mathcal{O}_X)$ is a geometrical space.

There is an other way to define the structure sheaf, namely set $\overline{\mathcal{O}_X}(D(f)) = A_f$. Then $\overline{\mathcal{O}_X}$ is a presheaf on the base $(D(f))_{f\in A}$. One can show that

$$\mathcal{O}_X = ext^X(\overline{\mathcal{O}_X}).$$

Functorial properties.

Theorem. *1) A ring homomorphism $\varphi : A \longrightarrow B$ induces a morphism of geometrical spaces*

$$(f, f^\natural) : (Y = \mathrm{Spec}B, \mathcal{O}_Y) \longrightarrow (X = \mathrm{Spec}A, \mathcal{O}_X).$$

2) Conversely, any morphism of geometrical spaces $(f, f^\natural) : (Y = \mathrm{Spec}B, \mathcal{O}_Y) \longrightarrow (X = \mathrm{Spec}A, \mathcal{O}_X)$ is induced by a ring homomorphism $\varphi : A \longrightarrow B$ as in 1).

Let us just construct $(f, f^\natural)$ in 1). The map $f : Y \longrightarrow X$ is given by $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$. It is continuous because, for each ideal $\mathfrak{a}$ in A, $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$. The definition of the structure sheaves gives a morphism of rings $f^\natural(V) : \Gamma(V, \mathcal{O}_X) \longrightarrow \Gamma(f^{-1}(V), \mathcal{O}_Y)$ for each open subset V in X, induced by the local homomorphism of local rings $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} \longrightarrow B_\mathfrak{p}$.

Schemes.

An *affine scheme* is a geometrical space $(X, \mathcal{O}_X)$ isomorphic (in the category $\mathfrak{GeomSp}$ of geometrical spaces) to the spectrum of some ring A. A *scheme* is a geometrical space $(X, \mathcal{O}_X)$ locally isomorphic (in the category $\mathfrak{GeomSp}$) to an affine scheme. This means that each point x in X has an open neighbourhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. A *morphism of (affine) schemes* is a morphism in the category $\mathfrak{GeomSp}$. One gets the category $\mathfrak{AffSch}$ of affine schemes and the category $\mathfrak{Sch}$ of schemes.

Let X be a scheme and $x \in X$. The *residue field* of x on X denoted by $\kappa(x)$ is the quotient

$$\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$$

where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$.

Let x be a point of a scheme X. The point x is called *closed* if the closure $\overline{\{x\}}$ of $\{x\}$ is $\{x\}$ itself. The point x is called *generic* if the closure $\overline{\{x\}}$ of $\{x\}$ is the whole space X.

Examples.

Example 1: Let k be a field. Then the topological space $\mathrm{Spec}k$ has just one point: (0). The structure sheaf is k. Suppose that $K \overset{\subset}{\neq} L$ are two fields. This inclusion yields a morphism of affine schemes $(\mathrm{Spec}L = \{*\}, L) \longrightarrow (\mathrm{Spec}K = \{*\}, K)$. On the level of the topological spaces it is of course an isomorphism (homeomorphism), but it is *not* an isomorphism of schemes!

Example 2: Spec$\mathbf{Z}$.

As a set, Spec$\mathbf{Z}$ is in bijection with $\{0\ ; prime\ numbers\ p \in \mathbf{Z}\}$. From the topological point of view, the closed subsets are $V(n) = \{p\ prime \in \mathbf{Z}; p \mid n\}$. So $p \in V(n) \iff n \in (p)$. The elementary open sets are $D(n) = \{p\ prime \in \mathbf{Z}; p \not| n\}$. The sheaf $\mathcal{O}_{\mathrm{Spec}\mathbf{Z}}$ is given by $\Gamma(D(n), \mathcal{O}_{\mathrm{Spec}\mathbf{Z}}) = \{\frac{a}{n^s}, a \in \mathbf{Z}, s \geq 0\} = \mathbf{Z}_n$ (localization with respect to n).

Stalks

i) at p prime.

$\mathcal{O}_{X,p} = \varinjlim_{p \not| n} \Gamma(D(n), \mathcal{O}_{\mathrm{Spec}\mathbf{Z}}) = \varinjlim_{p \not| n} \mathbf{Z}_n = \{\frac{a}{b}, p \not| b\} =: \mathbf{Z}_{(p)}$ (localization at the prime ideal (p))

Hence the residue field of p prime is $\kappa(p) = \mathbf{Z}_{(p)}/(p)\mathbf{Z}_{(p)} = \mathbf{Z}/p\mathbf{Z} = \mathbf{F}_p$.

ii) at (0).

$\mathcal{O}_{X,0} = \varinjlim_{n \neq 0} \mathbf{Z}_n = \mathbf{Q}$, hence the residue field of (0) is $\kappa(0) = \mathbf{Q}$.

It is easy to see that the closure $\overline{\{0\}}$ of $\{0\}$ is the whole space Spec$\mathbf{Z}$. The point 0 is generic.

One can prove, that Spec$\mathbf{Z}$ is a final object in the category $\mathfrak{Sch}$, *ie* each scheme admits a unique morphism to Spec$\mathbf{Z}$.

Example 3: Let R be a discrete valuation ring. Then the underlying topological space Spec$R$ has two points: one t_0 which is closed, with local ring R, and the other point t_1 which is open and generic with local ring K the field of fractions of R.

Example 4: Let k be an algebraically closed field. Set $\mathbb{A}_k^n = \mathrm{Spec}k[x_1, \ldots, x_n]$. The set of closed points of $\mathbb{A}_k^n$ with the induced topology is homeomorphic to the affine variety $\mathbb{A}^n$. There is also a generic point corresponding to the ideal (0) of $k[x_1, \ldots, x_n]$.

Example 5: **The schemes associated to the affine varieties.**

Let S be a scheme. A *scheme over* S is a scheme X together with a structure morphism $X \longrightarrow S$. A *morphism from the scheme* X *over* S *to the scheme* Y *over* S is a scheme morphism from X to Y which is compatible with the structure morphisms. The schemes over S constitute a category $\mathfrak{Sch}/S$ (the category of the objects over the object S). If A is a ring, we write $\mathfrak{Sch}/A$ instead of $\mathfrak{Sch}/\mathrm{Spec}A$. From example 2, we see that $\mathfrak{Sch}/\mathbf{Z} = \mathfrak{Sch}$.

Theorem. *Let k be an algebraically closed field. There is a natural fully faithfull functor*

$$t : \mathfrak{Var}/k \longrightarrow \mathfrak{Sch}/k.$$

Proof: If X is a variety, the underlying topological space of $t(X)$ is the set of (nonempty) irreducible closed subsets of X with the topology defined by taking as closed sets the subsets of the form $t(Y)$, where Y is a closed subset of X. One defines a continuous map

$$\begin{array}{rccc} \alpha : & X & \longrightarrow & t(X) \\ & P & \longmapsto & \overline{\{P\}} \end{array}$$

Then it remains to show, that $(t(X), \alpha_*(\mathcal{O}_X))$ is a scheme over k. ◇

The underlying topological space of X is homeomorphic to the set of closed points of $t(X)$, and the sheaf of regular functions of X is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.

Example: $t(\mathbb{A}^n) = \mathbb{A}_k^n = \mathrm{Spec}(k[x_1, \ldots, x_n])$.

Example 6: **The scheme Proj$S$.**

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and S_+ the ideal $\bigoplus_{d>0} S_d$.

Let Proj$S$ be the set of all homogeneous prime ideals $\mathfrak{p}$ which do not contain the ideal S_+. If we set for any homogeneous ideal $\mathfrak{a}$, $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}S; \mathfrak{p} \supseteq \mathfrak{a}\}$, then as in the affine case, the $V(\mathfrak{a})$ define the closed sets of a topology on Proj$S$. Define now a sheaf of rings on Proj$S$. Let $\mathfrak{p} \in \mathrm{Proj}S$. Let T be the multiplicative set consisting of all homogeneous elements of S which are not in $\wp$. Let $S_{(\wp)}$ be the ring of elements of degree zero in the localized ring $T^{-1}S$. Let $U \subseteq \text{Proj}S$ be an open set. Define

$$\mathcal{O}(U) = \{s : U \longrightarrow \coprod_{\wp \in U} S_{(\wp)}; \forall \wp \in U, \exists V \text{ neighbourhood of } \wp \text{ contained in } U,$$
$$\exists \textit{ homogeneous } a, b \in A \text{ such that } \forall \mathfrak{q} \in V, b \notin \mathfrak{q} \text{ and } s\mathfrak{q} = \frac{a}{b} \text{ in } S_{(\mathfrak{q})}\}.$$

Then $\mathcal{O}$ is a sheaf of rings.

Proposition. *The ringed space* $(\text{Proj}S, \mathcal{O})$ *is a scheme. The stalk* $\mathcal{O}_\wp$ *at* $\wp \in \text{Proj}S$ *is the local ring* $S_{(\wp)}$.

Example: Let A be a ring. The *projective n-space over A* is the scheme $\mathbf{P}^n_A := \text{Proj}A[x_1, \ldots, x_n]$. If k is an algebraically closed field, then the space of closed points of $\mathbf{P}^n_k$ is homeomorphic to the projective variety $\mathbf{P}^n$.

FIRST PROPERTIES OF SCHEMES.

Morphism of schemes.

Let A be a ring and $(X, \mathcal{O}_X)$ a scheme. One gets a morphism

$$\alpha : Mor_{\mathfrak{Sch}}(X, \text{Spec}A) \longrightarrow Hom_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X))$$

in the following way:
Let $f : X \longrightarrow \text{Spec}A$ be a morphism of schemes and $f^\sharp : \mathcal{O}_{\text{Spec}A} \longrightarrow f_*\mathcal{O}_X$ the corresponding morphism of sheaves. Then $\alpha(f)$ is obtained by taking global sections

$$f^\sharp(\text{Spec}A) : \mathcal{O}_{\text{Spec}A}(\text{Spec}A) = A \longrightarrow f_*\mathcal{O}_X(\text{Spec}A) = \Gamma(f^{-1}(\text{Spec}A), \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X).$$

Proposition. *The map* α *is bijective.*

As a corollary, if $X = \text{Spec}B$ and $Y = \text{Spec}A$ are two affine schemes, then

$$Mor_{\mathfrak{Sch}}(X, Y) = Hom_{\mathfrak{Rings}}(A, B).$$

Proposition. *The category* $\mathfrak{AffSch}$ *of affine schemes is opposite to the category* $\mathfrak{Rings}$ *of rings.*

Open and closed subschemes.

An *open subscheme* of a scheme X is a scheme U such that the underlying topological space of U is an open subset of X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X \mid_U$ of the structure sheaf of X.

A morphism of schemes $f : X \longrightarrow Y$ is called an *open immersion* if f induces an isomorphism of X with an open subscheme of Y.

A morphism of schemes $f : X \longrightarrow Y$ is called a *closed immersion* if:
1) f induces a homeomorphism from the underlying topological space of X onto a closed subset of Y
2) the morphism of sheaves on Y, $f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is surjective.

A *closed subscheme* of a scheme Y is an equivalence class of closed immersions. ($f : X \longrightarrow Y$ is said to be equivalent to $f' : X' \longrightarrow Y$ if there is an isomorphism $i : X' \longrightarrow X$ such that $f' = f \circ i$.)

Example: Let A be a ring and $\mathfrak{a}$ be an ideal of A. Set $X = \text{Spec}A/\mathfrak{a}$ and $Y = \text{Spec}A$. The canonical projection $A \longrightarrow A/\mathfrak{a}$ induces a morphism of schemes $f : X \longrightarrow Y$ which is a closed immersion. Thus we obtain a structure of closed subscheme on $V(\mathfrak{a})$.

Fibre products of schemes.

Let S be a fixed scheme.

Theorem. *In the category $\mathfrak{Sch}/S$, the fibre products exist. That is, for two schemes X and Y over S, there is a scheme denoted by $X \times_S Y$, called the fibre product of X and Y over S, verifying the universal property of fibre products. Moreover, if X, Y and S are affine, say $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$ and $S = \mathrm{Spec}R$ (thus A and B are R-algebras), then*

$$X \times_S Y = \mathrm{Spec}(A \otimes_R B).$$

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